Institut Camille Jordan
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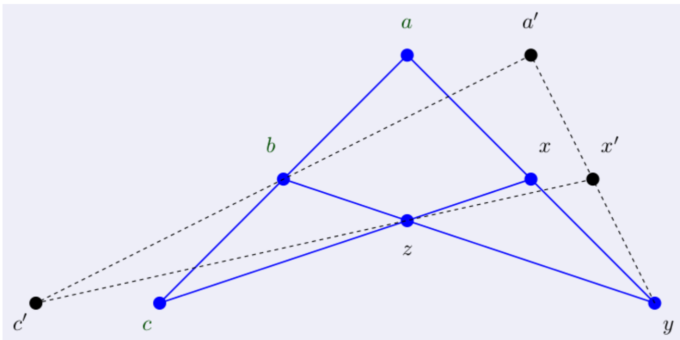
Groupes définissables dans des expansions de théories stables

Ampleur et notions relatives



Amador Martin-Pizarro

Habilitation à diriger des Recherches

Université Claude Bernard Lyon 1

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Introduction

One of the fundamental aspects of the study of definable groups in a given structure, which is a recurring topic in model theory, consists of deducing algebraic properties of definable groups in terms of the geometric nature of the ambient theory. A remarkable example of this approach is the *Algebraicity Conjecture*, which states that an uncountably categorical simple group must interpret an algebraically closed field so that the group becomes an algebraic group over it. An uncountably categorical simple group is a particular example of a group of finite Morley rank, that is, there is a dimension function, called *Morley rank* RM, defined on the collection of definable sets of a (sufficiently saturated) model M characterised by the following principle: If $X \subset M^k$ is definable, then $\mathrm{RM}(X) > n$ if and only if X contains an infinite family of pairwise disjoint non-empty definable sets Y_i with $\mathrm{RM}(Y_i) \geq n$. Morley rank extends to types by taking the smallest Morley rank of the formulae a type contains, and so it agrees with the Cantor-Bendixson rank on the space of types over an ω-saturated model, equipped with the Stone topology. For an algebraically closed field with no additional structure, definable sets are exactly the Zariski constructible ones, and Morley rank coincides with the Zariski dimension. For a group of finite Morley rank, this notion of dimension is well-behaved. For example, given a definable fibration $S \subset X \times Y \to Y$, the subset consisting of y's in Y such that the fibre over y has dimension k is definable for every k in $\mathbb{N}$. If all fibres have constant Morley rank k, then $\mathrm{RM}(X) = \mathrm{RM}(Y) + k$. A *bad group* is a minimal counterexample to the Algebraicity Conjecture, which still remains unsolved, and, in the author's humble opinion, with no definite answer in a foreseeable future. However, work on this area has become a solid discipline in its own, which combines techniques from group theory, in particular, the classification of finite simple simple groups, as well as ideas from algebraic groups and from model theory.

A different approach to the study of definable groups in model theory, already present in incidence geometry, consists of constructing, with various methods, some of them *ad hoc*, algebraic structures, such as groups or even fields, out of a pure abstract context, usually in terms of *germs of quasi-endogenies* or *generically given* maps. An example of this method already appeared in Weil's group chunk theorem [102], where a birational group law which was only partially defined can be extended to a full algebraic group. Weil's theorem has been adapted to various settings, such as topological spaces [92], and it is worth mentioning the generalisation given by Hrushovski with his *group configuration* [39] (cf. Section 1.5). Out of a configuration of six points satisfying certain conditions of independence and colinear algebraicity, a group is constructed in terms of germs of maps acting on a type. The group configuration lies at many of the fundamental applications of model theory, such as Pillay's theorem on differential algebraic groups [69], and it has become a recurrent tool in the work here presented.

In order to motivate the articles which constitute the present document, recall that Zilber identified the underlying geometry of an ω-categorical strongly minimal theory as either trivial or affine or projective over some finite field, in terms of *quasi-translations.* This led to the question whether the geometry of a general strongly minimal theory could be associated to two archetypal examples, either 1-*based* or *field-like*, in which case an infinite field had to be interpreted. In order to provide a negative answer thereupon, Hrushovski developed a method [40, 41] to construct ω-stable theories, and in particular new strongly minimal sets, with a prescribed geometry according to some dimension function. Pillay [71] and later Evans [31] refined the previous question on the geometry of strongly minimal sets, by introducing a whole hierarchy, called the *ample* hierarchy, on the complexity of non-forking independence with respect to the operator algebraic closure. According to this hierarchy, motivated by the incidence relation in euclidean space of the flags of affine subspaces of increasing dimension, from one point to a hyperplane, Hrushovski's *ab initio* construction is of low complexity, whereas algebraically closed fields or the free non-abelian group [65, 88] lie at the very top. The first two levels of this hierarchy are fairly well-understood: in the first level, definable ω-stable groups are virtually abelian and, in the second level, they are virtually nilpotent if the Morley rank is finite [44, 67].

Hrushovski's construction has been adapted by several authors in order to unexpectedly answer various questions on pure model theory. For example, related to Vaught's conjecture and stable non ω-categorical theories with only finitely many countable models, Herwig constructed [38] a stable non superstable theory with few types and a unique 1-type of infinite (pre-)weight with respect to itself. Baudisch's group [5] has finite Morley rank, yet it interprets no infinite field, since it is not 2-ample. Poizat [80] (see also the work of Baldwin and Holland [1]) constructed and *collapsed* an ω-stable field together with a distinguished predicate, obtaining thus an ω-stable field, which he called a *black field*, whose Lascar rank was not monomial, answering thus a question by Berline and Lascar. He constructed [81] furthermore two other ω-stable colored fields of infinite rank: the *red field* has positive characteristic anc comes equipped with a proper definable non-algebraic infinite additive subgroup. The *green field* has characteristic 0 and contains a proper divisible torsion-free multiplicative definable subgroup.

Poizat's colored fields were the starting point of a collaboration with Baudisch and Ziegler on Hrushovski's amalgams, producing, among others, a field of finite Morley rank in positive characteristic whose underlying additive group is not minimal [9], in contrast to the characteristic 0 case. However, an ω-stable differentially closed field in characteristic 0 with a definable non-differentially algebraic additive subgroup has been constructed [81, 15]. At the beginning of the author's stay at the Institut Camille Jordan, a collapse of Poizat's green field produced in [6] a *bad field*: an algebraically closed field of finite Morley rank with a definable proper divisible multiplicative subgroup, whose existence was long conjectured, though such an object should not exist in positive characteristic [99].

Poizat observed [80, Proposition 2.4] that no bad groups are intepretable in his black field, since every simple interpretable group is definably isogenous to an algebraic group. Similarly, in his fusion of two strongly minimal theories T_1 and T_2 into a new strongly minimal expansion T, Hrushovski [40, pp. 130] states that *"the geometry (of the fusion) can be seen as "relatively flat" over the geometries of the given strongly minimal sets"*. In particular, definable groups in T are isogenous to a product of groups, each interpretable in one of the theories T_i's. Poizat's result and Hrushovski's remark were the starting point to undertake a complete description of definable groups in the collapsed green field. Our first attempt combined the aforementioned approaches at the beginning of the introduction, in which we isolated certain properties of all known Hrushovski's amalgams in order to establish a close relation, referred to as *relative ampleness*, between a theory T and a stable reduct T_0 which controls, to a certain extent, non-forking independence in T. A definable group G in T yields a canonical group configuration in T_0 and hence a group homomorphism from G to a T_0-$*$-interpretable group H together with a definable isogeny $G \to H$. As a by-product, we recover the already-known results for algebraically closed fields with a generic automorphism [56] or for differentially closed fields in characteristic 0 [69], though in the latter Pillay obtains an actual embedding instead of a mere isogeny.

Theorem A. *[BMW15] Let $(T_i : i < n)$ be stable reducts with geometric elimination of imaginaries of a given simple theory T which is equipped with a closure operator $\langle . \rangle$ satisfying the two following technical conditions:*

- *If C is algebraically closed and $a \mathop{\smile\!\!\!\!\!\mid}_C b$ in T, then $\langle Cab \rangle \subset \bigcap_{i<n} \mathrm{acl}_{T_i}(\langle Ca \rangle, \langle Cb \rangle)$.*
- *Given b in $\bigcup_{i<n} \mathrm{acl}_{T_i}(A)$, then $\langle \mathrm{acl}_T(b), A \rangle \subset \bigcap_{i<n} \mathrm{acl}_{T_i}(\mathrm{acl}_T(b), \langle A \rangle)$.*

If T is relative 1-based over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$, then every type-definable group G is isogenous to a subgroup of a cartesian product of groups H_i, each T_i-type-interpretable.

If T is relatively CM-trivial over $(T_i : i < n)$ with respect to $\langle . \rangle$, then every type-definable group G has a type-definable subgroup of bounded index which definably maps to a cartesian product of T_i-type-interpretable groups, such that the kernel of the map is contained, up to finite index, in $\tilde{Z}(G)$.

A simple group definable in a colored field of finite Morley rank is linear.

The above results could (and should) be improved. Indeed, they should be adapted to the case where T only contains the universal part of some stable theories $(T_i : i < n)$, that is, every model of T embeds into a model of each T_i, which covers various theories of model-theoretic relevance, such as separably closed fields or perfect PAC fields with bounded Galois group. More generally, if, instead of a global homomorphism, we require a local one, the proof should adapt to some weakenings of simplicity for the theory T, such as NTP_2, mimicking the study of definable groups in local fields [45, 46] such as the reals or the p-adics. However, the notion of *generics* for NTP_2 theories still needs some development.

Fields of finite Morley rank eliminate imaginaries [98], so the previous theorem conveys little information if we consider abelian intepretable groups in a collapsed colored field, e.g., the multiplicative group of the field modulo the divisible green multiplicative subgroup. Therefore, an intensive study (in a span of several years with a certain emotional toll) on interpretable groups in the green field as well as on subgroups of algebraic groups in colored fields was conducted. The major difference with respect to Poizat's black fields is the existence of non-trivial transformations on the set of colored points, so that definable subgroups of algebraic groups decompose in a pure algebraic subgroup and non-trivial colored quotient. Unrelated but as a by-product of this effort, we provide a proof of Hrushovski's remark on groups definable in the fusion of two strongly minimal theories over equality.

Theorem B. *[BMW12a] Every interpretable group in a collapsed green field is isogenous to a quotient of a definable subgroup of an algebraic group by a central subgroup, which is itself isogenous to a cartesian power of the green predicate. In a (possibly uncollapsed) green field, every connected definable subgroup G of an algebraic group has a normal algebraic subgroup N such that the quotient G/N is definably isomorphic to a cartesian power of the green predicate.*

In a (possibly uncollapsed) red field, every connected definable subgroup G of an algebraic group has a normal algebraic subgroup N such that the quotient G/N is definably isogenous to the red points of an additive algebraic subgroup.

Every simple definable group in a colored field is definably isomorphic to an algebraic group. No bad groups are definable in a colored field.

Every definable group in the (possibly uncollapsed) fusion of two strongly minimal theories over equality is isogenous to a product of groups, each one interpretable in the base theories.

It is unclear to us how to generalise the last point to describe definable groups in the fusion of two strongly minimal theories over a common vector space over a finite field [7].

The previous work on colored fields, though intricate, did pay off. Indeed, using the techniques developed there, we could easily study groups definable in *belles paires* of stable theories, introduced by Poizat [78]. He isolated a strenthening of stability, the *non finite cover property* NFCP, developed by Keisler [52] in order to produce saturated ultraproducts and strongly related to the existence of a model companion TA for the theory $T_\sigma = T \cup \{$"σ is an automorphism"$\}$. NFCP ensures that a saturated model of the common theory T_P of belles paires of a stable theory is again a belle paire. In particular, algebraically closed fields have NFCP, and so do strongly minimal theories. However, the theory T_P need not eliminate imaginaries (modulo the imaginaries of T). It is the case if and only if no infinite group is interpretable in T [76]. Pillay introduced geometric sorts [72] to have geometric elimination of imaginaries for the theory of proper extensions of algebraically closed fields. Using the tools he thereupon developed, we could analyse interpretable groups in belles paires of algebraically closed fields.

Theorem C. *[BM14] Given a stable NFCP theory $T = T^{\mathrm{eq}}$, every group G type-definable in a belle paire (M, E) of T is isogenous to a subgroup of a T-type-definable group. Furthermore, the group G is,* *up to isogeny, the extension of the E-points of a T-type-definable group over E by a T-type-definable group.*

$$0 \to N \to G \to H(E) \to 0$$

An interpretable group G in a pair (K, F) of algebraically closed fields, with $F \subsetneq K$, is, up to isogeny, the extension of the F-rational points of an algebraic group H over F by group N, which is a quotient of an algebraic group V by a normal subgroup $N'(F)$, consisting of the F-rational points of an algebraic group over F.

$$0 \longrightarrow N(K) \longrightarrow G(K) \longrightarrow H(F) \longrightarrow 0$$

$$\textit{with}$$

$$0 \longrightarrow N'(F) \longrightarrow V(K) \longrightarrow N(K) \longrightarrow 0\,,$$

such that both H and N' are algebraic groups over F.

If G is interpretable over $k \not\subset F$, then V and N are defined over $\overline{kF}$.

Another application of the tools used in Theorem B is the description of bounded automorphisms of various theories of fields with operators. Lascar [61] showed that the group of strong automorphisms of a pure algebraically closed field in characteristic 0 is simple. The same holds for a differentially closed field in characteristic 0 [32], for there are no non-trivial bounded automorphisms [55]. An automorphism τ of a field is bounded if there is a finite set A such that, for every element b, the image $\tau(b)$ belongs to $\mathrm{cl}_{\mathrm{Gen}}(A \cup \{b\})$, where $\mathrm{cl}_{\mathrm{Gen}}(D)$ denotes the collection of elements whose type over D is coforeign to the generics of the field. If the field has monomial Lascar rank, then this closure consists of the collection of non-generic elements. Lascar notes that Ziegler had obtained a simpler proof of the triviality of bounded automorphisms for a pure algebraically closed field in characteristic 0, which could be generalised to the positive characteristic case, in order to show that such an automorphism is an integer power of Frobenius. Motivated by Lascar's remark, we obtained a uniform characterisation, probably not any different from Ziegler's proof, of bounded automorphisms in various theories of fields with operators, following the formalism introduced by Moosa and Scanlon [64]. A *field with operators* over a base subfield $\mathbb{F}$ is a structure

$$(K, 0, 1, +, -, \cdot, \{\lambda\}_{\lambda \in \mathbb{F}}, F_1, \ldots, F_n),$$

such that the operators $F_1, \ldots, F_n$ are $\mathbb{F}$-linear satisfy

$$F_k(xy) = \sum_{0 \le i,j \le n} a^k_{i,j} F_i(x) F_j(y),$$

for some constants $\{a^k_{i,j}\}_{0 \le i,j,k \le n}$ in $\mathbb{F}$. Consider the $\mathbb{F}$-algebra $D(\mathbb{F}) = \mathbb{F}\epsilon_0 \oplus \ldots \oplus \mathbb{F}\epsilon_n$ such that

$$\epsilon_i \star \epsilon_j = \sum_{0 \le k \le n} a^k_{i,j} \epsilon_k.$$

It is isomorphic to a product of local $\mathbb{F}$-algebras $B_0(\mathbb{F}), \ldots, B_t(\mathbb{F})$, whose residue fields are finite algebraic extensions of $\mathbb{F}$. If all these residue fields are $\mathbb{F}$, tensoring each local algebra with K, if θ_i, resp. ρ_i, denotes the projection of $D(K) = D(\mathbb{F}) \otimes_{\mathbb{F}} K$ onto $B_i(K)$, resp. the projection of $B_i(K)$ onto its residue K, we obtain the *associated endomorphisms*

$$\sigma_i = \rho_i \circ \theta_i \circ \varphi$$

of K.

Theorem D. *[BHM15] Consider a sufficiently saturated field with operators $(K, 0, 1, +, -, \cdot, F_1, \ldots, F_n)$ over a base subfield $\mathbb{F}$ such that all residue fields of the $\mathbb{F}$-algebra $D(\mathbb{F})$ are $\mathbb{F}$, and the associated endomorphisms are surjective and include both Frobenius and its inverse in case the characteristic is* *positive. Suppose that the theory T of K is simple and relatively 1-based over the reduct to pure algebraically closed fields with respect to the closure operator* acl*, which coincides with the field algebraic closure of the generated structure. If the family of words in $F_1, \ldots, F_n, \sigma_1^{-1}, \ldots, \sigma_t^{-1}$ does not satisfy, modulo T, any non-trivial linear relation over $\mathrm{acl}(\mathbb{F})$, then every bounded automorphism of K is a product of integer powers of the associated automorphisms (and Frobenius, in positive characteristic).*

This applies in particular to the following theories of fields with operators :

- *Algebraically closed fields (K, Id) in all characteristics with associated automorphism either identity or Frobenius in positive characteristic.*
- *Differentially closed fields with n commuting derivations $(K, \delta_1, \ldots, \delta_n)$ in characteristic 0 with associated automorphism the identity.*
- *Generic automorphisms (K, σ)in all characteristics with associated automorphism σ as well as Frobenius, in positive characteristic.*
- *Generic automorphisms of a differentially closed field (K, δ, σ) in characteristic 0 with associated automorphism σ, as considered by Bustamante-Medina [20].*
- *Fields with free operators $(K, F_1, \ldots, F_n)$ in characteristic 0 with associated automorphisms $\sigma_0, \ldots, \sigma_t$, as considered by Moosa and Scanlon [64].*

The characterisation of minimal types and definable sets in some of the above theories of fields with operators allowed Hrushovski to prove a functional version of Mordell-Lang in all characteristics [42], since the theories involved in his proof have a particular behaviour on definable sets, akin to the Zariski topology in classical algebraic geometry. Pillay and Ziegler [77] circumvented the use of *Zariski Geometries* [48] in Hrushovski's proof, by isolating a property, true in wider contexts, e.g., compact complex spaces [63], called the *Canonical Base Property* (CBP). In the case of a (saturated) differentially closed field, the CBP states that, given a definable set X of bounded differential degree and Morley degree 1, the field of definition of the constructible set determined by X is almost internal to the constant field over a generic realisation of X. The CBP generalises 1-basedness (that is, non-1-ampleness), since algebraic types are always almost internal to any invariant family Σ. Kowalski and Pillay [57], motivated by certain definability results valid for one-based groups, showed that a connected group definable in a stable theory with the CBP is central-by-(almost Σ-internal). Inspired by their result, we introduced a hierarchy of notions, generalising the ample counterpart, called *tightness*, of which 2-tight generalises CM-triviality (that is, non-2-ampleness), and proved the equivalent results in this setting, replacing *finite* by almost Σ-internal.

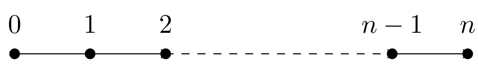
Theorem E. *[BMW12b] Let T be a stable 2-tight theory with respect to an invariant family Σ of types. An interpretable field is Σ-internal. An interpretable group of finite Lascar rank is nilpotent-by-(almost Σ-internal). In particular, an interpretable non-abelian simple group is Σ-internal.*

Back to the ample hierarchy, it was not known whether there were stable structures which were strictly n-ample, for n greater than 2. Generalising the construction of the *free pseudospace* by Baudisch and Pillay, for every natural number n we obtain (see also [89]) a *free n-dimensional pseudospace* satisfying the following:

Theorem F. *[BMZ14b] The free n-dimensional pseudospace is ω-stable of rank ω^{n+1} and n-ample yet not $n+1$-ample.*

As noticed by Tent, the above construction was bi-interpretable with the right-angled building with infinite residues of Coxeter graph

$[0, n]$



Similar to the use of trees to analyse free groups, buildings were introduced by Tits [91] in order to determine properties of semisimple algebraic groups. However, a building is a pure combinatorial structure which need not arise from a group, e.g., the incidence geometry of projective planes. A right-angled building of infinite residues with Coxeter group (W, Γ) is a set X equipped with a family of equivalence relations $(\sim_\gamma, \gamma \in \Gamma)$, satisfying that each $\sim_\gamma$-class is infinite, and that, for every pair x and y in X, there exists an element g in W such that there is a reduced path of type w from x to y if and only if the word w represents g. The associated *Coxeter graph* has vertex set Γ, the set of generators of W, where two vertices γ and δ are adjacent if they do not commute in W. There is a unique (up to isomorphism) countable building $\mathrm{B}_0(\Gamma)$ with infinite residues for each right-angled Coxeter group [36]. In order to make use of elementary extensions, non-standard paths between elements need to be considered, so we consider an expansion $\mathrm{B}^0(\Gamma)$ of the natural language of buildings. Adapting some of the techniques from the free pseudoespace, we obtained the following result.

Theorem G. *[BMZ14a] The theory* PS_Γ *of* $\mathrm{B}^0(\Gamma)$ *is* ω*-stable and equational with trivial forking. The model* $\mathrm{B}^0(\Gamma)$ *is the prime model. If* Γ *has no edges, then* PS_Γ *is* 1*-based. Otherwise, let* r *be the minimum of the valencies of the non-isolated vertices of* Γ*, and* n *in* $\mathbb{N}$ *be maximal such that the graph* $[0, n]$ *embeds fully in the Coxeter graph* Γ*, that is, as a substructure in the language of graphs. Then* PS_Γ *is* n*-ample but not* $(|\Gamma| - r + 1)$*-ample.*

Observe that these bounds are best possible, atttained for example by the graph $[0, n]$ itself, whose associated theory is n-ample but not $(n+1)$-ample. We have not attempted to consider more general, not necessarily right-angled, buildings, since a crucial feature of right-angled buildings is that a word is reduced if and only if no permutation has the form $w_1 \cdot \gamma \cdot \gamma \cdot w_2$, for some generator γ.

Introduction en français

En théorie des modèles, l'étude des groupes définissables dans une structure donnée est un sujet récurrent, dont un aspect fondamental consiste à isoler quelques propriétés algébriques des groupes définissables à partir de propriétés géométriques de la théorie ambiante. Un des exemples les plus notables de cette approche est la *Conjecture de l'Algebricité*, qui affirme que tout groupe simple $2^{\aleph_0}$-catégorique doit interpréter un corps algébriquement clos sur lequel le groupe est un groupe algébrique. Un groupe simple catégorique en puissance non-dénombrable est un groupe de rang de Morley fini, c'est-à-dire, il est muni d'une fonction de dimension, dite *rang de Morley*, sur la famille des ensembles définissables d'un modèle suffisamment saturé M, caractérisée par le principe suivant : le rang $\mathrm{RM}(X)$ d'un ensemble définissable $X \subset M^k$ est strictement plus grand que n si et seulement si X contient une famille infinie de sous-ensembles définissables deux-à-deux disjoints, chacun de rang de Morley au moins n. Le rang de Morley s'étend aux types en prenant le plus petit rang des formules l'appartenant. Ce rang ainsi défini coïncide avec le rang de Cantor-Bendixon sur l'espace des types au-dessus d'un modèle ω-saturé, muni de la topologie de Stone. Pour un pur corps algébriquement clos, les ensembles définissables correspondent aux ensembles constructibles de Zariski, et le rang de Morley coïncide avec la dimension de Zariski. Pour un groupe de rang de Morley fini, cette notion de dimension a de nombreuses propriétés remarquables. Par exemple, étant donnée une fibration définissable $S \subset X \times Y \to Y$, l'ensemble des éléments y dans Y tels que la fibre a dimension k est définissable, pour chaque entier k. En outre, si toutes les fibres ont même rang k, alors $\mathrm{RM}(X) = \mathrm{RM}(Y) + k$. Un *mauvais groupe* est un contre-exemple minimal à la Conjecture de l'Algebricité, qui reste encore ouverte et, de l'avis de l'auteur, sans réponse prévisible dans un futur proche. En revanche, cette direction de recherche est devenue une discipline bien établie et solide, qui combine des techniques de la théorie des groupes, et en particulier, de la classification de groupes simples finis, avec des méthodes de la théorie des modèles et des groupes algébriques.

