Quantum cohomology of the odd symplectic Grassmannian of lines

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Introduction
Motivation

Quantum cohomology has been extensively studied for

- homogeneous spaces ;
- toric varieties.

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- toric varieties.

But

- very few explicit formulas for non-homogeneous non-toric varieties ;
- quasi-homogeneous varieties (e.g odd symplectic Grassmannians) should provide interesting examples.

Studied by MIHAI (2007).

Definition

ω antisymmetric form of maximal rank on $\mathbb{C}^{2n+1}$.

$$\mathrm{IG}_\omega(m, 2n+1) := \{\Sigma \in \mathrm{G}(m, 2n+1) \mid \Sigma \text{ is isotropic for } \omega\}.$$

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Remarks

1. independant of the form ω ;
2. endowed with an action of the odd symplectic group :

   $$\mathrm{Sp}_{2n+1} := \{g \in \mathrm{GL}(2n+1) \mid \forall u, v \in V\ \omega(gu, gv) = \omega(u, v)\};$$

3. odd symplectic Grassmannians of lines are the $m = 2$ case.

Properties (of $\mathrm{IG}(m, 2n+1)$)

1. *smooth subvariety of dimension* $m(2n+1-m) - \frac{m(m-1)}{2}$ *of* $\mathrm{G}(m, 2n+1)$.
2. *two orbits under the action of* Sp_{2n+1} *:*
   - *closed orbit* $\mathbb{O} := \{\Sigma \in \mathrm{IG}(m, 2n+1) \mid \Sigma \supset K\}$, *isomorphic to* $\mathrm{IG}(m-1, 2n)$ *;*
   - *open orbit* $\{\Sigma \in \mathrm{IG}(m, 2n+1) \mid \Sigma \not\supset K\}$, *isomorphic to the dual of the tautological bundle over* $\mathrm{IG}(m, 2n)$ *;*

where $K = \mathrm{Ker}(\omega)$.

Classical cohomology

Schubert varieties for the symplectic Grassmannian

Schubert varieties of the symplectic Grassmannian $\mathrm{IG}(m, 2n)$

- are subvarieties defined by incidence conditions with respect to an isotropic flag ;
- can be indexed by k-strict partitions (cf BUCH-KRESCH-TAMVAKIS), i.e

  $$\lambda = (2n-m \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0) \text{ such that } \lambda_j > k \Rightarrow \lambda_j > \lambda_{j+1},$$

  with $k = n - m$;
- correspond to classes $\sigma_\lambda \in \mathrm{H}^{|\lambda|}(\mathrm{IG}, \mathbb{Z})$ generating the cohomology ring $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$ as a $\mathbb{Z}$-module.

Embedding in the symplectic Grassmannian :

- $\mathrm{IG}(m, 2n+1) \hookrightarrow \mathrm{IG}(m, 2n+2)$ identifies $\mathrm{IG}(m, 2n+1)$ with a Schubert variety of $\mathrm{IG}(m, 2n+2)$ (Mihai) ;
- hence "induced" Schubert varieties for $\mathrm{IG}(m, 2n+1)$ and decomposition $\mathrm{H}^*(\mathrm{IG}(m, 2n+1), \mathbb{Z}) = \bigoplus_\lambda \mathbb{Z}\sigma_\lambda$.

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For $\mathrm{IG}(2, 2n+1)$, Schubert varieties are indexed by

- "usual" $(n-2)$-strict partitions $\lambda = (2n-1 \geq \lambda_1 \geq \lambda_2 \geq 0)$;
- the "partition" $\lambda = (2n-1, -1)$ corresponding to the class of the closed orbit $\mathbb{O}$.

Classical cohomology

Presentation

$\mathrm{H}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})$ is generated as a ring by two sets of special Schubert classes :

1. "rows" σ_p for $1 \leq p \leq 2n-1$, plus the class $\sigma_{2n-1,-1}$;
2. "columns" σ_1 and $\sigma_{1,1}$.

