CLASSIFICATION OF THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS AND SYMPLECTIC SHEETS OF THE GEL'FAND–DIKII BRACKET

V. Yu. Ovsienko

Hill's equations $\psi''(x) + u(x)\psi = 0$ with a periodic potential u were classified for the first time in [1]. As it turned out later, this solved the problem of classification of orbits of a coadjoint representation of the Virasoro group, which was solved independently in [2, 3] (see also [4-6]). The authors of [7] classified the orbits of Lie superalgebras of the Neveu–Schwartz and Ramone types.

In this article we describe a relation between the classification of the symplectic sheets of the Gel'fand–Dikii bracket in the space of differential equations with periodic coefficients of the form

$$Ay = y'''(x) + u(x)\, y'(x) + v(x)\, y(x) = 0 \tag{1}$$

and calculations of homotopy classes of non-flattening curves on $S^2$. Our results are generalized in [8] to equations of higher orders.

1. A Tensor Interpretation of Third-Order Linear Differential Equations. In this paragraph we give a geometric interpretation of the second Gel'fand–Dikii bracket in the space of equations (1). Its centerpiece is the action of tensor fields of degree −2 on equations and locally convex curves in $\mathbb{R}P^2$.

1.1. Locally Convex Curves in $\mathbb{R}P^2$. To each ordinary linear differential equation there corresponds a non-flattening curve $\gamma(x)$ in the projective space. To every point x we assign a hyperspace in the space of solutions (and therefore, a straight line in its dual space) consisting of solutions which become zero at the point x. In the case of equations of the form (1) the curve $\gamma(x) \in \mathbb{R}P^2$ satisfies the following two properties:

1) it is quasiperiodic, i.e., $\gamma(x + 2\pi) = M\gamma(x)$, where the monodromy operator M defines a class of adjoint elements in the matrix group $SL(3, \mathbb{R})$ (which acts projectively on $\mathbb{R}P^2$), and

2) it is locally convex (does not contain points of inflection).

In turn, a nonflattening curve in the projective space uniquely defines a linear differential equation. This correspondence between curves and equations is described by multidimensional analogues of the Schwartz derivative. In addition, projectively equivalent curves correspond to the same equation. Let us carry out this calculation for Eq. (1). Suppose that the curve $\gamma(x) \in \mathbb{R}P^2$ can be written as $\gamma(x) = (f_1(x), f_2(x))$ in the affine chart. We lift it to a curve $r(x) = (f_1(x)\lambda(x), f_2(x)\lambda(x), \lambda(x))$ in $\mathbb{R}^3$ in such a way that the vector $r'''$ is a linear combination of vectors r and r'. The function $\lambda(x)$ is defined uniquely up to multiplication by scalars:

$$\lambda(x) = c \begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix}^{-1/3}. \tag{2}$$

The coordinates of the curve r(x) in $\mathbb{R}^3$, i.e., $\lambda$, $f_1\lambda$, $f_2\lambda$, are the solutions of the resulting equation (1).

1.2. Action of the Group $\mathrm{Diff}_+S^1$. Tensor Interpretation of Solutions. Define an action of the grup $\mathrm{Diff}_+S^1$ of diffeomorphisms of the circle which preserve its orientation on the space of equations (1) such that its action on the corresponding curves in $\mathbb{R}P^2$ is a change

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of variables: $g^*\gamma(x) = \gamma(g^{-1}(x))$. Eq. (2), which is used to determine the corresponding equation from a curve $\gamma(x)$, shows that under this action its solutions transform as follows:

$$g^*y(x) = y(g^{-1}(x))/(g^{-1}(x)'). \tag{3}$$

Thus, solutions of Eq. (1) can be regarded as a vector field on the line.

1.3. Action of the Space of Tensor Fields of Degree −2 on Equations and Curves in $\mathbb{R}P^2$. An action M of a tensor field $h = h(x)(dx)^{-2}$ on a certain geometrical object is a linear mapping which commutes with changes of variable and defines an infinitesimal deformation of these objects.

Proposition 1.1. An action of a tensor field $h = h(x)(dx)^{-2}$ on a locally convex $\gamma(x) = (f_1(x), f_2(x)$ in $\mathbb{R}P^2$ is defined by

$$V_h f_i = hf_i'' - (1/2)h'f_i' + vhf_i, \tag{4}$$

where $i = 1, 2$, $v = (\log\lambda)'/3 = (\log(f_1'f_2'' - f_2'f_1''))'/3$.