Une autre approche, présente en géométrie d'incidence, à l'étude de groupes définissables en théorie des modèles, consiste à construire, par des différentes méthodes, parfois particulières au contexte, des structures algébriques, groupes ou corps, à partir d'une situation purement abstraite, normalement en termes de *germes de quasi-endogénies* ou des applications *définies génériquement*. Un exemple de cette méthode apparaît déjà dans la configuration de groupe de Weil [102], qui étend à un groupe algèbrique une loi birationnelle de groupe partielle. La méthode de Weil a éte adaptée à des situations diverses, e.g., à des espaces topologiques [92]. Entre autres, le théorème de *configuration de groupe* dû à Hrushovski [39] (cf. la partie 1.5) mérite d'être mentionné. À partir d'une configuration de six points satisfaissant certaines conditions d'indépendance et d'algébricité colinéaire, il obtient un groupe en terme de germes des fonctions agissant sur un type donné. La configuration de groupe est à la base de nombreuses applications fondamentales de la théorie des modèles, comme par exemple le résultat de Pillay sur les groupes algébro-différentiels [69], et elle est devenue une technique récurrente dans les travaux présentés ici.

Afin de motiver les travaux consituant ce document, rappelons que Zilber identifie, à partir de l'ensemble des *quasi-translations*, la géométrie sous-jacente d'une théorie fortement minimale $\aleph_0$-catégorique comme soit triviale soit affine ou projective sur un corps fini. La question se pose si la géométrie d'une théorie fortement minimale quelconque peut être donc classifiée selon deux exemples canoniques : soit monobasée soit *corpique*, où un corps infini pouvait être interprété. Hrushovski donne une réponse négative en développant une technique d'amalgamation [40, 41] pour obtenir des théories ω-stables, et plus précisement de nouveaux ensembles fortement minimaux, avec de géométries exotiques déterminées au préalable par des fonctions de dimension données. Pillay [71] et Evans [31] réformulent cette classification sur la géométrie d'un ensemble fortement minimal, en introduisant toute une hiérarchie, dite *ample*, pour décrire la complexité de la non-déviation par rapport à l'opérateur de clôture algébrique. La hiérarchie ample s'inspire de la relation d'incidence dans l'espace euclidien des drapeaux des variétés affines de dimension croisante, d'un point à un hyperplan. La construction *ab initio* de Hrushovski est de basse complexité ample, tandis que les corps algébriquement clos ou le groupe libre non-abélien [65, 88] se placent au sommet de la hiérachie ample. Les groupes définissables ω-stables dans les premiers deux niveaux sont bien caractérisés : ils sont abélien-par-finis pour le premier niveau, et nilpotent-par-finis pour le deuxième niveau, lorsque le rang de Morley est fini [44, 67].

La méthode d'amalgamation de Hrushovski a été adoptée par plusieurs auteurs pour répondre, souvent de façon inattendue, à des questions diverses en théorie des modèles pure. En relation à la conjecture de Vaught et aux théories stables non-$\aleph_0$-catégoriques avec un nombre fini des modèles dénombrables, Herwig construit [38] une théorie stable non-superstable avec peu de types telle que le seule type unaire a (pré-)poids infini par rapport à lui-même. Le groupe de Baudisch [5] est de rang de Morley fini mais n'interprète aucun corps infini, car il n'est pas 2-ample. Poizat [80] (voir aussi le travail de Baldwin et Holland [1]) obtient un corps ω-stable coloré dont le rang de Lascar n'est pas un monôme, ce qui répond à une question due à Berline et Lascar. Il construit ensuite [81] deux autres corps colorés de rang infini. Le *corps rouge* de caractéristique positive est muni d'un sous-groupe additif propre infini non-algébrique. Le *corps vert* de caractéristique 0 possède un sous-groupe multiplicatif définissable propre divisible sans torsion.

Les corps colorés de Poizat [80, 81] marquent le début d'une collaboration personelle avec Baudisch et Ziegler autour des amalgames de Hrushovski. On construit, entre autres, un corps de rang de Morley fini et caractéristique positive dont le groupe additif sous-jacent n'est pas minimal, ce qui n'est pas possible en caractéristique nulle. Or, en caractéristique nulle, on obtient [81, 15], au début de l'installation de l'auteur à l'institut Camille Jordan, un corps ω-stable différentiellement clos muni d'un sous-groupe additif non-algébro-différentiel. De même, un collapse du corps vert de Poizat [81] produit un *mauvais corps*[6] : un corps algébriquement clos de rang de Morley fini avec un sous-groupe multiplicatif divisible propre. Un tel objet, dont l'existence était conjecturée depuis longtemps, ne devrait pas exister en caractéristique positive [99].

Poizat remarque [80, proposition 2.4] qu'aucun mauvais groupe n'est interprétable dans son corps noir. En effet, tout groupe simple qui y est interprétable est définissablement isogène à un groupe algébrique. En outre, dans la fusion fortement minimale T de deux théories fortement minimales T_1 et T_2 au-dessus de l'égalité, Hrushovski écrit [40, pp. 130] que *"la géométrie (de la fusion) est "relativement plate" sur les géométries des théories de départ"*. Ainsi, tout groupe définissable dans T est isogène à un produit des groupes, chacun interprétable dans une des théories T_i. Cette remarque et le résultat de Poizat se trouvent à la base de nos efforts pour décrire les groupes définissables dans le corps vert collapsé. Notre premier essaie combine les deux dites approches à l'étude des groupes définissables, en isolant certaines propriétés, valables dans tous les amalgames de Hrushovski connus, pour établir une relation directe entre une théorie T et un réduit stable T_0 qui contrôle l'indépendance dans T ; ce que l'on appelle *ampleur relative.* À partir d'un groupe définissable G dans T, l'on obtient canoniquement une configuration de groupe au sens T_0 et donc un groupe T_0-$*$-interprétable groupe H ainsi qu'un morphisme $G \to H$. Ceci s'étend aux corps différentiellement clos et aux corps aux différences génériques pour récupérer (partiellement) les résultats connus dans chacun de ces contextes.

Theorem A. *[BMW15] Soit T une théorie simple avec des réduit stables $(T_i : i < n)$, qui ont élimination géométrique des imaginaires. Supposons que T est munie d'un opérateur clôture $\langle . \rangle$ qui satisfait les deux conditions techniques suivantes :*

- *Si C est algébriquement clos et $a \mathop{\smile\!\!\!\!\!\mid}_C b$ au sens de T, alors $\langle Cab \rangle \subset \bigcap_{i<n} \mathrm{acl}_{T_i}(\langle Ca \rangle, \langle Cb \rangle)$.*
- *Pour b dans $\bigcup_{i<n} \mathrm{acl}_{T_i}(A)$, on a que $\langle \mathrm{acl}_T(b), A \rangle \subset \bigcap_{i<n} \mathrm{acl}_{T_i}(\mathrm{acl}_T(b), \langle A \rangle)$.*

Si la théorie T est relativement monobasée au-dessus des réduits $(T_i : i < n)$ par rapport à *l'opérateur clôture $\langle . \rangle$, alors tout groupe type-définissable G est isogène à un sous-groupe d'un produit cartesien de groupes H_i, chacun T_i-type-interprétable.*

Si la théorie T est relativement CM-triviale au-dessus des réduits $(T_i : i < n)$ par rapport à l'opérateur clôture $\langle . \rangle$, alors tout groupe type-définissable G a un sous-groupe type-définissable d'indice borné qui s'envoie définissablement en un produit cartesien des groupes, chacun T_i-type-interprétable. De plus, le noyau de cette application est presque contenu dans le centre approximatif $\tilde{Z}(G)$.

Tout groupe simple définissable dans un corps coloré de rang de Morley fini est linéaire.

Les résultats ci-dessus devraient pouvoir s'étendre lorsque T contient uniquement la partie universelle des théories stables $(T_i : i < n)$, et donc chaque modèle de T se plonge dans des modèles de chaque T_i. Ceci est le cas des corps séparablement clos ou des corps PAC parfaits avec groupe de Galois borné. Plus généralement, si l'on s'intéresse à un homomorphisme local, au lieu de global, il semble alors pertinent de considérer des généralisations de la simplicité, comme par exemple NTP_2, en imitant l'étude des groupes définissables dans des corps locaux [45, 46], comme les réels ou les p-adiques. Cependant, la notion de *géneriques* dans les groupes NTP_2 est encore en progrès.

Un corps de rang de Morley fini élimine les imaginaires [98]. Le théorème précédent apporte donc peu d'information sur un groupe interprétable abélien dans un corps coloré collapsé, e.g., le groupe multiplicatif du corps modulo le sous-groupe divisible multiplicatif vert. Une étude approfondie (dans une période de plusieurs années, ayant un impact considérable sur le moral de l'impétrant) sur les groupes interprétables dans le corps vert, ainsi que sur les sous-groupes des groupes algébriques dans les corps colorés, s'avère donc nécessaire. La principale différence par rapport au corps noir de Poizat est l'existence de transformations non-triviales sur l'ensemble des points colorés. Un sous-groupe définissable d'un groupe algébrique se décompose alors en un pur sous-groupe algébrique et un quotient coloré. Additionnellement, nous démontrons aussi la remarque de Hrushovski sur les groupes définissables dans la fusion de deux théories fortement minimales sur l'égalité.

Theorem B. *[BMW12a] Tout groupe interprétable dans un corps vert collapsé est isogène à un quotient d'un sous-groupe définissable d'un groupe algébrique par un sous-groupe central, qui est lui isogène à une puissance du groupe vert multiplicatif. Dans un corps vert (possiblement non-collapsé), tout groupe définissable connexe G d'un groupe algébrique a un sous-groupe algébrique distingué N tel que le quotient G/N est définissablement isomorphe à une puissance du groupe vert multiplicatif.*

Dans un corps rouge (possiblement non-collapsé), tout groupe définissable connexe G d'un groupe algébrique a un sous-groupe algébrique distingué N tel que le quotient G/N est définissablement isogène aux points rouges d'un sous-groupe algébrique additif.

Tout groupe simple définissable dans un corps coloré (possiblement non-collapsé) est définissablement isomorphe à un groupe algébrique. Aucun mauvais groupe n'est définissable dans un corps coloré.

Tout groupe définissable dans une fusion (possiblement non-collapsée) de deux théories fortement minimales au-dessus de l'égalité est isogène à un produit des groupes, chacun interprétable dans les théories de départ.

Nous ignorons comment généraliser ce dernier point aux groupes définissables dans une fusion de deux théories fortement minimales au-dessus d'un espace vectoriel commun sur un corps fini [7].

Le travail précédent sur les corps colorés, quoique alambiqué et moralement fatigant, se révèle payant pour l'auteur. En effet, grâce aux techniques développées précedemment, l'étude de groupes définissables dans des belles paires des théories stables, introduites par Poizat [78], s'avère fort accessible. Poizat isole une propriété qui entraîne la stabilité, la *négation de la propriété de recouvrement fini* NFCP, introduite par Keisler [52] afin de produire des ultraproduits saturés. La NFCP est fort liée à l'existence d'une modèle-compagne pour la théorie $T_\sigma = T \cup \{$"σ est un automorphisme"$\}$ et garantit qu'un modèle saturé de la théorie T_P commune à toute belle paire d'une théorie stable T soit aussi une belle paire. Les corps algébriquement clos, ainsi que toute théorie fortement minimale, ont la NFCP. Or, la théorie T_P n'élimine pas forcement les imaginaires (modulo ceux de T) dès qu'un groupe infini est interprétable dans la théorie T [76]. Pillay introduit des sortes géométriques pour avoir élimination géométrique des imaginaries pour la théorie des belles paire de corps algébriquement clos. À partir des outils qu'il y utilise, nous pouvons facilement analyser les groupes interprétables.

Theorem C. *[BM14] Étant donnée une belle paire (M, E) d'une théorie stable $T = T^{\mathrm{eq}}$ ayant la NFCP, tout groupe T_P-type-définissable G est isogène à un sous-groupe d'un groupe T-type-définissable. En outre, le groupe G est, à isogénie près, l'extension des points E-rationnels d'un groupe T-type-définissable sur E par un groupe T-type-définissable.*

$$0 \to N \to G \to H(E) \to 0$$

Un groupe G interprétable dans une paire de corps algébriquement clos (K, F), où $F \subsetneq K$, est, à isogénie près, l'extension des points F-rationnels d'un groupe algébrique H défini sur F par un groupe N, qui est lui un quotient d'un groupe algébrique V par un sous-groupe distingué $N'(F)$, consistué des points F-rationnels d'un groupe algébrique N', le tout défini sur F.

$$0 \longrightarrow N(K) \longrightarrow G(K) \longrightarrow H(F) \longrightarrow 0$$
$$\textit{with}$$
$$0 \longrightarrow N'(F) \longrightarrow V(K) \longrightarrow N(K) \longrightarrow 0\,,$$

Si G est interprétable sur $k \not\subset F$, alors V et N sont définis sur $\overline{kF}$.

Une autre application des techniques provenantes du Theorem B porte sur la déscription des automorphismes bornés de nombreuses théories des corps munis d'opérateurs. Lascar [61] montre que le groupe des automorphismes forts d'un corps algébriquement clos en caractéristique 0 est simple. Le même résultat est valable pour un corps différentiellement clos en caractéristique 0 [32], car il n'a pas d'automorphismes bornés non-triviaux [55]. Un automorphisme τ d'un corps est borné s'il existe un ensemble fini A tel que l'image $\tau(b)$ de tout élément b appartient à $\mathrm{cl}_{\mathrm{Gen}}(A \cup \{b\})$, où $\mathrm{cl}_{\mathrm{Gen}}(D)$ est la classe d'éléments dont les types sur D sont co-étrangers aux génériques du corps. Si le corps a pour rang de Lascar un monôme, alors cette clôture correspond aux éléments non-génériques. Lascar note que Ziegler avait obtenu une démonstration plus simple sur la trivialité des automorphismes bornés d'un pur corps algébriquement clos en caractéristique 0, qui s'étend à toute caractéristique positive pour montrer qu'un tel automorphisme doit être une puissance entière du Frobenius. Motivés par cette remarque de Lascar, nous donnons une caractérisation unforme, qui probablement ne diffère pas de la démonstration originale de Ziegler, des automorphismes bornés de certaines théories des corps munis d'opérateurs, selon le formalisme de Moosa and Scanlon [64]. Un *corps munis d'operateurs* sur un sous-corps de base $\mathbb{F}$ est une structure

$$(K, 0, 1, +, -, \cdot, \{\lambda\}_{\lambda \in \mathbb{F}}, F_1, \ldots, F_n),$$

telle que les opérateurs $F_1, \ldots, F_n$ sont $\mathbb{F}$-linéaires et satisfont

$$F_k(xy) = \sum_{0 \le i,j \le n} a^k_{i,j} F_i(x) F_j(y),$$

pour certaines constantes structurelles $\{a^k_{i,j}\}_{0 \le i,j,k \le n}$ dans $\mathbb{F}$. La $\mathbb{F}$-algèbre $D(\mathbb{F}) = \mathbb{F}\epsilon_0 \oplus \ldots \oplus \mathbb{F}\epsilon_n$, avec

$$\epsilon_i \star \epsilon_j = \sum_{0 \le k \le n} a^k_{i,j} \epsilon_k,$$

est isomorphe à un produit de $\mathbb{F}$-algèbres locales $B_0(\mathbb{F}), \ldots, B_t(\mathbb{F})$, dont les corps résiduels sont des extensions finies algébriques de $\mathbb{F}$. Si tous ces corps résiduels sont $\mathbb{F}$, alors en tensorisant chaque algèbre locale avec K, l'on obtient ainsi les *automorphismes associés* de K, en posant θ_i, resp. ρ_i, la projection de $D(K) = D(\mathbb{F}) \otimes_{\mathbb{F}} K$ sur $B_i(K)$, resp. la projection de $B_i(K)$ sur le corps résiduel K, et

$$\sigma_i = \rho_i \circ \theta_i \circ \varphi$$

.

Theorem D. *[BHM15] Soit $(K, 0, 1, +, -, \cdot, F_1, \ldots, F_n)$ un corps suffisament saturé algébriquement clos muni d'opérateurs sur un sous-corps de base $\mathbb{F}$ tel que tous les corps résiduels de la $\mathbb{F}$-algèbre $D(\mathbb{F})$ sont $\mathbb{F}$, les automorphismes associés sont surjectifs. De plus, si la caractéristique est positive, on inclut dans la collection des automorphismes associés le Frobenius et son inverse. Supposons que la théorie T de K est simple et relativement mononbasée sur le réduit de pur corps algebriquement clos par rapport à l'opérateur clôture algébrique* acl, *qui coïncide avec la clôture algébrique corpique de la structure engendrée. Si la famille de mots en $F_1, \ldots, F_n, \sigma_1^{-1}, \ldots, \sigma_t^{-1}$ ne satisfait aucune relation linéaire non-triviale sur* $\mathrm{acl}(\mathbb{F})$*, modulo T, alors tout automorphisme borné de K est un produit de puissances entières des automorphismes associés (et Frobenius, en caractéristique positive).*

Ceci s'applique aux théories suivantes de corps munis d'opérateurs :

- *les corps algébriquement clos (K, Id) en toute caractéristique avec automorphisme associé soit l'identité soit Frobenius en caractéristique positive.*
- *les corps différentiellement clos $(K, \delta_1, \ldots, \delta_n)$ en caractéristique nulle avec n dérivations qui commutent avec automorphisme associé l'identité ;*
- *les corps aux différences génériques (K, σ) en toute caractéristique avec automorphismes associés σ et le Frobenius, si la caractéristique est positive ;*
- *les corps différentiels aux différences (K, δ, σ) en caractéristique nulle avec automorphisme associé σ ;*
- *les corps $(K, F_1, \ldots, F_n)$ munis d'opérateurs libres en caractéristique nulle, avec automorphismes associés $\sigma_0, \ldots, \sigma_t$, introduits par Moosa et Scanlon [64].*

La caractérisation des ensembles définissables et des types dans certaines des théories ci-dessus des corps munis d'opérateurs permet à Hrushovski de démontrer une version fonctionelle de Mordell-Lang en toute caractéristique [42], car le comportement des ensembles définissables dans ces théories ressemble à la topologie de Zariski en géométrie algébrique. Pillay et Ziegler [77] contournent les *Géométries de Zariski* [48] dans la démonstration de Hrushovski, en isolant une propriété, valable dans plusieurs contextes, e.g., les espaces compacts complexes [63], dite *propriété de la base canonique* CBP. Pour un corps différentiellement clos en caractéristique nulle (suffisament saturé), la CBP affirme que, étant donné un ensemble définissable X de degré de transcendance differentiel borné et de degré de Morley 1, alors son corps de définition est presqu'interne au corps de constantes au-dessus d'une réalisation générique de X. Puisqu'un type algébrique est interne à toute famille invariante des types, la CBP généralise la monobasitude. Kowalski et Pillay [57] s'appuient sur des résultats de définissabilité des groupes monobasés pour montrer qu'un groupe connexe définissable dans une théorie stable ayant la CBP est central-par-(presque-Σ-interne). Inspirés de leurs résultats, nous introduisons une hiérarchie de notions, la *hiérarchie serrée*, généralisant la hiérarchie ample, selon laquelle 2-serré coïncide avec CM-triviale (c'est-à-dire, non-2-ample). Nous montrons donc les résultats équivalents dans ce contexte lorsque l'on réplace *fini* par *presque-Σ-interne*.

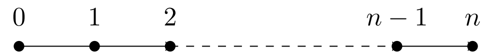
Theorem E. *[BMW12b] Soit T une théorie stable 2-serrée par rapport à une famille invariante Σ de types. Tout corps interprétable est Σ-interne. Tout groupe interprétable de rang de Lascar fini est nilpotent-par-(presque-Σ-interne). En particulier, tout groupe simple non-abélien est Σ-interne.*

Pour finir cette introduction, revenons à la hiérarchie ample. Quoique le pseudoplan libre est un exemple naturel d'une théorie strictement 1-ample, l'existence des théories de degré d'ampleur strictement n, pour $n \geq 2$, restait ouverte. Nous avons repris la construction du pseudo-espace libre, due à Baudisch et Pillay, pour obtenir (voir aussi [89]), pour chaque entier naturel n, un *pseudo-espace libre n-dimensionnel* avec les propriétés suivantes :

Theorem F. *[BMZ14b] Le pseudo-espace libre n-dimensionnel est ω-stable de rang ω^{n+1} et n-ample mais non-$n+1$-ample.*

Tent remarque que le pseudo-espace libre n-dimensionnel précedent est bi-intreprétable avec l'immeuble à angles-droits et résidus infinis de graphe de Coxeter

$[0, n]$



De façon analogue à l'utilisation d'arbres pour analyser les groupes libres, les immeubles furent introduits par Tits [91] pour déterminer des propriétés de groupes algébriques semisimples. Or, un immeuble est une structure de nature combinatoire qui ne doit pas forcement être liée à un groupe, e.g., la géométrie d'incidence des plans projectifs. Un immeuble à angles droits et résidus infinis de groupe de Coxeter (W, Γ) est la donnée d'un ensemble X muni d'une famille de relations d'équivalence $(\sim_\gamma,\ \gamma \in \Gamma)$, telle que chaque $\sim_\gamma$-classe est infinie et que, pour x et y dans X, il existe un élément g dans W réprésentant exactement tous les chemins réduits possibles entre x et y avec mot w. Le *graphe de Coxeter* asssocié à l'immeuble X a pour ensemble de sommets Γ, la famille génératrice de W, où γ et δ appartient à une arête s'ils ne commutent pas dans W. À isomorphisme près, il existe un seul immeuble dénombrable à angles droits et résidus infinis $\mathrm{B}_0(\Gamma)$ pour chaque groupe de Coxeter (W, Γ) [36]. Nous nous intéressons à l'étude modèle-théorique de $\mathrm{B}_0(\Gamma)$. Or, afin de pouvoir prendre des extensions élémentaires, nous avons besoin de considérer des chermins non-standards, ce qui nous oblige à introduire une expansion $\mathrm{B}^0(\Gamma)$ du langage naturel des immeubles. Une adaptation directe des techniques utilisées pour traiter le pseudoespace libre donne le résultat suivant.

Theorem G. *[BMZ14a] La théorie* PS_Γ *de* $\mathrm{B}^0(\Gamma)$ *est ω-stable et équationnelle avec déviation triviale. Le modèle* $\mathrm{B}^0(\Gamma)$ *est le modèle premier. Si* Γ *n'a aucune arête, alors* PS_Γ *est monobasée. Sinon, parmi les points non-isolés de* Γ*, soient r la valence minimale et notons par n le plus grand entier tel que le graphe* $[0, n]$ *se plonge comme sous-structure (au sens des graphes) du graphe de Coxeter* Γ*. Alors, la théorie* PS_Γ *est n-ample mais non-*$(|\Gamma| - r + 1)$*-ample.*

Ces bornes sont otpimales et atteintes par le graphe $[0, n]$ même, dont la théorie est n-ample mais non-$(n + 1)$-ample. Nous ignorons si nos méthodes peuvent s'appliquer dans un contexte plus large, puisqu'une propriété fondamentale des immeubles à angles-droits est qu'un mot est réduit si et seulement si aucune permutation ne peut s'écrire sous la forme $w_1 \cdot \gamma \cdot \gamma \cdot w_2$, pour un des générateurs γ.

"Begin at the beginning," the King said, gravely, "and go on till you come to an end; then stop."

Lewis Carroll – *Alice in Wonderland*

I
Preliminaries

Groups are ubiquitous in mathematics, materialising even in abstract contexts. Zilber's celebrated result shows that an ω-categorical strongly minimal set whose geometry is non-trivial interprets an infinite group, in terms of germs of *quasi-translations*. This allowed him to describe the associated geometry of an ω-categorical strongly minimal set as either trivial, or affine or projective over some finite field. Similarly, within an ambient stable theory, given a stationary type p internal to an invariant collection of types Σ, the group of permutations of p induced by those permutations fixing all realisations of Σ becomes type-definable, which has remarkable consequences in generalised differential Galois theory [70].

In this chapter, we will provide an overview of the techniques and notions required for later sections. Most of the material originates from Casanovas [23] and Wagner [97] for the simple case, and Poizat [79], Pillay [68] and Tent-Ziegler [90] for the stable case. However, the presentation here does not correspond to the historical timeline in which the results were obtained. We will assume that the reader is familiar with (the proof of) Morley's theorem and the techniques used thereupon.

Notation

Given a complete first-order theory T with infinite models in some possibly uncountable language $\mathcal{L}$, we will work inside a sufficiently saturated model of T, referred to as *the monster*, so that all sets are seen as small subsets of the monster model, where by *small* we means of size strictly less than the saturation of the monster. Similarly, by a model of T we mean a small elementary substructure of the monster.

