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QUASIPERIODIC MOTION FOR THE PENTAGRAM MAP

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ABSTRACT. The pentagram map is a projectively natural iteration defined on polygons, and also on a generalized notion of a polygon which we call *twisted polygons*. In this note we describe our recent work on the pentagram map, in which we find a Poisson structure on the space of twisted polygons and show that the pentagram map relative to this Poisson structure is completely integrable in the sense of Arnold-Liouville. For certain families of twisted polygons, such as those we call *universally convex*, we translate the integrability into a statement about the quasi-periodic motion of the pentagram-map orbits. We also explain how the continuous limit of the pentagram map is the classical Boussinesq equation, a completely integrable P.D.E.

1. Introduction and main results

The pentagram map, T, is a natural operation one can perform on polygons. This map was considered for pentagons as early as 1945 – see [3]. In general, see [5], [6] and [4]. Though this map can be defined for an essentially arbitrary polygon over an essentially arbitrary field, it is easiest to describe the map for convex polygons contained in $\boldsymbol{R}^2$. Given such an n-gon P, the corresponding n-gon $T(P)$ is the convex hull of the intersection points of consecutive shortest diagonals of P. Figure 1 shows two examples.

Thinking of $\boldsymbol{R}^2$ as a natural subset of the projective plane $\boldsymbol{RP}^2$, we observe that the pentagram map commutes with projective transformations. That is, $\phi(T(P)) = T(\phi(P))$, for any $\phi \in \mathrm{PGL}(3, \boldsymbol{R})$. Let $\mathcal{C}_n$ be the space of *convex n-gons modulo projective transformations*. The pentagram map induces a self-diffeomorphism $T : \mathcal{C}_n \to \mathcal{C}_n$. T is the identity map on $\mathcal{C}_5$ and an involution on $\mathcal{C}_6$, cf. [5]. For $n \geq 7$, the map T exhibits quasi-periodic properties. Experimentally, the orbits of T on $\mathcal{C}_n$ exhibit the kind of quasiperiodic motion associated to a *completely integrable system*. Indeed, as we will discuss below, the continuous limit of the pentagram map is the classical Boussinesq equation, a completely integrable P.D.E. A conjecture [6] that T is completely integrable on $\mathcal{C}_n$ is still open. However, our recent paper [4] very nearly proves this result.

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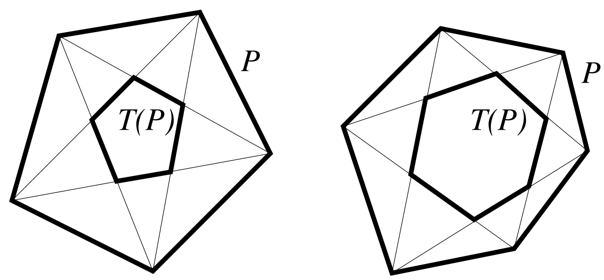

FIGURE 1. The pentagram map defined on a pentagon and a hexagon

Rather than work directly with $\mathcal{C}_n$, we work with a slightly larger space. A *twisted n-gon* is a map $\phi : \boldsymbol{Z} \to \boldsymbol{RP}^2$ such that

$$\phi(n+k) = M \circ \phi(k); \qquad \forall k \in \boldsymbol{Z},$$

for some fixed element $M \in \mathrm{PGL}(3, \boldsymbol{R})$ called the monodromy. We let $v_i = \phi(i)$ and assume that v_{i-1}, v_i, v_{i+1} are *in general position* for all i. We denote by $\mathcal{P}_n$ the space of twisted n-gons modulo projective equivalence.

Let $\mathcal{A}_n$ denote the algebra of smooth functions on $\mathcal{P}_n$. A *Poisson bracket* is an antisymmetric and bilinear map $\{\cdot,\cdot\} : \mathcal{A}_n \times \mathcal{A}_n \to \mathcal{A}_n$ that satisfies both the Leibniz and Jacobi identites:

$$(1) \qquad \{f, g_1 g_2\} = g_1\{f, g_2\} + g_2\{f, g_1\}; \qquad \sum \{f_1, \{f_2, f_3\}\} = 0.$$

The second sum is a cyclic sum. Two functions f and g are said to *Poisson commute* if $\{f, g\} = 0$. In the presence of a Poisson bracket, the space $\mathcal{P}_n$ is (generically) foliated into symplectic leaves. The *co-rank* of the bracket is the dimension of a generic leaf. Here is our main algebraic result.