Proof: a) We first prove the commutability with the group of diffeomorphisms. A mapping $k \mapsto g(x)$ maps $h(x)$ to $h(g^{-1}(x))(g^{-1}(x)')^{-2}$, $g^*f_i(x) = f_i(g^{-1}(x))$. Moreover, v is mapped to $g^*v(x) = v(g^{-1}(x))g^{-1}(x)' + g''/g'$, and an expression $V_h f_i$ is transformed like a function: $g^*V_h f_i(x) = V_h f_i(g^{-1}(x))$.

b) It remains to show that Eq. (4) does not depend on the choice of the affine chart $(f_1, f_2)$. While doing so, we derive the formula of action of a tensor field of degree −2 on solutions of Eq. (1) corresponding to (4), which will be used later.

LEMMA 1.1. The tensor field $h(x)(dx)^{-2}$ acts on solutions of Eq. (1) as follows:

$$V_h y = hy'' - (1/2)h'y' + (1/6)h''y + (2/3)huy^{\dagger}, \tag{4'}$$

where $u = u(x)$ is the potential of Eq. (1).

Proof. Every basis in the space of solutions defines a curve $Y(x) = (y_1, y_2, y_3)$ in $\mathbb{R}^3$. When projecting onto $\mathbb{R}P^2$, i.e., $f_1 = y_1/y_3$, $f_2 = y_2/y_3$, Eq. (4) is replaced by (4'), and the coefficient $h''/6 + 2bu/3$ of y is uniquely determined from the conservation of the Wronskian $W_3(y_1, y_2, y_3)$ for a small deformation $y\varepsilon = y + \varepsilon V_h y$. The latter implies that the action of the tensor field h does not change the form of Eq. (1). The commutativity of the action of (4') with the change in the variable x follows from the same commutativity for Eq. (4). Q.E.D.

Now the invariance of (4) with respect to the choice of an affine chart follows from the fact that Eq. (4') is linear in y. This proves the proposition.

1.4. The Gel'fand–Dikii Bracket. Under the action of the group $\mathrm{Diff}_+S^1$ given by (3), the coefficient of u is transformed as follows: $g^*u = u(g^{-1}(x))(g^{-1}(x)')^2 + 2S(g)$, where $S(g) = g'''/g' - (3/2)(g''/g')^2$ is the Schwartz derivative. The coefficient of v is transformed in a rather complicated manner, but $\tilde{v} = v - u'/2$ has a third-order tensor density: 3: $g^*\tilde{v} = \tilde{v}(g^{-1}(x))(g^{-1}(x)')^3$. Therefore, the action of the tensor field $f = f(x)d/dx$ on the coefficients of Eq. (1) is given by $L_f u = fu' + 2uf' + 2f'''$, $L_f\tilde{v} = f\tilde{v}' + 3f'\tilde{v}$. Let $H_f^1$ and $H_h^2$ be linear functionals of the form $\int_{S^1} f(x)u(x)\,dx$, and $\int_{S^1} h(x)\tilde{v}(x)\,dx$, respectively.

LEMMA (Definition). An operator of the second Gel'fand–Dikii Hamiltonian structure is an operator which maps linear functionals $H_f^1$ and $H_h^2$ to Hamiltonian vector fields $L_f$ and $V_h$, respectively, in the space of equations (1).

Remark. The Gel'fand–Dikii bracket defined by this operator is quadratic. Linear functionals on the space of equations (1) generate an algebra with quadratic relations which is isomorphic to Zamolodchikov's algebra, which has independently appeared in the two-dimensional conformal field theory (see [9]).

§2. Solution of the Homological Equation. Recall that symplectic sheets of the Poisson bracket are submanifolds which are tangent to all Hamiltonian fields. The restriction to these

† This equation has been derived earlier from the explicit form of the Gel'fand–Dikii bracket in a dissertation by T. G. Khovanova.

brackets is nondegenerate and defines a symplectic structure. The Poisson manifold is fibered into symplectic sheets (one and only one sheet passes through each point) (see [10]). Two equations of the form (1) belong to the same symplectic sheet of the Gel'fand–Dikii bracket if they can be joined by a path whose tangent vector at each point can be obtained by an action of a vector field and a tensor field of degree −2 on the circle defined by Eqs. (3) and (4).

THEOREM 1. The monodromy operator and the homotopy class of equations (1) with respect to homotopies which preserve it are the only invariants of symplectic sheets of the second Gel'fand–Dikii bracket.

