Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

A Landau-Ginzburg model for Lagrangian Grassmannians, Langlands duality and relations in quantum cohomology

C. Pech and K. Rietsch

Abstract

In [**Rie08**], the second author defined a Landau-Ginzburg model for homogeneous spaces G/P, as a regular function on an affine subvariety of the Langlands dual group. In this paper, we reformulate this LG-model in the case of the Lagrangian Grassmannian $\mathrm{LG}(m)$ as a function W_t on the complement $\check{X}^\circ$ of an anticanonical divisor in a Langlands dual orthogonal Grassmannian $\check{X}$, in the spirit of work by R. Marsh and the second author [**MR13**] for type A Grassmannians. This LG model, $(\check{X}^\circ, W_t)$, has some very interesting features, which are not visible in the type A case, to do with the non-triviality of Langlands duality. Moreover $\check{X}$ is embedded in $\mathbb{P}(H^*(\mathrm{LG}(m),\mathbb{C})^*)$ and W_t is expressed in terms of coordinates which identify with Schubert classes of $\mathrm{LG}(m)$. We make use of this identification to formulate a precise conjecture relating the Gauss-Manin system for our superpotential with the small Dubrovin connection of $\mathrm{LG}(m)$. Finally, our expression for W_t leads us to conjecture new formulas in the quantum Schubert calculus of $\mathrm{LG}(m)$.

1. *Introduction*

For a complex simple, simply connected algebraic group G and parabolic subgroup P, the homogeneous space G/P has a Landau-Ginzburg model defined by the second author [**Rie08**], which is a regular function on an affine subvariety of the Langlands dual group and is shown in [**Rie08**] to recover the Peterson variety presentation [**Pet97**] of the quantum cohomology of G/P. In the case of type A Grassmannians R. Marsh and the second author [**MR13**] reformulated this Landau-Ginzburg model as a rational function on a Langlands dual Grassmannian, and used this formulation to prove a version of the mirror symmetry conjecture about flat sections of the A-model connection stated in [**BCFKvS00**].

In this paper we formulate an LG-model $(\check{X}^\circ, W_t)$ for G/P in the case of a Lagrangian Grassmannian $X = \mathrm{LG}(m)$, where $\check{X}$ is a minimal co-orthogonal Grassmannian naturally embedded into $\mathbb{P}(H^*(X,\mathbb{C})^*)$, and $\check{X}^\circ$ is the complement of a particular anticanonical divisor inside $\check{X}$. Moreover, the coordinate ring of the affine variety $\check{X}^\circ$ is endowed with a cluster algebra structure (see [**GLS11**]). We prove that this LG model is isomorphic to the LG-model from [**Rie08**], and therefore recovers the quantum cohomology ring of X. This LG-model has some very interesting features, which are not visible in the case of type A Grassmannians. These are to do with the non-triviality of Langlands duality. Our methods are mostly representation-theoretic, making use of the geometric Satake correspondence (see Remark 1) and of the Clifford algebra to construct maps between representations of Spin_{2m+1}.

We conjecture that our superpotential gives rise to integral solutions of the quantum differential equations of $LG(m)$. Our expression for W_t also leads us to conjecture new formulas in the quantum Schubert calculus of $LG(m)$.

2000 *Mathematics Subject Classification* 14J33, 14M17 (primary), 14N35, 20G10(secondary).

This work was supported by Leverhulme Trust grant F07040AW - A Lie theoretic approach to derived categories of flag varieties G/P.

To give an idea of our result, which is very explicit, we give the first two interesting examples here. Note that the Schubert basis of $H^*(LG(m),\mathbb{C})$ is indexed by strict partitions λ fitting in an $m\times m$ box and can be identified with coordinates p_λ on the Grassmannian $\check{X} = \mathrm{OG}^{\mathrm{co}}(m+1,2m+1)$ of $(m+1)$-dimensional co-isotropic subspaces of $\mathbb{C}^{2m+1}$ endowed with a non-degenerate quadratic form. Note that $\mathrm{OG}^{\mathrm{co}}(m+1,2m+1)$ is canonically isomorphic to the maximal orthogonal Grassmannian $\mathrm{OG}(m,2n+1)$. Moreover, it is related to X by Langlands duality. The goal of this paper is to give an explicit description of a Landau-Ginzburg model for $LG(m)$ as a regular function on an open dense affine subvariety $\check{X}^\circ$ of $\mathrm{OG}^{\mathrm{co}}(m+1,2m+1)$. As an example, for $LG(2)$ our Landau-Ginzburg model is

$$W_t = \frac{p_{\ydiagram{1}}}{p_\emptyset} + \frac{p_{\ydiagram{2}}^2}{p_{\ydiagram{1}}p_{\ydiagram{2}} - p_\emptyset p_{\ydiagram{2,1}}} + e^t\,\frac{p_{\ydiagram{1}}}{p_{\ydiagram{2,1}}},$$

which is regular on $\check{X}^\circ = \left\{p_\emptyset(p_{\ydiagram{1}}p_{\ydiagram{2}} - p_\emptyset p_{\ydiagram{2,1}})p_{\ydiagram{2,1}} \neq 0\right\} \subset \mathrm{OG}^{\mathrm{co}}(3,5)$. For $LG(3)$ we obtain

$$W_t = \frac{p_{\ydiagram{1}}}{p_\emptyset} + \frac{p_{\ydiagram{2}}p_{\ydiagram{3}} - p_\emptyset p_{\ydiagram{3,2}}}{p_{\ydiagram{1}}p_{\ydiagram{3}} - p_\emptyset p_{\ydiagram{3,1}}} + \frac{p_{\ydiagram{3,1}}p_{\ydiagram{3,2}} - p_{\ydiagram{3}}p_{\ydiagram{3,2,1}}}{p_{\ydiagram{2,1}}p_{\ydiagram{3,2}} - p_{\ydiagram{2}}p_{\ydiagram{3,2,1}}} + e^t\,\frac{p_{\ydiagram{2,1}}}{p_{\ydiagram{3,2,1}}},$$

which is regular on $\check{X}^\circ = \left\{p_\emptyset(p_{\ydiagram{1}}p_{\ydiagram{3}} - p_\emptyset p_{\ydiagram{3,1}})(p_{\ydiagram{2,1}}p_{\ydiagram{3,2}} - p_{\ydiagram{2}}p_{\ydiagram{3,2,1}})p_{\ydiagram{3,2,1}} \neq 0\right\}$ inside $\mathrm{OG}^{\mathrm{co}}(4,7)$. We generalise these definitions of $\check{X}^\circ$ and W_t in Section 3.2 and construct an isomorphism of $\check{X}^\circ$ with the open Richardson variety appearing in [**Rie08**]. In Section 3.6 we prove that the pullback of the LG-model on the Richardson variety from the same paper agrees with W_t.

Notice how the above formulas have $3,4$ summands, these numbers being the index of $X = LG(2)$, $LG(3)$, respectively. Indeed this comes from the fact that in all of the cases W_t represents the anticanonical class of X in a natural sense (in the Jacobi ring for example), and each summand represents a hyperplane class. On the other hand, because one expects W_t to be regular in the complement of an anticanonical divisor, and indeed the degrees of the denominators of W_t add up to the index of $\check{X}$. That is, in the above two cases to 4 and 6, these being the index of $\mathrm{OG}^{\mathrm{co}}(3,5)$ and $\mathrm{OG}^{\mathrm{co}}(4,7)$, respectively. This is exactly what is achieved by the quadratic terms in the $LG(m)$ cases, with $1+2+1=4$, and $1+2+2+1=6$ (and so forth, in our general formula).

For usual Grassmannians X and $\check{X}$ are isomorphic so have the same index. Therefore numerators and denominators in W_t are allowed to be sections of $\mathcal{O}(1)$, in agreement with the formulas in [**MR13**].

Acknowledgements. The second author thanks Dale Peterson for his inspiring work and lectures.

2. *Background*

In [**Rie08**], the second author gave a Lie-theoretic construction of a Landau-Ginzburg model of any complete homogeneous space X of a simple complex algebraic group. The LG-model $(\mathcal{R},\mathcal{F})$ is set in the world of the Langlands dual group.

2.1. *Notation*

Let X be a complete homogeneous space for a simple complex algebraic group. For the purposes of this paper we will denote the group acting on X by $G^\vee$ and assume that $G^\vee$ is simply connected, and we will denote its Langlands dual group by G, which is therefore an adjoint group. For $G^\vee$ we may fix Chevalley generators $(e_i^\vee)_{1\leq i\leq m}$ and $(f_i^\vee)_{1\leq i\leq m}$ and correspondingly Borel subgroups $B_+^\vee = T^\vee U_+^\vee$ and $B_-^\vee = T^\vee U_-^\vee$. We may assume that $X = G^\vee/P^\vee$ for a parabolic subgroup $P^\vee$ which contains $B_+^\vee$. The parabolic $P^\vee$ is determined by a choice of subset of the $(f_i^\vee)_{1\le i\le m}$. This set also determines a parabolic subgroup P of G, where we also have the analogous Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ and Chevalley generators $(e_i)_{1\le i\le m}$ and $(f_i)_{1\le i\le m}$. Let $\Pi = \{\alpha_i \mid i \in I\}$ denote the set of simple roots. The set of all roots is $R = R^+ \sqcup R^-$, where R^+ is the subset of positive roots and R^- the subset of negative roots.

Denote by W the Weyl group of G (canonically identified with the Weyl group of $G^\vee$), and let W_P be the Weyl group of the parabolic subgroup P. Let T^{W_P} be the W_P-fixed sub-torus. If α is a positive root, we denote by $s_\alpha \in W$ the associated reflection. Let R_P^+ be the set of all positive roots α such that $s_\alpha \in W_P$, Π_P be the set of simple roots in R_P^+, and $\Pi^P = \Pi \setminus \Pi_P$. When $\alpha = \alpha_i$ is a simple root, we set $s_i := s_{\alpha_i}$. Moreover, we denote the length of $w \in W$ by $\ell(w)$. It is equal to the minimum number of simple reflections whose product is w. We also let w_0 and w_P, be the longest elements in W and W_P, respectively, and define W^P to be the set of minimal length coset representatives for W/W_P. The minimal length coset representative for w_0 is denoted by w^P, so that $w_0 = w^P w_P$. Let $\dot{w}$ denote a representative of $w \in W$ in G. Later on we will make a specific choice for $\dot{w}$.

Using the exponential map we may think of U_+ and U_- as being embedded in the completed universal enveloping algebra $\hat{\mathcal{U}}_+$, respectively $\hat{\mathcal{U}}_+$. Accordingly $e_i^*(u)$ will denote the coefficient of e_i in $u \in U_+$ after this embedding, and analogously for f_i^* and $\bar{u} \in U_-$.

2.2. *Quantum cohomology of G/P*

The quantum cohomology ring of a smooth complex projective variety X is a deformation of its cohomology ring. While the cohomology ring of X encodes the way its subvarieties intersect each other, the quantum cohomology ring encodes the way they are connected by rational curves. The structure constants of the (small) quantum cohomology ring are called Gromov-Witten invariants. When $X = G/P$ is homogeneous, Gromov-Witten invariants count the number of rational curves of given degree intersecting three given Schubert varieties of X.

The quantum cohomology rings of a full flag variety was first described by Givental and B. Kim [**GK95**, **Kim99**], who related it to a degenerate leaf of the Toda lattice of the Langlands dual group. Soon after, Dale Peterson came up with a new point of view in which all of the quantum cohomology rings of complete homogeneous spaces for one group are encoded in terms of strata of one remarkable subvariety of the Langlands dual full flag variety. This so-called *Peterson variety* $\mathbb{Y}$ is defined as follows. In our conventions Peterson's variety $\mathbb{Y}$ encoding the quantum cohomology rings of $G^\vee$-homogeneous spaces is a subvariety of G/B_-. Denote by $\mathfrak{n}_-$ the Lie algebra of U_-, and by $[\mathfrak{n}_-, \mathfrak{n}_-]$ its commutator subalgebra. The annihilator in $\mathfrak{g}^*$ of a subspace $\mathfrak{l}$ of $\mathfrak{g}$ is denoted by $\mathfrak{l}^\perp$. Consider the coadjoint action of G on $\mathfrak{g}^*$ and the 'principal nilpotent' element $F = \sum e_i^*$ in $\mathfrak{g}^*$. Then

$$\mathbb{Y} := \{gB_- \mid g^{-1} \cdot F \in [\mathfrak{n}_-, \mathfrak{n}_-]^\perp\}.$$

First note that this variety has an open stratum $Y_B = \mathbb{Y} \cap (B_+B_-/B_-)$ which is isomorphic to the degenerate leaf of the Toda lattice for G via the map $Y_B \hookrightarrow \mathfrak{g}^*$ defined by $u_+B_- \mapsto u_+^{-1} \cdot F$. By Peterson's theory, the quantum cohomology rings for all other $G^\vee/P^\vee$ are described by the coordinate rings of the smaller strata $Y_P = \mathbb{Y} \cap (B_+\dot{w}_P B_-/B_-)$, where we take the intersection in the possibly non-reduced sense.