Theorem 1.1. *The space $\mathcal{P}_n$ admits an invariant Poisson structure having co-rank 2 when n is odd and co-rank 4 when n is even. At the same time, there exists $2\lfloor n/2 \rfloor + 2$ algebraically independent functions that are invariant under the pentagram map and commute with the Poisson structure. In particular, the pentagram map is a completely integrable system on $\mathcal{P}_n$, in the sense of Arnold-Liouville.*

Here $\lfloor n/2 \rfloor$ denotes the floor of $n/2$. Below, we will define both the Poisson structure and the invariants. The space $\mathcal{C}_n$ is naturally a subspace of $\mathcal{P}_n$, and our algebraic results say something about the action of the pentagram map on $\mathcal{C}_n$, but not quite enough for us to get the complete integrability on $\mathcal{C}_n$. To get a crisp geometric result, we work with a related space, which we describe next.

We say that a twisted n-polygon is *universally convex* if the map ϕ is such that $\phi(\boldsymbol{Z}) \subset \boldsymbol{R}^2 \subset \boldsymbol{RP}^2$ is convex and contained in the positive quadrant. We also require that the monodromy $M : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a linear transformation having the form

$$(2) \qquad M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}; \qquad a < 1 < b.$$

The image of ϕ looks somewhat like a "polygonal hyperbola". We say that two universally convex twisted n-gons ϕ_1 and ϕ_2 are *equivalent* if there is a positive diagonal matrix μ such that $\mu \circ \phi_1 = \phi_2$. Let $\mathcal{U}_n$ denote the space of universally convex twisted n-gons modulo equivalence. It turns out that $\mathcal{U}_n$ is a pentagram-invariant and open subset of $\mathcal{P}_n$. Here is our main geometric result.

Theorem 1.2. *Almost every pont of $\mathcal{U}_n$ lies on a smooth torus that has a T-invariant affine structure. Hence, the orbit of almost every universally convex n-gon undergoes quasi-periodic motion under the pentagram map.*

2. Ingredients in the proofs

In this section we will sketch the main ideas in the proofs of Theorem 1.1 and 1.2. We refer the reader to [4] for more results and details.

2.1. **Coordinates.** Recall that the *cross ratio* of 4 collinear points in $\boldsymbol{RP}^2$ is given by

$$[t_1, t_2, t_3, t_4] = \frac{(t_1 - t_2)\,(t_3 - t_4)}{(t_1 - t_3)\,(t_2 - t_4)},$$

where t is an (arbitrary) affine parameter. We use the cross ratio to construct coordinates on the space of twisted polygons. We associate to every vertex v_i two numbers:

$$x_i = [v_{i-2},\, v_{i-1},\, ((v_{i-2}, v_{i-1}) \cap (v_i, v_{i+1}))\,,\, ((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2}))]$$

$$y_i = [((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2}))\,,\, ((v_{i-1}, v_i) \cap (v_{i+1}, v_{i+2}))\,,\, v_{i+1},\, v_{i+2}]$$

called the left and right corner cross-ratios, see Figure 2. We call our coordinates the *corner invariants*.

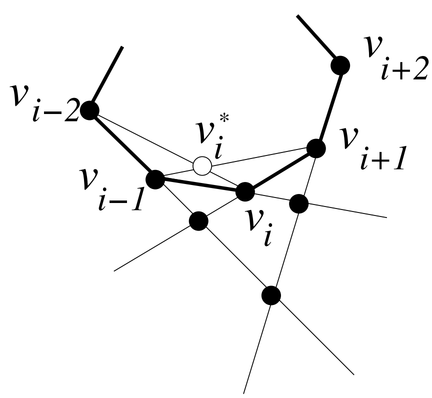

Figure 2. Points involved in the constructions

This construction is invariant under projective transformations, and thus gives us coordinates on the space $\mathcal{P}_n$. At generic points, $\mathcal{P}_n$ is locally diffeomorphic to $\boldsymbol{R}^{2n}$.

We will work with generic elements of $\mathcal{P}_n$, so that all constructions are well-defined. Let $\phi^* = T(\phi)$ be the image of ϕ under the pentagram map. The points of $T(\phi)$ are given by v_1^*, v_2^*, etc., as indicated in Figure 2.