Greek letters κ, λ, etc. will denote infinite cardinals. Letters a, b, etc. will denote possibly infinite subtuples of the monster, unless specified. Given two sets A and B, we will denote their union by AB, and by $\mathcal{P}(A)$ the collection of all subsets of A.

The word type refers, unless specified, to a complete type consistent with T, that is, a type p over a subset A is a consistent set of formulae over A and maximal such. We will often not specify the arity of the type. The space of n-types over A, denoted by $\mathrm{S}_n(A)$, is a compact Hausdorff totally disconnected 0-dimensional space with the logic topology, where the basic clopen sets are of the form :

$$[\varphi] = \{p \in \mathrm{S}_n(A) \,|\, \varphi(x) \in p\},$$

for some formula φ over A. The notation $a \equiv_A b$ means that the tuples a and b have the same type over the set A, or equivalently, that there is some automorphism of the monster fixing A pointwise and mapping a to b.

1.1 Forking and Imaginaries

The class of simple theories, as introduced by Shelah [86] and later considered by Kim and Pillay [54], comprises the class of stable theories and allows an adaptation of many of the key tools from geometric stability theory.

Recall that a formula $\varphi(x, b)$ *divides over* A if there exist a natural number k and a sequence $\{b_i\}_{i<\omega}$ such that $b_i \equiv_A b$ and the collection $\{\varphi(x, b_i)\}_{i<\omega}$ is *k-inconsistent*, that is, every subcollection of size k is inconsistent. A Ramsey-style argument shows that $\varphi(x, b)$ divides over A if and only if there is some A-indiscernible sequence $\{b_i\}_{i<\omega}$ such that $b_0 = b$ and $\{\varphi(x, b_i)\}_{i<\omega}$ is inconsistent.

A formula $\varphi(x, b)$ *forks over* A if it belongs to the ideal generated by those formulae, with parameters in the monster, dividing over A, that is, the formula $\varphi(x, b)$ forks over A if there are formulae $\psi_1(x, c_1), \dots, \psi_n(x, c_n)$, each dividing over A, such that

$$\varphi(x, b) \models \bigvee_{i=1}^{n} \psi_i(x, c_i).$$

A type p over B *divides*, resp. *forks*, over A if it contains a formula which does.

Notice that every dividing formula forks. The collection of types in $\mathrm{S}(B)$ which do not fork over A is a closed subset, and thus a compact space with the subset topology.

Given a partial type π, a natural number k and a finite set of formulae Δ, we define the *local rank* $D(\pi, \Delta, k)$, by induction, as follows:

- $D(\pi, \Delta, k) \geq 0$ whenever π is consistent.
- $D(\pi, \Delta, k) \geq \alpha + 1$ if there is some formula $\varphi(x, y)$ in Δ and a sequence $\{b_i\}_{i<\omega}$ such that the collection $\{\varphi(x, b_i)\}_{i<\omega}$ is k-inconsistent and $D(\pi \cup \{\varphi(x, b_i)\}, \Delta, k) \geq \alpha$ for every $i < \omega$.
- $D(\pi, \Delta, k) \geq \alpha$ if $D(\pi, \Delta, k) \geq \beta$ for $\beta < \alpha$, for a limit ordinal α.

Observe that $D(\pi, \Delta, k) \geq \alpha + 1$ if and only if there is some formula $\varphi(x, y)$ in Δ and a tuple b such that $\varphi(x, b)$ k-divides over the parameters of π and $D(\pi \cup \{\varphi(x, b)\}, \Delta, k) \geq \alpha$.

The theory T is simple if and only if $D(x = x, \varphi, k) < \omega$ for every formula φ and every integer k. This is equivalent to require that for every 1-type p over some subset B, there exists some subset B_0 of size at most $|T|$ such that p does not divide, or equivalently it does not fork, over some subset B_0 of B of size at most $|T|$.

Given a type p in $\mathrm{S}(A)$ in a simple theory, it does not fork over $B \subset A$ if and only if

$$D(p, \varphi, k) = D(p{\upharpoonright}B, \varphi, k)$$

for every formula φ and every integer k.

Given subsets A, B and C, we say that A is *independent* of B over C if, for every finite tuple a in A, the type $\mathrm{tp}(a/BC)$ does not fork over C.

Theorem 1.1. *In a simple theory T, the above independence relation between triples of sets satisfies the following conditions :*

Invariance *If $ABC \equiv A'B'C'$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $A' \mathop{\smile\!\!\!\!\!\!\mid}_{C'} B'$.*

Symmetry *$A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$.*

Monotonicity and Transitivity *$A \mathop{\smile\!\!\!\!\!\!\mid}_D BC$ if and only if $A \mathop{\smile\!\!\!\!\!\!\mid}_D C$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_{CD} B$.*

Extension *For all a, B and C, there exists some $a' \equiv_C a$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}_C B$.*

Finite Character *$A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$ for every finite tuples a in A and b in B.*

Local Character *For every finite tuple a and every set B, there exists some subset B_0 of size at most $|T|$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} B$.*

Independence Theorem over Models *Given tuples a and b, sets A and B and a model M such that :*

- *$a \equiv_M b$,*
- *$A \mathop{\smile\!\!\!\!\!\!\mid}_M B$,*
- *$a \mathop{\smile\!\!\!\!\!\!\mid}_M A$,*
- *$b \mathop{\smile\!\!\!\!\!\!\mid}_M B$,*

then there is some $c \mathop{\smile\!\!\!\!\!\!\mid}_M BC$ such that $c \equiv_{AM} a$ and $c \equiv_{MB} b$.

Furthermore, if the theory T is equipped with some independence relation between triples of sets satisfying the above conditions, then T is simple and the independence relation agrees with non-forking (cf. [54, Theorem 4.2]).

In a simple theory, every type is an *extension base* (cf. [29, Definition 2.7]): for every tuple a and every set B, we have that $a \mathop{\smile\!\!\!\!\!\!\mid}_B B$. In particular, every type admits a *global non-forking* extension to the monster model. However, there can be many different such extensions. A type p is *stationary* if it only admits a unique non-forking extension to every superset. Notice that in a simple theory, non-algebraic types have Morley sequences of arbitrary length.

A simple theory T is *stable* if types over models M are stationary, that is, each type over a model M admits a unique non-forking extension to every superset of parameters. Equivalently, if $|\mathrm{S}(A)| \leq \lambda$ for any cardinal λ with $\lambda^{|T|} = \lambda$ and any set A of size at most λ. Among stable theories, a distinguished subclass consists of ω-stable theories, those theories for which $|\mathrm{S}(A)| \leq \omega$ for any countable set A. If the language $\mathcal{L}$ is countable, a theory is ω-stable if and only if the Morley rank $\mathrm{RM}(x = x)$ is some ordinal. Recall that the *Morley rank* of a formula $\varphi(x, a)$ is defined by transfinite induction as follows: we say that $\mathrm{RM}(\varphi(x, a)) > \alpha$ if there is an infinite family $\{\psi_n(x)\}_{n\in\mathbb{N}}$ of disjoint consistent formulae, possibly with additional parameters from the monster model (or equivalently from some ω-saturated model containing a), such that each $\psi_n \vdash \varphi(x, a)$ and $\mathrm{RM}(\psi_n(x)) \geq \alpha$, for each n in $\mathbb{N}$. If $\mathrm{RM}(\varphi(x, a)) \not> \alpha$ for some ordinal α, then set $\mathrm{RM}(\varphi(x, a))$ to be the least ordinal β such that $\mathrm{RM}(\varphi(x, a)) \not> \beta$ but $\mathrm{RM}(\varphi(x, a)) \geq \beta$.

Two definable sets of Morley rank α are *α-equivalent* if their symmetric difference has rank strictly less than α. A formula of Morley rank α has *Morley degree* 1 if it does not contain two pairwise inconsistent formulae of rank α. Each formula φ of rank α can be written as a finite union of formulae of Morley rank α and Morley degree 1, in a unique way, up to permutation (and α-equivalence). The number of formulae appearing in the union is the *Morley degre* $\mathrm{dM}(\varphi)$ of the formula φ. We define $\mathrm{RM}(b/A) = \mathrm{RM}(\mathrm{tp}(b/A))$ as the minimum of the Morley rank of the formulae it contains.

A particular example of which are *strongly minimal* theories, that is, those of Morley rank and degree 1, or equivalently, those theories for which every definable set in one variable, with parameters in the monster, is either finite or cofinite. For strongly minimal, Morley rank is additive :

$$\mathrm{RM}(a, b/A) = \mathrm{RM}(a/A) + \mathrm{RM}(b/Aa),$$

and it corresponds to the dimension function of the pregeometry associated to the closure operator acl, which satisfies the exchange principle (cf. [62, Proposition 1.2] and [2, Lemma 2]). In strongly minimal theories, Morley rank is definable: for any formula $\varphi(x_1, \ldots, x_n, y)$, the collection

$$\{b \mid \mathrm{RM}(\varphi(x_1, \ldots, x_n, b)) = k\}$$

is definable for every integer $k \leq n$.

A theory is stable if and only if no formula $\varphi(x, y)$ has the *order property*: there are infinite sequences $\{a_i, b_j\}_{i,j \in \mathbb{N}}$ of tuples such that

$$\varphi(a_i, b_j) \text{ if and only if } i < j.$$

In order to check the stability of T, it suffices to show that no formula $\varphi(x, y)$, where x is a single variable, has the order property.

One of the many remarkable features of stability is the definability of types. A type p in $\mathrm{S}(A)$ is *definable* over $B \subset A$ if, for any $\mathcal{L}$-formula $\varphi(x, y)$, there is some formula $d_p\varphi(y)$ over B such that, given a in A,

$$\varphi(x, a) \in p \Leftrightarrow a \models d_p\varphi(y).$$

The type p is *definable* if it is definable over A. Stability is equivalent to types over models being definable. If T is stable, given a type p over a model, there is, for every $A \supset M$, one and only one extension q in $\mathrm{S}(A)$ of p satisfying any (all) of the following conditions:

- q does not fork over M.
- q is definable over M.
- q is an *heir* over M, that is, for every $\mathcal{L}$-formula $\varphi(x, y)$, if $\varphi(x, a)$ belongs q for some a, then $\varphi(x, m)$ belongs to p for some m in M.
- q is a *coheir* over M, that is, every formula in q has a realisation in M.

The foundation rank associated to the forking relation on types in a simple theory is called the *Lascar rank* SU, defined by transfinite induction as follows:

- $\mathrm{SU}(p) \geq 0$,
- $\mathrm{SU}(p) \geq \alpha + 1$ if and only if p has some extension q over $B \supset A$ with $\mathrm{SU}(q) \geq \alpha$ such that q forks over A,
- $\mathrm{SU}(p) \geq \alpha$ if $\mathrm{SU}(p) \geq \beta$ for $\beta < \alpha$, for a limit ordinal α .

Observe that, if $\alpha < \mathrm{SU}(p) < \infty$, then there is some extension q of p with $\mathrm{SU}(q) = \alpha$. However, Lascar rank need not be continuous. If the theory T is simple and $\mathrm{SU}(p)$ is defined, then an extension $q \supset p$ is non-forking if and only if $\mathrm{SU}(q) = \mathrm{SU}(p)$.

Recall the *Lascar inequalities* for a simple theory T. Given a set A and two finite tuples a and b such that $\mathrm{SU}(ab/A) < \infty$, then

$$\mathrm{SU}(a/Ab) + \mathrm{SU}(b/A) \leq \mathrm{SU}(a, b/A) \leq \mathrm{SU}(a/Ab) \oplus \mathrm{SU}(b/A),$$

where $\alpha \oplus \beta = \sum_n^0 \omega^{\gamma_i} \cdot (m_i + k_i)$ is the direct sum of

$$\alpha = \omega^{\gamma_n} \cdot m_n + \cdots + \omega^{\gamma_0} \cdot m_0$$

and

$$\beta = \omega^{\gamma_n} \cdot k_n + \cdots + \omega^{\gamma_0} \cdot k_0,$$

with m_i, k_i integers numbers and ordinals $\gamma_n \geq \ldots \geq \gamma_0 \geq 0$. In particular, if both $\mathrm{SU}(a/A)$ and $\mathrm{SU}(b/A)$ are finite, then $\mathrm{SU}(a, b/A) = \mathrm{SU}(a/Ab) + \mathrm{SU}(b/A)$.

The simple theory T is *supersimple* if the Lascar rank of every n-type is an ordinal, or equivalently, if every type p does not fork over some finite subset of the parameter set. A stable supersimple theory is called *superstable*. In this case, the above rank agrees with Lascar's original definition of the unique connected rank [58], so we will denote it by $\mathrm{U}(p)$ instead.

Morley rank bounds from above the Lascar rank : $\mathrm{SU}(p) \leq \mathrm{RM}(p)$, so ω-stable theories are superstable.

Though types in a stable theory are definable, there need not be a smallest subset of the monster over which they are definable. Likewise, definable sets need not have canonical sets of definition. For that reason, we need to consider the expansion T^{eq} of T by adding *imaginaries*. In the case of simple theories, imaginaries alone are not sufficient in order to obtain canonical bases for types, even over models, so that type-definable equivalence relations and *hyperimaginaries* need to be considered, despite that negation in the expansion T^{heq} is no longer first-order.

A tuple d is a *canonical parameter* for the definable set X if, given any automorphism f of the monster, $f(X) = X$ setwise if and only if f fixes pointwise the tuple d.

An easy compactness argument shows that, if d is a canonical parameter for the definable set X, then X is definable by a formula with parameters in d. Furthermore, the tuple d is unique, up to interdefinability, so we will denote it by $d = \ulcorner X \urcorner$.

The theory T *eliminates imaginaires* if each class a/E of every 0-definable equivalence relation E has a canonical parameter. Equivalently, if every definable set has a canonical parameter.

Lemma 1.2. *If T has elimination of imaginaries, given a global definable type p, it is then invariant over some set B: that is, given any automorphism σ of the monster, the type $\sigma(p)$ equals p if and only if σ fixes pointwise B.*

If such a set B exists, then it is unique, up to interdefinability, so we will denote it by $\mathrm{Cb}(p)$, the *canonical base* of p. In particular, the type p does not fork over $\mathrm{Cb}(p)$.

As long as the language $\mathcal{L}$ contains at least two constants, the theory T has elimination of imaginaries if and only if every 0-definable equivalence relation E fibers through a 0-definable function f: that is, two elements lie in the same class modulo E if and only if their images under f coincide. Thus, in order to obtain elimination of imaginaires, we will consider a multi-sorted expansion of T, denoted by T^{eq}, in a language $\mathcal{L}^{\mathrm{eq}}$ with sorts $\{S_{E_i}\}_{i\in I}$, where $\{E_i\}_{i\in I}$ is an enumeration, up to T-equivalence, of all possible 0-definable equivalence relations. The sort $S_=$ is the original universe and will be called the *real sort*, whereas elements from the other sorts are called *imaginaries*. We expand any $\mathcal{L}$-structure A by interpreting S_{E_i} on A as $A_i^{n_i}/E_i$, equipped with the natural projections $\pi_i : A^{n_i} \to A_i^{n_i}/E_i$.

In order to stress out that a set is $\mathcal{L}^{\mathrm{eq}}$-definable, we will say it is *interpretable*. The theory T^{eq} is then axiomatised by adding to T the following axioms schemes:

$$\forall y \, \exists x \text{ with } \pi_i(x) = y \text{ if } y \text{ lies in } S_i,$$

and

$$\forall x \, \forall y \ (xEy \Leftrightarrow \pi_i(x) = \pi_i(y)) \, .$$

Every $\mathcal{L}^{\mathrm{eq}}$-formula can be retranslated into an $\mathcal{L}$-formula, so there are no new 0-definable relations on the real sort on T^{eq}. Hence, the theory T^{eq} has itself elimination of imaginaries, and is is simple, resp. stable, if and only if T is. Furthermore, the theory T has elimination of imaginaries if and only if every imaginary element is interdefinable with a real tuple.

Notation. We denote by $\mathrm{dcl}^{\mathrm{eq}}$ and $\mathrm{acl}^{\mathrm{eq}}$ the definable and algebraic closures in the expansion T^{eq}.

The theory T has *weak elimination of imaginaries* if every imaginary α is definable over some real tuple a, which is itself algebraic over α.

The theory T has elimination of imaginaries if and only if it has weak elimination of imaginaries and eliminates those imaginaries which encode finite sets. A result of Lascar and Pillay shows that a strongly minimal theory T with infinite $\mathrm{acl}(\emptyset)$ has weak elimination of imaginaries.

The theory T has *geometric elimination of imaginaries* if every imaginary is interalgebraic with some real tuple a.

For a stable theory T, types over algebraically closed sets in T^{eq} are stationary, so we can refer to their canonical bases as the canonical bases of the global non-forking extension each of the types determine. A stationary type p over A does not fork over $B \subset A$ if and only if $\mathrm{Cb}(p)$ lies in $\mathrm{acl}^{\mathrm{eq}}(B)$. Recall that two tuples have the same type over $\mathrm{acl}^{\mathrm{eq}}(B)$ if and only if they have the same strong type over B. The *strong type* $\mathrm{stp}(a/B)$ is the collection of all elements which lie in the same class as a modulo all B-definable equivalence relations with only finitely many classes. In particular, two elements with the same strong type over B have the same type over B. Strong types in a stable theory are hence stationary, so we will often write $\mathrm{Cb}(a/A)$ to denote $\mathrm{Cb}(\mathrm{stp}(a/A))$.

Observe that, by compactness, a definable equivalence relation with a bounded number of classes has only finitely many classes. If the simple theory T is unstable, we need to additionally consider, possible not definable, equivalence relations with boundedly many classes. In general, since a small intersection of bounded equivalence relations is again bounded, there is a smallest A-invariant bounded equivalence relation, each of whose classes is called a *Lascar strong type* over A. Two tuples have the same Lascar strong type over A if and only if they lie in the transitive closure of the relation of having the same type over some model containing A. Simple theories are *G-compact* [59, 53], that is, equality of Lascar strong types is type-definable and coincides with the smallest bounded equivalence relation type-definable over A, for two tuples a and b in a simple theory T have the same Lascar strong type over A if there is some c such that both a, c and b, c start A-indiscernible sequences over A.

Type-definable equivalence relations allow us to describe canonical bases for simple theories. A *hyperimaginary* a_E is the equivalence class of a (possibly infinite) tuple a modulo some type-definable equivalence relation E. Clearly, every imaginary is a hyperimaginary. Given a collection A of hyperimaginaries, its bounded closure $\mathrm{bdd}(A)$ is the collection of all hyperimaginaries α having a bounded orbit under all automorphisms of the monster fixing A pointwise. The collection $\mathrm{bdd}(A)$ may be a proper class. Observe that, if A consists exclusively of imaginaries and α is some imaginary lying in $\mathrm{bdd}(A)$, then α lies in $\mathrm{acl}^{\mathrm{eq}}(A)$.

Notation. We denote by $\mathrm{dcl}^{\mathrm{heq}}$ and $\mathrm{acl}^{\mathrm{heq}}$ the hyperimaginary definable and algebraic closures, that is, the collection of hyperimaginaries whose orbit under the automorphisms fixing the base set is a singleton, resp. finite.

Given hyperimaginaries a_E and b_F and an $\mathcal{L}$-formulae φ, set

$$\Phi_\varphi(x, y) = \exists x' \exists y' \left(E(x, x') \wedge F(y, y') \wedge \varphi(x, y)\right).$$

We define the type $\mathrm{tp}(a_E/b_F)$ as the union, over all $\mathcal{L}$-formulae φ, of all partial types $\Phi_\varphi(x, b)$ such that $\models \varphi(a', b')$ for some $a' \models E(x, a)$ and $b' \models F(y, b)$. If we let b vary among its F-class, such a union gives an equivalent partial type. Notice that, though $\mathrm{tp}(a_E/b_F)$ is a partial type, any two realisations can be mapped one to another by an automorphism fixing b_F.

This allows us to define hyperimaginary dividing and forking, which inherit most of the properties of forking for real tuples, replacing acl by bdd, when T is simple. In particular, two tuples have the same Lascar strong type over a hyperimaginary e if and only if they have the same type over $\mathrm{bdd}(e)$. Over a model M, Lascar strong types and ordinary types coincide. Lascar strong types in a simple theory are *amalgamation bases*, for which the independence theorem holds: any two non-forking extensions of a common Lascar strong type over independent sets of parameters can be glued into a common non-forking extension. Hence, we define the canonical base of $\mathrm{Cb}(a/A)$ of a Lascar strong type $p = \mathrm{tp}(a/\mathrm{bdd}(A))$ to be the smallest hyperimaginary α in $\mathrm{bdd}(A)$ such that p does not fork over α and $p\restriction\alpha$ remains an amalgamation base. In the stable case, this definition agrees with the previous one, since type-definable equivalence relations are intersections of definable ones, so hyperimaginaries are interdefinable with infinite sequences of imaginary elements.

Canonical bases allow us then to prove the following fact, which will be used repeatedly.

Fact 1.3. In a simple theory, given subsets A, B, C and D with $\mathrm{bdd}(A) \cap \mathrm{bdd}(B) = \mathrm{bdd}(C)$ and $D \mathop{\smile\!\!\!\!\!\!\;|}_C AB$, then

$$\mathrm{bdd}(DA) \cap \mathrm{bdd}(DB) = \mathrm{bdd}(DC).$$

Proposition 1.4. *Let p be a Lascar strong type in a simple theory. Its canonical base $\mathrm{Cb}(p)$ is definable (as a hyperimaginary) over any Morley sequence $\{a_i\}_{i<\omega}$ of p. If the theory is supersimple and p is of real sort, then $\mathrm{Cb}(p)$ is definable over some initial segment $\{a_i\}_{i\leq n}$.*

In supersimple theories, hyperimaginaries need not be considered.

Theorem 1.5 (Buechler, Pillay, Wagner [19])**.** *In a supersimple theory, every hyperimaginary is interdefinable with a (possibly infinite) sequence of imaginary elements.*

1.2 NFCP and Equationality

This section presents two strengthenings of stability, which will play a decisive role in Chapters IV and VI. In order to produce saturated ultraproducts, Keisler [52] introduced the finite cover property, which somewhat unexpectedly relates to various constructions in geometric model theory.

Definition 1.6. A theory T *does not have the finite cover property* (or T *has NFCP*) if for every formula $\varphi(x, y)$, there is a natural number n such that for any sequence $\{a_i\}_{i\in\mathbb{N}}$ where $\{\varphi(x, a_i)\}_{i\in\mathbb{N}}$ is inconsistent, then there is a subset $J \subset \mathbb{N}$ of size n such that $\{\varphi(x, a_i)\}_{i\in J}$ is inconsistent.

In order to show that T has NFCP, it suffices to consider formulae $\varphi(x, y)$, where x is a single variable. NFCP theories are stable [87, Theorem 4.2]. A stable theory T has NFCP if and only if whenever $E(x, y, z)$ defines a family of equivalence relations, parametrised by z, then there is a uniform bound on the size of those classes which are finite [87, Theorem 4.4].

Given a complete theory T, expand its language by adding a new symbol σ. A *generic automorphism* (M, σ) is an existentially closed model of the incomplete theory

$$T_\sigma = T \cup \{\text{“}\sigma \text{ is an automorphism.”}\}$$

If the class of existentially closed models of T_σ is elementary, we denote it by TA. If T is stable and TA exists, then T has NFCP. For a stable theory, the existence of TA is equivalent to NFCP and a certain technical condition [73, Theorem 1.1], which holds for strongly minimal theories with the *definable multiplicity property* (DMP): given a formula $\varphi(x, b)$ of rank k and degree m, there is a formula θ in $\mathrm{tp}(b)$ such that, whenever $b' \models \theta$, then so does $\varphi(x, b')$ have rank k and degree m. In particular, the theory ACFA of existentially closed difference fields exists [27]. It is supersimple of Lascar rank ω.

Another remarkable expansion of the theory of algebraically closed fields is the theory of an algebraically closed field equipped with a definable algebraically closed proper subfield. This theory, already considered by Keisler [51] after work of Robinson, is complete once the characteristic is fixed. It is a particular example of the theory of *belle paires* [78] of models of a stable theory. Assume that T is stable with elimination of quantifiers and imaginaries, to ease the presentation. A pair of models $E \precneqq M$ of T is a *belle paire* if E is $|T|^+$-saturated and M realises every type over $A \cup E$, whenever $A \subset M$ has cardinality strictly less than $|T|^+$. Any two belle paires are elementarily equivalent, so we denote by T_P their common theory in the language $\mathcal{L}_P = \mathcal{L} \cup \{P\}$. However, a $|T|^+$-saturated model of T_P need not be itself a belle paire. Poizat showed that it is the case if and only if T has NFCP. If so, the theory T_P has NFCP as well, so it is, in particular, stable.