THEOREM 2.1 Peterson. *The quantum cohomology of $G^\vee/P^\vee$ is isomorphic to the coordinate ring $\mathbb{C}[Y_P]$ of the stratum Y_P of the Peterson variety $\mathbb{Y}$.*

In [**FW04**], Fulton and Woodward proved a quantum Chevalley formula for $X = G/P$, i.e. a formula giving the product of an arbitrary Schubert class by any Schubert class associated to a Schubert divisor. Here we state this formula, which we will refer to in Section 4. Note that for $P = B$, the formula is a result of Peterson [**Pet97**].

If s_i is a simple reflection, we denote by $\Gamma_i \in H_2(X, \mathbb{Z})$ the associated dimension 1 Schubert cycle, and we define, for $\alpha \in R^+ \setminus R_P^+$:

$$d(\alpha) := \sum_{i=1}^{m} \alpha^\vee(\omega_i)\Gamma_i.$$

Now set $q^{d(\alpha)} := \prod_{i=1}^m q_i^{\alpha^\vee(\omega_i)}$, where q_i is the quantum parameter associated to Γ_i. Finally, for $\alpha \in R^+ \setminus R_P^+$, we define $n_\alpha := \int_{\Gamma_\alpha} c_1(TX)$, where $\Gamma_\alpha \in H_2(X, \mathbb{Z})$ is the dimension 1 cycle associated to the reflection s_α (it is a linear combination of the Γ_i).

THEOREM 2.2 [**FW04**]. *For $1 \leq i \leq m$ and $w \in W^P$ we have*

$$\sigma_{s_i} \star \sigma_w = \sum_\alpha \alpha^\vee(\omega_i)\sigma_{ws_\alpha} + \sum_\alpha q^{d(\alpha)}\alpha^\vee(\omega_i)\sigma_{ws_\alpha},$$

where the first sum is over roots $\alpha \in R^+ \setminus R_P^+$ such that $l(ws_\alpha) = l(w) + 1$, and the second sum over roots $\alpha \in R^+ \setminus R_P^+$ such that $l(ws_\alpha) = l(w) + 1 - n_\alpha$.

2.3. *The Lie-theoretic LG model construction*

We recall how the mirror Landau-Ginzburg models are defined in [**Rie08**]. Let us fix a parabolic P. We consider the open Richardson variety $\mathcal{R} := R_{w_P,w_0} \subset G/B_-$, namely

$$\mathcal{R} := R_{w_P,w_0} = (B_+\dot{w}_P B_- \cap B_-\dot{w}_0 B_-)/B_-.$$

Instead of the whole stratum Y_P of the Peterson variety the LG-model is related to the open dense subset $Y_P^* := \mathbb{Y} \cap \mathcal{R}$, whose coordinate ring in Peterson's theory encodes the quantum cohomology ring $qH^*(G^\vee/P^\vee)$ with quantum parameters made invertible. We note that in this setting if $g = u_1 d\dot{w}_P \bar{u}_2 = b_-\dot{w}_0$ represents an element $gB_- \in \mathcal{R}$ lying in Y_P^*, then the values of the functions on Y_P^* corresponding to the quantum parameters are just the values $\alpha_j(d)$ for the simple roots $\alpha_j \in \Pi^P$. Indeed, fixing $d \in T^{W_P}$ determines a finite subscheme of $Y_P^* = \mathbb{Y} \cap \mathcal{R}$ which we denote by $Y_{P,d}^* = Y_P^* \times_{T^{W_P}} \{d\}$ and for which the non-reduced coordinate ring $\mathbb{C}[Y_{P,d}^*]$ becomes identified with the quantum cohomology ring of $G^\vee/P^\vee$ with quantum parameters fixed to the values $\alpha_j(d)$ in Peterson's theory.

Now let us define

$$Z = Z_{G^\vee/P^\vee} := B_-\dot{w}_0 \cap U_+ T^{W_P}\dot{w}_P U_-.$$

There is an isomorphism

$$\begin{array}{ccc} Z & \to & \mathcal{R} \times T^{W_P}, \\ g = u_1 d\dot{w}_P\bar{u}_2 = b_-\dot{w}_0 & \mapsto & (gB_-, d). \end{array}$$

Observe that $gB_- = b_-\dot{w}_0 B_- = u_1\dot{w}_P B_-$. Note that our conventions differ from [**Rie08**] in that we have translated the original definition of the variety Z by $\dot{w}_0$. The mirror superpotential to $X = G^\vee/P^\vee$ is now defined to be the regular function $\mathcal{F} : Z \to \mathbb{C}$ defined by

$$\mathcal{F}(u_1 d\dot{w}_P\bar{u}_2) = \sum_{i=1}^{m} e_i^*(u_1) + \sum_{i=1}^{m} f_i^*(\bar{u}_2). \tag{2.1}$$

Although u_1 and $\bar{u}_2$ are not uniquely determined for $g \in Z$, the function $\mathcal{F}$ is well-defined, as was shown in [**Rie08**]. Actually, there is another small difference with [**Rie08**], in that in

[**Rie08**] the group on the mirror side is assumed adjoint, whereas here we have assumed G to be simply connected. However we could have carried out the above definitions for $G/\mathrm{Center}(G)$, and in the following it will not matter.

The superpotential $\mathcal{F}$ may also be interpreted as a family of functions $\mathcal{F}_h : \mathcal{R} \to \mathbb{C}$ depending holomorphically on a parameter $h \in \mathfrak{h}^{W_P}$, by setting

$$\mathcal{F}_h(u_1 \dot{w}_P B_-) = \sum_{i=1}^{m} e_i^*(u_1) + \sum_{i=1}^{m} f_i^*(\bar{u}_2) \tag{2.2}$$

where $u_1 \in U_+$ and $u_1 \dot{w}_P B_- \in \mathcal{R}$, and where $\bar{u}_2 \in U_-$ is related to u_1 by $u_1 e^h \dot{w}_P \bar{u}_2 \in Z$. Equivalently the relationship between u_1 and $\bar{u}_2$ can be expressed as

$$\bar{u}_2 \cdot B_+ = e^{-h} \dot{w}_P^{-1} u_1^{-1} \cdot B_-.$$

where $g \cdot B$ denotes the conjugation action of $g \in G$ on a Borel subgroup B.

The main result in [**Rie08**] describes the critical point scheme of $\mathcal{F}_h$ as subscheme of $\mathcal{R}$ lying inside the Peterson variety. We denote by Y^*_{P,e^h} the (non-reduced) fiber over e^h of the Peterson variety, namely

$$Y^*_{P,e^h} = Y^*_P \times_{T^{W_P}} \{e^h\}.$$

THEOREM 2.3 [**Rie08**]. *The critical point scheme of $\mathcal{F}_h$ agrees with Y^*_{P,e^h}.*

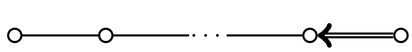
Putting this together with Peterson's presentation this result can be interpreted as follows. Suppose $h \in \mathfrak{h}^{W_P}$ represents a Kähler class $[\omega_h]$ under the identification $\mathfrak{h}^{W_P} = H^2(G^\vee/P^\vee)$.

COROLLARY 2.4. *The Jacobi ring $\mathbb{C}[Z_h]/(\partial \mathcal{F}_h)$ of $\mathcal{F}_h : Z_h \to \mathbb{C}$ is isomorphic to the quantum cohomology ring of the Kähler manifold $(G^\vee/P^\vee, [\omega_h])$ in its presentation due to Dale Peterson[*__Pet97__*].*

In [**MR13**], R. Marsh and the second author gave an expression of the Landau-Ginzburg model of the Grassmannian in terms of Plücker coordinates and then described the A-model connection. Here we will express the Landau-Ginzburg model of the Lagrangian Grassmannian in terms of *generalized Plücker coordinates*, i.e the coordinates of its minimal embedding $\mathrm{OG^{co}}(m+1, 2m+1) \hookrightarrow \mathbb{P}(V^*_{\mathrm{Spin}})$.

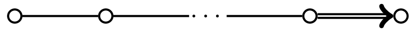
3. *The Lagrangian Grassmannian and its LG model*

Let $G^\vee = \mathrm{PSp}_{2m}(\mathbb{C})$, the adjoint group of type C_m, with Dynkin diagram



Let $P^\vee := P_{\omega_m^\vee}$ be the parabolic subgroup associated to the m-th fundamental weight $\omega_m^\vee$ of $G^\vee$. The quotient $G^\vee/P^\vee$ is the homogeneous space called the *Lagrangian Grassmannian*, which parametrizes Lagrangian subspaces in a $2m$-dimensional complex symplectic vector space. It is also denoted by $X = \mathrm{LG}(m)$ and will play the role of the A-model for us.

Now the Langlands dual group G is the simply connected group of type B_m, namely the spin group $\mathrm{Spin}_{2m+1}(\mathbb{C})$,



The parabolic subgroup of $\mathrm{Spin}_{2m+1}(\mathbb{C})$ associated to the m-th fundamental weight is denoted $P = P_{\omega_m}$. In this (B-model) setting we consider the quotient from the left $\check{X} := P\backslash G$. This quotient may be interpreted as the co-isotropic Grassmannian $\mathrm{OG}^{\mathrm{co}}(m+1, 2m+1)$ in a vector space of row vectors. We consider it in its minimal embedding, namely the homogeneous space $\check{X} := P\backslash G$ is embedded in $\mathbb{P}(V^*_{\omega_m})$ as right G-orbit of the highest weight vector $w^*_{\emptyset}$. We will express the mirror Landau-Ginzburg model to $LG(m)$ as a rational function on the orthogonal Grassmannian $\check{X}$ in the homogeneous coordinates of this embedding.

Remark 1. *Note that the Lagrangian Grassmannian $X = LG(m)$ is a cominuscule homogeneous space of type C_m, and therefore its cohomology appears in geometric Satake correspondence [*__Lus83__, __MV07__, __Gin97__*] as*

$$H^*(LG(m)) = IH^*(\overline{Gr^{\omega_m}_{G^\vee}}) = V^{\mathrm{Spin}_{2m+1}}_{\omega_m}.$$

*In other words it is canonically identified with the unique minuscule representation, the spin representation $V^{\mathrm{Spin}_{2m+1}}_{\omega_m}$ also denoted V_{Spin}, of the Langlands dual group, $G = \mathrm{Spin}_{2m+1}$. Therefore, essentially tautologically, $\mathbb{P}(V^*_{\mathrm{Spin}})$ has homogeneous coordinates given by the Schubert basis of $H^*(LG(m))$. This works of course for the other cominuscule homogeneous spaces. For instance, our recent preprint [*__PR13__*] deals with the case of odd-dimensional quadrics.*

3.1. *Notations and conventions*

Let $v_1, \dots, v_{2m+1}$ be the standard basis of $V = \mathbb{C}^{2m+1}$, and fix the symmetric non-degenerate bilinear form

$$\langle v_i, v_{2m+2-j}\rangle = 2\Phi(v_i, v_{2m+2-j}) = (-1)^{m+1-i}\delta_{i,j}.$$

We may also use the notation $\bar{v}_j = v_{2m+2-j}$ (with decreasing j) for the basis elements $v_{m+2}, \dots, v_{2m+1}$ and set $\epsilon(i) := (-1)^{m+1-i}$ so that $\Phi(v_i, \bar{v}_i) = \epsilon(i)$. The subspace of V spanned by the first m basis vectors $v_1, \dots, v_m$ is maximal isotropic and denoted by W.

We let $G = \mathrm{Spin}(V) = \mathrm{Spin}(V, \Phi)$, which is the universal covering group of $SO(V, \Phi)$. The Lie algebra of $G = \mathrm{Spin}(V)$ is therefore $\mathfrak{so}(V) = \mathfrak{so}(V, \Phi)$ which we view as lying in $\mathfrak{gl}(V)$. We have explicit Chevalley generators e_i, f_i given by

$$\begin{aligned} e_i &= E_{i,i+1} + E_{2m+1-i,2m+2-i} \quad \text{for } i = 1, \dots, m-1,\\ e_m &= \sqrt{2}E_{m,m+1} + \sqrt{2}E_{m+1,m+2},\\ f_i &= e_i^T \quad \text{for } i = 1, \dots, m. \end{aligned}$$

Here $E_{i,j}$ is the $(2m+1)\times(2m+1)$-matrix with (i,j)-entry 1 and all other entries 0. We also define the corresponding group homomorphisms $x_i : \mathbb{C} \to G$ and $y_i : \mathbb{C} \to G$, namely $x_i(a) := \exp(ae_i)$ and $y_i(a) := \exp(af_i)$. For Weyl group elements we can now choose specific representatives by setting $\dot{s}_i := x_i(1)y_i(-1)x_i(1)$, and $\dot{w} = \dot{s}_{i_1} \dots \dot{s}_{i_n}$ where $w = s_{i_1} \dots s_{i_n}$ is a reduced expression.

