Quantum cohomology of the odd symplectic Grassmannian of lines

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- homogeneous spaces ;
- toric varieties.

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But

- very few explicit formulas for non-homogeneous non-toric varieties ;
- quasi-homogeneous varieties (e.g odd symplectic Grassmannians) should provide interesting examples.

Studied by Mihai (2007).

Definition

ω antisymmetric form of maximal rank on $\mathbb{C}^{2n+1}$.

$$\mathrm{IG}_\omega(m, 2n+1) := \{\Sigma \in \mathrm{G}(m, 2n+1) \mid \Sigma \text{ is isotropic for } \omega\}.$$

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Remarks

1. independant of the form ω ;
2. endowed with an action of the odd symplectic group :

   $$\mathrm{Sp}_{2n+1} := \{g \in \mathrm{GL}(2n+1) \mid \forall u, v \in V \ \omega(gu, gv) = \omega(u, v)\};$$

3. odd symplectic Grassmannians of lines are the $m = 2$ case.

Properties (of $\mathrm{IG}(m, 2n+1)$)

1. *smooth subvariety of dimension* $m(2n+1-m) - \frac{m(m-1)}{2}$ *of* $\mathrm{G}(m, 2n+1)$.
2. *two orbits under the action of* Sp_{2n+1} *:*
   - *closed orbit* $\mathbb{O} := \{\Sigma \in \mathrm{IG}(m, 2n+1) \mid \Sigma \supset K\}$, *isomorphic to* $\mathrm{IG}(m-1, 2n)$ *;*
   - *open orbit* $\{\Sigma \in \mathrm{IG}(m, 2n+1) \mid \Sigma \not\supset K\}$, *isomorphic to the dual of the tautological bundle over* $\mathrm{IG}(m, 2n)$ *;*

where $K = \mathrm{Ker}(\omega)$.

Schubert varieties of the symplectic Grassmannian $\mathrm{IG}(m, 2n)$

- are subvarieties defined by incidence conditions with respect to an *isotropic flag* ;
- can be indexed by *k-strict partitions* (cf BUCH-KRESCH-TAMVAKIS), i.e

  $$\lambda = (2n - m \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0) \text{ such that } \lambda_j > k \Rightarrow \lambda_j > \lambda_{j+1},$$

  with $k = n - m$;
- correspond to classes $\sigma_\lambda \in \mathrm{H}^{|\lambda|}(\mathrm{IG}, \mathbb{Z})$ generating the cohomology ring $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$ as a $\mathbb{Z}$-module.

Classical cohomology

Schubert varieties for $\mathrm{IG}(m, 2n+1)$

Embedding in the symplectic Grassmannian :

- $\mathrm{IG}(m, 2n+1) \hookrightarrow \mathrm{IG}(m, 2n+2)$ identifies $\mathrm{IG}(m, 2n+1)$ with a Schubert variety of $\mathrm{IG}(m, 2n+2)$ (Mihai) ;
- hence "induced" Schubert varieties for $\mathrm{IG}(m, 2n+1)$ and decomposition $\mathrm{H}^*(\mathrm{IG}(m, 2n+1), \mathbb{Z}) = \bigoplus_\lambda \mathbb{Z}\sigma_\lambda$.

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For $\mathrm{IG}(2, 2n+1)$, Schubert varieties are indexed by

- "usual" $(n-2)$-strict partitions $\lambda = (2n-1 \geq \lambda_1 \geq \lambda_2 \geq 0)$;
- the "partition" $\lambda = (2n-1, -1)$ corresponding to the class of the closed orbit $\mathbb{O}$.

Classical cohomology

Special Schubert classes ; Pieri and Giambelli formulas

$\mathrm{H}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})$ is generated as a ring by two sets of special Schubert classes :

1. "rows" σ_p for $1 \leq p \leq 2n-1$, plus the class $\sigma_{2n-1,-1}$;
2. "columns" σ_1 and $\sigma_{1,1}$.

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Definition

- *A Pieri formula is a rule for multiplying any Schubert class with a special class ;*
- *A Giambelli formula is a rule expressing any Schubert class as a polynomial in special classes.*

Knowing both formulas, all cup-products of Schubert classes can be computed.