Assume now that T has NFCP, and work inside a sufficiently saturated model (M, E) of T_P. The index P will refer to T_P. Thus, by $\mathrm{acl}_P(A)$ we denote the real elements algebraic over A in the theory T_P and $\mathop{\smile\!\!\!\!\!\!|}^P$ denotes non-forking independence in T_P. A subset A of M is *P-independent* if

$$A \mathop{\smile\!\!\!\!\!\!|}_{E_A} E,$$

where $E_A = E \cap A = P(A)$. Two P-independent substructures having the same quantifier-free type in T_P are elementarily equivalent. Since every subset of E is P-independent, there is in particular no additional structure on E induced by T_P.

The following properties follow from [11, Remark 7.2 and Proposition 7.3]

Fact 1.7. Let A and B be subsets of M.

- The algebraic closure $\mathrm{acl}_P(A)$ is P-independent.
- If A is P-independent, then $A \mathop{\smile\!\!\!\!\!\!|}^P_{E_A} E$. In this case, we have that $\mathrm{acl}_P(A) = \mathrm{acl}(A)$ and $E_{\mathrm{acl}_P(A)} = \mathrm{acl}(E_A) = \mathrm{acl}_P(E_A)$.
- If A and B are P-algebraically closed, then

$$A \mathop{\smile\!\!\!\!\!\!|}^P_{A\cap B} B \quad \text{if and only if} \quad \begin{cases} A \mathop{\smile\!\!\!\!\!\!|}_{A\cap B, E} B \\ \text{and} \\ E_A \mathop{\smile\!\!\!\!\!\!|}_{E_{A\cap B}} E_B \end{cases}$$

Another strengthening of stability which resonates with NFCP is equationality [75]. A parameter-free formula $\varphi(x, y)$, where the tuple x has length n, is an *n-equation* if the family of finite intersections of instances $\varphi(x, a)$ has the descending chain condition (DCC). A complete theory T is *n-equational* if every definable set in n variables is a Boolean combination of instances of n-equations. A theory is *equational* if it is n-equational for every n in $\mathbb{N}$.

Typical examples of equational theories are the theory of an equivalence relation with infinite many infinite classes, completions of the theory of R-modules or algebraically closed fields. If $\varphi(x, y)$ is an equation, then so is $\varphi^{-1}(y, x) = \varphi(x, y)$. Finite conjunctions and disjunctions of equations are again equations.

Similar to stability, equationality is preserved under naming parameters and bi-interpretability [49]. However, it is unknown whether equationality follows from 1-equationality, which itself implies stability for formulae $\varphi(x, y)$, where x is a single variable, and thus stability [75]. Otherwise, since the order property is preserved by Boolean combinations, suppose that there is a formula $\varphi(x, y)$, where x is a single variable x, with the order property, witnessed by sequences $\{a_i, b_j\}_{i,j\in\mathbb{N}}$ such that

$$\models \varphi(a_i, b_j) \text{ if and only if } j < i.$$

The sequence of finite intersections $\varphi(x, b_1) \supsetneq \varphi(x, b_1) \cap \varphi(x, b_2) \supsetneq \dots$ is strictly decreasing, so φ is not a 1-equation.

Besides an unpublished construction of Hrushovski, the only known *natural* example of a stable non-equational theory so far is the free non-abelian group [84, 85]. However, proving that a particular stable theory is equational is far from obvious, in general.

Remark 1.8. If T is equational, given a type p over a model M, its definition scheme is particularly easy to obtain: Since $d_p(\neg\varphi) = \neg d_p\varphi$, we may assume that $\varphi(x, y)$ is an equation. The intersection

$$\bigcap_{\varphi(x,m)\in p} \varphi(x, m)$$

is a formula $\xi(x)$ over M contained in p. Set hence

$$d_p\varphi(y) = \forall x \left(\xi(x) \Rightarrow \varphi(x, y)\right).$$

Instances of equations have a fundamental property: they are Srour-closed. A definable set X is *Srour-closed* if the family of finite intersections of conjugates of X by automorphisms of the monster has the DCC. We have an equivalence between these two notions: Srour-closed definable sets are exactly those, which are instances of equations.

Recall that a set X is closed in the indiscernible topology, introduced by Junker and Lascar [50], or *indiscernible-closed* if, whenever $\{a_i\}_{i\in\mathbb{N}}$ is an indiscernible sequence such that a_i lies in X for $i > 0$, then so does a_0.

The indiscernible topology refines the Srour topology, though they agree on definable sets [50, Theorem 3.16]. Indeed, if $\varphi(x, b)$ is not Srour-closed, compactness and a Ramsey-style argument imply that there is a sequence of pairs $\{a_i, b_i\}_{i\in\mathbb{Z}}$ such that $\{a_i\}_{i\in\mathbb{Z}}$ is indiscernible and all $b_i's$ realise $\mathrm{tp}(b)$ such that

$$\models \varphi(a_i, b_j) \text{ if and only if } i < j.$$

Equipped with the inverse order, the sequence $\{a_i\}_{i\leq 0}$ is indiscernible, but witnesses that $\varphi(x, b_0)$ is not indiscernible-closed. Hence neither is $\varphi(x, b)$. Suppose now that the sequence $\{a_i\}_{i\in\mathbb{N}}$ witnesses that the definable set $X_b = \varphi(x, b)$ is not indiscernible-closed. For every $i > 0$, there is an automorphism σ_i which sends the sequence $\{a_j\}_{j\in\mathbb{N}}$ to $\{a_{i+j}\}_{j\in\mathbb{N}}$. In particular, for every $i > 0$, we have that

$$a_i \in \bigcap_{j<i} X_{\sigma_j(b)} \setminus X_{\sigma_i(b)}.$$

We obtain hence a decreasing chain

$$X_b \supsetneq X_b \cap X_{\sigma_1(b)} \supsetneq X_b \cap X_{\sigma_1(b)} \cap X_{\sigma_2(b)} \supsetneq \ldots,$$

so X_b is not Srour-closed.

1.3 Generics, Stabilisers and Isogenies

For this section, assume that the complete theory T is simple. Recall that a *type-definable group* G is a type-definable set with a group law given by a (relatively) definable binary operation $\cdot$, as well as an identity element. We denote by S_G the closed subspace of types extending the partial type “ $x \in G$ ”.

An element g in G is *(left) generic* over A if, whenever

$$h \mathop{\smile\!\!\!\!\!|}_A g,$$

then

$$h \cdot g \mathop{\smile\!\!\!\!\!|} A, h.$$

A type is *generic* if its realisations are. One can smilarly introduce right generic elements and types, though these notions agree with the previous ones. Genericity is preserved under non-forking extension and restriction, as well as by taking inverses. Furthermore, if g is generic over A and h is an element of G which is algebraic over A, then $h \cdot g$ is again generic over A. The product of two generic independent elements is generic and independent of each factor.

In order to show the existence of generic types over any given subset of parameters, we modify the local ranks to witness dividing in terms of the group action. Given an integer k and formula φ, the *stratified* (φ, k)-rank $D^*(\pi, \varphi, k)$ of a partial type π is characterised by the following principle:

- $D^*(\pi, \varphi, k) \geq n+1$ if and only if there are elements $\{g_i\}_{i\in\mathbb{N}}$ in G and parameters $\{a_i\}_{i\in\mathbb{N}}$ such that the family $\{\varphi(g_i \cdot x, a_i)\}_{i\in\mathbb{N}}$ is k-inconsistent and $D^*(\pi \cup \{\varphi(g_i \cdot x, a_i)\}, \varphi, k) \geq n$ for every i in $\mathbb{N}$.

Observe that the stratified (φ, k)-rank of any G-conjugate of π equals the stratified (φ, k)-rank of π and is bounded from above by the corresponding local rank of π. Every partial type in $\mathrm{S}_G(A)$ admits a completion of the same stratified rank. The sequence of stratified local ranks witnesses forking for types in S_G. An element is generic over A if and only if the values of all its stratified local ranks are maximal among types in $\mathrm{S}_G(A)$. Thus, generic types over any set of parameters exist and they correspond to *f-generic* types, that is, those types whose global non-forking extensions p satisfy that no translate

$$g \cdot p = \{\varphi(x) \,|\, p \models \varphi(g \cdot x)\}$$

forks over A (or over $\emptyset$).

The existence of generic types implies in particular that every element of G can be written as the product of two, not necessarily independent, generic elements.

A relatively definable subset X of G is *(left) generic* if there is a finite set G_0 of G such that $G = G_0 \cdot X = \bigcup_{g\in G_0} g \cdot X$.

We can define similarly the notions of right or bilateral generic definable subsets. If T is stable, then either a relatively definable subset is generic or its complement is, so generic types are exactly those containing only generic formulae. Thus, in the ω-stable case, a type is generic if and only if its Morley rank is maximal possible in S_G, since Morley rank is preserved by definable bijections, and hence by translations. The same holds for supersimple theories : a type is generic if its Lascar rank is maximal possible in S_G. Lascar inequalities imply the following :

$$\mathrm{SU}(H) + \ SU(G/H) \leq \mathrm{SU}(G) \leq \mathrm{SU}(H) \oplus + \mathrm{SU}(G/H).$$

For an ω-stable group, whenever $K \lneq H$ are definable subgroups, either

$$\mathrm{RM}(K) < \mathrm{RM}(H),$$

or the index $[H : K]$ is finite and

$$\mathrm{dM}(K) \cdot [H : K] = \mathrm{dM}(H).$$

In particular, since type-definable subgroups in stable theories are the intersection of definable groups, they become hence the intersection of finitely many in the ω-stable case, and thus definable. If the theory T is simple, type-definable subgroups need to be hence considered: given a set A, the *connected component* G^0_A of G over A equals the intersection of all type-definable subgroups of G over A of bounded index. The group G is *connected over* A if and only if $G = G^0_A$. It is *(absolutely) connected* if $G = G^0_A$ for any set of parameters A.

Note that $G^0_A \trianglelefteq G$ and is again type-definable over A of bounded index. In the stable case, the Baldwin-Saxl chain condition [3] yields that the connected component does not depend on the set of parameters and is type-definable over the same set of parameters as G. The Baldwin-Saxl chain condition implies furthermore that, given a possibly non-definable subgroup H of a stable group G, the generics of its *definable hull* [94] in G, which is the intersection of all definable groups in G containing G, are exactly the types in S_G which are generic for H, that is, those types which contains only formulae which are generic for H: finitely many translates cover H.

For stable groups, there is a close relation between generic types and G^0. Each coset of G^0 contains a unique generic type, hence there is a correspondance between the G/G^0 and the collection of generic types, on which G acts by translation. A generic type in $G^0_{\mathrm{bdd}(A)}$ is called *principal generic*. If T is simple, the connected component may have several generic types, though a partial result was already observed in [74, Proposition 2.2] :

Proposition 1.9. *Given three generic Lascar strong types p, q and r over A in $G^0_{\mathrm{bdd}(A)}$, there are realisations $g \models p$ and $h \models q$ such that $g \cdot h \models r$ and g, h and $g \cdot h$ are pairwise A-independent.*

A Lascar strong type p over A in S_G determines a particular subgroup of $G^0_{\mathrm{bdd}(A)}$, its stabiliser. Given a set of parameters A, the *(left) stabiliser* of an element g in G over A is the subgroup $\mathrm{Stab}(g/A)$, type-definable over $\mathrm{bdd}(A)$, generated by

$$\mathrm{St}(g/A) = \{h \in G \ : \ \exists x \models \mathrm{Lstp}(g/A) \ (hx \models \mathrm{Lstp}(g/A) \ \wedge x \mathop{\smile\!\!\!\!\!\mid}_A h)\}.$$

The independence theorem implies that $\mathrm{Stab}(g/A) = \mathrm{St}(g/A) \cdot \mathrm{St}(g/A)$. All generics of $\mathrm{Stab}(g/A)$ are contained in $\mathrm{St}(g/A)$. Equality $\mathrm{Stab}(g/A) = G^0_{\mathrm{bdd}(A)}$ holds if and only if g is generic over A. If $g \mathop{\smile\!\!\!\!\!\mid}_A B$, then $\mathrm{Stab}(g/B) \leq \mathrm{Stab}(g/A)$ has bounded index.

If G is stable and R is either Morley rank, Lascar rank or the local ranks, then given a strong type $p = \mathrm{stp}(g/A)$ over $A = \mathrm{acl}^{\mathrm{eq}}(A)$, we have that

$$R(\mathrm{Stab}(p)) \leq R(p).$$

If equality above holds, e.g., if the coset $\mathrm{Stab}(p) \cdot g$ is definable over A, then the stabiliser of p is connected and p is the generic type of the coset $\mathrm{Stab}(p) \cdot g$.

Ziegler [104, Theorem 1] noticed a sort of of converse to Proposition 1.9, which can be easily generalised to non-abelian groups type-definable in a simple theory [BMW12a, lemme 1.2]. Recall that two groups H and K are *commensurable* if $H \cap K$ has bounded index in both H and K.

Lemma 1.10. *Given a type-definable group H over $A = \mathrm{bdd}(A)$ in a simple theory, and two elements h and h' such that h, h' and $h \cdot h'$ are pairwise independent over A, then $\mathrm{Stab}(hh'/A) = \mathrm{Stab}(h/A)$ is connected over A and commensurable with some conjugate of $\mathrm{Stab}(h'/A)$, which is connected over A as well. Furthermore, the element h is generic in the coset $\mathrm{Stab}(h/A) \cdot h$, which is definable over $\mathrm{bdd}(A)$. Likewise for h and $h \cdot h'$.*

Ziegler's lemma has played a fundamental role in several of the results which are presented here, specially in the analysis of groups interpretable in Poizat's *green fields* [BMW12a] and groups definable in belles paires [BM14], as well as to generalise Lascar's description of bounded automorphisms of algebraically closed fields to various theories of fields equipped with operators [BHM15]. Furthermore, Göral [34] adapted some of the results in [BM14] in order to describe definable and interpretable groups in the structure $\mathbb{C}$ equipped with a predicate for the multiplicate group of roots of unity, which has the *Mann property*.

Given two type-definable groups G and H, a *definable isogeny* between G and H is a type-definable subgroup $S \leq G \times H$ such that :

- the projection G_S on G, resp. H_S on H, is a subgroup of finite index, and
- both the *kernel* $\ker(S) = \{g \in G : (g, 1) \in S\}$ and the *cokernel* $\mathrm{coker}(S) = \{h \in H : (1, h) \in S\}$ are bounded.

Such an isogeny induces a group isomorphism between $G_S/\ker(S)$ and $N_H(\mathrm{coker}(S))/\mathrm{coker}(S)$. Thus, the isogeny relation is an equivalence relation. Every type-definable group is isogenous to its connected component. Therefore, an isogeny between G and H induces an isogeny between their connected components, and vice versa.

Combined, Lemma 1.10 and [BMW12a, Lemma 1.5] provide a useful criteria to construct isogenies between two given groups in a simple theory (cf. [BM14, lemme 2.4]):

Lemma 1.11. *Suppose G_1 and G_2 are two type-definable (or even type-interpretable) groups in a simple theory. Given a boundedly closed set of parameters C, and elements a_1 and b_1 of G_1, and a_2 and b_2, of G_2 such that:*

1. *the elements a_1 and a_2) are pairwise interbounded over C. Likewise for b_1 and b_2, and for $a_1 \cdot b_1$ and $a_2 \cdot b_2$,*
2. *the elements a_1, b_1 and $a_1 \cdot b_1$ are pairwise independent over C.*

Then, the element a_1, resp. a_2, is generic in a unique, up to commensurability, translate of a type-definable subgroup H_1 of G_1, resp. H_2 de G_2, where everything is definable over C. The stabiliser $\mathrm{Stab}(a_1, a_2/C)$ *induces a definable isogeny between H_1 and H_2.*

In case we only have that a_2 is bounded over C, a_1, the element b_2 is bounded over C, b_1, and $a_2 \cdot b_2$ is bounded over $C, a_1 \cdot b_1$ in condition (1)*, then there is a type-definable projection from H_1 to a quotient of H_2 by a bounded subgroup, relatively definable over C.*

1.4 Internality, Analysability and P-closure

All throughout this section, suppose T is simple and let Σ be an invariant family of types. We do not require that the size of Σ is bounded.

Given a partial type π over A, we say that π is Σ-*finitely generated* if there is some $B \supset A$ such that for every realisation a of π, there is a tuple of realisations c of types in Σ over B and a is definable over Bc.

The type π is Σ-*internal*, resp. *almost* Σ-*internal*, if for every realisation a of π, there are $B \mathop{\smile\!\!\!\!\!\!|}_A a$ and a tuple of realisations c of types in Σ over B such that a is definable, resp. bounded, over Bc.

We say that π is Σ-*analysable* if every realisation a of π is interbounded over A with a sequence $\{a_i\}_{i<\alpha}$ such that each type $p(a_i/A\{a_j\}_{j<i})$ is almost Σ-internal.

A complete type p over A is *foreign to* Σ if, whenever a realises a non-forking extension of p over $B \supset A$ and c is any tuple of realisations of types of Σ over B, then $a \mathop{\smile\!\!\!\!\!\!|}_B c$.

An extension of an internal, resp. almost internal, type remains so. A non-forking restriction of an internal, resp. almost internal, type is again internal, resp. almost internal. (Almost) Internality is transitive: Given an $\emptyset$-invariant family Σ' of types, if every type in Σ is (almost) Σ'-internal, then every partial (almost) Σ-internal type is (almost) Σ'-internal. In particular, if a is bounded over Ab and $\mathrm{tp}(b/A)$ is almost Σ-internal, then so is $\mathrm{tp}(a/A)$.

As observed by Hrushovski, some of the above notions are related. For a stable theory, the notion of Σ-internality agrees with being Σ-finitely generated. In the simple case, we have the following result.

Lemma 1.12. *If* $\mathrm{tp}(a/A)$ *is not foreign to* Σ, *then there is some hyperimaginary* a_0 *in* $\mathrm{dcl}^{\mathrm{heq}}(Aa)\backslash\mathrm{bdd}(A)$ *such that* $\mathrm{tp}(a_0/A)$ *is* Σ-*internal.*

Two types p and q are *orthogonal* if any two realisations of their corresponding non-forking extensions to the same base set are independent.

A non-bounded type p is *regular* if it is foreign to the collection of all its forking extensions.

Since regular types are particular examples of types of weight 1, it is easy to see that non-orthogonality is an equivalence relation among regular types. Lascar inequalities yield that a type of monomial Lascar rank ω^α is always regular. If $\mathrm{SU}(p) = \beta + \omega^\alpha n$ for some integer $n > 0$, then p is non-orthogonal to some (possibly hyperimaginary) type of Lascar rank ω^α. If T is supersimple, every non-bounded type is non-orthogonal to a regular type of a real element, though no estimate on the rank of the latter can be given in general.

The above notions have interesting consequences in the presence of a definable group structure. Lascar's analysis [60] uses strongly Zilber's indecomposables in order to show that a group of finite Morley rank admits a normal series such that each quotient is almost strongly minimal. In particular, a group of finite Morley rank is finite-dimensional and Lascar rank agrees with Morley rank. However, Hrushovski's analysis [39] provides a finer decomposition of a group in terms of its generic types, even in the mere simple case. Nevertheless, for the purpose of this work, we will present this analysis only for stable groups (cf. [96, Theorem 3.1.1 and Corollary 3.1.2]):

Lemma 1.13. *If the generic type of a stable type-definable group G is not foreign to an $\emptyset$-invariant family Σ, then there is a relatively definable normal subgroup N of infinite index such that G/N is Σ-internal.*

In particular, if the generic type of a definably simple stable group is not foreign to Σ, then the group must be Σ-internal. Thus, if its generic type is almost Σ-internal then G is Σ-internal. Likewise for stable division rings.

Hrushovski's analysis of a superstable group G of finite rank goes as follows: take a type of rank 1 which is not orthogonal to a principal generic. Taking the family of its conjugates, there is a normal subgroup of infinite index G_1 such that the quotient G/G_1 is internal to types of rank 1. Since $\mathrm{SU}(G_1) < \mathrm{SU}(G)$, this process must stop and we obtain a finite analysis for G, in terms of types of rank 1.

Let us now conclude by introducing a certain closure operator which will play a fundamental role in Chapter V.

Definition 1.14. Fix some principal generic type p of G over $\emptyset$. Given a set of parameters D, we define its p-closure $\mathrm{cl}_p(D)$ as the collection of elements g in G *co-foreign* over D to p, that is, such that, whenever $D_1 \supset D$ and $h \models p$ is independent from D_1 over D, then $g \mathop{\smile\!\!\!\!\!\!\mid}_{D_1} h$.

Observe that this closure does not depend on the generic type p. Let q be another generic type and g be in $\mathrm{cl}_q(D)$. Choose some $D_1 \supset D$ and $h \models p$, generic over D_1. Given $b \models q$ generic over $D \cup \{h\}$, then h and $b^{-1} \cdot h$ are independent over D_1. Likewise b and $b^{-1} \cdot h$ are independent over D_1, so b is generic over $D_1 \cup \{b^{-1} \cdot h\}$. Hence

$$b \mathop{\smile\!\!\!\!\!\!\mid}_{D_1 \cup \{b^{-1}\cdot h\}} g,$$

thus

$$h \mathop{\smile\!\!\!\!\!\!\mid}_{D_1 \cup \{b^{-1}\cdot h\}} g \text{ and therefore } h \mathop{\smile\!\!\!\!\!\!\mid}_{D_1} g.$$

We will therefore denote the closure of a set D by $\mathrm{cl}_{\mathrm{Gen}}(D)$. Since any element of G can be written as the product of two generic elements, the above definition agrees with [97, Definition 3.5.1], thanks to [97, Remark 5.1.19].

In contrast to the closure operators acl or bdd, the cardinality of $\mathrm{cl}_{\mathrm{Gen}}(D)$ may be comparable to the saturation of the monster, even if D is finite. For example, if G is a differentially closed field in characteristic 0, the generic has Lascar rank ω, so it is regular. The closure $\mathrm{cl}_{\mathrm{Gen}}(\emptyset)$ contains all elements which are not generic, that is, differentially algebraic over $\mathbb{Q}$. It contains in particular all the constants. The same holds more generally for any type-definable group of monomial Lascar rank. Despite that the closure $\mathrm{cl}_{\mathrm{Gen}}(A)$ of a set may be unbounded in the monster, it is compatible with non-forking independence (cf. [97, Lemma 3.5.5]).

Lemma 1.15. *If $A \mathop{\smile\!\!\!\!\!\mid}_C B$, then $\mathrm{cl}_{\mathrm{Gen}}(A) \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{cl}_{\mathrm{Gen}}(C)} \mathrm{cl}_{\mathrm{Gen}}(B)$. More precisely, given any subset A_0 of $\mathrm{cl}_{\mathrm{Gen}}(A)$, set $C_0 = \mathrm{bdd}(A_0 C) \cap \mathrm{cl}_{\mathrm{Gen}}(C)$. Then*

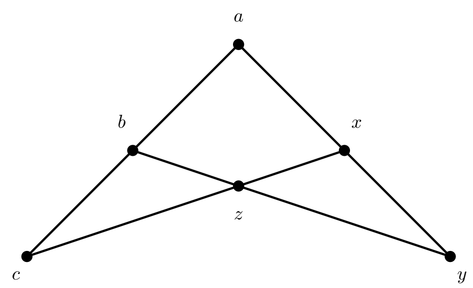
$$A_0 \mathop{\smile\!\!\!\!\!\mid}_{C_0} \mathrm{cl}_{\mathrm{Gen}}(B).$$

1.5 The Group Configuration

Hrushovski's group configuration [39] produces, out of a certain incidence relation, a $*$-*definable group*, that is, a partial type π, possibly in infinitely many variables, equipped with both a type-definable equivalence relation E and a type-definable binary relation $\cdot$ compatible with E such that the quotient $(\pi/E, \cdot)$ has a natural group structure. Technically, we should say $*$-interpretable, but we hope the reader forgives this *abus de langage*. The basic idea is to obtain, out of the aforementioned incidence relation, an infinite collection of invertible definable functions acting on the realisations of some strong type. The group law will then be the composition of germs.