Next we introduce notations for the Clifford algebra $\mathrm{Cl}(V)$ and the Spin representation V_{Spin}, see also [**Var04**] whose conventions we follow for the most part. The Clifford algebra $\mathrm{Cl}(V)$ is the algebra quotient of the tensor algebra $T(V)$ by the ideal generated by the expressions

$$v \otimes v' + v' \otimes v - 2\Phi(v, v').$$

So it is the algebra with generators v_{m+1} and $v_i, \bar{v}_i$ for $i = 1, \dots, m$, with relations

$$v_i\bar{v}_i + \bar{v}_i v_i = \epsilon(i), \quad v_{m+1}^2 = \frac{1}{2},$$

and where all other generators anti-commute. The Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$-graded, as the relations are in even degrees only, and the even part of $\mathrm{Cl}(V)$ is denoted by $\mathrm{Cl}^+(V)$.

Since $\mathrm{Spin}(V)$ acts on V, it acts on $\bigwedge^\bullet V$, and because it preserves the bilinear form Φ, it also acts on $\mathrm{Cl}(V)$. The anti-symmetrization map

$$\begin{aligned}\bigwedge\nolimits^k V &\to \mathrm{Cl}(V)\\ v_{i_1}\wedge\ldots\wedge v_{i_k} &\mapsto \frac{1}{k!}\Big(\sum_{\sigma\in S_k} v_{i_{\sigma(1)}}v_{i_{\sigma(2)}}\cdots v_{i_{\sigma(k)}}\Big).\end{aligned}$$

is an embedding of representations, and we will usually identify elements of $\bigwedge^k V$ with their images, as we are mainly interested in the algebra structure of the Clifford algebra. The representation $\bigwedge^2 V$ is isomorphic to the adjoint representation. Moreover the image of $\bigwedge^2 V$ in $\mathrm{Cl}(V)$ is indeed a Lie algebra under the commutator Lie bracket of $\mathrm{Cl}(V)$, and it is isomorphic to $\mathfrak{so}(V)$ as such. In particular our generators e_i, f_i can be identified with elements of $\bigwedge^2 V$ and their images in $\mathrm{Cl}(V)$. Under this identification they are given by

$$\begin{aligned}e_i &= \epsilon(i+1)v_i\wedge\bar v_{i+1} = \epsilon(i+1)v_i\bar v_{i+1} \text{ for } i=1,\ldots,m-1\\ e_m &= \sqrt{2}v_m\wedge v_{m+1} = \sqrt{2}v_m v_{m+1},\\ f_i &= \epsilon(i)v_{i+1}\wedge\bar v_i = \epsilon(i)v_{i+1}\bar v_i \text{ for } i=1,\ldots,m-1,\\ f_m &= \sqrt{2}\bar v_m\wedge v_{m+1} = \sqrt{2}\bar v_m v_{m+1}.\end{aligned}$$

Putting all of the anti-symmetrization maps together gives an isomorphism of $\mathfrak{so}(V)$-modules

$$\bigwedge\nolimits^\bullet V \longrightarrow \mathrm{Cl}(V).$$

Moreover the even wedge powers map to the even part $\mathrm{Cl}^+(V)$ of the Clifford algebra and odd ones to the odd part, $\mathrm{Cl}^-(V)$. Therefore we have two isomorphisms of $\mathfrak{so}(V)$-modules

$$\alpha_+ : \bigwedge\nolimits^{\text{even}} V \longrightarrow \mathrm{Cl}^+(V), \tag{3.1}$$

$$\alpha_- : \bigwedge\nolimits^{\text{odd}} V \longrightarrow \mathrm{Cl}^-(V). \tag{3.2}$$

The Spin representation, as a vector space, is $V_{\mathrm{Spin}} = \bigwedge^\bullet W$. Its standard basis elements are the elements $w_I := v_{i_1}\wedge\ldots\wedge v_{i_k}$ with $i_1<i_2<\cdots<i_k$, where $I=\{i_1,\ldots,i_k\}$ is any subset of $\{1,\ldots,m\}$. We sometimes write $[v_{i_1}\wedge\ldots\wedge v_{i_k}]$ instead of $v_{i_1}\wedge\ldots\wedge v_{i_k}$ when we mean the element of V_{Spin}. Note that if $I=\emptyset$ then $w_\emptyset=[1]$.

The subsets I are also in one-to-one correspondence with strict partitions λ contained in an $m\times m$ square, by sending the empty set to the empty partition, and

$$I=\{i_1,\ldots,i_k\} \quad\mapsto\quad \lambda=(m+1-i_1, m+1-i_2,\ldots,m+1-i_k).$$

In this correspondence the k-row partitions correspond to the basis elements in the k-th graded component, $\bigwedge^k W$, of V_{Spin}. We may denote w_I also by w_λ. If λ is a strict partition contained in an $m\times m$ rectangle, then we denote by $|\lambda|$ the sum of all its parts and by $\mathrm{PD}(\lambda)$ the Poincaré dual partition.

The Spin representation of $\mathfrak{so}(V)$ extends to a representation of the Clifford algebra, which can be defined on generators by

$$v_i\cdot w_I = v_i\wedge w_I,\quad v_{m+1}\cdot w_I = \frac{(-1)^{|I|}}{\sqrt{2}}w_I,\quad \bar v_j\cdot w_I = i_{\bar v_j}(w_I),$$

where $i_{\bar v_i}$ is the insertion operator on $\bigwedge^\bullet W$, for $\bar v_i$ identified with the linear form $2\Phi(\bar v_i,\quad)$ on W.

We recall the important fact that the even subalgebra $\mathrm{Cl}^+(V)$ of the Clifford algebra is isomorphic to $\mathrm{End}(V_{\mathrm{Spin}})$ via the action just defined. Combined with the map (3.1) we obtain an isomorphism of $\mathfrak{so}(V)$-modules

$$\kappa_+ : \bigwedge^{\text{even}} V \longrightarrow \mathrm{End}(V_{\mathrm{Spin}}). \tag{3.3}$$

Moreover there is also an isomorphism of $\mathfrak{so}(V)$-modules,

$$\kappa_- : \bigwedge^{\text{odd}} V \longrightarrow \mathrm{End}(V_{\mathrm{Spin}}) \tag{3.4}$$

given by antisymmetrization, $\alpha_- : \bigwedge^{\text{odd}} \to \mathrm{Cl}^-(V)$ followed by the action of $\mathrm{Cl}^-(V)$ on V_{Spin}.

The standard basis $\{w_I\}$ of V_{Spin} defined above is also precisely the integral weight basis obtained by successively applying generators e_i to the lowest weight vector $w_\emptyset = [1]$, and it agrees with the MV-basis of V_{Spin}, which in this case is one and the same as the Schubert basis of $H^*(LG(m))$. We will use the notation σ_λ for the Schubert basis element naturally identified with w_λ.

The generalized Plücker coordinates on our $\mathrm{OG}^{\mathrm{co}}(m+1, 2m+1) = P\backslash G$ are the sections of $\mathcal{O}(1)$ in the embedding $P\backslash G \hookrightarrow \mathbb{P}(V_{\mathrm{Spin}}^*)$ which are given by the basis elements w_λ of V_{Spin} described above. Explicitly, we define

$$p_\lambda(g) := \langle w_\emptyset^* \cdot g, w_\lambda\rangle = w_\emptyset^*(g \cdot w_\lambda),$$

where $w_\emptyset^*$ is the dual basis vector to $w_\emptyset$, which is therefore a highest weight vector of $V_{\omega_m}^*$, and where w_λ is as above. We may think of an element $Pg \in \mathrm{OG}^{\mathrm{co}}(m+1, V^*) = P\backslash G$ as specified by its homogeneous coordinates $(p_{\lambda_1}(g) \colon p_{\lambda_2}(g) \colon \dots \colon p_{\lambda_{2^m}}(g))$, where $\lambda_1, \dots, \lambda_{2^m}$ are the strict partitions in $m \times m$ in some ordering.

To summarize, associated to strict partitions $\lambda \subset m \times m$, or equivalently subsets I of $\{1, \dots, m\}$, we have elements

$$\sigma_\lambda \in H^*(LG(m)), \quad w_\lambda \in V_{\mathrm{Spin}}, \quad \text{and} \quad p_\lambda \in \Gamma[\mathcal{O}_{\mathrm{OG}^{\mathrm{co}}(m+1,V^*)}(1)],$$

all canonically identified. We may also denote them by σ_I, w_I and p_I, respectively.

For a later section we will also require an explicit isomorphism $V \cong V^*$. Since V has on it a quadratic form, we have that $V \cong V^*$ by $v \mapsto \langle v, \ \rangle$ and V^* has basis $v_1^*, \dots, v_{m+1}^*, v_{m+2}^*, \dots, v_{2m+1}^*$. Under the isomorphism with V this basis corresponds to

$$\begin{array}{ccc}
v_1^* = \epsilon(1)\bar{v}_1 & & v_{2m+1}^* = \bar{v}_1^* = \epsilon(1)v_1 \\
v_2^* = \epsilon(2)\bar{v}_2 & & v_{2m}^* = \bar{v}_2^* = \epsilon(2)v_2 \\
\vdots & & \vdots \\
v_m^* = -\bar{v}_m & & v_{m+2}^* = \bar{v}_m^* = -v_m \\
& v_{m+1}^* = v_{m+1}. &
\end{array}$$

This also enables us to describe an equivariant isomorphism between $\bigwedge^m V$ and $\bigwedge^{m+1} V$, which is the composition of

$$c\colon \ \bigwedge^m V \to \bigwedge^{m+1} V^*$$

obtained by contraction with $(-1)^{\frac{m(m+1)}{2}} v_1^* \wedge \dots \wedge v_{2m+1}^*$ and of

$$d\colon \ \bigwedge^{m+1} V^* \to \bigwedge^{m+1} V$$

defined using the isomorphism $V \cong V^*$ given by the quadratic form.

3.2. *Definition of W_t and $\check{X}^\circ$*

We will now explain our formula for $W_t : \mathrm{OG}^{\mathrm{co}}(m+1, V^*) \dashrightarrow \mathbb{C}$ in terms of the coordinates p_λ. Here are some particular partitions which will play an important role. Let $\rho_l := (l, l-1, \dots, 2, 1)$ be the length l staircase partition and let $\mu_l := (m, m-1, \dots, m+1-l)$ be the maximal strict partition with l lines contained in an $m \times m$ rectangle. For ρ_l with $l < m$ there is a unique strict partition obtained by adding a single box to the Young diagram. It is obtained by adding one box to the first line, and we denote it by $\rho_{l,+}$. If J is any subset of $\{1,\ldots,l\}$, we denote by ρ_l^J the partition obtained after removing for every $j \in J$ the j-th line from the Young diagram of ρ_l (and similarly for $\rho_{l,+}^J$). On the other hand we denote by μ_l^J the partition obtained by adding for each $j \in J$ a row of $l+1-j$ boxes to the bottom of μ_l. Similarly, $\mu_{l,+}^J$ is obtained by adding for each $j \in J$ a row of $l+1-j+\delta_{j,1}$ boxes to the bottom of μ_l. If the resulting Young diagram does not give a strict partition, then we set $\mu_l^J = 0$, respectively $\mu_{l,+}^J = 0$. Finally, set $s(J) := \sum_{j\in J} j$ for any subset J of $\{1,\ldots,m\}$.

Using the above notations, we define $W_t : \mathrm{OG^{co}}(m+1, V^*) \to \mathbb{C}$ by

$$W_t := \frac{p_{\rho_{0,+}}}{p_{\rho_0}} + \sum_{l=1}^{m-1} \frac{\sum\limits_{J\subset\{1,\ldots,l\}} (-1)^{s(J)} p_{\rho_{l,+}^J} p_{\mu_{l,+}^J}}{\sum\limits_{J\subset\{1,\ldots,l\}} (-1)^{s(J)} p_{\rho_l^J} p_{\mu_l^J}} + e^t \frac{p_{\rho_{m-1}}}{p_{\rho_m}}. \tag{3.5}$$

This is a rational function on $\check{X} = \mathrm{OG^{co}}(m+1, V^*)$. Inside $\check{X}$ the denominators in W_t give rise to divisors

$$D_0 := \{p_\emptyset = 0\}, \quad D_m := \{p_{\rho_m} = 0\}$$

and

$$D_l := \left\{ \sum_{J\subset\{1,\ldots,l\}} (-1)^{s(J)} p_{\rho_l^J} p_{\mu_l^J} = 0 \right\}, \quad \text{where } l = 1,\ldots,m-1.$$

Then

$$D := D_0 + D_1 + \ldots + D_{m-1} + D_m$$

is an anticanonical divisor. Indeed, the index of $\check{X} = \mathrm{OG^{co}}(m+1, V^*)$ is $2m$. We define $\check{X}^\circ := \check{X} \setminus D$. The restriction of our rational function W_t to $\check{X}^\circ$ is regular, and is again denoted W_t.