Classical cohomology

A Pieri formula for $\mathrm{IG}(2, 2n+1)$

- $i : \mathrm{IG}(m, 2n+1) \hookrightarrow \mathrm{IG}(m, 2n+2)$ induces a restriction map in cohomology, which happens to be surjective.
- For $m = 2$ the map and its "inverse" can be explicitely computed.
- So Pieri rules for $\mathrm{IG}(2, 2n+2)$ (cf Pragasz-Ratajski, BKT) can be "pulled back" to $\mathrm{IG}(2, 2n+1)$, hence

Proposition (Pieri formula for $\mathrm{IG}(2, 2n+1)$)

$$\sigma_{a,b} \cdot \sigma_1 = \begin{cases} \sigma_{a+1,b} + \sigma_{a,b+1} & \text{if } a+b \neq 2n-3, \\ \sigma_{a,b+1} + 2\sigma_{a+1,b} + \sigma_{a+2,b-1} & \text{if } a+b = 2n-3. \end{cases}$$

$$\sigma_{a,b} \cdot \sigma_{1,1} = \begin{cases} \sigma_{a+1,b+1} & \text{if } a+b \neq 2n-4, 2n-3, \\ \sigma_{a+1,b+1} + \sigma_{a+2,b} & \text{if } a+b = 2n-4 \text{ or } 2n-3. \end{cases}$$

To find a Giambelli formula for $\mathrm{IG}(2, 2n+1)$:

- use the well-known Giambelli formula on $\mathrm{G}(2, 2n+1)$;
- "pull it back" to $\mathrm{IG}(2n+1)$ by the natural embedding $\mathrm{IG}(2, 2n+1) \hookrightarrow \mathrm{G}(2, 2n+1)$;
- get an explicit formula.

Proposition (Presentation of $\mathrm{H}^*\left(\mathrm{IG}(2, 2n+1), \mathbb{Z}\right)$)

The ring $\mathrm{H}^*\left(\mathrm{IG}(2, 2n+1), \mathbb{Z}\right)$ *is generated by the classes* σ_1, $\sigma_{1,1}$ *and the relations are*

$$\det\left(\sigma_{1^{1+j-i}}\right)_{1 \leq i,j \leq 2n} = 0$$

$$\frac{1}{\sigma_1} \det\left(\sigma_{1^{1+j-i}}\right)_{1 \leq i,j \leq 2n+1} = 0$$

Quantum cohomology

Definition

Goal : compute the small quantum product

$$\sigma_{\alpha,\beta} \star \sigma_{\gamma,\delta} = \sum_{d \geq 0} \sum \underbrace{I_d(\sigma_{\alpha,\beta} \cdot \sigma_{\gamma,\delta} \cdot \check{\sigma}_{\epsilon,\zeta})}_{\text{Gromov-Witten invariant}} \sigma_{\epsilon,\zeta} q^d,$$

where

- q is the quantum parameter and has degree $2n$;
- $\sigma_{\epsilon,\zeta}$ runs through the Schubert classes ; $\check{\sigma}_{\epsilon,\zeta}$ runs through the corresponding dual basis.

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- q is the quantum parameter and has degree $2n$;
- $\sigma_{\epsilon,\zeta}$ runs through the Schubert classes ; $\check{\sigma}_{\epsilon,\zeta}$ runs through the corresponding dual basis.

Idea : to compute the GW invariants, use their enumerative interpretation.

What does it mean ?

$I_d(\gamma_1, \gamma_2, \gamma_3) =$ number of degree d rational curves through $\Gamma_1, \Gamma_2, \Gamma_3$,

where Γ_i's are cycles Poincaré dual to the classes γ_i.

What are the obstructions ?

1. moduli space may not have the expected dimension ;
2. maybe Γ_i's can't be made to intersect transversely ;
3. stable maps with reducible source may contribute ;
4. a curve may cut one of the Γ_i's in several points, contributing several times to the invariant ;
5. similarly a curve may cut one of the Γ_i's with multiplicities.

Proposition

The moduli spaces $\overline{\mathcal{M}}_{0,2}(\mathrm{IG},1)$ *and* $\overline{\mathcal{M}}_{0,3}(\mathrm{IG},1)$ *are smooth (as stacks) and of the expected dimension.*

Quantum cohomology

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Proposition

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Idea of proof : We prove that $\mathrm{H}^1(f^*\mathrm{T}\,\mathrm{IG}) = 0$ for each stable f.

- If no irreducible component of the source of f is entirely mapped into $\mathbb{O}$, use the generic global generation of $f^*\mathrm{T}\,\mathrm{IG}$ due to the transitive Sp_{2n+1}-action on $\mathrm{IG} \setminus \mathbb{O}$;
- Else use the tangent exact sequence of the closed orbit and prove that $\mathrm{H}^1(f^*\mathrm{T}\,\mathcal{N}_{\mathbb{O}}) = 0$.

For homogeneous varieties, enumerativity of GW invariants comes from Kleiman's lemma. For quasi-homogeneous spaces there is a version by Graber :

Lemma

- *G a connected algebraic group ;*
- *X a quasi-G-homogeneous variety ;*
- *$f : Z \to X$ a morphism from an irreducible scheme ;*
- *$Y \subset X$ intersecting the orbit stratification properly.*

Then there exists a dense open subset U of G such that $\forall g \in U$, $f^{-1}(gY)$ is either empty or has pure dimension $\dim Y + \dim Z - \dim X$.