We will include a short overview of Hrushovski's group configuration, to stress out that the construction can be carried over even when the departing tuples are infinite. In this section, the theory T is assumed stable, though a variant of the group configuration exists for simple theories [12]. Also, more general closure operators, similar to the one introduced in Definition 1.14, could be considered [16], instead of algebraic closure.

A *group configuration* over A is a given by the, possibly infinite, tuples $\{a, b, c, x, y, z\}$ and the following quadrangle:



satisfying the following conditions:

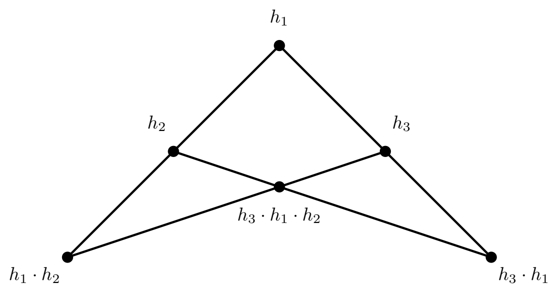
- On each line, every point is algebraic over the other two together with A.
- Every three non-colinear points are A-independent.

Observe that, if we replace any of the above points by a tuple which is interalgebraic over A, we obtain again a group configuration. Furthermore, the tuple a is interalgebraic with $\mathrm{Cb}(b, c/Aa)$, and likewise for each other point.

In order to construct a $*$-definable group out of the above configuration, the idea is to enlarge the set of parameters A by carefully adding independent realisations of some of the types of the tuples, in order to render z and y interdefinable over b. Hence, the realisations of $\mathrm{stp}(b/A)$ can now be seen as partial functions from $\mathrm{stp}(z/A)$ to $\mathrm{stp}(y/A)$. Pairs of independent realisations of $\mathrm{stp}(b/A)$ induce a germ of a function on $\mathrm{stp}(y/A)$, as desired.

Theorem 1.16. *Suppose T admits a group configuration over A, where the corresponding tuples are possibly infinite. Possibly after adding independent parameters B to A, there exist a connected $*$-definable group G and generic independent elements g and g' over B such that both a and g are interalgebraic over B, and b and g' are interalgebraic over B, as well as c and $g \cdot g'$ are interalgebraic over B.*

Remark 1.17. Suppose that H is a type-definable group over A. Choose generic independent elements h_1, h_2 and h_3 in H over A. The following diagram yields a group configuration:



However, we do not obtain any new groups. The $*$-definable group obtained in Theorem 1.16 to the above group configuration is isogenous to H, by Lemma 1.11, and hence, type-definable.

"Je üppiger die Pläne blühen,
um so verzwickter wird die Tat."

Erich Kästner – Spruch in der Silvesternacht

II

Ampleness

The dichotomy principle, formulated by Zilber, establishes a division line on the geometry of the minimal sets in a given theory: Either the lattice of algebraically closed sets (in T^{eq}) is modular or an algebraically closed field can be interpreted. At the base of many key applications of Geometric Model Theory to Diophantine Geometry [42, 83] lie *Zariski Geometries* [48], for which this division principle holds. The dichotomy principle does not hold for strongly minimal sets, as shown by Hrushovski, who developed a general method [40, 41] to produce ω-stable theories with prescribed geometries in terms of underlying dimension functions, which agree with Morley rank on the resulting theories. Despite the exotic behaviour of the geometry of his *ab initio* example, it satisfies a weakening of the modularity principle, which in itself prevents an infinite field to be interpretable [67]. Motivated by this, Pillay [71] and Evans [31] introduced the ample hierarchy of stable theories, in order to provide finer division lines on the analysis of the geometry of strongly minimal sets. Little is known about definable groups in theories with large degree of ampleness, or whether there are such examples of finite rank which do not interpret an infinite field.

2.1 Amalgams and Collapse

Notation. Though some of the notions presented in this section can be adapted to a wider context [82], we will assume that all underlying theories are stable.

Since Hrushovski-Fraïssé's amalgamation construction will be a recurring topic in this document (and in the author's own research), we will first introduce a general overview of the main features of this construction, extracted (almost) verbatim from [BMW15, Section 6].

The amalgams we will consider fall into two different categories:

Fusion (cf. [40, 7, 103]) We consider several theories T_i, together with a common reduct T_{com}. Let $\mathcal{F}$ denote the class of models of all $T_i^{\forall}$.

Colored (cf. [41, 5, 80, 81, 1, 8, 9, 6]) Given a underlying theory T_0 and a new predicate P, whose points are called *colored*, we denote by $\mathcal{F}$ the class of colored structures whose $\mathcal{L}$-restriction are models of $T_0^{\forall}$.

Remark 2.1. The *ab initio* construction (or rather, constructions [95]) in a pure relational language can be seen as a colored expansion of the theory of equality, in which a collection of tuples are colored if they lie in one of the distinguished relations.

All model-theoretic notions in the sense of the reduct T_i, such as algebraic or definable closure, types, independence or canonical basis, will be denoted by the index i, that is, dcl_i, acl_i, tp_i, Cb_i, $\mathop{\smile\!\!\!\!\!\!|}^{i}$. We assume that all the base theories T_i's have elimination of quantifiers.

In order to construct a certain structure $\mathcal{M}$ with a prescribed geometry, we first consider a predimension function δ, defined on the class of finitely generated models of $\mathcal{F}$, satisfying the submodular inequality:

$$\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B),$$

where we write $\delta(A)$ in order to denote the value of the predimension of the structure generated by the set A. Since every theory T_i has elimination of quantifiers, the i-diagram $\mathrm{Diag}_i(A)$ of a structure A determines its i-type, so

(δ) $\bigcup_i \mathrm{Diag}_i(A)$, together with the color of A (in case there is), determine $\delta(A)$.

Thus, the fusion of two theories T_1 and T_2 of finite Morley rank with the definable multiplicity property (DMP) uses the following predimension:

$$\delta(A) = n_1 \,\mathrm{RM}_1(A) + n_2 \,\mathrm{RM}_2(A) - n|A|,$$

with $n = n_1 \,\mathrm{RM}(T_1) = n_2 \,\mathrm{RM}(T_2)$.

The predimension for the fusion of two strongly minimal theories over an infinite $\mathbb{F}_p$-vector space is given by

$$\delta(A) = \mathrm{RM}_1(A) + \mathrm{RM}_2(A) - \mathrm{ldim}_{\mathbb{F}_p}(A).$$

Poizat's colored fields are algebraically closed fields equipped with a predicate P for a distinguished subset and predimension

$$\delta(k) = 2 \,\mathrm{degtr}(k) - \dim_P(P(k)).$$

In the *black* field, the predicate P denotes a subset N, such that $\dim_P(N) = |N|$. The predicate of the *red* field in positive characteristic is a proper additive subgroup R with $\dim_P(R) = \mathrm{ldim}_{\mathbb{F}_p}(R)$. In the *green* fields, whose collapse produced a *bad field* of characteristic 0 [6], the predicate P denotes a divisible torsion-free multiplicative subgroup $\ddot{U}$ (considered as a $\mathbb{Q}$-vector space) with $\dim_P(\ddot{U}) = \mathrm{ldim}_{\mathbb{Q}}(\ddot{U})$, though the requirement on the torsion can be removed [25].

According to the negative factor which is substracted when computing the predimension, we distinguish two types:

Degenerated The negative factor corresponds to the cardinality of some predicate (more generally, to the dimension of some degenerated pregeometry), e.g., the fusion over equality or the black field.

Modular There is an abelian group, which is $\emptyset$-definable in a language common to all theories, and which is either an $\mathbb{F}_p$-vector space (e.g., the fusion of two theories over an $\mathbb{F}_p$-vector space or the red field) or divisible of finite n-torsion for every natural number n. (e.g., the green field). Every structure is then equipped of the aforementioned group law. The collection of colored points (if they exists) forms a subgroup, which is divisible if the group is.

In the modular case, we will assume, in order to simplify the notation, that the abelian $\emptyset$-definable group has as underlying set the same universe as the structure (For the green field, this is verified if we virtually add to the multiplicative group of the field the element 0). Up to Morleyisation, we may also assume that the given theories (T_i, T_{com}) eliminate quantifiers in a purely relational language, except for the group law in the modular case, e.g., addition for the red fields, multiplication for the green fields and vector addition for the fusion over a vector space.

If the group is divisible with finite n-torsion for every natural number n, we will refer to the divisible closure of the group generated by the set B as the structure generated by B. Thus, a finitely generated structure corresponds hence to a divisible group of finite rank in the group-theoretical sense, that is, a finite-dimensional vector space modulo the torsion.

We define a relative predimension $\delta(./A)$ working over a substructure A. Submodularity implies that

$$\delta(\bar{a}/A) = \lim_{A_0 \to A} \{\delta(\bar{a}/A_0)\} = \inf\{\delta(\bar{a} \cup A_0) - \delta(A_0) : A_0 \subseteq A \text{ f.g } \}.$$

Thus, the limit exists and the relative predimension is again submodular. Again, for a set A, we write $\delta(./A)$ to denote the relative predimension over the substructure generated by A. So that the predimension becomes meaningful, restrict $\mathcal{F}$ to the subclass $\mathcal{K}$ of structures such that every finitely generated substructure has non-negative predimension. Given M in $\mathcal{K}$, a substructure A of M is *self-sufficient* in M, denoted by $A \leq M$, if

$$0 \leq \delta(b/A)$$

for every finite tuple b in M.

By submodularity, the intersection of two self-sufficient structures of M in $\mathcal{K}$ is again self-sufficient in M, so for every subset A of M, there is a smallest self-sufficient structure in M containing A, which we call the *self-sufficient closure* A of M and denote by $\langle A\rangle_M$. If $M \leq N$ are two structures in $\mathcal{K}$, then $\langle A\rangle_M = \langle A\rangle_N$, for the relation of self-sufficiency is clearly transitive. Hence, we will simply refer to the self-sufficient closure of A by $\langle A\rangle$. Uniqueness implies that $\langle . \rangle$ is contained in the algebraic closure of A and has finite character:

$$\langle A\rangle = \bigcup\{\langle A_0\rangle : A_0 \subset A \text{ f.g. } \},$$

Remark 2.2. The red differential field [81, 15] does not fit into our treatment, since the algebraic closure does not contain the self-sufficient closure, due to the floating constants. However, it does fit in spirit into this general presentation.

For a finite tuple a, we have that

$$\delta(\langle a\rangle) = \liminf\{\delta(b) \,|\, a \subset b \text{ finite } \}.$$

Observe that, if δ is integer-valued (which is the case in all of the aforementioned examples, except [95]), then the self-sufficient closure of a finitely generated structure is again finitely generated.

Remark 2.3. Given a set A and a finite tuple b such that $\delta(b/A) < 0$, but $\delta(b'/A) \geq 0$ for every proper subtuple b' of b, then b lies in $\langle A\rangle$.

Indeed, if $b' = b \cap \langle A\rangle$ were a proper subtuple of b, then

$$\delta(b/\langle A\rangle) \leq \delta(b/A \cup b') < 0,$$

by submodularity.

The following property:

$(\star)_1$ $\delta(b/A) \leq 0$ whenever b lies in $\bigcup_i \mathrm{acl}_i(A)$

holds for all predimension functions considered so far. Indeed, in the colored case, there is a unique reduct T_0 and the positive part of the predimension is 0 when b lies in $\mathrm{acl}_0(A)$. For a fusion, notice that each of the positive factors in the predimension function is bounded from above by the negative part.

By choosing a subclass $\mathcal{K}_0 \subset \mathcal{K}$ of finitely generated structures having the amalgamation property with respect to self-sufficient embeddings, Fraïssé's amalgamation method yields a countable amalgam $\mathcal{M}$ which is universal and strongly homogeneous for self-sufficient substructures. This generic model $\mathcal{M}$ is a model of $\bigcup_{i<n} T_i$, and it is stable. If δ is integer-valued, it is superstable, and it is ω-stable if the base-theories have the DMP [100]. *Collapsing* renders it of finite Morley rank, by restricting to a subclass of $\mathcal{K}$ where (certain) elements of predimension 0 become algebraic. In both cases, collapsed and uncollapsed, the quantifier-free type of a self-sufficient structure determines its type.

The independence in the theory T of $\mathcal{M}$ is characterised as follows for two tuples a and b over an algebraically closed set C:

$$a \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$\mid$}}_C b$$

if and only if

$$\begin{array}{c}\langle \bar{a} \cup C\rangle \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$\mid$}}^i_C \langle \bar{b} \cup C\rangle \\ \text{for each } i\end{array} \quad \text{and} \quad \begin{cases} \textbf{Degenerated } \langle \bar{a}\bar{b} \cup C\rangle = \langle \bar{a} \cup C\rangle \cup \langle \bar{b} \cup C\rangle. \\ \textbf{Modular } \langle \bar{a}\bar{b} \cup C\rangle \text{ equals the subgroup generated by } \langle \bar{a} \cup C\rangle \text{ and } \langle \bar{b} \cup C\rangle, \text{ and its colored points (if any) are the products of those of } \langle \bar{a} \cup C\rangle \text{ and of } \langle \bar{b} \cup C\rangle. \end{cases}$$

All known amalgamation classes $\mathcal{K}_0$ so far satisfy furthermore that the free amalgam of two structures A and B of $\mathcal{K}_0$ over a common self-sufficient substructure C lies in $\mathcal{K}_0$ whenever C is maximal self-sufficient in B and $\delta(B/C) > 0$. In particular, we have the following property:

$(\star)_2$ Any finite tuple b in $\mathrm{acl}(A)$ can be extended to some finite b' in $\mathrm{acl}(A)$ with $\delta(b'/\langle A\rangle) = 0$.

The above properties allows us to isolate two crucial features of the self-sufficient closure over the given data.

Lemma 2.4. *The operator $\langle\rangle$ satisfies the following condition with respect to the given theories T_i:*

(†) *If C is algebraically closed and $a \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$\mid$}}_C b$, then $\langle Cab\rangle \subset \bigcap_{i<n} \mathrm{acl}_i(\langle Ca\rangle, \langle Cb\rangle)$.*

(‡) *Given b in $\bigcup_{i<n} \mathrm{acl}_i(A)$, then $\langle \mathrm{acl}(b), A\rangle \subset \bigcap_{i<n} \mathrm{acl}_i(\mathrm{acl}(b), \langle A\rangle)$.*

Property (†) is clear in the degenerated case. In the modular case, it follows from the fact that the group law is defined in the common sublanguage.

For property (‡), suppose A is self-sufficient and let b be in $\bigcup_i \mathrm{acl}_i(A)$. Set $B = \mathrm{acl}(b)$. It suffices to show that the structure generated by $A \cup B$ is already self-sufficient. We will show that it is an increasing union of self-sufficient substructures, each being the self-sufficient closure of a finite tuple.

Let a in A be finite such that b lies in $\bigcup_i \mathrm{acl}_i(\bar{a})$. By $(\star)_1$,

$$\delta(b/\langle a\rangle) = 0,$$

Hence, the structure C generated by $b \cup \langle a\rangle$ is self-sufficient.

Choose now some finite b' in B extending b. By $(\star)_2$, we may assume that $\delta(b'/\langle b\rangle) = 0$, so $\delta(\langle b'\rangle) = \delta(\langle b\rangle)$. In particular,

$$\langle b\rangle \subset C \cap \langle b'\rangle.$$

Let us now show that the structure generated by $\langle a\rangle \cup \langle b'\rangle$ is self-sufficient. Observe that it equals the structure generated by $C \cup \langle b'\rangle$, so it suffices to show that $\delta(\langle b'\rangle/C) = 0$. Submodularity implies that

$$0 \leq \delta(\langle b'\rangle/C) \leq \delta(\langle b'\rangle/C \cap \langle b'\rangle).$$

Together with $\delta(\langle b'\rangle) = \delta(\langle b\rangle) \leq \delta(C \cap \langle b'\rangle)$, we have that $\delta(\langle b'\rangle/C) = 0$, as desired.

2.2 Ampleness and Variants

We first recall the definition of 1-basedness, CM-triviality and n-ampleness [71, 31] for a simple theory T, though they were originally formulated in the stable context.

Definition 2.5. The theory T is 1-*based* if for every pair of boundedly closed subsets $A \subset B$ and every real tuple c, we have that $\mathrm{Cb}(c/A)$ is bounded over $\mathrm{Cb}(c/B)$. Equivalently, for every boundedly closed set A and every real tuple c, the canonical base $\mathrm{Cb}(c/A)$ is bounded over c.

The theory T is *CM-trivial* if for every pair of boundedly closed subsets $A \subset B$ and every real tuple c, if $\mathrm{bdd}(Ac) \cap B = A$, then $\mathrm{Cb}(c/A)$ is bounded over $\mathrm{Cb}(c/B)$.

The theory T is called n-*ample* if there are $n+1$ real tuples satisfying the following conditions (possibly working over parameters):

(a). $\mathrm{bdd}(a_0, \ldots, a_i) \cap \mathrm{bdd}(a_0, \ldots, a_{i-1}, a_{i+1}) = \mathrm{bdd}(a_0, \ldots, a_{i-1})$ for every $0 \leq i < n$,

(b). $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0, \ldots, a_{i-1}$ for every $1 \leq i < n$,

(c). $a_n \not\mathop{\smile\!\!\!\!\!\!|} a_0$.

By inductively choosing models $M_i \supset a_i$ such that

$$M_i \mathop{\smile\!\!\!\!\!\!|}_{a_i} M_0, \ldots, M_{i-1}, a_{i+1}, \ldots, a_n,$$

Fact 1.3 allows us to replace, in the definition of n-ampleness, all tuples by models. This was already remarked in [67, Corollary 2.5] in the case of CM-triviality. Likewise, if the theory T is 1-based, resp. CM-trivial, the corresponding conclusion holds whenever the tuple c is a hyperimaginary. Likewise for n-ample.

Given a definable subset X of a model M of T such that the theory of X equipped with the induced structure is n-ample, then so is T [BMZ14a, Lemma 8.3].

Every 1-based theory is CM-trivial. A theory is 1-based if and only if it is not 1-ample; it is CM-trivial if and only if it is not 2-ample [71]. Observe that n-ampleness implies $(n-1)$-ampleness: by construction, if $a_0, \ldots, a_n$ witness that T is n-ample, then $a_0, \ldots, a_{n-1}$ witness that T is $(n-1)$-ample. In order to see this, we need only show that

$$a_{n-1} \not\mathop{\smile\!\!\!\!\!\!|} a_0,$$

which follows from

$$a_n \not\mathop{\smile\!\!\!\!\!\!|} a_0$$

and

$$a_n \mathop{\smile\!\!\!\!\!\!|}_{a_{n-1}} a_0,$$

by transitivity. Ampleness establishes thus a hierarchy among simple theories, according to which both (pure) algebraically closed fields [71] and the free non-abelian group [65, 88] are n-ample for every natural number.

The full strength of ampleness was not required in [BMZ14b, BMZ14a], but a weaker notion, further studied by Carmona [22, Definition 2.3.2]:

Remark 2.6. If a theory is n-ample, then there are (possibly infinite) tuples $a_0, \dots, a_n$ such that:

(a). $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_{i-1}$ for every $1 \leq i < n$.

(b). $\mathrm{bdd}(a_i, a_{i+1}) \cap \mathrm{bdd}(a_i, a_n) = \mathrm{bdd}(a_i)$ for every $0 \leq i < n-1$.

(c). $a_n \mathop{\not\!\smile\!\!\!\!\!\!|}_{\mathrm{bdd}(a_i)\cap\mathrm{bdd}(a_{i+1})} a_i$ for every $0 \leq i < n-1$.

A definable set X is *weakly normal* if its canonical parameter is algebraic over any element in X. Equivalently, if, given any collection $\{X_i\}_{i\in\mathbb{N}}$ of pairwise distinct conjugates of X, then

$$\bigcap_{i\in\mathbb{N}} X_i = \emptyset.$$

A stable theory T is 1-based if and only if every definable set is a Boolean combination of weakly normal definable ones. Note that weakly normal definable sets are Srour-closed, so 1-based stable theories are in particular equational. Hrushovski and Pillay showed [44] that in, a stable one-based group, definable sets are finite unions of cosets of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable subgroups and every connected subgroup is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Thus, the group itself is abelian-by-finite. In the finite Lascar rank context, the notion of 1-basedness agrees with both local modularity and k-linearity (i.e. the canonical parameter of any uniformly definable family of curves has Lascar rank at most k) for any $k > 0$. In particular, all these notions agree for different k's.

The simplest example of a CM-trivial theory that is not 1-based is the *free pseudoplane:* an infinite forest with infinite branching at every node. Hrushovski's *ab initio* strongly minimal set does not interpret any infinite field, since a CM-trivial stable theory interprets neither infinite fields nor bad groups, as shown by Pillay [67], who deduces that a CM-trivial group of finite Morley rank must be nilpotent-by-finite. Thus, the new uncountably categorical group due to Baudisch [5], being CM-trivial and connected, had to be nilpotent. If T is stable CM-trivial with continuous finite Lascar rank (e.g., a CM-trivial strongly minimal theory or a CM-trivial group of finite Morley rank), then T is equational [50, Corollary 4.21]. This may partially justify why the first unpublished example of a stable non-equational theory [47] happens to be an expansion of the free pseudospace [10], whose theory is not CM-trivial.

Inspired by Hrushovski's original proof of functional Mordell-Lang, but avoiding the use of Zariski Geometries, Pillay and Ziegler [77] reproved the function field case of the Mordell-Lang conjecture in characteristic zero. Instead, motivated by work of Campana [21] in bimeromorphic geometry, they isolated a crucial property, the *Canonical Base Property* (in short, CBP), on the collection of types of finite rank of several theories of fields with operators, in particular, the theory of differentially closed fields in characteristic zero, as well as the theory of existentially closed difference fields in any characteristic.

Definition 2.7. ([BMW12b, Definitions 2.1 and 3.1] and [BMZ14b, Definition 2.6])

Fix an $\emptyset$-invariant family Σ of partial types in a simple theory T.

A collection $\mathcal{F}$ of partial types in T is 1-*tight* with respect to Σ if, for every set A_0 of parameters, every realisation c of a tuple of types in $\mathcal{F}$ with parameters in A_0 and every boundedly closed set A containing A_0, the type $\mathrm{tp}(\mathrm{Cb}(c/A)/A_0c)$ is almost Σ-internal.

A collection $\mathcal{F}$ of partial types in the theory T is 2-*tight* with respect to Σ if for every set A_0 of parameters, every realisation c of a tuple of types in $\mathcal{F}$ with parameters in A_0 and every pair of boundedly closed sets $A \subset B$ containing A, if $\mathrm{bdd}(Ac) \cap \mathrm{bdd}(B) = \mathrm{bdd}(A)$, then $\mathrm{tp}(\mathrm{Cb}(c/A)/A_0\,\mathrm{Cb}(c/B))$ is almost Σ-internal.

The theory T is 1-*tight*, resp. 2-*tight*, if the collection of all types is.

The theory T is called *n-tight* with respect to the family Σ if, whenever there are $n+1$ real tuples $a_0, \ldots, a_n$ satisfying (possibly over parameters) the following conditions:

1. $\mathrm{bdd}(a_0, \ldots, a_i) \cap \mathrm{bdd}(a_0, \ldots, a_{i-1}, a_{i+1}) = \mathrm{bdd}(a_0, \ldots, a_{i-1})$ for every $0 \leq i < n$.
2. $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0, \ldots, a_{i-1}$ for every $1 \leq i < n$,

then $\mathrm{Cb}(a_n/\mathrm{bdd}(a_0))$ is almost Σ-internal over a_1.

Since algebraic types are always internal to any invariant family Σ, a theory is n-tight whenever it is not n-ample. Clearly, if a theory is n-tight, it is $n+1$-tight: Indeed, suppose $a_0, \ldots, a_{n+1}$ satisfy the hypothesis of $n+1$-tightness. Then, the tuples $a_0, \ldots, a_n$ satisfy the conditions of n-tightness, so $\alpha = \mathrm{Cb}(a_n/\mathrm{bdd}(a_0))$ is almost Σ-internal over a_1. Since $a_{n+1} \mathop{\smile\!\!\!\!\!\!|}_{a_n} a_0$, transitivity implies that $a_{n+1} \mathop{\smile\!\!\!\!\!\!|}_{\alpha} a_0$. Thus $\mathrm{Cb}(a_{n+1}/\mathrm{bdd}(a_0))$ is bounded over α and hence Σ-internal over a_1, as well.