In coordinates, the pentagram map has the following form.

$$T^* x_i = x_i \,\frac{1 - x_{i-1}\, y_{i-1}}{1 - x_{i+1}\, y_{i+1}}, \qquad T^* y_i = y_{i+1} \,\frac{1 - x_{i+2}\, y_{i+2}}{1 - x_i\, y_i}. \tag{3}$$

2.2. **The invariants.** From the formula in Equation 3, one sees rather easily the functions

$$O_n = \prod_{i=1}^{n} x_i; \qquad E_n = \prod_{i=1}^{n} y_i \tag{4}$$

are invariants of the pentagram map for any n, and the functions

$$O_{n/2} = \prod_{i \text{ even}} x_i + \prod_{i \text{ odd}} x_i; \qquad E_{n/2} = \prod_{i \text{ even}} y_i + \prod_{i \text{ odd}} y_i. \tag{5}$$

are invariants of the pentagram map when n is even. In all cases, the products in this last equation run from 1 to n.

Let M be the monodromy of ϕ. We lift M to an element of $\mathrm{GL}_3(\boldsymbol{R})$. By slightly abusing notation, we also denote this matrix by M. The two quantities

$$\Omega_1 = \frac{\mathrm{trace}^3(M)}{\det(M)}; \qquad \Omega_2 = \frac{\mathrm{trace}^3(M^{-1})}{\det(M^{-1})};$$

are only dependent on the conjugacy class of M.

We define

$$\widetilde{\Omega}_1 = O_n^2 E_n \Omega_1; \qquad \widetilde{\Omega}_2 = O_n E_n^2 \Omega_2.$$

In [6] (and again in [4]) it is shown that $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are polynomials in the corner invariants. Since the pentagram map preserves the monodromy, and O_n and E_n are invariants, the two functions $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are also invariants. One immediate consequence of Equation 3 is that the pentagram map commutes with the following *scaling operation*.

$$R_t : \quad (x_1, y_1, ..., x_n, y_n) \to (tx_1, t^{-1}y_1, ..., tx_n, t^{-1}y_n). \tag{6}$$

We say that a polynomial in the corner invariants has *weight* k if

$$R_t^*(P) = t^k P.$$

For instance, O_n has weight n and E_n has weight $-n$. In [6] it shown that

$$\widetilde{\Omega}_1 = \sum_{k=1}^{[n/2]} O_k; \qquad \widetilde{\Omega}_2 = \sum_{k=1}^{[n/2]} E_k$$

where O_k has weight k and E_k has weight $-k$. Since the pentagram map commutes with the rescaling operation and preserves $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$, it also preserves their "weighted homogeneous parts". That is, the functions $O_1, E_1, O_2, E_2, ...$ are also invariants of the pentagram map. These are the monodromy invariants. They are all nontrivial polynomials. In [6] it is shown that the monodromy invariants are algebraically independent.

The explicit formulas for the monodromy invariants was obtained in [6]. Introduce the monomials

$$X_i := x_i \, y_i \, x_{i+1}.$$

(1) We call X_i and X_j consecutive if $j \in \{i-2,\, i-1,\, i,\, i+1,\, i+2\}$;
(2) we call X_i and x_j consecutive if $j \in \{i-1,\, i,\, i+1,\, i+2\}$;
(3) we call x_i and x_{i+1} consecutive.

Let $O(X,x)$ be a monomial obtained by the product of the monomials X_i and x_j, i.e.,

$$O = X_{i_1} \cdots X_{i_s}\, x_{j_1} \cdots x_{j_t}.$$

Such a monomial is called admissible if none of the indices are consecutive. For every admissible monomial, we define the weight $|O| = s+t$ and the sign $\mathrm{sign}(O) = (-1)^t$. One then has

$$O_k = \sum_{|O|=k} \mathrm{sign}(O)\, O; \qquad k \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}.$$

The same formula works for E_k, if we make all the same definitions with x and y interchanged.

2.3. The Poisson bracket.

In [4] we introduce the Poisson bracket on $C^\infty(\mathcal{P}_n)$. For the coordinate functions we set

$$\{x_i, x_{i\pm1}\} = \mp x_i\, x_{i+1}, \qquad \{y_i, y_{i\pm1}\} = \pm y_i\, y_{i+1} \tag{7}$$

and all other brackets vanish. Once we have the definition on the coordinate functions, we use linearity and the Leibniz rule to extend to all rational functions. An easy exercise shows that our bracket satisfies all the axioms.

A function f is said to be a *Casimir* (relative to the Poisson structure) if f Poisson commutes with all other functions. We have already mentioned that a Poisson structure induces a (generic) foliation of a manifold by symplectic leaves. These symplectic leaves can be locally described as levels $f_i = \mathrm{const}$ of the Casimir functions.

The main lemmas of [4] concerning our Poisson bracket are as follows.