3.3. *Statement of results*

We would like to compare $W_t : \check{X}^\circ \to \mathbb{C}$ with the known super-potential of $X = LG(m)$ defined as a special case of (2.2). Explicitly recall that $LG(m) = G^\vee/P^\vee$ for $G^\vee = PSp(2m)$ with $P^\vee$ the parabolic corresponding to the m-th node of the Dynkin diagram C_m. The function $\mathcal{F}_h$ for $h \in \mathfrak{h}^{W_P}$ is therefore defined on the open Richardson variety $\mathcal{R} = B_+ w_P B_- \cap B_- \dot{w}_0 B_-/B_-$ inside the full flag variety of $G = \mathrm{Spin}(V)$, where P is the parabolic corresponding to the m-th node of B_m. So we would like to relate our variety $\check{X} = P\backslash G = \mathrm{OG^{co}}(m+1, V^*)$, or rather its open part $\check{X}^\circ$, with this open Richardson variety. The parameter t in W_t and the $h \in \mathfrak{h}^{W_P}$ appearing in $\mathcal{F}_h$ should be thought of as equivalent, by the relation $h = t\omega_m^\vee$.

For fixed parameter t we define the following maps

$$\begin{array}{rcccl} \mathrm{OG^{co}}(m+1, V^*) = P\backslash G & \xleftarrow{\Psi_L} & B_- \dot{w}_0 \cap U_+ e^{t\omega_m^\vee} \dot{w}_P \dot{U}_- & \xrightarrow{\Psi_R} & \mathcal{R}, \\ Pg & \longleftarrow & g & \to & gB_-. \end{array}$$

given by taking left and right cosets, respectively. Note that $g = b_- \dot{w}_0$ in our previous notation and factorizes as

$$g = u_1 e^{t\omega_m^\vee} \dot{w}_P \bar{u}_2,$$

Moreover Ψ_R is an isomorphism, so we can define

$$\Psi := \Psi_L \circ \Psi_R^{-1} : \mathcal{R} \to \mathrm{OG^{co}}(m+1, V^*).$$

Our main goals here are to prove the two following theorems.

Theorem 3.1. *Let $X = LG(m)$ and $t \in \mathbb{C}$. The regular function W_t on $\check{X}^\circ$ defined in (3.5) pulls back under Ψ to the Landau-Ginzburg model $(\mathcal{R}, \mathcal{F}_h)$ from Theorem 2.4, where h and t are related by $h = t\omega_m^\vee$.*

Theorem 3.2. *Ψ defines an isomorphism from $\mathcal{R}$ to $\check{X}^\circ$.*

3.4. *Outline of the proof of Theorems 3.1 and 3.2*

Let $h = t\omega_m^\vee$ as in Theorem 3.1, and define $Z_h := B_-\dot{w}_0 \cap U_+ e^h \dot{w}_P \dot{w}_0^{-1} U_-$. The superpotential $\mathcal{F}_h$ pulls back under $Z_h \to G/B_-$ to $\tilde{\mathcal{F}}_h : Z_h \to \mathbb{C}$ where

$$\tilde{\mathcal{F}}_h(u_1 e^h \dot{w}_P \bar{u}_2) = \sum_{i=1}^m e_i^*(u_1) + \sum_{i=1}^m f_i^*(\bar{u}_2).$$

To prove Theorem 3.1 we need to show that W_t pulls back to $\tilde{\mathcal{F}}_h$ under $Z_h \dot{w}_0 \xrightarrow{\Psi_L} P\backslash G = \mathrm{OG}^{\mathrm{co}}(m+1, 2m+1)$. We will do this in two steps.

We consider two related projective embeddings of $\check{X} = \mathrm{OG}^{\mathrm{co}}(m+1, V^*)$, the standard one corresponding to $\bigwedge^{m+1} V^* = V^*_{2\omega_m}$, and the minimal one corresponding to the (right) representation $V^*_{\mathrm{Spin}} = V^*_{\omega_m}$ of $G = \mathrm{Spin}(V)$ composed with its Veronese embedding. So

$$\begin{aligned} \pi_1 : P\backslash G &\hookrightarrow \mathbb{P}(\textstyle\bigwedge^{m+1} V^*), \\ Pg &\mapsto \left\langle v^*_{m+1} \wedge v^*_{m+2} \wedge \cdots \wedge v^*_{2m+1} \cdot g \right\rangle, \\ \pi_2 : P\backslash G &\hookrightarrow \mathbb{P}(\mathrm{Sym}^2(V^*_{\mathrm{Spin}})), \\ Pg &\mapsto \left\langle (w^*_\emptyset \cdot w^*_\emptyset) \cdot g \right\rangle. \end{aligned}$$

The interesting numerators and denominators in W_t are made up of sections in $\Gamma[\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(V^*_{\mathrm{Spin}}))}(1)] = \mathrm{Sym}^2(V_{\mathrm{Spin}})$. However the pullback of $\tilde{\mathcal{F}}_h$ to $\check{X}$ is not easy to reformulate directly in those terms. It can be more easily expressed in terms of sections in $\Gamma[\mathcal{O}_{\mathbb{P}(\bigwedge^{m+1} V^*)}(1)] = \bigwedge^{m+1} V$, which correspond to $(m+1) \times (m+1)$-minors. The two embeddings are however related by an embedding of projective spaces coming from the inclusion of representations

$$\textstyle\bigwedge^{m+1} V^* \hookrightarrow \mathrm{Sym}^2(V^*_{\mathrm{Spin}}),$$

Therefore dually we have a surjection of representations

$$\mathrm{Sym}^2(V_{\mathrm{Spin}}) \twoheadrightarrow \textstyle\bigwedge^{m+1} V, \tag{3.6}$$

which is the restriction map $\Gamma[\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(V^*_{\mathrm{Spin}}))}(1)] \to \Gamma[\mathcal{O}_{\mathbb{P}(\bigwedge^{m+1} V^*)}(1)]$.

The first step of the proof of Theorem 3.1 is to express $\tilde{\mathcal{F}}_h$ in terms of $(m+1) \times (m+1)$-minors, which is done in Section 3.5. The conclusion of the proof of Theorem 3.1 involves the explicit construction of the map (3.6) of representations in terms of the Clifford Algebra. This is done in Section 3.6. The expression of W_t in terms of minors is then used in Section 3.7 to prove Theorem 3.2.

3.5. *A formula for $\tilde{\mathcal{F}}_h$ in terms of minors*

Definition 3.1. *If $g \in \mathrm{Spin}(V)$ we consider it as acting from the right on $\bigwedge^n V^*$ and from the left on $\bigwedge^n V$ for any $n = 1, \ldots, 2m+1$. The bases $\{v_i^*\}$ and $\{v_i\}$ give rise to bases of $\bigwedge^n V^*$ and $\bigwedge^n V$, and we use the following notation for the matrix coefficients (minors of g acting in the representation V). Let $I = \{i_1 < \ldots < i_r\}$ be a set indexing rows, and $J = \{j_1 < \cdots < j_r\}$ a set indexing columns, then*

$$D^I_J(g) := \langle v^*_{i_1} \wedge \cdots \wedge v^*_{i_r} \cdot g \ , \ v_{j_1} \wedge \cdots \wedge v_{j_r} \rangle.$$

We begin by arguing that $\bar{u}_2$ appearing in $u_1 e^h \dot{w}_P \bar{u}_2 \in Z_h$ can be assumed to lie in $U_- \cap B_+(\dot{w}^P)^{-1}B_+$. This is because we have two birational maps

$$\begin{aligned} \Psi_1 : U_- \cap B_+(\dot{w}^P)^{-1}B_+ \to P\backslash G : & \qquad \bar{u}_2 \mapsto P\bar{u}_2, \\ \Psi_2 : B_- \cap U_+ e^h \dot{w}^P U_- \to P\backslash G : & \; b_- = u_1 e^h \dot{w}_P \bar{u}_2 \mapsto Pb_-, \end{aligned}$$

which compose to give $\Psi_1^{-1} \circ \Psi_2 : b_- \mapsto \bar{u}_2$. This gives a birational map

$$\Psi_1^{-1} \circ \Psi_2 : Z_h \to U_- \cap B_+(\dot{w}^P)^{-1}B_+.$$

Now a generic element $\bar{u}_2$ in $U_- \cap B_+(\dot{w}^P)^{-1}B_+$ can be assumed to have a particular factorisation. Let $N := \binom{m+1}{2}$. The smallest representative w^P in W of $[w_0] \in W/W_P$ has the following reduced expression :

$$w^P = (s_m)(s_{m-1}s_m)\dots(s_1 s_2 \dots s_m) = s_{i_1}\dots s_{i_N},$$

It follows that as a generic element of $U_- \cap B_+(\dot{w}^P)^{-1}B_+$, the element $\bar{u}_2$ can be assumed to be written as:

$$\big(y_m(a_{m,m})y_{m-1}(a_{m-1,m})\dots y_1(a_{1,m})\big)\dots\big(y_m(a_{m,2})y_{m-1}(a_{m-1,2})\big)y_m(a_{m,1}).$$

where $a_{i,j} \neq 0$, or equivalently as

$$\bar{u}_2 = y_m(b_N)\dots y_2(b_{N-m+2})y_1(b_{N-m+1})\dots y_m(b_3)y_{m-1}(b_2)y_m(b_1). \tag{3.7}$$

with nonzero b_i. Note that the k-th factor here is $y_{i_{N-k+1}}(b_{N-k+1})$.

We may think of the Plücker coordinate p_λ as a function on G. Then we have the following standard expression for the p_λ on factorized elements.

LEMMA 3.3. *Fix λ a strict partition in an $m \times m$ square, and $w \in W^P$ the corresponding Weyl group element. Note that the length $\ell(w)$ equals $|\lambda|$. Then if $\bar{u}_2$ is of the form (3.7) we have*

$$p_\lambda(\bar{u}_2) = \sum_J b_{j_1}\dots b_{j_m}.$$

where the sum is over subsets $J = \{j_1 < j_2 < \dots < j_m\}$ of $\{1,\dots,N\}$ for which $s_{i_{j_1}}\dots s_{i_{j_m}}$ is a reduced expression of w.

Proof. Recall that by definition $p_\lambda(\bar{u}_2) = \langle w_\emptyset^* \cdot \bar{u}_2, w_\lambda\rangle = w_\emptyset^*(\bar{u}_2 \cdot w_\lambda)$ and $w_\lambda = e_{i_{j_1}}\dots e_{i_{j_m}} \cdot w_\emptyset$ if $w = s_{i_{j_1}}\dots s_{i_{j_m}}$ is a reduced expression. So in an expansion for $\bar{u}_2$ the coefficients of $f_{i_{j_m}}\dots f_{i_{j_1}}$ will contribute a summand of $b_{j_1}\dots b_{j_m}$ to $p_\lambda(\bar{u}_2)$. $\square$

PROPOSITION 3.4. *If u_1 and $\bar{u}_2$ are as above then we have the following identities*

$$f_m^*(\bar{u}_2) = \frac{p_{\rho_{0,+}}(\bar{u}_2)}{p_{\rho_0}(\bar{u}_2)}, \tag{3.8}$$

$$e_i^*(u_1) = 0 \textit{ for all } 1 \le i \le m-1, \tag{3.9}$$

$$e_m^*(u_1) = e^t \frac{p_{\rho_{m-1}}(\bar{u}_2)}{p_{\rho_m}(\bar{u}_2)}, \tag{3.10}$$

where $\rho_0 = \emptyset$ and $\rho_{0,+} = \square$.