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Proposition (Presentation of $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$)

The ring $\mathrm{H}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$ *is generated by the classes* σ_1, $\sigma_{1,1}$ *and the relations are*

$$\det\left(\sigma_{1^{1+j-i}}\right)_{1\le i,j\le 2n} = 0$$
$$\frac{1}{\sigma_1}\det\left(\sigma_{1^{1+j-i}}\right)_{1\le i,j\le 2n+1} = 0$$

Quantum cohomology

Moduli space of stable maps

Consider

- X smooth projective Fano variety over $\mathbb{C}$ with Picard rank 1
- n, d integers.

A stable map of degree d with n marked points is a map $f : (C; p_1, \ldots, p_n) \to X$, where

- C is a tree of projective curves with n smooth marked points $p_1, \ldots, p_n$;
- $f_*[C] = d \cdot [\text{hyperplane}]$;

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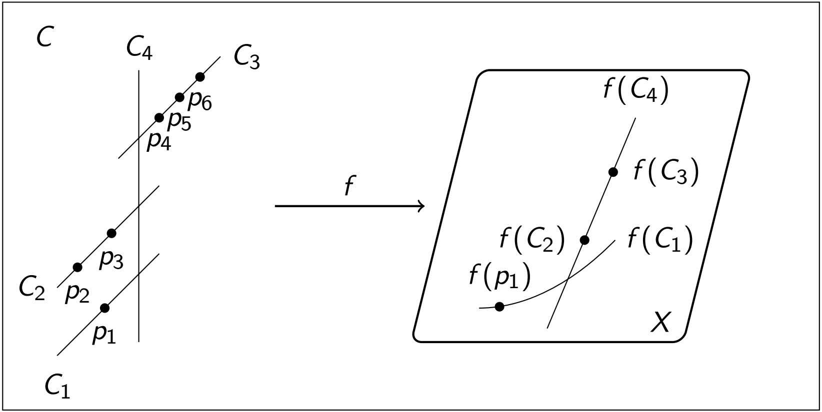
- C is a tree of projective curves with n smooth marked points $p_1, \ldots, p_n$;
- $f_*[C] = d \cdot [\text{hyperplane}]$;
- stability condition : each contracted component of C has at least 3 special points.

The corresponding moduli space is denoted by $\overline{\mathcal{M}}_{0,n}(X, d)$.

Quantum cohomology

An example of a stable map



Evaluation maps

$$
\begin{array}{rccc}
ev_i : & \overline{\mathcal{M}}_{0,n}(X,d) & \longrightarrow & X \\
 & [f : (C; p_1, \dots, p_n) \to X] & \mapsto & f(p_i)
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Definition

The degree d GW invariant associated to classes $\gamma_1,\ldots,\gamma_n$ is

$$I_d(\gamma_1,\ldots,\gamma_n) = \int_{[\overline{\mathcal{M}}_{0,n}(X,d)]^{vir}} ev_1^*\gamma_1 \cup \cdots \cup ev_n^*\gamma_n,$$

where $\left[\overline{\mathcal{M}}_{0,n}(X,d)\right]^{vir}$ is the virtual fundamental cycle.

Remark : GW invariants are integers.

Quantum cohomology

Quantum product

The small quantum product of classes γ_1 and γ_2 is

$$\gamma_1 \star \gamma_2 = \sum_{\beta} \sum q^d \underbrace{I_d(\gamma_1 \cdot \gamma_2 \cdot \check{\gamma}_3)}_{\text{Gromov-Witten invariant}} \gamma_3,$$

- q is the quantum parameter and has degree the index of X ;
- γ_3 runs over a basis of $\mathrm{H}^*(X, \mathbb{C})$; $\check{\gamma}_3$ runs over the corresponding dual basis.

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Properties

1. *The quantum product is commutative, degree-preserving, associative, with unit $1 \in \mathrm{H}^*(X, \mathbb{C})$.*
2. *It is a deformation of the cup-product.*

What does it mean ?

$I_d(\gamma_1, \gamma_2, \gamma_3) =$ number of degree d rational curves through $\Gamma_1, \Gamma_2, \Gamma_3$,

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What are the obstructions ?