The CBP, as defined in [63], states that the collection of all types of finite Lascar rank is 1-tight with respect to the family of types of Lascar rank one. Chatizidakis [26] showed that, for a supersimple theory of finite rank, the CBP is equivalent to a strengthening of this notion, called UCBP, introduced by Moosa and Pillay in their study of compact complex spaces. Hrushovski, Palacín and Pillay [43], elaborating on an example of Hrushovski, have constructed a theory of finite Morley rank without the CBP. We have not attempted to determine whether their theory is 2-tight.

Generalising some of the results for groups definable in a 1-based theory, Kowalski and Pillay proved the following:

Fact 2.8. [57, Theorems 4.2 and 4.3] Let G be a type-definable group in a stable 1-tight theory with respect to the invariant family Σ.

1. Given a connected type-definable subgroup $H \leq G$, the type of its canonical parameter $\mathrm{tp}(\ulcorner H \urcorner)$ is almost Σ-internal.
2. If G is connected, then $G/Z(G)$ is almost Σ-internal.

The previous result motivated us to prove the following (cf. [BMW12b, Theorem 3.6]).

Theorem 2.9. *Let T be a stable 2-tight theory with respect to the invariant family Σ.*

(a). An interpretable field K is Σ-internal.

(b). An interpretable group G of finite Lascar rank is nilpotent-by-(almost Σ-internal). In particular, an interpretable non-abelian simple group is Σ-internal.

Part (a) of 2.9 is a straight-forward adaptation of the proof of [67, Proposition 3.2], by taking a generic point p in a generic line l inside a generic plane P, and observing that $\mathrm{Cb}(p/\mathrm{acl}^{\mathrm{eq}}(P)$ is almost Σ-internal over $\mathrm{Cb}(p/\mathrm{acl}^{\mathrm{eq}}(l))$, which implies that the generic type of K is almost Σ-internal, and thus Σ-internal, by the remark after Lemma 1.13. Part (b) reduces to treat several cases separately: If G is a simple non-abelian group of finite Lascar rank, which is unidimensional, we may suppose, by induction on the rank, that all proper connected type-definable subgroups of G are nilpotent, a situation which mimics the structure of a *bad group*, with a slight modification of the configuration exhibited in [67, Lemma 3.4]. For a general group G of finite Lascar rank, Baudisch's analysis [4] yields a increasing chain of definable groups, such that the quotients are either finite, simple or abelian. In the latter case, we can subsequently split this quotient into Σ-internal ones and eventually Σ-foreign ones. If C denotes the centraliser in G of all the Σ-internal quotients, which is solvable, observe that G/C embeds into a product of Σ-internal groups and thus it is Σ-internal, as well. It suffices hence to show that the connected component C^0 of C is nilpotent. Otherwise, a field can be interpreted as a section in C. The field is then Σ-internal, by Part (a), which yields the desired contradiction, for the scalar multiplication of K^* on K^+ arises from conjugation in the ambient group G, but C acted trivially on the internal quotients.

2.3 Relative Ampleness

Pillay [69] showed that every differentially algebraic group could be embedded into an algebraic group. answering thus a question of Kolchin. A similar result, modulo a finite kernel, was obtained by Kowalski and Pillay [56] for connected constructible groups in an existentially closed difference field. Hrushovski and Pillay [45, 46] showed that a semialgebraic connected affine Nash group is Nash isogenous to the semialgebraic connected component of the group $H(\mathbb{R})$ of real points of some algebraic group H defined over $\mathbb{R}$. The proofs to these three examples follow a similar approach, replacing the given groups with a purely algebraic group configuration, in order to obtain an algebraic group.

Hrushovski's *ab initio* strongly minimal set has a *flat geometry*, which prohibits the existence of infinite groups. He remarks in the introduction of [40] that the fusion T of two strongly minimal theories T_1 and T_2 over a trivial reduct is flat over the given theories, and thus, any definable group in T is isogenous to a product $H_1 \times H_2$, where H_1 is definable in the reduct T_i. Motivated by this remark, we started in [BMW15] an analysis of groups definable in Hrushovski's amalgams, which led us in [BMW12a] to a proof of his assertion, as well as, among other various results, a description of definable groups in the collapsed green fields. As a by-product of this study, we develop a general approach to the study of definable groups in differentially closed fields or existentially closed difference fields. However, our approach requires the underlying universe of both theories to be the same, so it does not cover interesting cases, such as definable groups in local fields [45] or definable groups in separably closed fields [17].

Notation. For this section, fix a simple theory T in a language $\mathcal{L}$ together with a family of stable reducts $(T_i : i < n)$ in sublanguages $\mathcal{L}_i$. All model-theoretic notions, such as definable, algebraic or bounded closure, types, independence or canonical bases, are to be understood with respect to the theory T. If we want to consider any of them with respect to the reduct T_i, we will denote them with the index i, e.g., dcl_i, acl_i, tp_i, Cb_i, $\mathop{\smile\hspace{-0.9em}\mid}^{i}$.

Since imaginaries in the sense of T have no meaning in the reducts, we will, unless specified, only consider real elements when taking the operators dcl or acl, even if applied to (hyper-)imaginary sets. However, for the bounded closure, we mean as usual the collection of hyperimaginaries in T with a bounded orbit. We will furthermore assume that all the reducts have geometric elimination of imaginaries.

Assume furthermore that the theory T comes equipped with a finitary invariant closure operator $\langle . \rangle$ on the real sort such that $A \subset \langle A \rangle \subseteq \mathrm{acl}(A)$ for every subset A (cf. Section 2.1).

Definition 2.10. The theory T is *relatively* 1-*based* over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$ if, given real algebraically closed sets $A \subset B$ and a real tuple c such that:

$$\langle A\bar{c} \rangle \mathop{\smile\hspace{-0.9em}\mid}^{i}_{A} B \text{ for each } i < n,$$

then the canonical base $\mathrm{Cb}(c/\mathrm{bdd}(B))$ is bounded over A.

The theory T is *relatively CM-trivial* over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$ if, given real algebraically closed $A \subset B$ and a real tuple c such that:

$$\langle A\bar{c} \rangle \mathop{\smile\hspace{-0.9em}\mid}^{i}_{A} B \text{ for each } i < n,$$

then the canonical base $\mathrm{Cb}(c/\mathrm{bdd}(A))$ is bounded over $\mathrm{Cb}(c/\mathrm{bdd}(B))$.

The theory T is *relatively* 1-*ample* over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$ if there are real tuples a, b and c such that

$$\mathrm{acl}(a,b) \mathop{\smile\hspace{-0.9em}\mid}^{i}_{\mathrm{acl}(a)} \langle \mathrm{acl}(a), c \rangle \text{ for each } i < n,$$

but $b \not\mathop{\smile\hspace{-0.9em}\mid}_{a} c$.

The theory T is *relatively* 2-*ample* over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$ if there are real tuples a, b and c such that

$$\mathrm{acl}(a,b) \mathop{\smile\hspace{-0.9em}\mid}^{i}_{\mathrm{acl}(a)} \langle \mathrm{acl}(a), c \rangle \text{ for each } i < n$$

and

$$c \mathop{\smile\!\!\!\!\!\!\,|}_b a,$$

but $c \not\mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{bdd}(a)\cap\mathrm{bdd}(b)} a$.

Remark 2.11. Note that T is relatively 1-based, resp. relatively CM-trivial, over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$ if it is not relatively 1-ample, resp. not relatively 2-ample, over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$

If T is relatively 1-based, then it is relatively CM-trivial over the same reducts with respect to the same closure operator.

Due to the distinguished role of $\langle . \rangle$, there is an inherent asymmetry in the definition of relative ampleness, which prevents us from given a suitable definition for larger values. A possible definition is that the theory T is *relatively k-ample* over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$ if there are real tuples $a_0, \ldots, a_k$ satisfying the following conditions:

- $\mathrm{acl}(a_0, \ldots, a_j) \mathop{\smile\!\!\!\!\!\!\,|}^i_{\mathrm{acl}(a_0,\ldots,a_{j-1})} \langle \mathrm{acl}(a_0, \ldots, a_{j-1}), a_k \rangle$ for each $i < n$ and $j < k$.
- $a_k \mathop{\smile\!\!\!\!\!\!\,|}_{a_j} a_0, \ldots, a_{j-1}$ for each $j < k$.
- $a_k \not\mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{bdd}(a_0)\cap\mathrm{bdd}(a_1)} a_0$.

However, this definition gives a special role to the tuple a_k, somewhat similar in spirit to Pillay's original definition of ampleness. We have not yet been able to obtain a relative version of ampleness which corresponds to Evans' definition.

Remark 2.12. Every theory is relatively 1-based, resp. relatively CM-trivial, over itself with respect to acl. Similarly, if T is relatively 1-based, resp. relatively CM-trivial, over the reduct to equality with respect to acl, then T is 1-based, resp. CM-trivial. The converse holds if T eliminates geometrically hyperimaginaries.

Whenever the closure operator $\langle . \rangle$ satisfies Property (†) of Lemma 2.4, the above notions are preserved by adding or removing parameters.

In the general situation of a simple theory T with a stable reduct T_0, which has geometric elimination of imaginaries, with no further assumptions, we can canonically construct, out of a connected T-type-definable group G, a T_0-$*$-definable group H together with a definable morphism $G \to H$. However, in this full generality, the above map could well be the trivial one. The following result was first stated in [BMW15, Theorem 3.1], though slightly reformulated in [BMW12a, Theorem 1.10].

Theorem 2.13. *Let G be a type-definable group in T over $\emptyset$. Over an algebraically closed set of parameters A containing a Morley sequence of a principal generic type of G, there are a definable homomorphism ϕ with domain a type-definable subgroup of G of bounded index, and target a T_0-$*$-definable group H over A, and two independent generic elements a and b of G over A, such that*

$$\mathrm{acl}(b, A), \mathrm{acl}(ab, A) \mathop{\smile\!\!\!\!\!\!\,|}^0_{\phi(a), A} \mathrm{acl}(a, A).$$

Let us describe the different steps of the proof. Given two independent elements a and b of some principal generic type in G, we want to capture the behaviour of each one of the elements a, b and ab over the other two. Therefore, we will need to work with a particular Morley sequence D of some principal generic type in G, independent from the pair a, b. Set:

$$\begin{aligned} a_1 &= \mathrm{acl}_0(\mathrm{acl}(b, D), \mathrm{acl}(ab, D)) \cap \mathrm{acl}(a, D), \\ b_1 &= \mathrm{acl}_0(\mathrm{acl}(a, D), \mathrm{acl}(ab, D)) \cap \mathrm{acl}(b, D), \\ (ab)_1 &= \mathrm{acl}_0(\mathrm{acl}(a, D), \mathrm{acl}(b, D)) \cap \mathrm{acl}(ab, D). \end{aligned}$$

The tuples a_1, b_1 and $(ab)_1$ are pairwise independent. By a finitary argument, taking a finite tuple d in D, we consider $\mathrm{Cb}_0(\mathrm{acl}(b,d), \mathrm{acl}(ab,d)/\mathrm{acl}(a,d))$, which is 0-algebraic over any Morley sequence (in the sense of T_0) of $\mathrm{tp}_0(\mathrm{acl}(b,d), \mathrm{acl}(ab,d)/\mathrm{acl}(a,d))$. It suffices to take (see [BMW15, Lemma 2.1]) a Morley sequence (in the sense of T) of $\mathrm{tp}(\mathrm{acl}(b,d), \mathrm{acl}(ab,d)/\mathrm{acl}(a,d))$. A suitable choice of D ensures that enough such Morley sequences lie in $\mathrm{acl}(b,D) \cup \mathrm{acl}(ab,D)$, so a_1 contains the canonical base $\mathrm{Cb}_0(\mathrm{acl}(b,d), \mathrm{acl}(ab,d)/\mathrm{acl}(a,d))$. Running over all possible finite tuples of D, we conclude that $\mathrm{Cb}_0(\mathrm{acl}(b,D), \mathrm{acl}(ab,D)/\mathrm{acl}(a,D))$ coincides with a_1. So each of the infinite tuples $a_1, b_1, (ab)_1$ is 0-algebraic over the other two, and

$$\mathrm{acl}(b,D), \mathrm{acl}(ab,D) \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{a_1} \mathrm{acl}(a,D).$$

Choosing now a third generic c independent from a and b over D, one shows that the a_1 coincides with the set

$$\mathrm{acl}_0(\mathrm{acl}(c,D), \mathrm{acl}(ca,D)) \cap \mathrm{acl}(a,D),$$

so the tuples $(a_1, b_1, (ab)_1, c_1, (ca)_1, (cab)_1)$ yield a 0-group configuration, and thus a 0-$*$-definable group H_1, possibly over new parameters A_1, which may be supposed independent from a, b, c, D, together with a definable endogeny ϕ_1 from G to H_1 by Lemma 1.11. Replacing H_1 by

$$N_{H_1}(\mathrm{coker}(\phi_1))/\mathrm{coker}(\phi_1),$$

we may assume that ϕ_1 is an actual homomorphism. Iterating this process countably many times yields an algebraically closed set set $A \supset D$ and definable homomorphism ϕ from a type-definable subgroup of G to a 0-$*$-definable group H over A such that

$$\mathrm{acl}(b,A), \mathrm{acl}(ab,A) \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{\phi(a),A} \mathrm{acl}(a,A),$$

as desired.

Remark 2.14. The above theorem, as stated, will not suffice in order to describe definable and interpretable groups. Indeed, notice that if we require to add parameters $B \supset A$ independent of a, B over A, it need no longer be the case that

$$\mathrm{acl}(b,B), \mathrm{acl}(ab,B) \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{\phi(a),B} \mathrm{acl}(a,B),$$

for we cannot ensure the independence in T_0 over the set $\mathrm{acl}_0(\phi(a), B)$, which is not algebraically closed in the sense of T (cf. [BMW15, Lemma 2.1]).

A stronger version of the theorem is required, which allows to consider extensions of the base set of parameters: Assume that there is some algebraically closed set B, over which there is a bound on the 0-rank (either Lascar or Morley) of $\psi(a)$, where ψ is a definable homomorphism from G to a T_0-definable group, then the above group H can be chosen to be T_0-interpretable. Furthermore, whenever $C \supset B$ is algebraically closed, and a and b are two two independent generic elements of G over C, we have that

$$\mathrm{acl}(b,C), \mathrm{acl}(ab,C) \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{\phi(a),C} \mathrm{acl}(a,C).$$

For the sake of the presentation, when we quote Theorem 2.13, we mean this stronger form, except for the relative 1-based case, where the weaker form suffices.

Notation. In order to use the strength of relative non-ampleness, we will assume that the closure operator $\langle . \rangle$ satisfies Properties (†) and (‡) of Lemma 2.4. Notice that Property (†) does not hold for the algebraic closure over the reduct to equality, whenever the theory T defines an infinite group.

Let us list below some examples of theories and operators satisfying the above definitions:

- The theory DCF_0 of differentially closed fields is relatively 1-based over its reduct ACF_0 with respect to the model-theoretic algebraic closure acl_δ, which satisfies (†) and (‡).
- If the stable theory T_0 eliminates $\exists^\infty$, the expansion T of T_0 by a generic predicate [28] is relatively 1-based over T_0 with respect to the algebraic closure, which satisfies (†) and (‡). Likewise, the model-completion T, if it exists, of a stable NFCP theory T_0 by a generic automorphism is relatively 1-based over T_0 with respect to the algebraic closure acl_σ, which satisfies (†) and (‡). Thus, the theory ACFA of an existentially closed difference field is relatively 1-based over its reduct to a pure algebraic closed field with respect to acl_σ.
- The theory T of a Hrushovski-Fraïssé amalgam, as described in Section 2.1, is relatively CM-trivial [BMW15, Theorem 6.4] over the given theories $(T_i : i < n)$ with respect to the self-sufficient closure $\langle . \rangle$, which satisfies (†) and (‡).

In the degenerated case, the proof of the last point is immediate and resonates with Hrushovski's proof [41, Lemma 13] that the *ab initio* construction is CM-trival. Indeed, we need only consider two tuples a and b enumerating small models in the amalgam with $\mathrm{acl}^{\mathrm{eq}}(a) \cap \mathrm{acl}^{\mathrm{eq}}(b) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ and a tuple c such that:

- $\mathrm{acl}(a,b) \mathop{\smile\!\!\!\!\!\!|}^{i}_{\mathrm{acl}(a)} \langle \mathrm{acl}(a), c\rangle$,
- $c \mathop{\smile\!\!\!\!\!\!|}_{b} a$.

In order to show that $c \mathop{\smile\!\!\!\!\!\!|} a$,observe that, by the characterisation of independence in amalgams, the sets $\mathrm{acl}(a,b)$ and $\mathrm{acl}(c,b)$ are freely amalgamated over $\mathrm{acl}(b)$, and their union is self-sufficient. Thus, so is its intersection with $\langle \mathrm{acl}(a), c\rangle$. We conclude that $\langle \mathrm{acl}(a), c\rangle \cap \mathrm{acl}(b,c)$ is independent from $\mathrm{acl}(a)$ over $\emptyset$, and hence $c \mathop{\smile\!\!\!\!\!\!|} a$, as desired.

Thw above result allows us to recover Kowalski and Pillay's result for groups definable in existentially closed difference fields and, partially, Pillay's result on differentially algebraic groups.

Theorem 2.15. *[BMW15, Theorem 4.9] If the simple theory T is* relatively 1-based *over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$, satisfying (†) and (‡), then every type-definable group G is isogenous to a subgroup of a cartesian product of groups H_i, each T_i-interpretable.*

Name the required parameters and assume that G is connected. For each reduct T_i, we have a definable homomorphism $\phi : G \to H_i$, where each H_i is T_i-$*$-definable. Observe that, by stability, each H_i is an inverse limit of T_i-interpretable groups. Set

$$\phi = \prod_{i<n} \phi_i : G' \to H = \prod_{i<n} H_i.$$

Choose two generic independent elements a and b of G. Since $\phi_i(a)$ is algebraic over $\phi(a)$, which is algebraic over a, we conclude that, for each $i < n$,

$$\mathrm{acl}(b), \mathrm{acl}(ab) \mathop{\smile\!\!\!\!\!\!|}^{i}_{\mathrm{acl}(\phi(a))} \mathrm{acl}(a).$$

Observe that b and ab are independent, so the closure $\langle b, ab\rangle$ is contained in

$$\bigcap_i \mathrm{acl}_i(\mathrm{acl}(b), \mathrm{acl}(ab)),$$

by Property (†). Since $\phi_i(a)$ is interalgebraic with $a_1 \subset \mathrm{acl}_i(\mathrm{acl}(b), \mathrm{acl}(ab))$, we conclude that

$$\langle \mathrm{acl}(\phi(a)), \mathrm{acl}(b), \mathrm{acl}(ab)\rangle \underset{\mathrm{acl}(\phi(a))}{\overset{i}{\perp\!\!\!\!\perp}} \mathrm{acl}(a),$$

by Property (‡). Thus, relatively 1-basedness implies that

$$\mathrm{acl}(b), \mathrm{acl}(ab) \underset{\mathrm{acl}(\phi(a))}{\perp\!\!\!\!\perp} \mathrm{acl}(a),$$

so a and $\phi(a)$ are interalgebraic. By Lemma 1.11, we may replace each group H_i by a T_i-interpretable one.

In the case of relative CM-trivial theories, the kernel may well be non-trivial, but, as we will see below, it is almost central. Recall that, for a type-definable group in a stable theory, the center is relatively definable, since it agrees with a finite intersection of centralisers. This no longer holds for a group G type-definable in a simple theory, but its approximate center [97, Definition 4.4.9]

$$\tilde{Z}(G) = \{g \in G : [G : C_G(g)] < \infty\}$$

is relative definable. It agrees with the usual center whenever G is stable and connected.

Theorem 2.16. *[BMW15, Theorem 5.7] If T is* relatively CM-trivial *over the reducts $(T_i : i < n)$ with respect to $\langle . \rangle$, satisfying (†) and (‡), then every type-definable group G has a type-definable subgroup of bounded index which definably maps to a cartesian product of T_i-interpretable groups, such that the kernel of the map is contained, up to finite index, in $\tilde{Z}(G)$.*

As in the previous case, working over some parameters containing a Morley sequence of G, we have that

$$\langle \phi(a), \mathrm{acl}(b), \mathrm{acl}(ab)\rangle \underset{\mathrm{acl}(\phi(a))}{\overset{i}{\perp\!\!\!\!\perp}} \mathrm{acl}(a).$$

Choosing generic elements h, h' and e in the Morley sequence contained in the base parameters, we have that

$$\langle b, ab, hbe, e^{-1}abh', \phi(a)\rangle \underset{\mathrm{acl}(\phi(a))}{\overset{i}{\perp\!\!\!\!\perp}} \mathrm{acl}(a).$$

Relative CM-triviality implies that, for $c = (b, ab, hbe, e^{-1}abh')$, the canonical base $\mathrm{Cb}(c/\phi(a))$ is bounded over $\mathrm{Cb}(c/a)$. Let N be the kernel of the map ϕ and consider $Z = \tilde{Z}(G) \cap N$. If N were not almost contained in $\tilde{Z}(G)$, then hZ would lie in $\mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(c/\phi(a))) \setminus \mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(c/a))$, providing the final contradiction.

Though the above theorem conveys little information on abelian groups, if the type-definable group G is simple, then it is a definably isomorphic to a type-definable subgroup of a cartesian product of T_i-interpretable groups [BMW15, Corollary 5.10]. Likewise, if K is a type-definable field, then it is a definably isomorphic to a subfield of an interpretable field in one of the reducts [BMW15, Corollary 5.11].

Since Hrushovski amalgams are CM-trivial over the base theories with respect to the self-sufficient closure, we deduce the following:

Corollary 2.17. *[BMW15, Corollary 6.6] Every simple definable in a colored field is linear.*

"Es grünt so grün / Wenn Spaniens Blüten blühen; / ... "
Robert Gilbert (German version of the text by Alan Jay Lerner)

III

Colored Fields

According to the *Algebraicity Conjecture*, a group of finite Morley rank is an algebraic group over an algebraically closed field, which is itself interpretable in the group structure. A first approach to the conjecture consisted in a characterisation of the Borel subgroups, that is, maximal solvable definable subgroups, which are then of the form $K^+ \rtimes T$, where T stands for *torus* and is a divisible definable subgroup of K^*. The study of a Borel subgroups, if they arise from a field, simplifies considerably if the torus T happens to be trivial. Thus, the notion of a *bad field* so appeared: a field of finite Morley rank with a proper definable divisible multiplicative subgroup. Though the existence of a bad field in positive characteristic is unlikely [99], it does exist [6] in characteristic 0, obtained by collapsing Poizat's green field [81].

Based on the results stated in the previous chapter, a finer analysis [BMW12a] of definable groups in colored fields, and in particular, in the bad field so obtained allows us to conclude that every definable simple group of finite Morley rank is an algebraic group. Furthermore, we describe definable groups in the fusion of two strongly minimal theories over equality. They are isogenous to a cartesian product of groups interpretable in the base theories.

3.1 Green Fields

Additionally to the general presentation of amalgams given in Section 2.1, let us now recall some specific properties of the bad field obtained in [6]. It consists of an algebraically closed field K of characteristic 0 of Morley rank 2, together with a proper definable divisible torsion-free multiplicative subgroup of Morley rank 1, which we will denote by Ü and whose elements are *green points*. Elements in $K \setminus$ Ü are *white*.

Though the condition on Ü being torsion-free can be relaxed [25], in our case the group Ü can be seen as a $\mathbb{Q}$-vector space, which is strongly minimal with the induced structure from K. The language consists solely of the green predicate, elements 0 and 1, as well as the mutiplicative group law (and the multiplicative inverse map outside of 0) and the scalar multiplication by rationals on the green points. Removing the 0 element of K, a structure is therefore a multiplicative subgroup of K, whose subgroup of green points is divisible torsion-free. Given two structures A and B, we denote by $A * B$ the structure they generate: modulo the 0 element, it is the smallest multiplicative subgroup containing them and closed under extracting green roots.

The corresponding predimension of a finitely generated structure A is

$$\delta(A) = 2\,\mathrm{tr}(A) - \mathrm{ldim}_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)).$$

For every finitely generated substructure $A \subset K$, the predimension $\delta(A)$ is non-negative. A self-sufficient extension $A \leq B$ is *minimal* if there is no proper intermediate structure self-sufficient in B. Given a self-sufficient structure A and a minimal self-sufficient extension B of A, there is a self-sufficient embedding of B in K over A. Furthermore, if $\delta(B/A) = 0$, then there are only finitely many such embeddings. This implies that every white point is sum of two green ones. Each of these couples of green points is algebraic over the given white point. The collection of all these couples is finite.