Proof. A vector field $f = f(x)d/dx$ and a tensor field $h = h(x)(d/dx)^2$ act on solutions of Eq. (1) as follows (see §1):

$$\dot{y}_{f,h} = yf' - fy' + hy'' - h'y'/2 + (h''/6 + 2hu/3)\,y, \tag{5}$$

The above formula is an explicit expression of an action defined in [11]. Because of the linearity of this action, it preserves the monodromy operator M. From the connectivity of symplectic sheets it is clear that, for a fixed monodromy operator, the homotopy class of Eq. (1) is also invariant.

To prove the theorem, we have to show that every infinitesimally small deformation of Eq. (1) preserving M can be obtained by action (5).

The Basic Formula. The tangent vector A to the space of equations (1) is obtained by the following action of a vector field $f(x)d/dx$ and a tensor field $h(x)(d/dx)^2$ on an equation $Ay = 0$:

$$f = -\frac{1}{2}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ \dot{y}_1' & \dot{y}_2' & \dot{y}_3' \end{vmatrix} + \frac{1}{2}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \end{vmatrix}, \quad h = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \end{vmatrix}, \tag{6}$$

where $y_1$, $y_2$, $y_3$ are linearly independent solutions of Eq. (1) normalized by a condition $W_3(y_1, y_2, y_3) = 1$ and $\dot{y}_1$, $\dot{y}_2$, $\dot{y}_3$ is their small deformation ($A\dot{y} + \dot{A}y = 0$) which preserves the Wronskian

$$(W_3(\dot{y}_1, y_2, y_3) + W_3(y_1, \dot{y}_2, y_3) + W_3(y_1, y_2, \dot{y}_3) = 0).$$

Remark 3.1. Mappings $(y, \dot{y}) \to h$ and $(y, \dot{y}) \to f$ defined by Eq. (6) are invariant differential operators (see [12]) on the line: $h\colon \bigwedge^3(F_1 \oplus F_1) \to F_2$, $f\colon \bigwedge^3(F_1 \oplus F_1) \to F_1$, where $F_k$ is the space of tensor fields of degree $-k$ on the line. This means that if $y_i$ and $\dot{y}_i$ transform as vector fields by diffeomorphisms then h and f transform as a tensor field of degree −2 and a vector field, respectively. The above statement is obvious for h, but it requires further proof in the case of f.

Proof of the Basic Formula: Consider a system of linear equations $\dot{y}_i = y_i''h + y_i'(-f - h'/2) + y_i(f' + h''/6 - 2uh/3)$ $(i = 1, 2, 3)$. Let X, Y, and Z be the coefficients of $y_i''$, $y_i'$, and $y_i$, respectively. Using Kramer's rule to solve this system (with a condition $W_3(y_1, y_2, y_3) = 1$, we obtain

$$X = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \end{vmatrix}, \quad Y = -\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \end{vmatrix}, \quad Z = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \end{vmatrix}.$$

From the first two equations follows Eq. (6). A condition $[W_3(y_1, y_2, y_3)]^{\cdot} = 0$ implies that the last equation for Z coincides with the expression for Z in terms of f and h. To see this, we differentiate each line of the Wronskian, obtaining

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ \dot{y}_1'' & \dot{y}_2'' & \dot{y}_3'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ \dot{y}_1' & \dot{y}_2' & \dot{y}_3' \end{vmatrix} - \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 \end{vmatrix}.$$

The rest is a straightforward calculation.

We note that if A is a small deformation preserving the monodromy operator of Eq. (1) then we can choose a deformation of solutions $\dot{y}_1$, $\dot{y}_2$, and $\dot{y}_3$ such that the monodromy matrix M remains the same with respect to the basis $y_i$. In that case, as a result of a translation by the period, functions f and h are multiplied by det M = 1, i.e., these functions are periodic. This concludes the proof of the theorem.

COROLLARY. In the complex case, when Eqs. (1) are defined on a Riemann surface of arbitrary type with deleted points, every infinitesimally small deformation of Eq. (1) with a constant monodromy group can be obtained by applying a certain holomorphic vector field and a holomorphic tensor field of degree −2.

§3. Versal Deformations. We define a versal deformation of Eq. (1) similarly to the versal deformation of a point of a manifold acted on by a Lie group (see [13]).

A deformation of Eq. (1) Ay = 0 is the germ of a smooth mapping D of a finite-dimensional manifold $\Lambda$ (the basis of the deformation) into the space of Eqs. (1) at the point 0 of $\Lambda$, where the mapping is such that $D(0) = \{Ay = 0\}$. Two deformations $D_1$ and $D_2$ of the same equation are called equivalent if there exists a homotopy $D_t$ between $D_1$ and $D_2$ such that the tangent vector $D_t(\lambda)$ to the space of Eqs. (1) can be obtained from a point $D_t(\lambda)$ by an action of a vector field $f(\lambda)$ and a tensor field $h(\lambda)$ of degree −2 on the circle, both of which depend smoothly on $\lambda \in \Lambda$. A deformation D' is said to be induced by D if it is obtained from D by a mapping of the basis onto some manifold $\Lambda'$ (the basis of D').

