WHAT IS... the Schwarzian Derivative?

Valentin Ovsienko and Sergei Tabachnikov

Almost every mathematician has encountered, at some point of his or her education, the following rather intimidating expression and, most likely, tried to forget it right away:

$$S\left(f(x)\right) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{1}$$

Here $f(x)$ is a function in one (real or complex) variable and $f'(x), f''(x), \ldots$ are its derivatives. This is the celebrated Schwarzian derivative, or the *Schwarzian*, for short. It was discovered by Lagrange in his treatise "Sur la construction des cartes géographiques" (1781); the Schwarzian also appeared in a paper by Kummer (1836), and it was named after Schwarz by Cayley.

Expression (1) is ubiquitous and tends to appear in seemingly unrelated fields of mathematics: classical complex analysis, differential equations, and one-dimensional dynamics, as well as, more recently, Teichmüller theory, integrable systems, and conformal field theory. Leaving these numerous applications aside, we focus on the basic properties of the Schwarzian itself.

Two examples. a) The first example is perhaps the oldest one. Consider the simplest second-order differential equation, the Sturm-Liouville equation,

$$\varphi''(x) + u(x)\,\varphi(x) = 0 \tag{2}$$

where the potential $u(x)$ is a (real or complex valued) smooth function. The space of solutions is two-dimensional and spanned by any two linearly independent solutions, φ_1 and φ_2. Suppose that we know the quotient $f(x) = \varphi_1(x)/\varphi_2(x)$; can one reconstruct the potential? The reader can carry out the relevant computations to check that $u = \frac{1}{2}S(f)$. The geometrical meaning of this formula is as follows. The quotient $t = \varphi_1/\varphi_2$ is an *affine coordinate* on the projective line $\mathbb{P}^1$ so that $t = f(x)$ is a parametrized curve in $\mathbb{P}^1$. This curve has non-vanishing speed, i.e., $f' \neq 0$, since the Wronski determinant of two solutions of (2) is a non-zero constant. The Schwarzian then reconstructs a Sturm-Liouville equation from such a curve.

b) The next example is due to C. Duval, L. Guieu, and the first author (2000). Consider the Lorentz plane with the metric $g = dxdy$ and a curve $y = f(x)$. If $f'(x) > 0$, then its Lorentz curvature can be easily computed: $\varrho(x) = f''(x)\,(f'(x))^{-3/2}$, and the Schwarzian enters the game when one computes $\varrho' = S(f)/\sqrt{f'}$. Thus, informally speaking, *the Schwarzian derivative is curvature.*

The following beautiful theorem of E. Ghys (1995) is a manifestation of this principle: *for an arbitrary diffeomorphism f of the real projective line, its Schwarzian derivative $S(f)$ vanishes at least at 4 distinct points.* Ghys' theorem is analogous to the classical 4 vertex theorem of Mukhopadhyaya (1909): *the Euclidean curvature of a smooth closed convex curve in $\mathbb{R}^2$ has at least 4 distinct extrema.*

Not a function. A surprise hidden in formula (1) is that the Schwarzian is actually not a function. The difference between a function and a more complicated tensor field is in its behavior

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under coordinate changes. Choose another coordinate y. What is the formula for $S(f)(y)$? For a function the answer is simply $f(y) = f(x(y))$, but for its derivative, due to the Chain Rule, it is different: $f'(y) = f'(x(y))\, x'(y)$; this is why the invariant (geometric) quantity is not the derivative but the differential $df = f'(x)\, dx$—a distinction that tortures countless calculus students. The geometric quantity corresponding to (1) is the quadratic differential: $S(f) = S\,(f(x))\,(dx)^2$. In other words, $S(f)$ is a quadratic function on the tangent space $T\mathbb{R}$, just like a metric but without the non-vanishing condition. We denote by $\mathcal{Q}_2(\mathbb{RP}^1)$ the space of the quadratic differentials on the real projective line $\mathbb{RP}^1$.

Main properties. 1. $S(f) = S(g)$ if and only if $g(x) = (af(x) + b)/(cf(x) + d)$, where a, b, c, d are (real or complex) constants with $ad - bc \neq 0$. In particular, $S(f) = 0$ if and only if f is a linear-fractional transformation:

$$f(x) = \frac{ax + b}{cx + d}. \tag{3}$$

Note that $f(-d/c) = \infty$, but $f(x)$ is well-defined for $x = \infty$. We can understand f as a *diffeomorphism* of $\mathbb{RP}^1$. The transformations (3) with *real* coefficients form a group of projective symmetries of $\mathbb{RP}^1$, which is $\mathrm{SL}(2, \mathbb{R})/\{\pm 1\}$. It follows that the Schwarzian is a *projective invariant.*

2. Given two diffeomorphisms f, g of $\mathbb{RP}^1$, one has: $S\,(g \circ f) = S(g) \circ f + S(f)$, where the first summand is the action of f on a quadratic differential, $(u \circ f)(x) = u(f(x))\,(f'(x))^2$.