1. moduli space may not have the expected dimension ;
2. maybe Γ_i's can't be made to intersect transversely ;
3. stable maps with reducible source may contribute ;
4. a curve may cut one of the Γ_i's in several points, contributing several times to the invariant ;
5. similarly a curve may cut one of the Γ_i's with multiplicities.

Proposition

The moduli spaces $\overline{\mathcal{M}}_{0,2}(\mathrm{IG}, 1)$ *and* $\overline{\mathcal{M}}_{0,3}(\mathrm{IG}, 1)$ *are smooth (as stacks) and of the expected dimension.*

Quantum cohomology

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Idea of proof : We prove that $\mathrm{H}^1(f^*\mathrm{T}\,\mathrm{IG}) = 0$ for each stable f.

- If no irreducible component of the source of f is entirely mapped into $\mathbb{O}$, use the generic global generation of $f^*\mathrm{T}\,\mathrm{IG}$ due to the transitive Sp_{2n+1}-action on $\mathrm{IG} \setminus \mathbb{O}$;
- Else use the tangent exact sequence of the closed orbit and prove that $\mathrm{H}^1(f^*\mathrm{T}\,\mathcal{N}_{\mathbb{O}}) = 0$.

Quantum cohomology

Graber's lemma

For homogeneous varieties, enumerativity of GW invariants comes from Kleiman's lemma. For quasi-homogeneous spaces there is a version by Graber :

Lemma

- *G a connected algebraic group ;*
- *X a quasi-G-homogeneous variety ;*
- *$f : Z \to X$ a morphism from an irreducible scheme ;*
- *$Y \subset X$ intersecting the orbit stratification properly.*

Then there exists a dense open subset U of G such that $\forall g \in U$, $f^{-1}(gY)$ is either empty or has pure dimension $\dim Y + \dim Z - \dim X$.

Theorem

- *$r = 2$ or 3 ;*
- *$Y_1, \dots, Y_r$ cycles in IG representing $\gamma_1, \dots, \gamma_r$ and intersecting $\mathbb{O}$ generically transversely ;*
- *$\deg \gamma_i \geq 2$ for all i ;*
- *$\sum_{i=1}^{r} \deg \gamma_i = \dim \overline{\mathcal{M}}_{0,r}(\mathrm{IG}, 1)$.*

Then there exists a dense open subset $U \subset \mathrm{Sp}_{2n+1}^r$ such that for all $g_1, \dots, g_r \in U$, the Gromov-Witten invariant $I_1(\gamma_1, \dots, \gamma_r)$ is equal to the number of lines of IG incident to the translates $g_1 Y_1, \dots, g_r Y_r$.

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Idea of proof : We get rid of the last three obstructions to enumerativity using Graber's lemma.

Quantum cohomology

Finding subvarieties with transverse intersection

Problem :

- To compute an invariant with the enumerativity theorem we need transverse cycles.
- Schubert varieties can never be made to intersect transversely.

Quantum cohomology

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Solution :

- Use pullbacks of the Schubert varieties of the type A Grassmannian $\mathrm{G}(2, 2n+1)$;
- They can be made to intersect transversely on the *homogeneous* space $\mathrm{G}(2, 2n+1)$;
- Corresponding pullbacks to $\mathrm{IG}(2, 2n+1)$ stay transverse.

Quantum cohomology

Quantum presentation

Proposition (Presentation of $\mathrm{QH}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$)

The ring $\mathrm{QH}^*(\mathrm{IG}(2,2n+1),\mathbb{Z})$ is generated by the classes σ_1, $\sigma_{1,1}$ and the quantum parameter q. The relations are

$$\det\left(\sigma_{1^{1+j-i}}\right)_{1\leq i,j\leq 2n} = 0$$

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Corollary

1. $\mathrm{QH}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})_{q \neq 0}$ *is semisimple ;*
2. *hence Dubrovin's conjecture holds for* $\mathrm{IG}(2, 2n+1)$.

Conclusion

Other results :

- Quantum Pieri formula ;
- J-function.

Next step :

- The $m > 2$ case ?