1. The Poisson bracket (7) is invariant with respect to the pentagram map.
2. The monodromy invariants Poisson commute.
3. The invariants in Equations (4) and (in the even case) (5) are Casimirs.
4. The Poisson bracket has corank 2 if n if odd and corank 4 if n is even.

A dimension count shows that, after we exclude the Casimirs, the number of independent invariants – e.g. $n-1$ in the odd case – coincides with the dimension of the symplectic leaves defined by the Poisson bracket. This gives the complete integrability of Theorem 1.1.

Remarks: (i) We think that perhaps there is another invariant Poisson structure on $\mathcal{P}_n$, compatible with the one we have defined. The existence of a second structure would allow us to bring bi-Hamiltonian techniques to bear on the analysis of the pentagram map.
(ii) The space $\mathcal{P}_n$ is naturally a cluster manifold, and our Poisson bracket bears a striking resemblence to the canonical Poisson bracked defined on cluster manfolds [2]. It would be nice to work out this connection in detail.

2.4. The universally convex case.

Now we specialize our algebraic result to the space $\mathcal{U}_n$ of universally convex twisted n-gons. We check that $\mathcal{U}_n$ is an open and invariant subset of $\mathcal{P}_n$. The invariance is pretty clear. The openness result derives from three facts.

1. Local convexity is stable under perturbation.
2. The linear transformations in Equation 2 extend to projective transformations whose type is stable under small perturbations.

(3) A locally convex twisted polygon that has the kind of hyperbolic monodromy given in Equation 2 is actually globally convex.

As a final ingredient in our proof, we show that the leaves of $\mathcal{U}_n$, namely the level sets of the monodromy invariants, are compact. We don't need to consider all the invariants; we just show in a direct way that the level sets of E_n and O_n together are compact.

The rest of the proof is the usual application of Sard's theorem and the definition of integrability. We explain the main idea in the odd case. The space $\mathcal{U}_n$ is locally diffeomorphic to $\boldsymbol{R}^{2n}$, and foliated by leaves which generically are smooth compact symplectic manifolds of dimension $2n-2$. A generic point in a generic leaf lies on an $(n-1)$-dimensional smooth compact manifold, the level set of our monodromy invariants. On a generic leaf, the symplectic gradients of the monodromy functions are linearly independent at each point of the leaf.

The $n-1$ symplectic gradients of the monodrony invariants give a natural basis of the tangent space at each point of our generic leaf. This basis is invariant under the pentagram map, and also under the Hamiltonian flows determined by the invariants. This gives us a smooth compact $(n-1)$-manifold, admitting $n-1$ commuting flows that preserve a natural affine structure. From here, we see that the leaf must be a torus. The pentagram map preserves the canonical basis of the torus at each point, and hence acts as a translation. This is the quasi-periodic motion of Theorem 1.2.

3. Connection to the Boussinesq equation

The classical Boussinesq equation is one of the best known infinite-dimensional integrable systems. We will explain below how the Boussinesq equation is the continuous limit of the pentagram map. This limit was noted in [6] and used systematically in [4]. We discovered the Poisson structure on $\mathcal{P}_n$ by comparing the pentagram map (in suitable coordinates) and the Boussinesq equation.

Discretization of the Boussinesq equation is an interesting and well-studied subject, see [7] and references therein. However, the known versions of the discrete Boussinesq equation lack geometric interpretation.

3.1. Difference equations and global coordinates.

To highlight the analogy between the Poisson structure on $\mathcal{P}_n$ and the Poisson structure associated to the Boussinesq equation, we first discuss a second coordinate system for the space $\mathcal{P}_n$. For technical reasons we assume throughout this section that $n \neq 3\,m$.

Let $\{v_i\}$ be the vertices of a twisted polygon. We think of these vertices as points in $\boldsymbol{RP}^2$. Now, we choose lifts $V_i \in \boldsymbol{R}^3$ of these vertices so that we have $\det(V_i, V_{i+1}, V_{i+2}) = 1$. Then

$$V_{i+3} = a_i\, V_{i+2} + b_i\, V_{i+1} + V_i \tag{8}$$

where a_i, b_i are n-periodic sequences of real numbers. Conversely, given two arbitrary n-periodic sequences (a_i) and (b_i), the difference equation (8) determines a projective equivalence class of a twisted polygon. This provides a global coordinate system $(a_i,\, b_i)$ on the space of twisted n-gons. If $n \neq 3m$, then $\mathcal{P}_n$ is isomorphic to $\boldsymbol{R}^{2n}$. When $n = 3m$, the topology is trickier so we ignore this case.