From discrete projective invariants to differential ones. The reader may be familiar with projective invariants; see F. Labourie's article [1]. Recall that a quadruple of points in $\mathbb{P}^1$ has a numerical invariant. Choose an affine coordinate that represents the points by four numbers t_1, t_2, t_3, and t_4; the *cross-ratio*

$$[t_1, t_2, t_3, t_4] = \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_2)(t_3 - t_4)}$$

is invariant under the projective transformations of the projective line. What is the relation of this discrete invariant to the Schwarzian?

Consider a diffeomorphism f of $\mathbb{RP}^1$. The Schwarzian measures how f changes the cross-ratio of infinitesimally close points. Let t be a point in $\mathbb{RP}^1$ and v a tangent vector to $\mathbb{RP}^1$ at t. Extend v to a vector field in a vicinity of t and denote by ϕ_s the corresponding local flow. Consider four points: $t, t_1 = \phi_\varepsilon(t), t_2 = \phi_{2\varepsilon}(t), t_3 = \phi_{3\varepsilon}(t)$. The cross-ratio does not change in the first order in ε:

$$[f(t), f(t_1), f(t_2), f(t_3)] =$$
$$[t, t_1, t_2, t_3] - 2\varepsilon^2 S(f)(t) + O(\varepsilon^3).$$

The coefficient of ε^2 depends on the diffeomorphism f, the point t, and the tangent vector v, but not on its extension to a vector field. It is homogeneous of degree 2 in v, and therefore $S(f)$ is indeed a quadratic differential.

Schwarzian as a cocycle. Let G be a group acting on a vector space V, i.e., there is a homomorphism $\rho : G \to \mathrm{End}(V)$. A map $c : G \to V$ is a 1-cocycle on G with coefficients in V if

$$\widetilde{\rho}_g : (v, \lambda) \mapsto (\rho_g v + \lambda\, c(g),\, \lambda)$$

is again a G-action on $V \oplus \mathbb{R}$. The cocycle c is *nontrivial* if this action is not isomorphic to that with $c = 0$. In this case, c defines a class of cohomology of G, the notion that plays a fundamental role in geometry, algebra, and topology.

Diffeomorphisms of $\mathbb{RP}^1$ form an infinite-dimensional group, $\mathrm{Diff}(\mathbb{RP}^1)$, which acts on all tensor fields on $\mathbb{RP}^1$. Property 2 means precisely that the Schwarzian is a 1-cocycle with coefficients in the space of quadratic differentials $\mathcal{Q}_2(\mathbb{RP}^1)$. Moreover, one can prove uniqueness: *the Schwarzian is the only projectively invariant 1-cocycle on* $\mathrm{Diff}(\mathbb{RP}^1)$. This serves as a good intrinsic definition.

The algebra of vector fields, $\mathrm{Vect}(\mathbb{RP}^1)$, is the Lie algebra of the group $\mathrm{Diff}(\mathbb{RP}^1)$. Every differentiable map on $\mathrm{Diff}(\mathbb{RP}^1)$ corresponds to a map on $\mathrm{Vect}(\mathbb{RP}^1)$, its infinitesimal version. The infinitesimal version of the Schwarzian is easy to compute (substitute $f(x) = x + \varepsilon\, X(x)$ in (1) and differentiate with respect to ε at $\varepsilon = 0$): $s(X(x)\, d/dx) = X'''(x)\,(dx)^2$. This is a projectively invariant 1-cocycle on $\mathrm{Vect}(\mathbb{RP}^1)$ with coefficients in $\mathcal{Q}_2(\mathbb{RP}^1)$. Moreover, the invariant pairing between quadratic differentials and vector fields yields a 2-cocycle on $\mathrm{Vect}(\mathbb{RP}^1)$ with trivial coefficients:

$$\omega\left(X(x)\,\frac{d}{dx},\, Y(x)\,\frac{d}{dx}\right) = \int_{\mathbb{RP}^1} X'''(x)\, Y(x)\, dx.$$

This is the Gelfand-Fuchs cocycle (1967); it defines a central extension of $\mathrm{Vect}(\mathbb{RP}^1)$ called the Virasoro algebra, perhaps the most famous infinite-dimensional Lie algebra, defined on the space $\mathrm{Vect}(\mathbb{RP}^1) \oplus \mathbb{R}$ by the commutator

$$[(X, \alpha),\, (Y, \beta)] = ([X, Y],\, \omega(X, Y))\,,$$

where $[X, Y]$ is the commutator of vector fields.

The Schwarzian contains complete information about the Gelfand-Fuchs cocycle; for instance, the 2-cocycle condition follows from Property 2. The relations between the Schwarzian and the Virasoro algebra were discovered, independently, by A. Kirillov and G. Segal (1980).

Multi-dimensional versions of the Schwarzian. Here is a "universal method" of discovering a multidimensional Schwarzian:
a) choose a group of diffeomorphisms and a subgroup G that has a nice geometrical meaning,
b) find a G-invariant 1-cocycle on the group of diffeomorphisms,
c) (the most important step) check that no one did it before.