Proof. For (3.8) notice that in fact $p_\emptyset(\bar{u}_2) = 1$ and

$$p_\square(\bar{u}_2) = \langle w_\emptyset^* \cdot \bar{u}_2, w_\square\rangle = w_\emptyset^*(\bar{u}_2 \cdot w_\square).$$

Then (3.8) is apparent since $f_m \cdot w_\Box = w_\emptyset$. In fact (3.8) does not depend on the special form of u_1 and $\bar{u}_2$. The equations (3.9) and (3.10) are consequences of the Lemmas A.2 and A.3, respectively, as well as the Lemma 3.3. $\square$

PROPOSITION 3.5.

$$f_j^*(\bar{u}_2) = \frac{D^{m+1,\dots,2m+1}_{j,j+2,\dots,j+m+1}(\bar{u}_2)}{D^{m+1,\dots,2m+1}_{j+1,\dots,j+m+1}(\bar{u}_2)} \quad \textit{for all } 1 \le j \le m-1 \tag{3.11}$$

Proof. The result is a consequence of the vanishing of the following minor of $\bar{u}_2$:

$$D^{j+1,m+1,\dots,2m+1}_{j,j+1,\dots,j+m+1}(\bar{u}_2),$$

which is equal to

$$\langle v^*_{j+1} \wedge v^*_{m+1} \cdots \wedge v^*_{2m+1} \cdot g \ , \ v_j \wedge v_{j+1} \cdots \wedge v_{j+m+1} \rangle.$$

Define an element in the enveloping algebra

$$\underline{e} := \left(e_m^{(a_{1,m})} e_{m-1}^{(a_{1,m-1})} \dots e_1^{(a_{1,1})}\right) \dots \left(e_m^{(a_{m-1,m})} e_{m-1}^{(a_{m-1,m-1})}\right) e_m^{(a_{m,m})},$$

where $a_{i,j} \in \{0,1,2\}$ if $j = m$ and $a_{i,j} \in \{0,1\}$ otherwise. Here $e_i^{(a)} = \frac{1}{a!} e_i^a$. Due to the shape of $\bar{u}_2$, the minor is zero if for any such $\underline{e}$, $v^*_{j+1} \wedge v^*_{m+1} \cdots \wedge v^*_{2m+1} \cdot \underline{e}$ has zero $v^*_j \wedge v^*_{j+1} \cdots \wedge v^*_{j+m+1}$-component. Assume by contradiction that there exists an $\overline{e}$ such that this component is nonzero.

First suppose $j = m-1$. Then since $v^*_m \wedge v^*_{m+1} \cdots \wedge v^*_{2m+1} \cdot e_m = 0$, the exponent $a_{1,m}$ in $\underline{e}$ has to be zero. Now the v^*_{2m+1} has to be moved to v^*_{2m}, which means that v^*_m needs to be moved *before* to v^*_{m-1} by an e_{m-1}. Since only one e_1 appears in the expression of $\underline{e}$, it means that $a_{1,m-1} = 1$. Hence $v^*_m \wedge v^*_{m+1} \cdots \wedge v^*_{2m+1} \cdot \underline{e}$ is equal to

$$v^*_{m-1} \wedge v^*_{m+1} \wedge \cdots \wedge v^*_{2m+1} \cdot \left(e_{m-2}^{a_{1,m-2}} \dots e_1^{a_{1,1}}\right) \dots \left(e_m^{a_{m-1,m}} e_{m-1}^{a_{m-1,m-1}}\right) e_m^{a_{m,m}}.$$

Since $v^*_{m-1} \wedge v^*_{m+1} \wedge \cdots \wedge v^*_{2m+1} \cdot e_i = 0$ for all $1 \le i \le m-2$, it follows that $a_{1,m-2} = \cdots = a_{1,2} = a_{1,1} = 0$, which means that the v^*_{2m+1} can never be moved to v^*_{2m}. Hence there exists no $\underline{e}$ such that $v^*_{j+1} \wedge v^*_{m+1} \cdots \wedge v^*_{2m+1} \cdot \underline{e}$ has nonzero $v^*_j \wedge v^*_{j+1} \cdots \wedge v^*_{j+m+1}$-component.

Now suppose $j < m-1$. v^*_{2m+1} has to be moved to v^*_{2m} by the only e_1 in the expression of $\underline{e}$, hence $a_{1,1} = 1$. But $v^*_{m+1}, \dots, v^*_{2m}$ need to be moved before, hence $a_{1,i} = 1$ for $1 \le i \le m-1$ and $a_{1,m} = 2$. It follows that $v^*_{j+1} \wedge v^*_{m+1} \cdots \wedge v^*_{2m+1} \cdot \underline{e}$ is equal to

$$\left(v^*_{j+1} \wedge v^*_m \cdots \wedge v^*_{2m} + v^*_1 \wedge v^*_m \cdots \wedge v^*_{m-j} \wedge v^*_{m+2-j} \wedge \cdots \wedge v^*_{2m+1}\right) \cdot \underline{e}',$$

where

$$\underline{e}' := \left(e_m^{a_{2,m}} e_{m-1}^{a_{2,m-1}} \dots e_2^{a_{2,2}}\right) \dots \left(e_m^{a_{m-1,m}} e_{m-1}^{a_{m-1,m-1}}\right) e_m^{a_{m,m}}.$$

Then

$$v^*_1 \wedge v^*_m \cdots \wedge v^*_{m-j} \wedge v^*_{m+2-j} \wedge \cdots \wedge v^*_{2m+1} \cdot \underline{e}'$$

has clearly no non-zero $v^*_j \wedge v^*_{j+1} \cdots \wedge v^*_{j+m+1}$-component, hence we focus on $v^*_{j+1} \wedge v^*_m \cdots \wedge v^*_{2m} \cdot \underline{e}'$.

If $j = m-2$, then v^*_{2m} has to be moved to v^*_{2m-1} by the only e_2 in $\underline{e}'$. Hence $a_{2,2} = 1$. But $v^*_{m-1} \wedge v^*_m \cdots \wedge v^*_{2m} \cdot e_m = 0$, which means that $a_{2,m} = 0$. It follows that v^*_{m+1} cannot be moved to v^*_m before having to move the v^*_{2m}, and hence that a suitable $\underline{e}$ does not exist.

Finally if $j \le m-3$, then $v^*_{j+1} \wedge v^*_m \cdots \wedge v^*_{2m} \cdot e_i = 0$ for all $j+1 \le i \le m$, hence $a_{2,j+1} = \cdots = a_{2,m} = 0$. It follows that the v^*_{m+1-j} cannot be moved before the v^*_{2m} has to be by the only remaining e_2 in $\underline{e}'$. This concludes the proof of the minor vanishing.

To prove the proposition, we only need to expand this vanishing minor with respect to the $(j+1)$-st row. Indeed, due to $\bar{u}_2$ being lower triangular, this row has only two non-zero entries : 1 on the $(j+1)$-st column and $f_j^*(\bar{u}_2)$ on the j-th column. □

3.6. *The Clifford Algebra and homogeneous coordinates*

3.6.1. *Setting* In this section we study the surjection of representations from (3.6), that is

$$\pi : \mathrm{Sym}^2(V_{\mathrm{Spin}}) \to \bigwedge{}^{m+1} V,$$

which is also interpreted as the restriction map of homogeneous coordinates

$$\Gamma[\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2(V^*_{\mathrm{Spin}}))}(1)] \to \Gamma[\mathcal{O}_{\mathbb{P}(\bigwedge^{m+1} V^*)}(1)].$$

Of course in representation-theoretic terms the map π exists just because $\bigwedge^{m+1} V$ is irreducible with highest weight $2\omega_m$ and this highest weight also occurs in $\mathrm{Sym}^2(V_{\mathrm{Spin}})$ with multiplicity one. But in order to compute with this map we will need to use a more intrinsic construction. We first note the following auxiliary lemma, whose proof is straightforward.

Lemma 3.6. *The isomorphism*

$$\begin{array}{rcl} \delta : V_{\mathrm{Spin}} & \to & V^*_{\mathrm{Spin}} \\ v_\lambda & \mapsto & (-1)^{|\lambda|} v_{\mathrm{PD}(\lambda)} \end{array}$$

is $\mathfrak{so}(V)$-equivariant.

For the construction of the map π first we define an equivariant embedding

$$\iota_{V_{\mathrm{Spin}}} : \mathrm{Sym}^2(V_{\mathrm{Spin}}) \hookrightarrow V_{\mathrm{Spin}} \otimes V_{\mathrm{Spin}} \xrightarrow{\delta \otimes \mathrm{id}_{V_{\mathrm{Spin}}}} V^*_{\mathrm{Spin}} \otimes V_{\mathrm{Spin}} = \mathrm{End}(V_{\mathrm{Spin}}).$$

Then there are two subtly different cases to distinguish.

Case 1: If m is odd then we construct π as follows. Applying the constructions from Section 3.1 we have an isomorphism of representations (3.4),

$$\kappa_-^{-1} : \mathrm{End}(V_{\mathrm{Spin}}) \to \mathrm{Cl}^-(V) \to \bigoplus_{k=0}^{m} \bigwedge{}^{2k+1} V.$$

Because m is odd we have a projection onto the summand with $k = \frac{m-1}{2}$,

$$\mathrm{pr}_{\wedge^m} : \bigoplus_{k=0}^{m} \bigwedge{}^{2k+1} V \to \bigwedge{}^{m} V.$$

Recall also the equivariant isomorphisms from Subsection 3.1 :

$$c : \bigwedge{}^{m} V \to \bigwedge{}^{m+1} V^*$$

and

$$d : \bigwedge{}^{m+1} V^* \to \bigwedge{}^{m+1} V.$$

Composing $\iota_{V_{\mathrm{Spin}}}$ with these four maps gives us our homomorphism of representations

$$\pi : \mathrm{Sym}^2(V_{\mathrm{Spin}}) \longrightarrow \bigwedge{}^{m+1} V.$$

Case 2: Suppose m is even. In this case we use the even part of the Clifford algebra of V, namely we use the inverse of the isomorphism from (3.3)

$$\kappa_+^{-1} : \mathrm{End}(V_{\mathrm{Spin}}) \to \mathrm{Cl}^+(V) \to \bigoplus_{k=0}^{m} \bigwedge^{2k} V.$$

Since m is even we have a projection onto the middle summand, $k = \frac{m}{2}$,

$$\mathrm{pr}_{\wedge^m} : \bigoplus_{k=0}^{m} \bigwedge^{2k} V^* \to \bigwedge^m V.$$

Finally we use the isomorphism of representations c as in Case 1,

$$c : \bigwedge^m V \xrightarrow{\sim} \bigwedge^{m+1} V^*$$

as well as the map

$$d : \bigwedge^{m+1} V^* \to \bigwedge^{m+1} V.$$

Composing $\iota_{V_{\mathrm{Spin}}}$ with these four maps gives us our homomorphism of representations

$$\pi : \mathrm{Sym}^2(V_{\mathrm{Spin}}) \longrightarrow \bigwedge^{m+1} V$$

in the case where m is even.

3.6.2. *Statement*

Definition 3.2. *Corresponding to the quadratic denominators in W_t we define elements of $\mathrm{Sym}^2(V_{\mathrm{Spin}})$ by*

$$\mathcal{D}_{(j)} := \sum_I (-1)^{s(I)} w_{\rho^I_{m+1-j}} w_{\mu^I_{m+1-j}}$$

and

$$\mathcal{N}_{(j)} := \sum_I (-1)^{s(I)} w_{\rho^I_{m+1-j,+}} w_{\mu^I_{m+1-j,+}}$$

where the sums are over all subsets $I \subset \{1, \dots, m+1-j\}$ and $j = 2, \dots, m$.

We will prove the following formulas.

Proposition 3.7. *Let $j = 2, \dots, m$. We have*

$$\sum_I (-1)^{s(I)} p_{\rho^I_{m+1-j}}(\bar u_2) p_{\mu^I_{m+1-j}}(\bar u_2) = D^{m+1,\dots,2m+1}_{m+2-j,\dots,2m+2-j}(\bar u_2)$$

and

$$\sum_I (-1)^{s(I)} p_{\rho^I_{m+1-j,+}}(\bar u_2) p_{\mu^I_{m+1-j,+}}(\bar u_2) = D^{m+1,\dots,2m+1}_{m+1-j,m+3-j\dots,2m+2-j}(\bar u_2)$$

where the sums are over all subsets $I \subset \{1, \dots, m+1-j\}$.

Note that this proposition gives us an alternative definition of $\check X^\circ$.

Corollary 3.8. *The affine open subset $\check X^\circ$ of $\check X = P\backslash G$ is*

$$\left\{ Pg \in \check X \mid D^{m+1,\dots,2m+1}_{m+1-j,m+3-j\dots,2m+2-j}(g) \neq 0,\ j = 1, \dots, m-1, p_\emptyset(g) \neq 0,\ p_{\rho_m}(g) \neq 0 \right\}.$$

We remark that the Plücker coordinates $p_{\emptyset}(g)$ and $p_{\rho_m}(g)$ are Pfaffians of g.