The relation between the coordinates here and the ones in §2 is as follows.

$$x_i = \frac{a_{i-2}}{b_{i-2}\, b_{i-1}}, \quad y_i = -\frac{b_{i-1}}{a_{i-2}\, a_{i-1}}.$$

Here is the formula for the pentagram map in the new coordinates, when $m \neq 3m$.

(9)
$$T^* a_i = a_{i+2} \prod_{k=1}^{m} \frac{1 + a_{i+3k+2}\, b_{i+3k+1}}{1 + a_{i-3k+2}\, b_{i-3k+1}}, \qquad T^* b_i = b_{i-1} \prod_{k=1}^{m} \frac{1 + a_{i-3k-2}\, b_{i-3k-1}}{1 + a_{i+3k-2}\, b_{i+3k-1}};$$

The monodromy invariants also have explicit formulas in these coordinates. See [4]. Finally, here is the formula for the Poisson structure in the new coordinates.

(10)
$$\begin{aligned} \{a_i, a_j\} &= \sum_{k=1}^{m} (\delta_{i,j+3k} - \delta_{i,j-3k})\, a_i\, a_j, \\ \{a_i, b_j\} &= 0, \\ \{b_i, b_j\} &= \sum_{k=1}^{m} (\delta_{i,j-3k} - \delta_{i,j+3k})\, b_i\, b_j. \end{aligned}$$

3.2. **The continuous limit.** We understand the $n \to \infty$ continuous limit of a twisted n-gon as a smooth parametrized curve $\gamma : \boldsymbol{R} \to \boldsymbol{RP}^2$ with monodromy:

$$\gamma(x+1) = M(\gamma(x)),$$

for all $x \in \boldsymbol{R}$, where $M \in \mathrm{PGL}(3, \boldsymbol{R})$ is fixed. The assumption that every three consequtive points are in general position corresponds to the assumption that the vectors $\gamma'(x)$ and $\gamma''(x)$ are linearly independent for all $x \in \boldsymbol{R}$. A curve γ satisfying these conditions is usually called *non-degenerate*. As in the discrete case, we consider classes of projectively equivalent curves.

The space of non-degenerate curves is very well known in classical projective differential geometry. There exists a one-to-one correspondence between this space and the space of linear differential operators on $\boldsymbol{R}$:

$$A = \left(\frac{d}{dx}\right)^3 + \frac{1}{2}\left(u(x)\,\frac{d}{dx} + \frac{d}{dx}\,u(x)\right) + w(x),$$

where u and w are smooth periodic functions.

We are looking for a continuous analog of the map T. The construction is as follows. Given a non-degenerate curve $\gamma(x)$, at each point x we draw the chord $(\gamma(x-\varepsilon), \gamma(x+\varepsilon))$ and obtain a new curve, $\gamma_\varepsilon(x)$, as the envelop of these chords, see Figure 3. Let u_ε and w_ε be the respective periodic functions. It turns out that

$$u_\varepsilon = u + \varepsilon^2 \widetilde{u} + (\varepsilon^3), \quad w_\varepsilon = w + \varepsilon^2 \widetilde{w} + (\varepsilon^3),$$

giving the curve flow: $\dot{u} = \widetilde{u},\ \dot{w} = \widetilde{w}$. We show that

$$\dot{u} = w', \quad \dot{w} = -\frac{u\,u'}{3} - \frac{u'''}{12},$$

or

$$\ddot{u} + \frac{\left(u^2\right)''}{6} + \frac{u^{(IV)}}{12} = 0,$$

which is nothing else but the classical Boussinesq equation.

Consider the space of functionals of the form

$$H(u,w) = \int_{S^1} h(u, u', \ldots, w, w', \ldots)\, dx,$$

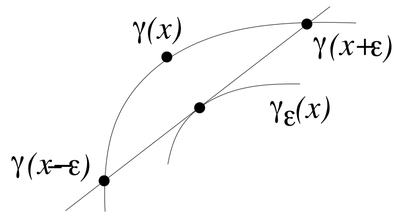

FIGURE 3. Evolution of a non-degenerate curve

where h is a polynomial. The *first Poisson bracket* on the above space of functionals is defined by

$$\{G, H\} = \int_{S^1} \left(\delta_u G\, (\delta_w H)' + \delta_w G\, (\delta_u H)'\right) dx, \tag{11}$$

where $\delta_u H$ and $\delta_w H$ are the standard variational derivatives. The Poisson bracket (10) is a discrete version of the bracket (11).

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