The self-sufficient closure $\langle A\rangle$ of a set A is A-invariant and algebraic over A in the sense of the theory T of K. If A is finitely generated, then so is $\langle A\rangle$. Furthermore,

$$\mathrm{RM}(b/A) = \delta(\langle Ab\rangle/\langle A\rangle).$$

Following the terminology of Section 2.1, the theory T of K is of modular type over the base theory T_0 of algebraically closed fields in characteristic 0: Two tuples a and b are independent over $C = \langle aC\rangle \cap \langle bC\rangle$ if and only if the following conditions hold:

- $\delta(\langle abC\rangle/\langle bC\rangle) = \delta(\langle aC\rangle/C)$
- $\langle aC\rangle \mathop{\smile\!\!\!\!\!\!\,|}^{0}_{C} \langle bC\rangle$, $\langle abC\rangle = \langle aC\rangle * \langle bC\rangle$ and $\ddot{\mathrm{U}}(\langle abC\rangle) = \mathrm{U}(\langle aC\rangle) \cdot \ddot{\mathrm{U}}(\langle bC\rangle)$.

Notice that, whenever A and B are self-sufficient and independent over their intersection, then $A * B$ equals, modulo 0, the product of A and B.

Given structures $A \subset B$, a *green basis* of B over A is any tuple in $\ddot{\mathrm{U}}(B)$ which completes a linear basis of $\ddot{\mathrm{U}}(A)$ in a linear basis of $\ddot{\mathrm{U}}(B)$. Since any linear combination of green points remains green, a green basis is in particular linearly independent over A. Hence, any green basis of B is a basis of $\ddot{\mathrm{U}}(B)$. The following remark will be useful for the description of definable groups in T.

Remark 3.1. Suppose that the substructure A is self-sufficient. Then

1. So is the substructure $\mathrm{acl}_0(A)$, which has no new green points.
2. The algebraic closure satisfies condition $(\star)_2$ of Section 2.1: The element b is algebraic over A if and only if there is some $B \supset Ab$ finitely generated over A with $\delta(B/A) = 0$. If x is a green basis of B over A, then $\mathrm{tr}(B/A) = \mathrm{tr}(x/A) = |x|/2$. In particular, the set B is contained in $\mathrm{acl}_0(A, \ddot{\mathrm{U}}(B))$. If A is green, then $B \subset \mathrm{acl}_0(\ddot{\mathrm{U}}(B))$.

The following result provides a criterion to show that a definable subgroup of an algebraic group is itself algebraic. Stability of the base theory T_0 is required (in the proof), for we use that a type is generic if and only if it only contains generic formulae.

Lemma 3.2. *[BMW12a, Remark 4.1] Fix some algebraically closed set A, and consider a generic element b of a translate, defined over A, of a connected A-definable subgroup H of an algebraic group G, defined over A. If $\ddot{\mathrm{U}}(\langle Ab\rangle) = \ddot{\mathrm{U}}(A)$, then H is algebraic.*

After translation, we may assume that b realises the generic type p of H over A. Observe that $H = \mathrm{Stab}(p)$. Let $H_0 = \mathrm{Stab}_0(p\restriction T_0)$. The characterisation of the independence yields that $H \subset H_0$. Actually H_0 is the T_0-definable envelope of H in G, which is therefore connected and $p\restriction T_0$ is its unique T_0-generic type.

The condition $\ddot{\mathrm{U}}(\langle Ab\rangle) = \ddot{\mathrm{U}}(A)$ implies that p is the unique generic type of H_0, so $H = H_0$ is algebraic, as desired.

Since the green bad field is a field of finite Morley rank, it eliminates imaginaries [98]. Therefore, the quotient $K^*/\ddot{\mathrm{U}}$ is definably isomorphic to a definable group. Given two real tuples a and b in bijection with two independent generics of $K^*/\ddot{\mathrm{U}}$, the corresponding tuple a_1 from Theorem 2.13 is empty, so the associated morphism would be trivial. We require hence a finer analysis in order to describe definable groups.

Theorem 3.3. *[BMW12a, Theorem 3.14] Every interpretable group in a collapsed green field is isogenous to a quotient of a definable subgroup of an algebraic group by a central subgroup, which is itself isogenous to a cartesian power of* $\ddot{\mathrm{U}}$.

The idea behind the proof consists in taking two generic independent elements a and b of G, and studying a set of predimension 0 over a, b containing the element ab, which is algebraic over the latter. However, we have *a priori* little control on the green basis of this extension. Thus, the actual proof has several intricate steps. Starting from three generic points a, b and c of a definable connected group G over a Morley sequence of its generic, we replace the point a by an interalgebraic tuple $\bar{a}$, which is self-sufficient and equals the T_0-algebraic closure of a finitely generated tuple over a_1. The reason to do so is to reduce our study to a finitely generated situation (so that the predimension is defined), since neither a_1 nor the full algebraic closure $\mathrm{acl}(a)$ are. Decompose the tuple $\bar{a}$ so obtained into a T_0-transcendental part and a green part. Completing these green bases to a green basis of $\langle \bar{a}, \bar{b}, \bar{ab}\rangle$, we obtain a green tuple t, which is T-generic over $\bar{a}$. Hence, up to isogeny, the locus t over $\bar{a}$ is some cartesian power of $\ddot{\mathrm{U}}$.

We obtain hence a connected pro-algebraic group H, which projects onto the algebraic group H_1 obtained in 2.13 from the corresponding points a_1, b_1, etc. The kernel N of this projection is central in H, since N is the T_0-definable envelope of its torsion, which is finite and thus centralised by H, being connected. Inside H, we produce an element k, which is T_0-interalgebraic over $\bar{a}$ with the green tuple t. Lemma 1.11 provides an isogeny S between N and a certain cartesian power of K^*. Set Γ to be the preimage under this isogeny of the corresponding cartesian power of $\ddot{\mathrm{U}}$. The group Γ is again central in H, since N is. The final step consists in showing that the element a is interalgebraic with the coset $k\Gamma$. In particular, we may assume that H is an algebraic group, by compactness.

3.2 Definable Subgroups of Algebraic Groups and in the Fusion

From the result in the previous section and Theorem 2.16, we are bound to study definable subgroups of algebraic groups. They are indeed quite tame.

Theorem 3.4. *[BMW12a, Theorems 4.2 and 4.3, and Corollary 4.4]*

- *In a (possibly uncollapsed) green field, every connected definable subgroup G of an algebraic group has a normal algebraic subgroup N such that the quotient G/N is definably isomorphic to a cartesian power of* $\ddot{\mathrm{U}}$.
- *In a (possibly uncollapsed) red field, every connected definable subgroup G of an algebraic group has a normal algebraic subgroup N such that the quotient G/N is definably isogenous to the red points of an algebraic subgroup of some cartesian power of the additive group K^+.*
- *Every simple definable group in a colored field is definably isomorphic to an algebraic group. No bad groups are definable in a colored field.*

The proof of the above is drastically simpler than the proof of Theorem 3.3, for the group law inside a definable subgroup of an algebraic group is already field-algebraic. Hence, given generic points a and b of a definable subgroup G of an algebraic group, it follows that $\langle a \cdot b\rangle$ is contained in $\mathrm{acl}_0(\langle a\rangle * \langle b\rangle)$, by Remark 3.1(1). Up to linear transformation, we may assume that the green basis t of $\langle a \cdot b\rangle$ equals $s \cdot r$, where s, resp. r, is the green basis of $\langle a \rangle$, resp. $\langle b \rangle$. An easy application of Lemmas 1.10 and 1.11 yields an endogeny between G and the colored points of some cartesian power of (K^*), or (K^+), according to whether the color is green or red. In the former case, since Ü has no torsion, it is easy to see that the cokernel of this map is trivial. As the endogeny S maps a tuple to its colored basis, Lemma 3.2 implies that the connected component of the kernel of S is an algebraic group, and so is the kernel itself.

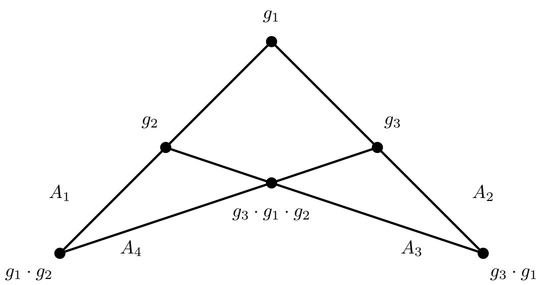
The third point was already observed by Poizat for the black field [80, Proposition 2.4 et conclusion p.1354]. For red and green fields, it follows from Corollary 2.17 and the previous two parts of the above result, since the quotient G/N is abelian, so $G = N$ must be algebraic.

We will finish this chapter with the study of that groups definable in the (possibly uncollapsed) fusion T of two strongly minimal theories T_1 and T_2 over equality. As remarked by Hrushovski, they are isogenous to a cartesian product of groups interpretable in the base theories.

Recall that no infinite group is interpretable in Hrushovski's new strongly minimal set since its theory is *flat*: whenever $A_1, \ldots, A_n$ are finitely generated structures, then

$$\sum_{s \subset \{1,\ldots,n\}} (-1)^{|s|} \operatorname{RM}(A_s) \leq 0,$$

where $A_\emptyset = \bigcup_1^n A_i$ and $A_s = \bigcap_{i \in s} A_i$, for non-empty $s \subset \{1, \ldots, n\}$. Indeed, if an infinite group G were definable, let A_1, A_2, A_3 and A_4 list all four lines in the corresponding group configuration given by G:



In a group of finite Morley rank, Lascar rank and Morley rank coincide, so it is additive. Thus, the rank $\operatorname{RM}(A_\emptyset) = 3\operatorname{RM}(G)$, whereas

$$\begin{aligned} \operatorname{RM}(A_i) &= 2\operatorname{RM}(G), \text{ for } i = 1, \ldots, 4 \\ \operatorname{RM}(A_{\{i,j\}}) &= \operatorname{RM}(G), \text{ for } i \neq j \in \{1, \ldots, 4\} \\ \operatorname{RM}(A_s) &= 0, \text{ for } s \subset \{1, \ldots, 4\} \text{ with } |s| \geq 3, \end{aligned}$$

so

$$\operatorname{RM}(G) = \sum_{s \subset \{1,\ldots,n\}} (-1)^{|s|} \operatorname{RM}(A_s) \leq 0$$

implies that G is finite.

Some of the ideas behind the notion of flatness allow us to show that a definable group in the fusion T of dimension 0 must be then finite, which is not obvious in the uncollapsed case. If we define a subset of imaginary sorts S_1 of T_1 and S_2 of T_2 to be *intepretable* if it is the projection of a definable set in the real sort, we obtain the following result.

Lemma 3.5. *Let H_i be a T_i-interpretable group, for $i = 1, 2$. Every interpretable connected subgroup K of $H_1 \times H_2$ has the form $K_1 \times K_2$, where each $K_i \leq H_i$ is T_i-interpretable. A generic $g = (g^1, g^2)$ of K is an independent pair of generics of each K_i. Furthermore,*

$$\operatorname{RM}(g^i) = \operatorname{RM}_i(g^i) \text{ and } \operatorname{RM}(K) = \operatorname{RM}_1(K_1) + \operatorname{RM}_2(K_2).$$

Choosing a Morley sequence $\{g_i = (g_i^1, g_i^2)\}$ of K, and studying the asymptotical behaviour of the Morley rank of $\{g_i \cdot g_j^{-1}\}$, we deduce that g_i^1 and g_i^2 are independent. Lemma 1.11 yields projections $K \to H_i$ with image $K_i \leq H_i$. A straight-forward adaptation of Lemma 3.2 implies that each K_i is T_i-definable.

Notice that, if $\phi : G \to H$ is a definable homomorphism of a connected definable group G to a T_i-interpretable group H, then $\mathrm{RM}_i(\phi(a)) \leq \mathrm{RM}(a)$, whenever a is a generic of G. Combining this with Theorem 2.13 and Lemma 3.5, we have that, given a definable group G, there are surjections $\phi_k : G \to H_k$, for $k = 1, 2$, where each H_k is T_k-interpretable such that

$$\mathrm{acl}(b), \mathrm{acl}(ab) \underset{\mathrm{acl}(\phi_k(a))}{\overset{k}{\perp\!\!\!\!\perp}} \mathrm{acl}(a).$$

The kernel of the map $\phi = (\phi_1, \phi_2)$ is of dimension 0, so finite, which gives the following.

Theorem 3.6. *[BMW12a, Theorem 5.5] In the (possibly uncollapsed) fusion of two strongly minimal theories T_1 and T_2 over equality, every definable group is isogenous to a cartesian product of groups interpretable in the base theories T_1 and T_2.*

"Paris, ..., quelle belle ville !
– Je m'en fous, dit Zazie, moi ce que j'aurais voulu c'est aller dans le métro."

Raymond Queneau – Zazie dans le métro

IV

Groups in Belles Paires

Differentially closed fields in characteristic 0 play the universal role of algebraically closed fields in the differential setting, providing generic solutions to differential systems of equations. Differentially closed fields are in particular algebraically closed, and thus, the field of constants, consisting of the elements of derivation 0, forms a proper algebraically closed subfield. Therefore, the theory of differentially closed fields has, as a reduct, the theory of belle paires of algebraically closed fields in characteristic 0.

Recenlty, Delon [30] has exhibited a natural expansion of the language $\mathcal{L}_P$ to obtain quantifier elimination for belles paires of algebraically closed fields. Her idea consists in rendering structures in this language P-independent (cf. Section 1.2), by adding the coefficients witnessing any linear dependence over the predicate, mimicking the λ-functions for separably closed fields. During her talk in Antalya Algebra Days, we became aware of the similarities of the theory of belle paires with our previous work on relative geometries and definable groups in colored fields, and consequently described [BM14] definable and interpretable groups in belles paires.

Fix a complete theory T, with quantifier elimination and elimination of imaginaries in the language $\mathcal{L}$. Suppose furthermore that T has NFCP, so it is stable, and let (M, E) be a sufficiently saturated model of the theory T_P of belles paires of T. In contrast to the previous chapters, the index P will refer to the theory T_P and otherwise, we refer to the base reduct T.

Recall that T_P is again stable and non-forking independence of two P-algebraically closed subsets A and B over their intersection $C = A \cap B$ is described as follows:

$$A \mathop{\smile\!\!\!\!\!|}^{P}_{C} B \quad \text{if and only if} \quad \begin{cases} A \mathop{\smile\!\!\!\!\!|}_{C,E} B \\ \text{and} \\ E_A \mathop{\smile\!\!\!\!\!|}_{E_C} E_B \end{cases}$$

A straight-forward application of this implies the following:

Lemma 4.1. *[BM14, Lemma 1.2] Whenever A and B are two P-algebraically closed subsets independent over their intersection, we have that*

$$\mathrm{acl}_P(A, B) = \mathrm{acl}(A, B) \textit{ and } E_{\mathrm{acl}_P(A,B)} = \mathrm{acl}(E_A, E_B).$$

The above gives us an explicit description, for two independent sets, of their algebraic closure in the sense of T_P , which consists of the algebraic closure in the sense of T.

Lemma 3.2 can be directly adapted to this context, to prove the following:

Lemma 4.2. *[BM14, Lemma 2.1] Fix some T_P-algebraically closed set A, and consider a generic element b of a translate, defined over A, of a connected T_P-type-definable subgroup H over A of a T-type-definable group G, defined over A. If $E_{\mathrm{acl}_P(Ab)} = E_A$, then H is T-type-definable.*

All these ingredients suffice to easily prove the following result.

Proposition 4.3. *[BM14, Proposition 2.6] Every T_P-type-definable group G is isogenous to a subgroup of a T-type-definable group. Furthermore, the group G is, up to isogeny, the extension of the E-points of a T-type-definable group over E by a T-type-definable group.*

$$0 \to N \to G \to H(E) \to 0$$

In order to see that the T_P-type-definable group G is, up to isogeny, a subgroup of a T-type-definable ambient group, choose three independent principal generics a, b and c of G. Lemma 4.1 implies that $\bar{ab} = \mathrm{acl}_P(ab)$ is T-algebraic over $\bar{a}, \bar{b}$. Lemma 1.11 gives an isogeny between G and a subgroup of a type-definable group in T, as desired. We may thus assume that G is a subgroup of a T-type-definable ambient group.

Together with Lemma 1.11, Lemma 4.1 applied to the infinite tuples $E_{\mathrm{acl}}(a)$, $E_{\mathrm{acl}}(b)$ and $E_{\mathrm{acl}}(ab)$ yields a projection from G to the E-points of a $*$-definable group H over E. The connected component of the kernel N is T-type-definable, by Lemma 4.2. An easy compactness argument allows us to assume that H is type-definable, correcting the published proof. Since T_P induces no extra structure on E, the group H is T-type-definable, as desired.

Let us now consider interpretable groups. Though T has elimination of imaginaries, this need not be the case for the theory T_P, which eliminates imaginaries if and only if no infinite group is definable in T [76]. Indeed, if G is an infinite T-definable group over E, given any element g in G, the coset $g \cdot G(E)$ is a new imaginary.

Whenever T is strongly minimal theory with infinite $\mathrm{acl}(\emptyset)$, e.g., algebraically closed fields, Pillay [72] exhibited new sorts in order for T_P to have geometric elimination of imaginaries. He proves in particular the following:

Fact 4.4. [72, Lemmas 2.2, 2.4 and 2.5] Let T be a strongly minimal theory with infinite $\mathrm{acl}(\emptyset)$. Then, up to interalgebraicity, every imaginary α of T_P^{eq} is definable over a real tuple a such that $\mathrm{tp}_P(a/\alpha)$ is almost P-internal and $a \mathop{\smile\!\!\!\!\!\!|}_{\alpha} E$.

Let $F \subsetneq K$ denote a proper pair of algebraically closed fields.

Proposition 4.5. *[BM14, Theorem 3.5 and Corollary 3.6] Up to isogeny, every interpretable group G in the pair (K, F) is the extension of the F-rational points of an algebraic group H over F by a group N, which is a quotient of an algebraic group V by a normal subgroup $N'(F)$, consisting of the F-rational points of an algebraic group over F.*

$$0 \longrightarrow N(K) \longrightarrow G(K) \longrightarrow H(F) \longrightarrow 0$$

$$\textit{with}$$

$$0 \longrightarrow N'(F) \longrightarrow V(K) \longrightarrow N(K) \longrightarrow 0\,,$$

such that both H and N' are algebraic groups over F.

If G is interpretable over $k \not\subset F$, then V and N are defined over $\overline{kF}$, which is an elementary submodel of (K, F).

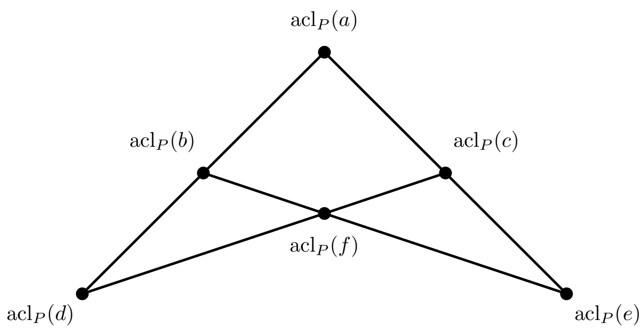
Assume that (K, F) is a sufficiently saturated model of the theory T_P of belles paires of algebraically closed fields. Let α, β and γ be three independent principal generics of G. The imaginary α is algebraic over some real tuple a, which is T_P-independent of F over α. Furthermore, the type $\mathrm{tp}_P(a/\alpha)$ is almost P-internal.

Observe that, for a real subset A, if the type $\mathrm{tp}_P(a/A)$ is almost P-internal, then a is algebraic over $A \cup E$. Indeed, assume A is P-algebraically closed and let $B = \mathrm{acl}_P(B)$ contain A such that $a \mathop{\smile\!\!\!\!\!\!\mid}^{P}_{A} B$ and $a \in \mathrm{acl}_P(B, E) = \mathrm{acl}(B, E)$. By the characterisation of non-forking independence in T_P, we have that $a \mathop{\smile\!\!\!\!\!\!\mid}_{A,E} B$, so a lies in $\mathrm{acl}(A, E)$.

Choosing now suitable non-forking extensions, we may find real tuples b, c, d, e and f such that, in the theory T_P, the pairs

$$(\alpha, a) \equiv (\beta, b) \equiv (\gamma, c) \equiv (\alpha \cdot \beta, d) \equiv (\gamma \cdot \alpha, e) \equiv (\gamma \cdot \alpha \cdot \beta, f),$$

have all the same type. Every colinear triple in the diagram



is T_P-independent. Likewise, a point outside a given line is T_p-independent of it. Since $a \mathop{\smile\!\!\!\!\!\!\mid}^{P}_{\alpha} F$, the set $F_{\mathrm{acl}_P(a)}$ equals the points in F which are algebraic over α. By the previous discussion, we have that $F_{\mathrm{acl}_P(d)}$ is algebraic over $F_{\mathrm{acl}_P(a)}, F_{\mathrm{acl}_P(b)}$. Lemma 1.11 yields a projection of G onto the F-rational points of an algebraic group, with kernel N. By the previous discussion, the above diagram yields a group configuration over F, and thus we obtain an algebraic group V over E which projects onto N. Lemma 4.2 shows that the connected component of the kernel of this projection is of the form $N'(F)$, where N' is an algebraic group over F, as desired.

"I am not bound to please thee with my answers."
William Shakespeare – The Merchant of Venice

V
Bounded Operators

The group of automorphisms of a countable structure, equipped with pointwise convergence, is a *Polish* group, a topological group homeomorphic to a complete separable metric space. The automorphism group $\mathrm{Aut}(M)$ of a (possibly uncountable) structure M contains a normal subgroup, consisting of the *strong automorphisms*, that is, those automorphisms which fix each class of every 0-definable equivalence relation with only finitely many classes, or equivalently, the automorphisms which fix $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ pointwise. By elimination of imaginaries, an automorphism of $\mathbb{C}$ is strong if and only if it fixes $\mathbb{Q}^{\mathrm{alg}}$. Thus, the quotient $\mathrm{Aut}(\mathbb{C})/\mathrm{Aut}_f(\mathbb{C})$ is isomorphic to the absolute Galois group of $\mathbb{Q}$. More generally, given a sufficiently saturated model M of a complete theory T, the quotient $\mathrm{Aut}(M)/\mathrm{Aut}_f(M)$ is an invariant of the theory T, called the *Galois group* of T.

Though the Galois group is far from being classified in general, the subgroup $\mathrm{Aut}_f(M)$ can be sometimes characterised. Lascar proved [61] that, for a countable saturated model M of an almost strongly minimal theory, the group $\mathrm{Aut}_f(M)$ is simple modulo the subgroup of bounded strong automorphisms of M. An automorphism τ of M is *bounded* if there is a finite set A such that $\tau(b)$ is algebraic over $A \cup \{b\}$ for every element b. The simplicity of $\mathrm{Aut}_f(M)$ modulo the bounded strong automorphisms has been generalised, to any structure equipped with a dimension function and a compatible stationary independance relation, e.g., Hrushovski's amalgams, by Evans, Ghadernezad and Tent [32], replacing algebraic by 0-dimensional in the definition of bounded automorphisms.

The group of bounded automorphisms may be described in the presence of some algebraic structure. Lascar shows [61] that the only bounded automorphism of $\mathbb{C}$ is the identity, so $\mathrm{Aut}_f(\mathbb{C})$ is simple. He mentions that Ziegler has a general description of bounded automorphisms of an algebraically closed field, valid in any characteristic. The only bounded automorphisms of an an algebraically closed field of positive characteristic are the integer powers of Frobenius. Ziegler, who did not publish his proof, provides instead in [104] an abelian version of Lemma 1.10. For a saturated differentially closed field of characteristic 0, Konnerth [55] shows that, replacing algebraic by differentially algebraic in the definition of bounded automorphisms, then the only such automorphism is the identity. His approach to proving so seems unrelated to Lascar's treatment of bounded automorphisms for algebraically closed fields in characteristic 0.

We present in this chapter work contained in [BHM15]. Motivated by Lascar and Konnerth's results, we give a uniform characterisation, probably not any different from Ziegler's approach, of bounded automorphisms in various theories of fields with operators. An automorphism τ of a group or a field, possibly with additional structure, is bounded if there is a finite set A such that, for every element b, the image $\tau(b)$ belongs to $\mathrm{cl}_{\mathrm{Gen}}(A \cup \{b\})$, where $\mathrm{cl}_{\mathrm{Gen}}$ is the closure operator introduced in Definition 1.14.