A deformation D of Eq. (1) is called versal if every deformation of this equation is equivalent to an induced one from D. Hereafter we consider only miniversal deformations (we assume that the dimension of the deformation basis is minimal).

The following result is a corollary of Eq. (6), as applied to the family of Eqs. (1) parametrized by $\lambda$.

THEOREM 2. A versal deformation of Eq. (1) is essentially a versal deformation of the corresponding class of adjoint elements of the group $SL(3, \mathbb{R})$ induced by the monodromy operator M. In particular, in the case where M = 1 the equation has an eight-dimensional versal deformation, while in general it has a two-dimensional one.

A similar theorem holds for Hill's equations (see [1]). In particular, the bifurcation diagram given in Fig. 1 in [1] is a well-known depiction of orbits of a coadjoint representation of the group $SL(2, \mathbb{R})$ (see, for example, [14]).

§4. Homotopy Classes of Locally Convex Curves on $\mathbb{R}P^2$ and Symplectic Sheets of the Gel'-fand–Dikii Bracket. In Sec. 1.1 we established a relation between Eqs. (1) and locally convex quasiperiodic curves in $\mathbb{R}P^2$. Theorem 1 implies that the symplectic sheets of the Gel'fand–Dikii bracket in the space of Eqs. (1) are identified with homotopy classes of these curves on $\mathbb{R}P^2$ with respect to homotopies for a fixed monodromy operator.

THEOREM 3. For a fixed monodromy operator there exist exactly three symplectic sheets of the second Gel'fand–Dikii on the space of Eqs. (1)

Proof: Up to an orientation, the homotopy classes of closed retractable locally convex curves in $\mathbb{R}P^2$ are in a one-to-one correspondence with the homotopy classes of these curves in $S^2$. Little calculated these in [15].

THEOREM (Little). A locally convex closed curve in $\mathbb{R}P^2$ is homotopic (up to an orientation) to one of the three curves pictured in Fig. 1.

Theorem 3 follows from Little's theorem, whose proof is lengthy and therefore omitted from this article. Figure 2 illustrates its most important, and at the same time its most beautiful argument: a homotopy between the curve 2) and a curve with four twists.

COROLLARY 1. Nonoscillating Eqs. (1) with periodic solutions form an isolated symplectic sheet of the second Gel'-fand–Dikii bracket.

Proof. Curves which are homotopic to the curve 1) pictured in Fig. 1 have the distinguishing property that they intersect every equator no more than twice. This means that a curve in $\mathbb{R}^3$ which is a lifting of this curve intersects with every hyperplane containing the origin no more than twice. Solutions of the equation of the form (1) which correspond to a given curve are restrictions of linear functionals on $\mathbb{R}^3$ to the curve obtained by a lifting to $\mathbb{R}^3$, so none of them are equal to zero more than twice, i.e., the obtained equation is nonoscillating.

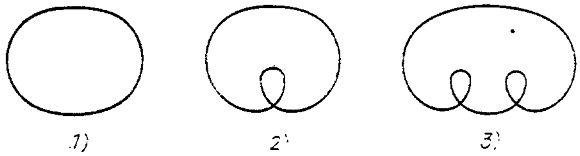

Fig. 1

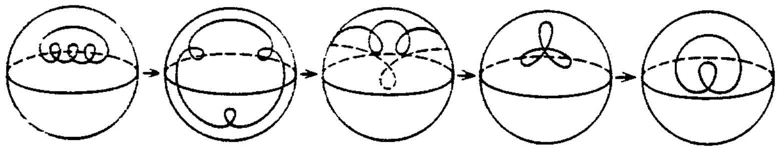

Fig. 2

This symplectic sheet can be of most use in the conformal field theory with additional symmetries (see [9]). Witten [6] quantized a similar orbit of the coadjoint representation in the case of the Virasoro algebra.

COROLLARY 2. A normal form of Eq. (1) with a fixed monodromy operator has the form

$$y'''(x) + n^2 y'(x) = 0, \text{ где } n = 1,\ 2,\ 3.$$

Indeed, the corresponding curves are homotopic to curves 1), 2), and 3), respectively.

Remark. The problem of classification of locally convex quasiperiodic curves in $S^2$ with an arbitrary monodromy operator has been solved recently by Khesin and Shapiro (see [8]), which completed the classification of symplectic sheets of the second Gel'fand–Dikii bracket in the space of equations (1). It turns out that for a general monodromy operator there are also exactly three homotopy classes.