3.6.3. *Proof of Proposition 3.7* To prove Proposition 3.7, we will need to compare $\mathcal{D}_{(j)}, \mathcal{N}_{(j)} \in \mathrm{Sym}^2(V_{\mathrm{Spin}})$ to the elements of $\bigwedge^{m+1} V$ defined below.

DEFINITION 3.3. *Inside the exterior power $\bigwedge^{m+1} V$, if $2 \leq j \leq m$ we consider the elements*

$$v^{\wedge}_{(j)} := v_j \wedge \cdots \wedge v_{j+m}$$
$$v^{\wedge}_{(j),+} := v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_{j+m}$$

of $\bigwedge^{m+1} V$.

We will show :

PROPOSITION 3.9. *The projection map $\pi : \mathrm{Sym}^2(V_{\mathrm{Spin}}) \longrightarrow \bigwedge^{m+1} V$ takes $\mathcal{D}_{(j)}$ to $v^{\wedge}_{(j)}$ and $\mathcal{N}_{(j)}$ to $v^{\wedge}_{(j),+}$.*

We will in fact prove this proposition only for the denominators $\mathcal{D}_{(j)}$, the case of the numerators $\mathcal{N}_{(j)}$ being extremely similar.

DEFINITION 3.4. *If $I = \{1 \leq i_1 < \cdots < i_r \leq 2m+1\}$, we define v_I to be the product $v_{i_1} \cdot \cdots \cdot v_{i_r}$ in $\mathrm{Cl}(V)$. For $I = \{j, j+1, \ldots, j+m\}$ we also denote v_I by $v_{(j)}$, so $v_{(j)} = v_j v_{j+1} \cdots v_{j+m}$. Moreover, if L is a subset of $\{j, \ldots, m\}$, we write $v^L_{(j)}$ for the Clifford algebra element obtained from the product $v_{(j)}$ by removing all of the factors v_l and $\bar{v}_l = v_{2m+2-l}$ for which $l \in L$.*

LEMMA 3.10. *The map $\iota_{V_{\mathrm{Spin}}} : \mathrm{Sym}^2(V_{\mathrm{Spin}}) \hookrightarrow \mathrm{End}(V_{\mathrm{Spin}})$ maps $\mathcal{D}_{(j)}$ to*

$$\beta_{m,j} \cdot \sum_I \left[w^*_{\mu^I_{j-1}} \otimes w_{\mu^I_{m+1-j}} + (-1)^{(m+1-j)(j-1)} w^*_{\rho^I_{j-1}} \otimes w_{\rho^I_{m+1-j}} \right] \tag{3.12}$$

where

$$\beta_{m,j} := \frac{(-1)^{\frac{(m+1-j)(m+2-j)}{2}}}{2}$$

and the sum is over all subsets I of $\{1, \ldots, m+1-j\}$.

Proof. First $w_{\rho^I_{m+1-j}} w_{\mu^I_{m+1-j}}$ maps to

$$\frac{1}{2}(w_{\rho^I_{m+1-j}} \otimes w_{\mu^I_{m+1-j}} + w_{\mu^I_{m+1-j}} \otimes w_{\rho^I_{m+1-j}}) \in V_{\mathrm{Spin}} \otimes V_{\mathrm{Spin}}.$$

Then according to Lemma 3.6 :

$$w_{\rho^I_{m+1-j}} \mapsto (-1)^{\frac{(m+1-j)(m+2-j)}{2} - s(I)} w^*_{\mu^I_{j-1}} \in V^*_{\mathrm{Spin}}$$
$$w_{\mu^I_{m+1-j}} \mapsto (-1)^{\frac{m(m+1)}{2} - \frac{j(j-1)}{2} + s(I)} w^*_{\rho^I_{j-1}} \in V^*_{\mathrm{Spin}},$$

hence the result. □

We now need to map the element (3.12) to the Clifford algebra of V.

PROPOSITION 3.11.

$$\mathcal{D}_{(j)} \mapsto \frac{(-1)^{\frac{m(m+1)}{2}}}{2}\Bigg[2v_{1,\dots,m+1-j,2m+3-j,\dots,2m+1} + \sum_{I\subsetneq\{1,\dots,m+1-j\}}\left(\prod_{l\in\{1,\dots,m+1-j\}\setminus I}(-1)^l\right)v_{I\cup\{2m+3-j,\dots,m+j\}\cup\overline{I}}\Bigg] \in \mathrm{Cl}(V). \quad (3.13)$$

Proof. We assume $j > \frac{m+1}{2}$, the other case being symmetric. For convenience, let us denote the right-hand side as $A_{(j)} \in \mathrm{Cl}(V)$. Because of the definition of the Clifford algebra :

$$v_{1,\dots,m+1-j,2m+3-j,\dots,2m+1} = (-1)^{m(m+1-j)}v_{\{1,\dots,m+1-j\}\cup\overline{\{1,\dots,m+1-j\}}}\,t_{(j)},$$

where $t_{(j)} = v_{2m+3-j}\dots v_{m+j}$. Similarly

$$v_{I\cup\{2m+3-j,\dots,m+j\}\cup\overline{I}} = (-1)^{m|I|}v_{I\cup\overline{I}}\,t_{(j)}.$$

We will use two lemmas :

LEMMA 3.12. *Let I be a subset of $\{1,\dots,m\}$. Then*

$$v_{I\cup\overline{I}} \mapsto \left(\prod_{i\in I}\epsilon(i)\right)\sum_L w_L^*\otimes w_L \in \mathrm{End}(V_{\mathrm{Spin}}),$$

where the sum is over all subsets L of $\{1,\dots,m\}$ containing I.

Proof Proof of lemma 3.12. First notice that

$$v_{\bar{i}}\cdot w_L = \begin{cases} 0 & \text{if } i\notin L\\ (-1)^{\#\{l\in L|l<i\}}\epsilon(i)w_{L\setminus\{i\}} & \text{otherwise,}\end{cases}$$

and

$$v_iv_{\bar{i}}\cdot w_L = \begin{cases} 0 & \text{if } i\notin L\\ \epsilon(i)w_L & \text{otherwise.}\end{cases}$$

Hence $v_{I\cup\overline{I}}$ is zero unless $L\supset I$. Now assume $L\supset I$ and write $I=\{i_1<i_2<\dots<i_r\}$. From the definition of the Clifford algebra, it follows that $v_{I\cup\overline{I}} = \prod_{p=1}^r v_{i_p}v_{\overline{i_p}}$. Hence :

$$v_{I\cup\overline{I}}\cdot w_L = \left(\prod_{p=1}^r\epsilon(i_p)\right)w_L.$$

The claim follows. □

LEMMA 3.13. *The element $t_{(j)} = v_{2m+3-j}\dots v_{m+j}$ of $\mathrm{Cl}(V)$ maps to*

$$\left(\prod_{p=m+2-j}^{j-1}\epsilon(p)\right)\sum_{K_1,K_2}(-1)^{m|K_1|}w^*_{K_1\cup\{m+2-j,\dots,j-1\}\cup K_2}\otimes w_{K_1\cup K_2}$$

in $\mathrm{End}(V_{\mathrm{Spin}})$, where K_1 is any subset of $\{1,\dots,m+1-j\}$ and K_2 is any subset of $\{j,\dots,m\}$.

Proof Proof of lemma 3.13. As in the proof of Lemma 3.12, we notice that $t_{(j)}\cdot w_L = 0$ if $L\not\supset\{m+2-j,\dots,j-1\}$. Now write $L = L_1\cup\{m+2-j,\dots,j-1\}\cup L_2$, where $L_1\subset$

$\{1, \ldots, m+1-j\}$ and $L_2 \subset \{j, \ldots, m\}$. We have

$$v_{m+j} \cdot w_L = (-1)^{m|L_1|}\epsilon(m+2-j) w_{L_1 \cup \{m+3-j, \ldots, j-1\} \cup L_2}.$$

Recursively, we obtain :

$$t_{(j)} \cdot w_L = (-1)^{m|L_1|} \left(\prod_{p=m+2-j}^{j-1} \epsilon(p) \right) w_{L_1 \cup L_2},$$

hence the lemma. □

Now to prove Proposition 3.11, first assume $L = \{1, \ldots, j-1\} \cup L_2$, where $L_2 \subset \{j, \ldots, m\}$. Then

$$v_{1,\ldots,m+1-j,2m+3-j,\ldots,2m+1} \cdot w_L = \left(\prod_{p=1}^{j-1} \epsilon(p) \right) w_{L_1 \cup L_2},$$

and

$$\left(\prod_{l \in \{1,\ldots,m+1-j\} \setminus I} (-1)^l \right) v_{I \cup \{2m+3-j,\ldots,m+j\} \cup \overline{I}} \cdot w_L$$

is equal to

$$\left(\prod_{p=1}^{j-1} \epsilon(p) \right) (-1)^{|I|} (-1)^{m+1-j} w_{L_1 \cup L_2}.$$

Hence

$$\begin{aligned} A_{(j)} \cdot w_L &= \left(\prod_{p=1}^{j-1} \epsilon(p) \right) \left[2 + (-1)^{m+1-j} \sum_{I \subsetneq \{1,\ldots,m+1-j\}} (-1)^{|I|} \right] w_{L_1 \cup L_2} \\ &= \left(\prod_{p=1}^{j-1} \epsilon(p) \right) w_{L_1 \cup L_2}. \end{aligned}$$

Now assume $L = L_1 \cup \{m+2-j, \ldots, j-1\} \cup L_2$, where $L_1 \subsetneq \{1, \ldots, m+1-j\}$ and $L_2 \subset \{j, \ldots, m\}$. Then

$$A_{(j)} \cdot w_L = \left(\prod_{p=1}^{j-1} \epsilon(p) \right) (-1)^{m|L_1|} (-1)^{(m+1)(m+1-j)} \sum_{I \subset L_1} (-1)^{|I|} w_{L_1 \cup L_2}.$$

Finally :

$$A_{(j)} \cdot w_L = \begin{cases} 0 & \text{if } L_1 \neq \emptyset \\ \left(\prod_{p=1}^{j-1} \epsilon(p) \right) (-1)^{(m+1)(m+1-j)} w_{L_2} & \text{otherwise.} \end{cases}$$

Looking precisely at the expression of $\mathcal{D}_{(j)}$ in $\mathrm{End}(V_{\mathrm{Spin}})$, this concludes the proof of the proposition. □

COROLLARY 3.14. *We have :*

$$\mathrm{pr}_{\wedge^m} \circ \kappa_{\pm}^{-1} \circ \iota_{V_{\mathrm{Spin}}}(\mathcal{D}_{(j)}) = (-1)^{\frac{m(m+1)}{2}} v_1 \wedge \cdots \wedge v_{m+1-j} \wedge v_{2m+3-j} \wedge \cdots \wedge v_{2m+1}$$

where $\kappa_{\pm}$ is κ_- if m is odd and κ_+ otherwise.

Proof. The result is a simple consequence of Proposition 3.11 and of the definition of the antisymmetrisation maps (3.1) and (3.2). □

We can now prove Proposition 3.9 :

Proof Proof of Proposition 3.9. From Corollary 3.14, we know that $\mathcal{D}_{(j)}$ maps to $(-1)^{\frac{m(m+1)}{2}} v_1 \wedge \cdots \wedge v_{m+1-j} \wedge v_{2m+3-j} \wedge \cdots \wedge v_{2m+1}$ in $\bigwedge^m V$. Now the latter element is mapped by the contraction c to

$$(-1)^{(m+1)(j-1)} v^*_{m+2-j} \wedge \cdots \wedge v^*_{2m+2-j}.$$

Then we map this to $\bigwedge^{m+1} V$ using the isomorphism d. The element $v^*_{m+2-j} \wedge \cdots \wedge v^*_{2m+2-j}$ maps to :

$$\left(\prod_{i=m+2-j}^{m} \epsilon(i)\right)\left(\prod_{k=j}^{m} \epsilon(k)\right) v_{j+m} \wedge v_{j+m+1} \wedge \cdots \wedge v_j (-1)^{j^2+m^2-mj+1} v^{\wedge}_{(j)}.$$

Now

$$\mathcal{D}_{(j)} \mapsto v^{\wedge}_{(j)},$$

which concludes the proof. □

3.7. *Isomorphism with the open Richardson variety*

Here we prove Theorem 3.2. We use the presentation of the coordinate ring of the open Richardson variety due to [**GLS11**], which we reformulate here using our notations. First notice that the open Richardson variety $\mathcal{R} \cong B_-\dot{w}_0 \cap U_+\dot{w}_P U_-$ is isomorphic to the 'unipotent cell' $U_- \cap B_+(\dot{w}^P)^{-1}B_+$, in the terminology of [**GLS11**], via the map

$$U_-^P := U_- \cap B_+(\dot{w}^P)^{-1}B_+ \to \mathcal{R} : \bar{u}_2 \mapsto \bar{u}_2\dot{w}_0 B_-.$$

To state the result we need to recall the definition of a generalized minor. In our setting $G = \mathrm{Spin}_{2m+1}(\mathbb{C})$.