Theorem

- *$r = 2$ or 3 ;*
- *$Y_1, \ldots, Y_r$ cycles in IG representing $\gamma_1, \ldots, \gamma_r$ and intersecting $\mathbb{O}$ generically transversely ;*
- *$\deg \gamma_i \geq 2$ for all i ;*
- *$\sum_{i=1}^{r} \deg \gamma_i = \dim \overline{\mathcal{M}}_{0,r}(\mathrm{IG}, 1)$.*

Then there exists a dense open subset $U \subset \mathrm{Sp}_{2n+1}^r$ such that for all $g_1, \ldots, g_r \in U$, the Gromov-Witten invariant $I_1(\gamma_1, \ldots, \gamma_r)$ is equal to the number of lines of IG incident to the translates $g_1 Y_1, \ldots, g_r Y_r$.

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Idea of proof : We get rid of the last three obstructions to enumerativity using Graber's lemma.

Quantum cohomology

Finding subvarieties with transverse intersection

Problem :

- To compute an invariant with the enumerativity theorem we need transverse cycles.
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Solution :

- Use pullbacks of the Schubert varieties of the type A Grassmannian $\mathrm{G}(2, 2n+1)$;
- They can be made to intersect transversely on the *homogeneous* space $\mathrm{G}(2, 2n+1)$;
- Corresponding pullbacks to $\mathrm{IG}(2, 2n+1)$ stay transverse.

Quantum cohomology

Quantum Pieri rule for $\mathrm{IG}(2, 2n+1)$

Theorem

$$\sigma_1 \star \sigma_{a,b} = \begin{cases} \sigma_{a+1,b} + \sigma_{a,b+1} & \text{if } a+b \neq 2n-3 \text{ and } a \neq 2n-1, \\ \sigma_{a,b+1} + 2\sigma_{a+1,b} + \sigma_{a+2,b-1} & \text{if } a+b = 2n-3, \\ \sigma_{2n-1,b+1} + q\sigma_b & \text{if } a = 2n-1 \text{ and } 0 \leq b \leq 2n-3, \\ q(\sigma_{2n-1,-1} + \sigma_{2n-2}) & \text{if } a = 2n-1 \text{ and } b = 2n-2. \end{cases}$$

$$\sigma_{1,1} \star \sigma_{a,b} = \begin{cases} \sigma_{a+1,b+1} & \text{if } a+b \neq 2n-4,\ 2n-3 \text{ and } a \neq 2n-1, \\ \sigma_{a+1,b+1} + \sigma_{a+2,b} & \text{if } a+b = 2n-4 \text{ or } 2n-3, \\ q\sigma_{b+1} & \text{if } a = 2n-1 \text{ and } b \neq 2n-3, \\ q(\sigma_{2n-1,-1} + \sigma_{2n-2}) & \text{if } a = 2n-1 \text{ and } b = 2n-3. \end{cases}$$

Quantum cohomology

Quantum Hasse diagrams

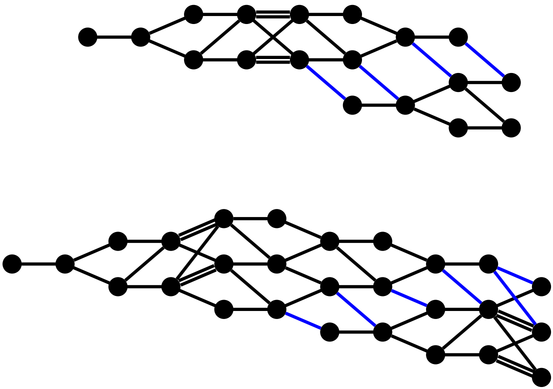

Figure: Quantum Hasse diagrams of $\mathrm{IG}(2,6)$ and $\mathrm{IG}(2,7)$

Proposition (Presentation of $\mathrm{QH}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})$)

The ring $\mathrm{QH}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})$ is generated by the classes σ_1, $\sigma_{1,1}$ and the quantum parameter q. The relations are

$$\det\left(\sigma_{1^{1+j-i}}\right)_{1 \leq i,j \leq 2n} = 0$$

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Corollary

1. $\mathrm{QH}^*(\mathrm{IG}(2, 2n+1), \mathbb{Z})_{q \neq 0}$ *is semisimple ;*
2. *hence Dubrovin's conjecture holds for* $\mathrm{IG}(2, 2n+1)$.

Conclusion

Main results :

- Enumerativity of GW invariants ;
- Quantum Pieri formula ;
- Quantum presentation and semisimplicity.

Next step :

- The $m > 2$ case ?