§5. Legendre's Involution. A geometric interpretation of linear differential equations as locally convex curves in $\mathbb{R}P^r$ suggests the possibility of applying Legendre's transformations defined on such curves to this space of equations. We carry this out for Eqs. (1).

Definition. Recall that the space $\mathbb{R}P^{2*}$, which is the dual space to $\mathbb{R}P^2$, is the set of lines in $\mathbb{R}P^2$. Legendre's transformation of a smooth locally convex curve in $\mathbb{R}P^2$ maps every point to a tangent at that point. The obtained curve in $\mathbb{R}P^{2*}$ is smooth and locally convex. Legendre's transformation is involutive, i.e., its repeated application is the identity transformation.

The projective space $\mathbb{R}P^2$ can be identified with $\mathbb{R}P^{2*}$. The various methods of identification differ by the projective map. In this way, Legendre's transformation identifies a curve in $\mathbb{R}P^2$ with a class of equivalent curves in $\mathbb{R}P^2$ and therefore defines an involution on the space of Eqs. (1).

Proposition 5.1. Legendre's transformation of Eq. (1) transforms its coefficients (u, v) into (u, u' − v), and every solution of the resulting equation is a commutator of two solutions of the original one treated as vector fields on the line: $[y_1, y_2] = y_1'y_2 - y_2'y_1$.

Proof: For convenience, identify $\mathbb{R}P^2$ with $\mathbb{R}P^{2*}$ using the Euclidean structure of $\mathbb{R}^3$. Lift curves in $\mathbb{R}P^2$ to $\mathbb{R}^3$ in such a way that the coordinates of the lifted curve satisfy a condition $W_3(y_1, y_2, y_3) = 1$ (see Sec. 1.1). Then the lifted curve, which is Legendre dual, is equal to $y \times y'$, the vector product of the curve by its velocity. Indeed, the dual curve in $\mathbb{R}P^2$ is orthogonal to the radius–vector of the curve and its velocity vector. It uniquely lifts to $\mathbb{R}^3$, and it is easy to see that the coordinates of the curve $y \times y$ satisfy a condition $W_3(y \times y')_1, (y \times y')_2, (y \times y')_3) = 1$. The coordinates of the curve $y \times y'$ have the form $([y_1, y_2], [y_3, y_1], [y_2, y_3])$, from which we deduce the second statement of the proposition. The first one is easy to check.

Remark 5.1. Legendre's transformation of Eq. (1) commutes with the action of the group of diffeomorphisms of $S^1$, and tensor fields of degree −2 invert. Indeed, they transform $\tilde{v} = v - u'2$ into $-\tilde{v}$, and therefore the Hamiltonian $V_h = \int h\tilde{v}$ into $-V_h$.

Consider the stationary points of Legendre's transformation. They are Eqs. (1) such that $u' - v = v$. They are given by a skew-self-adjoint operator $A_u = d^3/dx^3 + ud/dx + u'/2$, which satisfies the following remarkable properties:

a) $A_u$ is the Hamiltonian operator of the second Hamiltonian Gel'fand–Dikii structure (for the Korteweg–de Vries equation) defined on Hill's equations;

b) $A_u$ is the operator of a coadjoint representation of the Virasoro algebra;

c) Proposition 5.1 implies that solutions of an equation $A_u y = 0$ form a Lie algebra which is equal to $sl(2, \mathbb{R})$; these solutions (see [4]), together with solutions of Hill's equation $(-2d^2/dx^2 + u)\psi = 0$ form a Lie superalgebra $osp(1/2)$ with respect to a commutator $[y, \psi] = y\psi' - \psi y'/2$, $[\psi_1, \psi_2]_+ = \psi_1 \cdot \psi_2$.

Therefore, solutions of the equation for $A_u y$ can be written in terms of solutions of Hill's equation: $y_1 = \psi_1^2$, $y_2 = \psi_2^2$, $y_3 = \psi_1\psi_2$.

Remark 5.2. The operator $A_u$ induces a natural (commuting with the action of the group of diffeomorphisms) imbedding of the space of Hill's equations into Eqs. (1). Here nonequivalent equations can become equivalent. Consider, for example, Hill's equations with monodromy operators $\pm 1$. Their normal form is $-2d^2\psi/dx^2 + (n^2/2)\psi = 0$, where $n = 1, 2, 3, \ldots$ . After imbedding into Eqs. (1) they become equivalent for both even and odd $n > 1$.

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