Definition 3.5. *Let $v \in W$ and ω_j a fundamental weight of* $\mathrm{Spin}_{2m+1}(\mathbb{C})$. *Let V_{ω_j} denote the irreducible representation with highest weight ω_j and $v^+_{\omega_j}$ a fixed highest weight vector. Then for any $g \in \mathrm{Spin}_{2m+1}(\mathbb{C})$ we can set*

$$\Delta_{\omega_j, v\cdot\omega_j}(g) = \left\langle g\dot{v} \cdot v^+_{\omega_j}, v^+_{\omega_j} \right\rangle.$$

where the brackets $\left\langle v\, ,\, v^+_{\omega_j} \right\rangle$ refers to the coefficient of $v^+_{\omega_j}$ in the projection of v to the weight space of $v^+_{\omega_j}$.

*If $g\dot{v}$ has a decomposition into factors U_-TU_+ then this agrees with ω_j applied to the torus part of $g\dot{v}$, which is the definition given by Fomin and Zelevinsky [**FZ99**].*

Theorem 3.15 [**GLS11**, Section 8]. *For $P = P_{\omega_m}$, the coordinate ring of the unipotent cell $U_-^P := U_- \cap B_+(\dot{w}^P)^{-1}B_+$ is described as follows. Consider the reduced expression of $(w^P)^{-1}$ given by*

$$(w^P)^{-1} = s_{i_1} \dots s_{i_N} = (s_m \dots s_2 s_1) \dots (s_m s_{m-1})(s_m),$$

with length denoted $N = \frac{m(m+1)}{2}$. For every $1 \leq r \leq N$, we let

$$(w^P)^{-1}_{\leq r} := s_{i_1} \dots s_{i_r}$$

and consider indices $r_1, r_2, \dots r_m$ given by

$$r_k = \sum_{i=m+1-k}^{m} i = \binom{m+1}{2} - \binom{m-k+1}{2}.$$

Note that $i_{r_k} = k$. Then the coordinate ring of U_-^P is given by

$$\mathbb{C}\left[U_-^P\right] = \mathbb{C}\left[\Delta_{\omega_{i_r},(w^P)^{-1}_{\leq r}\cdot\omega_{i_r}}, \Delta^{-1}_{\omega_k,(w^P)^{-1}_{\leq r_k}\cdot\omega_k}\right]_{1\leq r\leq N, 1\leq k\leq m}.$$

Let us now reformulate the description of this coordinate ring for our purposes.

Corollary 3.16. *The coordinate ring $\mathbb{C}\left[U_-^P\right]$ is generated by the ordinary minors*

$$D^{1,\dots,m-j-t,m+2,\dots,m+2+t}_{1,2,\dots,m+1-j}, \quad 0\leq t\leq m-1, \quad 2\leq j\leq m-t \tag{3.14}$$

together with the Pfaffians

$$\Delta_{\omega_m,\frac{1}{2}[\epsilon_1+\cdots+\epsilon_{m-1-r}-(\epsilon_{m-r}+\cdots+\epsilon_m)]}, \quad 0\leq r\leq m-1 \tag{3.15}$$

the inverses of minors

$$\left(D^{m+2,\dots,m+1+r_k}_{1,\dots,r_k}\right)^{-1}, \quad 1\leq k\leq m-1, \tag{3.16}$$

and the inverse of Pfaffian

$$\Delta^{-1}_{\omega_m,-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_m)}, \tag{3.17}$$

Note that the $D^{1,\dots,m-j-t,m+2,\dots,m+2+t}_{1,2,\dots,m+1-j}$ are the minors defined in Section 3.6. We may now prove the isomorphism :

Proof Proof of Theorem 3.2. Our first step is to prove that the map $(U_-^P \to \check{X}; \bar{u}_2 \mapsto P\bar{u}_2)$ lands inside $\check{X}^\circ$. Recall from Corollary 3.8 the description of $\check{X}^\circ$ using minors,

$$\check{X}^\circ = \left\{Pg \mid D^{m+1,\dots,2m+1}_{j+1,\dots,j+m+1}(g)\neq 0 \text{ for all } 1\leq j\leq m-1, p_\emptyset(g)\neq 0, p_{\rho_m}(g)\neq 0\right\}.$$

Let $\bar{u}_2$ be in U_-^P. Using the isomorphism between $\bigwedge^m V$ and $\bigwedge^{m+1} V$ from Section 3.1, we get

$$D^{m+1,\dots,2m+1}_{j+1,\dots,j+m+1}(\bar{u}_2) = D^{m+2,\dots,2m+1}_{1,\dots,m-j,2m+2-j,\dots,2m+1}(\bar{u}_2).$$

Now using that $\bar{u}_2$ is in U_-, we get

$$D^{m+2,\dots,2m+1}_{1,\dots,m-j,2m+2-j,\dots,2m+1}(\bar{u}_2) = D^{m+2,\dots,2m+1-j}_{1,\dots,m-j}(\bar{u}_2).$$

By Cor. 3.16, for all $\bar{u}_2\in U_-^P$, the following minors do not vanish,

$$D^{m+2,\dots,m+1+r_k}_{1,\dots,r_k}(\bar{u}_2)$$

for $1\leq k\leq m-1$. Setting $k=m-j$ we find that $P\bar{u}_2$ is in $\check{X}^\circ$.

So we now have an algebraic map between affine varieties $U_-^P\to\check{X}^\circ$, which induces a pullback map $\mathbb{C}[U_-^P]\to\mathbb{C}[\check{X}^\circ]$ between their coordinate rings. We now prove that this map is a ring isomorphism. Injectivity is a simple consequence of the fact that the map $U_-^P\to\check{X}$ is dominant.

We now prove surjectivity. To do this, it is enough to find a pre-image for each of the functions (minors, Pfaffians and inverses of minors) generating $\mathbb{C}[U_-^P]$. We have already seen that the inverses of minors correspond to the inverses of denominators of W, which are by definition well-defined functions on $\check{X}^\circ$. Let us now consider the minors $D^{1,\dots,m-j-t,m+2,\dots,m+2+t}_{1,2,\dots,m+1-j}(\bar{u}_2)$ for $0\leq t\leq m-1$ and $2\leq j\leq m-t$. Since $\bar{u}_2$ is in U_-, we have :

$$\begin{aligned} D^{1,\dots,m-j-t,m+2,\dots,m+2+t}_{1,\dots,m+1-j}(\bar{u}_2) &= D^{1,\dots,m-j-t,m+2,\dots,2m+1}_{1,\dots,m+1-j,m+3+t,\dots,2m+1}(\bar{u}_2)\\ &= D^{m+2,\dots,2m+1}_{m-j-t+1,\dots,m+1-j,m+3+t,\dots,2m+1}(\bar{u}_2). \end{aligned}$$

Now using the isomorphism between $\bigwedge^m \mathbb{C}^{2m+1}$ and $\bigwedge^{m+1} \mathbb{C}^{2m+1}$, we get

$$D^{m+2,\dots,2m+1}_{m-j-t+1,\dots,m+1-j,m+3+t,\dots,2m+1}(\bar{u}_2) = D^{m+1,\dots,2m+1}_{m-t,\dots,m+j,m+j+t+2,\dots,2m+1}(\bar{u}_2).$$

Since the minor $D^{m+1,\dots,2m+1}_{m-t,\dots,m+j,m+j+t+2,\dots,2m+1}$ is a well-defined element of the homogeneous coordinate ring of $\check{X} = P\backslash G$, it gives in particular a well-defined function on $\check{X}^\circ$ after setting $p_\emptyset$ to 1.

Let us finally consider the Pfaffians $\Delta_{\omega_m, \frac{1}{2}[\epsilon_1+\cdots+\epsilon_{m-1-r}-(\epsilon_{m-r}+\cdots+\epsilon_m)]}$ $(0 \le r \le m-1)$. By definition

$$\Delta_{\omega_m, \frac{1}{2}[\epsilon_1+\cdots+\epsilon_{m-1-r}-(\epsilon_{m-r}+\cdots+\epsilon_m)]} = \langle w_\emptyset^* \cdot g \ , \ w_{\rho_{m-r}} \rangle,$$

where w_λ is the element of V_{Spin} associated to a strict partition λ as in Subsection 3.1. By definition of the Plücker coordinates, the right-hand side is equal to the Plücker coordinate $p_{\rho_{m-r}}(g)$ which gives well-defined functions on $\check{X}^\circ$. □

This concludes the proofs of Theorems 3.1 and 3.2. We now state some related conjectures.

4. *Relations in the quantum cohomology of LG(m)*

In [**Rie08**], the second author proved an isomorphism between the quantum cohomology ring of $X = G^\vee/P^\vee$ and the Jacobi ring of the LG-model $(\mathcal{R}, \mathcal{F}_h)$ (either at fixed quantum parameter $q = e^h$ as in Corollary 2.4 or over the ring $\mathbb{C}[q, q^{-1}]$). By Theorem 3.1 together with Theorem 3.2 our LG-model $(\check{X}, W_t)$ is isomorphic to this one, and is therefore related to the quantum cohomology ring of $LG(m)$ in the same way. Therefore we expect the denominators appearing in the expression of W_t, once written with Schubert classes replacing the Plücker coordinates, to represent invertible elements in this quantum cohomology ring. We have a precise conjecture for which elements these are.

Conjecture 4.1. *The following relation holds in the quantum cohomology of* $\mathrm{LG}(m)$ *for all* $1 \le l \le m-1$ *:*

$$\sum_{J \subset \{1,\dots,l\}} (-1)^{s(J)} \sigma_{\rho_l^J} \star \sigma_{\mu_l^J} = q^l. \tag{4.1}$$

Remark 2. *If* $l = 1$*, the relation* (4.1) *is a consequence of the quantum Chevalley formula 2.2. Indeed, this formula implies that*

$$\sigma_1 \star \sigma_m = \sigma_{m,1} + q,$$

which, rewritten as

$$\sigma_1 \star \sigma_m - \sigma_\emptyset \star \sigma_{m,1} = q,$$

is exactly the relation (4.1) *with* $l = 1$*. For* $l > 1$ *however, to the best of the authors' knowledge, the relations* (4.1) *are new.*

5. *The B-model connection*

Our expression for the LG-model W in terms of homogeneous coordinates coming from $\check{X}^\circ \subset \mathbb{P}(H^*(X,\mathbb{C})^*)$ makes it possible to state very concretely a mirror conjecture in the spirit of Dubrovin and Givental. Namely we conjecture an explicit isomorphism between a Gauss-Manin connection associated to $(\check{X}^\circ, W)$, and a D-module associated to X arising from the small Dubrovin connection, see [**Dub96**, **Giv96**, **CK99**].

Let $X = LG(m)$. Consider $H^*(X, \mathbb{C}[\hbar, e^t])$ as space of sections on a trivial bundle with fiber $H^*(X)$ and let

$$^A\nabla_{\partial_t} S := \frac{dS}{dt} + \frac{1}{\hbar}\sigma^{\square} \star_{e^t} S \tag{5.1}$$

$$^A\nabla_{\hbar\partial_\hbar} S := \hbar\frac{\partial S}{\partial \hbar} - \frac{1}{\hbar}c_1(TX) \star_{e^t} S + \mathrm{Gr}(S) \tag{5.2}$$

define a meromorphic flat connection on this bundle. Here Gr is the 'grading operator' defined as the diagonal map $H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$ which multiplies $\sigma \in H^{2k}(X, \mathbb{C})$ by k, and we are using the convenient notation e^t for q and ∂_t for $q\partial_q$. This is our A-model side.

For the B-model let $N = \frac{m(m+1)}{2}$ denote the dimension of $\check{X}$. Recall that $\check{X}^\circ$ is $\mathrm{OG}^{\mathrm{co}}(m + 1, 2m + 1)$ with an anticanonical divisor removed. Therefore there is an up to scalar unique non-vanishing holomorphic N-form on $\check{X}^\circ$ which we will fix and call ω. Let $\Omega^k(\check{X}^\circ)$ denote the space of all holomorphic k-forms.