One of the most interesting multi-dimensional Schwarzians is that of Osgood and Stowe (1992). Consider a Riemannian surface (M, g) and the group $\mathrm{Diff}_c(M)$ of all conformal transformations of M. The Riemann uniformization theorem implies that M is conformally flat. One can (locally) express the metric as $\mathrm{g} = (1/F)\psi^*\mathrm{g}_0$, where ψ is a conformal diffeomorphism of M, F is a smooth function, and g_0 is a metric of constant curvature. The conformal Schwarzian is given by

$$S(\psi) = \frac{\nabla dF}{F} - \frac{3}{2}\frac{dF \otimes dF}{F^2} + \frac{1}{4}\frac{\mathrm{g}^{-1}(dF, dF)\,\mathrm{g}}{F^2},$$

where ∇ is the covariant derivative corresponding to the Levi-Civita connection. This is a 1-cocycle on $\mathrm{Diff}_c(M)$, invariant with respect to the (local) Möbius subgroup $\mathrm{SO}(3,1)$ associated to the metric g_0. The construction also makes sense if $\dim M > 2$, but the conformal group is finite-dimensional in this case.

Among other generalizations, let us mention the "Lagrangian Schwarzian" modeled on symmetric matrices, Ovsienko (1989); a more general non-commutative Schwarzian of Retakh and Shander (1993); and various generalized Schwarzians with coefficients in the space of differential operators.

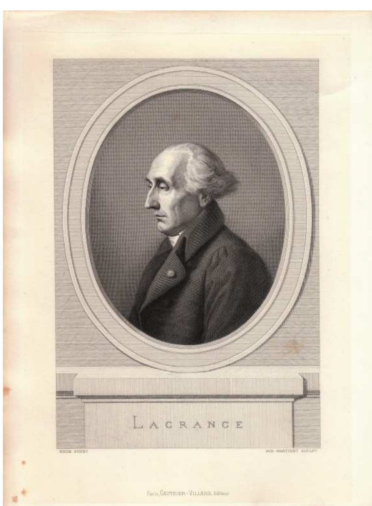
Last but not least, the Schwarzian derivative plays a key role in Teichmüller theory, namely, the Bers embedding of the Teichmüller space of a Riemann surface into an appropriate complex space. However, this vast topic deserves a separate treatment.

Further Reading

[1] F. Labourie, What is...a cross-ratio?, *Notices* **55** (2008), No 10. (November), 1234–1235.

[2] V. Ovsienko and S. Tabachnikov, *Projective Differential Geometry Old and New. From the Schwarzian Derivative to the Cohomology of Diffeomorphism Groups*, Cambridge University Press, Cambridge, 2005.

Lagrange and the Schwarzian Derivative



The name "Schwarzian derivative" was coined by Cayley, but he points out that Schwarz himself says that it occurs already, at least implicitly, in Lagrange's essay on *cartes géographiques*. Felix Klein learned about Lagrange's work through a private communication from Schwarz (noted in §III.5 of *Lectures on the Icosahedron*). It is not quite straightforward, however, in reading Lagrange, to see what he is doing, and it is not clear to what extent later mathematicians went to the original work. Joseph Sylvester, in "Method of reciprocants" records that he tried to track down Lagrange's use of the Schwarzian, but then only concludes that "There are two papers by Lagrange … but I have not been able to discover the Schwarzian derivative in either one of them." Even in modern times Lagrange's role has been missed—for example, George Heine in an article in the recent book *Euler at 300: An Appreciation* says that Lagrange's work had little influence on either mathematics or cartography.

Schwarz was, however, correct—Lagrange did introduce some version of the Schwarzian derivative $S(f)$, and for an interesting purpose. He considers the Earth as a general body of revolution, taking into account the known non-sphericity. He studies the maps given by a conformal mapping from a spherical region to the plane that takes all the meridians and all the parallels to arcs of circles (as does stereographic projection). This is equivalent to describing local conformal mappings $z \mapsto f(z)$ for which the image of each horizontal and each vertical line is a circular arc. He proves that the conditions on horizontals and verticals are equivalent, and that both are equivalent, in contemporary notation, to the equation $\mathrm{Im}\,S(f) = 0$. This implies in turn that $S(f) = \mathrm{const}$. He solves this equation and explicitly describes its solutions. The Schwarzian derivative $S(f)$ appears in his paper as ϕ''/ϕ where $\phi = 1/\sqrt{f'}$), which excuses Sylvester to some extent for missing it.

All this was found again much later, but apparently quite independently of Lagrange's original work. What is sometimes called *Arnold's Law* asserts, "Discoveries are rarely attributed to the correct person." One might add to this *(Michael) Berry's Law*, prompted by the observation that the sequence of antecedents under the previous law seems endless: "Nothing is ever discovered for the first time."

Lagrange's article on *cartes géographiques* is in Volume IV of his collected works. This is not, unfortunately, available at `http:gallica.bnf.fr` as are Volumes II and VIII, but we have made a scan of it available at

`http:www.math.ubc.ca/~cass/cartes.pdf`.

It is this volume, incidentally, that contains as frontispiece the well known portrait of Lagrange reproduced here.

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