Definition 5.1. *Define the $\mathbb{C}[\hbar, e^t]$-module*

$$G_0^{W_t} := \Omega^N(X)[\hbar, e^t]/(\hbar d + dW_t \wedge -)\Omega^{N-1}(X)[\hbar, e^t].$$

It has a meromorphic (Gauss-Manin) connection given by

$$^B\nabla_{\partial_t}[\alpha] = \frac{\partial}{\partial t}[\alpha] - \frac{1}{\hbar}[\frac{\partial W_t}{\partial t}\alpha], \tag{5.3}$$

$$^B\nabla_{\partial_\hbar}[\alpha] = \frac{\partial}{\partial \hbar}[\alpha] + \frac{1}{\hbar^2}[W_t\,\alpha]. \tag{5.4}$$

We conjecture that the function W_t is cohomologically tame [**Sab99**] and the elements $[p_\lambda\omega]$ freely generate $G_0^{W_t}$, where the p_λ's are the Plücker coordinate on $OG^{co}(m + 1, V^*)$ and λ runs through the strict partitions inside an $m \times m$ box.

Independently of this we conjecture the following.

Conjecture 5.1. *The differential operators $\hbar\,{}^B\nabla_{\partial t}$ and $\hbar\,{}^B\nabla_{\hbar\partial\hbar}$ preserve the $\mathbb{C}[\hbar, e^t]$-submodule $\bar{G}_0^{W_t}$ of $G_0^{W_t}$ generated by the $[p_\lambda\omega]$. Moreover the assignment $\sigma^\lambda \mapsto [p_\lambda\omega]$ defines an isomorphism of $H^*(X, \mathbb{C}[\hbar, e^t])$ with $\bar{G}_0^{W_t}$ under which ${}^A\nabla$ is identified with ${}^B\nabla$.*

Appendix A. *Laurent polynomial version of W_t*

Here we give a Laurent polynomial expression for the restriction of W_t to a particular torus, see Proposition A.1. The lemmas used in the proof are also used in the proof of Theorem 3.1. We observe that our Laurent polynomial formula for the superpotential of $LG(m)$ has similarities with the much simpler case of projective space.

Let us pull back W_t to the open subset of $\check{X}$ defined as the image of the map $(\mathbb{C}^*)^N \hookrightarrow P\backslash G$ which sends $(b_1, \ldots, b_N)$ to $P\bar{u}_2$, where as in Section 3.5

$$\bar{u}_2 = y_m(b_N)\ldots y_2(b_{N-m+2})y_1(b_{N-m+1})\ldots y_m(b_3)y_{m-1}(b_2)y_m(b_1). \tag{A.1}$$

Proposition A.1 Laurent polynomial restriction of W_t. *The Landau-Ginzburg model W_t of $X = \mathrm{LG}(m)$ defined in Theorem 2.4 restricts to the open torus defined above to give*

$$\tilde{W}_t(b_1, \ldots, b_N) = \sum_{j=1}^{N} b_j + e^t\ \frac{\mathcal{N}(b_1, \ldots, b_N)}{\prod_{j=1}^N b_j},$$

where

$$\mathcal{N}(b_1, \dots, b_N) := \sum b_{j_{i_1}} \dots b_{j_{i_{N-m}}},$$

and the sum is over all subsets $\{i_1 < \dots < i_{N-m}\}$ *of* $\{1, \dots, N\}$ *such that* $(s_{j_{i_1}} \dots s_{j_{i_N}})s_1 \dots s_m$ *is a reduced expression for* w^P.

Proof. We will rename the coordinates b_i when convenient by $a_{i,j}$, in terms of which $\bar{u}_2$ is given by

$$\big(y_m(a_{m,m})y_{m-1}(a_{m-1,m}) \dots y_1(a_{1,m})\big) \dots \big(y_m(a_{m,2})y_{m-1}(a_{m-1,2})\big)y_m(a_{m,1}).$$

As a consequence of the shape of $\bar{u}_2$ and the definition of the y_i, we immediately obtain :

$$f_i^*(\bar{u}_2) = \sum_{j=m+1-i}^{m} a_j^{(i)}. \tag{A.2}$$

We now need to compute the $e_i^*(u_1)$, where u_1 is such that $u_1 e^h \dot{w}_P \bar{u}_2 \in B_- \dot{w}_0$.

Lemma A.2.

$$e_i^*(u_1) = 0 \textit{ for all } 1 \le i \le m-1 \tag{A.3}$$

Proof Proof of Lemma A.2. From [**Rie08**], we know that

$$\begin{aligned} e_i^*(u_1) &= \frac{\langle u_1^{-1} v_{\omega_i}^-, e_i \cdot v_{\omega_i}^- \rangle}{\langle u_1^{-1} v_{\omega_i}^-, v_{\omega_i}^- \rangle} \\ &= \frac{\langle e^h \dot{w}_P \bar{u}_2 \dot{w}_0{}^{-1} v_{\omega_i}^-, e_i \cdot v_{\omega_i}^- \rangle}{\langle e^h \dot{w}_P \bar{u}_2 \dot{w}_0{}^{-1} v_{\omega_i}^-, v_{-\omega_i}^- \rangle} \\ &= \frac{\langle e^h \dot{w}_P \bar{u}_2 v_{\omega_i}^+, e_i \cdot v_{\omega_i}^- \rangle}{\langle e^h \dot{w}_P \bar{u}_2 v_{\omega_i}^+, v_{\omega_i}^- \rangle}. \end{aligned}$$

Now $e_i^*(u_1) = 0$ if and only if $\langle \overline{u}_2 v_{\omega_i}^+, \dot{w}_P^{-1} e_i \cdot v_{\omega_i}^- \rangle = 0$. The vector $\dot{w}_P^{-1} e_i \cdot v_{\omega_i}^-$ is in the μ-weight space of the i-th fundamental representation, where $\mu := w_P^{-1} s_i(-\omega_i)$. Moreover, $\overline{u}_2 \in B_+ (\dot{w}^P)^{-1} B_+$, hence it can only have non-zero components down to the weight space of weight $(w^P)^{-1}(\omega_i) = w_P^{-1}(-\omega_i)$. However, μ is lower than $w_P^{-1}(-\omega_i)$ when $i \ne m$. □

We are left with computing $e_m^*(u_1)$:

Lemma A.3.

$$e_m^*(u_1) = e^t \ \frac{\mathcal{N}(b_1, \dots, b_N)}{\prod_{j=1}^N b_j} \tag{A.4}$$

Proof Proof of Lemma A.3. As in the proof of Lemma A.2, we have

$$\begin{aligned} e_m^*(u_1) &= \frac{\langle e^h \dot{w}_P \bar{u}_2 v_{\omega_m}^+, e_m \cdot v_{\omega_m}^- \rangle}{\langle e^h \dot{w}_P \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle} \\ &= (\omega_m + \alpha_m - \omega_m)(e^h) \frac{\langle \dot{w}_P \bar{u}_2 v_{\omega_m}^+, e_m \cdot v_{\omega_m}^- \rangle}{\langle \dot{w}_P \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle} \\ &= e^t \frac{\langle \dot{w}_P \bar{u}_2 v_{\omega_m}^+, e_m \cdot v_{\omega_m}^- \rangle}{\langle \dot{w}_P \bar{u}_2 v_{\omega_m}^+, v_{\omega_m}^- \rangle}. \end{aligned}$$

Indeed, $\alpha_m(e^h) = e^t$. Moreover, $\langle \dot{w_P}\overline{u}_2 v^+_{\omega_m}, v^-_{\omega_m}\rangle = \langle \overline{u}_2 v^+_{\omega_m}, \dot{w_P}^{-1} v^-_{\omega_m}\rangle = \langle \overline{u}_2 v^+_{\omega_m}, v^-_{\omega_m}\rangle$. Now the only way to go from the lowest weight vector $v^-_{\omega_m}$ of the m-th fundamental representation to the highest $v^+_{\omega_m}$ is to apply w_0. Since $\overline{u}_2 \in B(w^P)^{-1}B$, it follows that we need to take all factors of $\overline{u}_2$, hence $\langle \dot{w_P}\overline{u}_2 v^+_{\omega_m}, v^-_{\omega_m}\rangle = \prod_{j=1}^N b_j$.

Now we prove that $\langle \dot{w_P}\overline{u}_2 v^+_{\omega_m}, e_m \cdot v^-_{\omega_m}\rangle = \mathcal{N}(b_1, \ldots, b_N)$. Indeed :

$$\langle \dot{w_P}\overline{u}_2 v^+_{\omega_m}, e_m \cdot v^-_{\omega_m}\rangle = \langle \overline{u}_2 v^+_{\omega_m}, \dot{w_P}^{-1} e_m \cdot v^-_{\omega_m}\rangle,$$

and the weight of the vector $\dot{w_P}^{-1} e_m \cdot v^-_{\omega_m}$ is $\mu' := \frac{1}{2}(\epsilon_1 - \epsilon_2 - \cdots - \epsilon_m)$. Now consider the Weyl group element

$$w' := s_m(s_{m-1}s_m)\ldots(s_2\ldots s_{m-1}s_m).$$

We have

$$w' \cdot \omega_m = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \cdots - \epsilon_m).$$

Hence the way to the μ'-weight space is through one of the reduced expression for w', which concludes the proof of the claim. □

Now the proof of Proposition A.1 follows immediately from Theorem 2.4 and the equations (A.2), (A.3) and (A.4). □

The expression for the Landau-Ginzburg model in Proposition A.1 is quite close to the usual expression for the Landau-Ginzburg model of projective space $\mathbb{P}^n$, which looks like :

$$W_t^{\mathbb{P}^n} = x_1 + x_2 + \cdots + x_n + \frac{e^t}{x_1 x_2 \ldots x_n}.$$

Indeed, it is the sum of as many parameters as the dimension of the variety, plus a more complicated e^t-term depending on those parameters. To the best of our knowledge, this expression is new for LG(m) with $m > 2$. However, for the three-dimensional quadric LG(2), we obtain :

$$W_t^{LG(2)} = a_{2,1} + a_{1,2} + a_{2,2} + e^t \frac{a_{2,1} + a_{2,2}}{a_{2,1}a_{1,2}a_{2,2}},$$

which, up to a toric change of coordinates, corresponds to one of the expressions of [**Prz07**].

References

BCFKvS00 Victor V. Batyrev, Ionuţ Ciocan-Fontanine, Bumsig Kim, and Duco van Straten. Mirror symmetry and toric degenerations of partial flag manifolds. *Acta Math.*, 184(1):1–39, 2000.

CK99 David Cox and Sheldon Katz. *Mirror Symmetry and Algebraic Geometry*. American Mathematical Soc., 1999.

Dub96 Boris Dubrovin. Geometry of 2D topological field theories. *Integrable Systems and Quantum Groups*, 1620:120 – 348, 1996.

FW04 W. Fulton and C. Woodward. On the quantum product of Schubert classes. *J. Algebraic Geom.*, 13(4):641–661, 2004.

FZ99 Sergey Fomin and Andrei Zelevinsky. Double Bruhat cells and total positivity. *J. Amer. Math. Soc.*, 12(2):335–380, 1999.

Gin97 V. Ginzburg. Perverse sheaves on a loop group and Langlands duality. preprint, 1997.

Giv96 Alexander B. Givental. Equivariant Gromov-Witten invariants. *IMRN*, 13:613–663, 1996.

GK95 Alexander Givental and Bumsig Kim. Quantum cohomology of flag manifolds and Toda lattices. *Comm. Math. Phys.*, 168(3):609–641, 1995.

GLS11 Christof Geiß, Bernard Leclerc, and Jan Schröer. Kac-Moody groups and cluster algebras. *Adv. Math.*, 228(1):329–433, 2011.

Kim99 Bumsig Kim. Quantum cohomology of flag manifolds G/B and quantum Toda lattices. *Annals of Mathematics, Second Series*, 149(1):129–148, 1999.

Lus83 G. Lusztig. Singularities, character formulas and a q-analog of weight multiplicities. *Astérisque*, 101-102:208–229, 1983.

MR13 R. Marsh and K. Rietsch. The B-model connection and T-equivariant mirror symmetry for Grassmannians. arXiv:1307.1085, 2013.

MV07 I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

Pet97 D. Peterson. Quantum cohomology of G/P. Lecture Course, MIT, Spring Term, 1997.

PR13 C.. Pech and K. Rietsch. The B-model connection and T-equivariant mirror symmetry for Grassmannians. arXiv:1306.4016, 2013.

Prz07 Victor Przyjalkowski. On Landau-Ginzburg models for Fano varieties. *Commun. Number Theory Phys.*, 1(4):713–728, 2007.

Rie08 Konstanze Rietsch. A mirror symmetric construction of $qH_T^*(G/P)_{(q)}$. *Adv. Math.*, 217(6):2401–2442, 2008.

Sab99 Claude Sabbah. Hypergeometric period for a tame polynomial. *C. R. Acad. Sci. Paris Ser. I Math.*, 328(7):603–608, 1999.

Var04 V. S. Varadarajan. *Supersymmetry for mathematicians: an introduction*, volume 11 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2004.