In particular, for the following fields with operators:

- Algebraically closed fields (K, Id) in all characteristics with *associated automorphism* (to be defined later on) either identity or Frobenius in positive characteristic.
- Differentially closed fields with n commuting derivations $(K, \delta_1, \ldots, \delta_n)$ in characteristic 0 with associated automorphism the identity.
- Generic automorphisms (K, σ) in all characteristics with associated automorphisms σ as well as Frobenius, in positive characteristic.
- Generic automorphisms of a differentially closed field (K, δ, σ) in characteristic 0 with associated automorphism σ, as considered by Hrushovski and later Bustamante-Medina [20].
- Fields with free operators $(K, F_1, \ldots, F_n)$ in characteristic 0 with associated automorphisms $\sigma_0, \ldots, \sigma_t$, as considered by Moosa and Scanlon [64].

every bounded automorphism is a product of the associated automorphisms and their inverses. Recently, Wagner generalised [BHM15] to show [101] that the bounded automorphisms of either a field or a simple group type-definable in a simple theory are definable.

Our proof reduces to isolate common properties to all the above examples, in order to study bounded automorphisms in a general set-up. Given a bounded automorphism, Lemma 1.10 gives us a generic action of a multiplicative subgroup on a type-definable subgroup of $\mathbb{G}_a^2$, a situation which resonates with [37]. The generalised Leibniz rule allows us to conclude that the graph of the bounded automorphism is (generically) an affine transformation composed with a product of the associated automorphisms and their inverses.

It shall be mentioned that we have not used that the underlying field is algebraically closed. As suggested by Chatzidakis, our treatment may most likely transfer to other theories of fields with operators, e.g., pseudofinite fields or separably closed fields.

Let us now start by introducting the objects in question. Moosa and Scanlon [64] developed a formalism to treat simultaneously various fields with operators, which play a relevant role in model theory, such as algebraically closed fields, differentially closed fields or fields with a generic automorphism.

Definition 5.1. A *field with operators* over a base subfield $\mathbb{F}$ is a structure

$$(K, 0, 1, +, -, \cdot, \{\lambda\}_{\lambda \in \mathbb{F}}, F_1, \ldots, F_n)$$

satisfying the following conditions:

- The operators $F_1, \ldots, F_n$ are $\mathbb{F}$-linear and

  $$F_k(xy) = \sum_{0 \le i,j \le n} a_{i,j}^k F_i(x) F_j(y),$$

  for some constants $\{a_{i,j}^k\}_{0 \le i,j,k \le n}$ in $\mathbb{F}$, where F_0 is the identity operator.
- The $\mathbb{F}$-vector space $D(\mathbb{F}) = \mathbb{F}\epsilon_0 \oplus \ldots \oplus \mathbb{F}\epsilon_n$ is a commutative $\mathbb{F}$-algebra, with

  $$\epsilon_i \star \epsilon_j = \sum_{0 \le k \le n} a_{i,j}^k \epsilon_k.$$

A field with operators is bi-interpretable with $(K, 0, 1, +, -, \cdot, \{\lambda\}_{\lambda \in \mathbb{F}}, D(K), \varphi)$, where the K-algebra

$$D(K) = D(\mathbb{F}) \otimes_F K = K\epsilon_0 \oplus \ldots \oplus K\epsilon_n$$

has linear dimension $n+1$, as a K-vector space, and it is equipped with a morphism of $\mathbb{F}$-algebras $\varphi : K \to D(K)$ such that the projection of $D(K)$ onto the first coordinate, composed with φ, is the identity. Indeed, it suffices to set

$$\varphi(x) = \sum_{0 \leq k \leq n} F_k(x)\epsilon_k,$$

which is the approach exhibited in [64].

The $\mathbb{F}$-algebra $D(\mathbb{F})$ is finite-dimensional and thus it is (isomorphic to) a product of local $\mathbb{F}$-algebras $B_0(\mathbb{F}), \ldots, B_t(\mathbb{F})$. Their residue fields are finite algebraic extensions of $\mathbb{F}$. We will suppose that all residue fields equal $\mathbb{F}$. Hence, tensoring each local algebra with K, if θ_i, resp. ρ_i, denotes the projection of $D(K)$ onto $B_i(K)$, resp. the projection of $B_i(K)$ onto its residue K, we obtain *associated endomorphisms*

$$\sigma_i = \rho_i \circ \theta_i \circ \varphi$$

of K. The endomorphism σ_0 is the identity. Observe that the morphisms so obtained are $\mathbb{F}$-linear combinations of the operators, and thus, they do not change whenever we apply an invertible $\mathbb{F}$-linear transformation on the operators.

For the sake of the presentation, we will assume that K is algebraically closed. Suppose as well that the associated endomorphisms $\sigma_1, \ldots, \sigma_t$ are field automorphisms. If the characteristic of K is positive, include Frobenius and its inverse among the list of associated automorphisms.

Let us denote by Θ the family of formal words on the operators $F_1, \ldots, F_n$ as well as on $\sigma_1^{-1}, \ldots, \sigma_t^{-1}$, equipped with the following lexicographic order:

$$F_i < F_j \text{ and } \sigma_i^{-1} < \sigma_j^{-1}, \text{ for } i < j,$$

and

$$\sigma_i^{-1} < F_j \text{ for any } i\,,\, j.$$

Each K-linear combination $S(x)$ of words in x has a *degree*, that is, the largest word occuring in S with non-trivial coefficient, which is called the *leading coefficient* of S. Set $\sigma_{F_j} = \sigma_{i_j}$ and $\sigma_{\sigma_j^{-1}} = \sigma_j^{-1}$, and extend this to every word θ in the obvious way.

An $\mathbb{F}$-linear transformation which renders simultaneously several nilpotent applications triangular yields the following result.

Lemma 5.2. *[BHM15, Proposition 1.4 and Corollary 1.5] Up to $\mathbb{F}$-linear transformation, the operators are* in triangular form*, that is,*

$$F_j(xy) = \sigma_{i_j}(x)F_j(y) + \sum_{l<j} R_{j,l}(x)F_l(y),$$

where each $R_{j,l}(x)$ is a polynomial over $\mathbb{F}$ in the variables $\{F_r(x)\}_{0 \leq r \leq j}$. In particular, given a K-linear combination $S(x)$ of words in x of degree θ and leading coefficient λ_θ, given g in K, we have that

$$S(gx) = \lambda_\theta \sigma_\theta(g)\theta(x) + R(x),$$

with $R(x)$ a K-linear combination of words in x of degree strictly less than θ.

In contrast to [64], where the operators considered are as free as possible, we are interested in theories of fields with operators which may impose some relations between the operators, such as commutation, e.g., differentially closed fields with n commuting derivations or generic automorphisms of differentially closed fields [20]. Among all K-linear combinations of words in Θ which are equivalent to a fixed one, there exists one of least degree.

Fix now a sufficiently saturated field K with operators and assume the following conditions on its theory T:

- The algebraic closure of a subset A coincides with the field algebraic closure of the structure it generates.
- The theory T of K is simple and eliminates both hyperimaginaries and imaginaries. It is relatively 1-based over the reduct of algebraically closed fields with respect to the operator algebraic closure in the sense of T, that is, given real algebraically closed sets A and B,

$$A \mathop{\smile\!\!\!\!\!\!\,|}_{A\cap B} B$$

if and only if A is linear disjoint from B over $A \cap B$.

As in the previous two sections, relative ampleness and the description of algebraic closure suffice to partially describe the nature of additive definable groups.

Corollary 5.3. *A connected subgroup H of some cartesian power $\mathbb{G}_a^k$ of unbounded index and type-definable over an algebraically closed subset D is contained in a subgroup of unbounded index of the form*

$$\{(x_1, \dots, x_k) \in \mathbb{G}_a^k \mid S(x_1, \dots, x_k) = 0\},$$

where S is some linear combination of words in the x_i's with coefficients in D.

Given two generic elements a and b of H independent over D, there is a finite sequence of words $\bar{\theta}$, common to a, b and $a + b$, witnessing an algebraic relation between each element over the other two. Considering the additive stabilisers, in the reduct to pure algebraically closed fields, of the tuples $\bar{\theta}a$, $\bar{\theta}b$ and $\bar{\theta}(a + b)$, an application of Lemma 1.10 yields the desired non-trivial linear combination S over D, which implies a new relation among the wods in the x_i's not imposed by the theory, since a is not a generic element of $\mathbb{G}_a^k$,

Recall (cf. Definition 1.14) that the closure operator $\mathrm{cl}_{\mathrm{Gen}}$ is the collection of all elements whose type is coforeign to the generics of K. An automorphism τ of the field with operators $(K, +, \cdot, F_1, \dots, F_n)$ is bounded if there is a finite set D such that, whenever a is a generic over D, then $\tau(a)$ lies in $\mathrm{cl}_{\mathrm{Gen}}(D \cup \{a\})$. In particular, the element $\tau(a)$ is not generic over $D \cup \{a\}$.

An easy application of Lemma 1.15 yields the following:

Lemma 5.4. *Given a bounded automorphism τ over a finite set D of parameters, and two generic elements a and b independent over D, the pairs $(a, \tau(a))$, $(b, \tau(b))$ and $(ab, \tau(ab))$ are pairwise independent over*

$$D_0 = \mathrm{cl}_{\mathrm{Gen}}(D) \cap \mathrm{acl}(D(a, b, \tau(a), \tau(b)).$$

Likewise, the pairs $(a, \tau(a))$, $(b, \tau(b))$ and $(a+b, \tau(a+b))$ are pairwise independent over D_0. Furthermore, the element a is generic over D_0.

We have now all the ingredients to prove the following:

Theorem 5.5. *[BHM15, Theorem 3.1] Given a sufficiently saturated algebraically closed field with operators $(K, +, \cdot, F_1, \dots, F_n)$ over a base subfield $\mathbb{F}$ satisfying the following conditions :*

- *The residue fields of all the local algebras associated to the $\mathbb{F}$-algebra $D(\mathbb{F})$ are $\mathbb{F}$.*
- *The associated endomorphisms are surjective and include both Frobenius and its inverse in case the characteristic is positive.*
- *The theory T of K is simple and eliminates imaginaries and hyperimaginaries. It is relatively 1-based over the reduct to pure algebraically closed fields with respect to the model theoretic algebraic closure* acl, *which coincides with the field algebraic closure of the generated structure. Hence*

$$A \mathop{\smile\!\!\!\!\!\!\,|}_{C} B \iff \text{acl}(A \cup C) \textit{ and } \text{acl}(B \cup C) \textit{ are linearly disjoint over } \text{acl}(C).$$

Every bounded automorphism of K is a product of integer powers of the associated automorphisms (and Frobenius, in positive characteristic).

The proof is quite straight-forward. We may assume that the operators are in triangular form, by Lemma 5.2, since the associated automorphisms do not change. Given a bounded automorphism τ of K over the finite set D, take two generic independent elements a and b over $D \cup \tau^{-1}(D)$. Thus, both a and $\tau(a)$ are generic over D. Lemmas 5.4 and 1.10, together with Corollary 5.3, yield a non-trivial linear combination

$$\lambda_{\theta_1}\theta_1(a) + \mu_{\theta_2}\theta_2(\tau(a)) + S_1(a) + S_2(\tau(a)) = \xi,$$

over $D_0 = \mathrm{cl}_{\mathrm{Gen}}(D) \cap \mathrm{acl}(D(a, b, \tau(a), \tau(b))$, where $\lambda_{\theta_1} \cdot \mu_{\theta_2} \neq 0$, and S_1, resp. S_2, is a linear combination of words in a, resp. $\tau(a)$, of degree less than θ_1, resp. θ_2.

Again, Lemmas 5.4 and 1.10 imply that the multiplicative stabiliser $\mathrm{Stab}_{\mathbb{G}_m^2}(a, \tau(a)/D_0)$ is infinite. By choosing a generic element of $\mathrm{Stab}_{\mathbb{G}_m^2}(a, \tau(a)/D_0)$ independent from $(a, \tau(a))$ over D_0, the generalised Leibniz rule and induction on the degrees θ_1 and θ_2 allow us to conclude that there is some $\lambda_a \neq 0$ in D_0 such that

$$\tau(a) = \lambda_a \cdot \sigma_\theta(a),$$

for some word θ. We need only show that $\lambda_a = 1$ for every generic element a over D, since every element of K is the product of two generics. Notice that

$$\lambda_{a+b}\sigma_\theta(a + b) = \tau(a + b) = \tau(a) + \tau(b) = \lambda_a\sigma_\theta(a) + \lambda_b\sigma_\theta(b),$$

so $\lambda_a = \lambda_{a+b} = \lambda_b$. The equality $\lambda_{a+b} = \lambda_a \cdot \lambda_b$ implies that $\lambda_a = 1$, as desired.

"Every building is like a person. Single and unrepeatable."
Ayn Rand – The Fountainhead

VI
Ample Buildings

The ample hierarchy, introduced in Section 2.2, is far from being well understood beyond the first two levels. The free pseudospace, constructed by Baudisch and Pillay [10] by glueing free pseudoplanes in a clever fashion, was the first example of a stable 2-ample structure which did not interpret an infinite field. In [89, BMZ14b], an n-ample but not $(n+1)$-ample stable structure was constructed: the free n-dimensional pseudospace, which is n-tight with respect to the family of Lascar rank 1 types. However, the free pseudospaces, which have trivial forking, provide no insight on the nature of definable groups of low ample degree.

Tent noticed that the free n-dimensional pseudospace could be understood as a building, which led us in [BMZ14a] to investigate the model-theoretic nature of right-angled buildings, for most of the techniques developped in [BMZ14b] can be adapted to this wider context. Note that we did not deal with buildings directly, but with certain biinterpretable graphs associated to them. In this chapter, we will provide a slightly different presentation of the results from [BMZ14a], without passing to the corresponding graphs.

6.1 Basics on Buildings

Buildings were introduced by Tits [91] in order to study certain groups of Lie type, akin to the role played by trees to capture some of the properties of free groups. A building is a combinatorial structure which generalises certain aspects of finite projective planes and Riemannian symmetric spaces. Although the theory of semisimple algebraic groups provided the initial motivation for the notion of a building, not all buildings arise from a group, e.g., the incidence geometry in projective planes satisfies the axioms of a building and has no connection to some underlying group. We will mostly concentrate on the combinatorial nature of buildings and refer the reader to [35] for a pleasant introduction to buildings.

A *Coxeter group* (W, Γ) consists of a group W with a fixed set Γ of generators and defining relations $(\gamma \cdot \delta)^{m_{\gamma,\delta}} = 1$, where $m_{\gamma,\gamma} = 1$ and $m_{\gamma,\delta} = m_{\delta,\gamma}$, for $\gamma \neq \delta$, is either ∞ or an integer larger than 1. For the presentation, we will exclusively consider finitely generated Coxeter groups, with Γ finite, though this is not a major obstruction.

A *word* w is a finite sequence on the generators from Γ. It is *reduced* if its length is minimal with respect to all words representing the same element of W. The word w' is a *permutation* of w, or w' is *equivalent* to w, if it can be obtained from w by a sequence of swaps of commuting consecutive generators.

The Coxeter group (W, Γ) is *right-angled* if for every pair $\gamma \neq \delta$, the value $m_{\gamma,\delta}$ is either 2 or ∞. Note that, for involutions γ and δ, the relation $(\gamma \cdot \delta)^2 = 1$ means that γ and δ commute. As a convention, no element γ *commutes* with itself. A right-angled Coxeter group (W, Γ) is determined by its *Coxeter diagram*: a graph with vertex set Γ such that γ and δ are *adjacent*, that is, they have an edge connecting them, if and only if $m_{\gamma,\delta} = \infty$. In an abuse of notation, we will denote this graph again by Γ. The elements of Γ will be referred to as *colours* or *levels*. In right-angled Coxeter groups, a word w is reduced if and only if no permutation of w has the form $w_1 \cdot \gamma \cdot \gamma \cdot w_2$, for some generator γ. Every element of W is represented by a unique reduced word, up to permutation.

A *chamber system* (X, W, Γ) for a Coxeter group (W, Γ), possibly not right-angled, is a set X, equipped with a family of equivalence relations $(\sim_\gamma,\ \gamma \in \Gamma)$. If $w = \gamma_1 \cdots \gamma_n$ is a reduced word, a *reduced path of type* w from x to y in X is a sequence $x = x_0, \ldots, x_n = y$ such that x_{i-1} and x_i are $\sim_{\gamma_i}$-related and different for every $1 \leq i \leq n$. A chamber system (X, W, Γ) is a *building* if each $\sim_\gamma$-class contains at least two elements, and so that for every pair x and y in X, there exists an element $g \in W$ such that there is a reduced path of type w from x to y if and only if the word w represents g. It follows that g is uniquely determined by x and y, and that the reduced path connecting x and y is uniquely determined by its type w. We will denote the existence of a path of type w connecting x to y by $x \xrightarrow{w} y$. In particular, we have that $x \xrightarrow{\gamma} y$ if and only if $x \neq y$ are $\sim_\gamma$-related. In a building (X, W, Γ), the class $x/\sim_\gamma$ is the γ*-residue* of x.

We will say that a building is *right-angled* if its corresponding Coxeter group is. From now on, all Coxeter groups in this chapter will be right-angled.

A right-angled Coxeter group admits a unique (up to isomorphism) countable building $\mathrm{B}_0(\Gamma)$ with infinite residues [36, Proposition 5.1], which we call *rich*. In the particular case of the free n-dimensional pseudospace, its associated Coxeter diagram is as follows:

$[0, n]$

Notice that, in order to study the model-theoretic properties of the above building $\mathrm{B}_0(\Gamma)$, elementary extensions involve *non-standard paths* between elements, so we will consider the following expansion of the natural language. Given a chamber system X for (W, Γ) and a subset s of Γ, set $\sim_\emptyset$ to be the diagonal in $X \times X$, and otherwise, let $\sim_s$, for $\emptyset \neq s \subset \Gamma$, denote the the transitive closure of all $\sim_\gamma$, with $\gamma \in s \subset \Gamma$. The $\sim_s$-class of an element is called its s*-residue*. In particular, its γ*-residue* is its $\sim_\gamma$-class, which is often called γ*-panel* in the literature. It is easy to see that $x \sim_s y$ if and only if $x \xrightarrow{w} y$ for some w with support in s, that is, the letters γ occuring in w belong to s.

For γ in Γ, set $\sim^\gamma = \sim_{\Gamma \setminus \{\gamma\}}$. The chamber system $(X, \sim^\gamma)_{\gamma \in \Gamma}$ is called the associated *dual chamber system* of X. Observe that:

$$\sim_{s_1 \cap s_2} = \sim_{s_1} \cap \sim_{s_2},$$

and, in particular,

$$\sim_s = \bigcap_{\beta \notin s} \sim^\beta .$$

Thus, setting $s = \{\gamma\}$, we conclude that the chamber system $(X, \sim_\gamma)_{\gamma \in \Gamma}$ of a building is definable in its associated dual chamber system $(X, \sim^\gamma)_{\gamma \in \Gamma}$. The converse holds only in $\mathcal{L}_{\omega_1,\omega}$, where countable disjunctions are allowed.

For a building $(X, \sim_\gamma)_{\gamma \in \Gamma}$, its associated dual chamber system has the following properties, which are elementary in the above expansion of the language:

- Given x and y in X with $x \sim^\gamma y$ for all $\gamma \in \Gamma$, then $x = y$.

- If $(x_\gamma)_{\gamma\in\Gamma}$ is a *coherent* sequence in X, i.e. whenever γ and δ are adjacent, there exists some $y_{\gamma,\delta}$ in X with $x_\gamma \sim^\gamma y_{\gamma,\delta} \sim^\delta x_\delta$, then there exists an element $z \in X$ with $z \sim^\gamma x_\gamma$ for all $\gamma \in \Gamma$.

A chamber system $(X, \sim^\gamma)_{\gamma\in\Gamma}$ satisfying the above properties is a *dual quasi-building*. The aim of this chapter is to study the complete theory of $\mathrm{B}^0(\Gamma)$, the dual quasi-building associated to $\mathrm{B}_0(\Gamma)$. This was done in [BMZ14a] by means of Γ-spaces.

A Γ-*graph* M is a coloured graph with colours $\mathcal{A}_\gamma(M)$ for γ in Γ, and no edges between elements of $\mathcal{A}_\gamma(M)$ and $\mathcal{A}_\delta(M)$ whenever γ and δ are not adjacent in Γ. A *(full) flag* of the Γ-graph M is a subgraph $F = \{f_\gamma\}_{\gamma\in\Gamma}$, where each f_γ lies in $\mathcal{A}_\gamma(M)$, such that the map $\gamma \mapsto f_\gamma$ induces a graph isomorphism between Γ and F.

A Γ-*space* M is a Γ-graph satisfying two additional properties:

- Every vertex belongs to a flag of M.
- Any two adjacent vertices in M can be expanded to a flag of M.

Out of a dual chamber system $(X, \sim^\gamma)_{\gamma\in\Gamma}$, we define the following Γ-graph $\mathcal{M}(X)$: for every $\gamma \in \Gamma$, the colour $\mathcal{A}_\gamma$ is $X/\sim^\gamma$, the set of $\sim^\gamma$-classes of elements in X. We consider the $\mathcal{A}_\gamma$'s as being pairwise disjoint. For the graph structure on $\mathcal{M}(X)$, we impose that two elements u and v are adjacent if u lies in $\mathcal{A}_\gamma$ and v in A_δ, with adjacent colours γ and δ in Γ, and there is some common z in X with $z \sim^\gamma u$ and $z \sim^\delta v$.

Every $x \in X$ gives rise to the flag $\phi(x) = \{x/\sim^\gamma \mid \gamma \in \Gamma\}$ of $\mathcal{M}(X)$. Thus, given a dual chamber system of a building X, we have that $\mathcal{M}(X)$ is a Γ-space. Actually, the class of Γ-spaces is bi-interpretable with the class of dual quasi-buildings [BMZ14a, Lemma 2.13 and Theorem 2.17]. However, the whole model-theoretical study of $\mathrm{B}^0(\Gamma)$ can be done without passing to its corresponding Γ-space.

6.2 Axiomatising Right-Angled Buildings

In order to study dual quasi-buildings $(X, \sim^\gamma)_{\gamma\in\Gamma}$, recall that *letters*, denoted usually by s, t, etc., are now non-empty connected subsets of the graph Γ. A *word* u is a finite sequence of letters. Two letters s and t *commute* if $s \cup t$ is not a letter, i.e. if all elements of s commute with the elements of t. In particular, no letter commutes with itself. Two words commute if their corresponding letters do. A word is *commuting* if it consists of pairwise commuting letters. We recover in this context the notion of a *permutation* of a word, or when two words are *equivalent*.

Given two elements x and y in a dual quasi-building $(X, \sim^\gamma)_{\gamma\in\Gamma}$, we say that there is a *weak path* between them with word $w = s_1 \cdots s_n$, denoted by $x \underset{w}{\longrightarrow} y$, if there is a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $x_i \sim_{s_{i+1}} x_{i+1}$ for $0 \leq i < n$. Two elements x and y are A-*equivalent*, for $A \subset \Gamma$, denoted by

$$x \sim_A y,$$

if the set of γ's such that their $\sim^\gamma$-classes differ is contained in $A \subset \Gamma$. By decomposing any subset of Γ as a disjoint union of its connected components, it is easy to see that two elements x and y are A-equivalent if and only if they can be connected by a weak path whose word is commuting and consists of letters contained in A. In particular, any two elements are connected by a weak path. Given a weak path $P : x \underset{u}{\longrightarrow} y$ and a permutation u' of u, there is a unique weak path $P' : x \underset{u'}{\longrightarrow} y$. Such a path P' is a *permutation* of P.

Weak paths between two given elements are far from being canonical, so we aim to isolate a property, similar to the case of buildings, which will ensure the existence of canonical paths between elements, up to permutation. This is where simple connectedness and reduced (strong) paths appear. A *splitting* of a letter s is a (possibly trivial) word, whose letters are properly contained in s. Given words u and v, we write $u \prec v$ if u is equivalent to a word obtained from v by replacing at least one occurrence of a letter in v by a splitting. The relation $\prec$ is transitive, irreflexive and well-founded. The notation $u \preceq v$ stands for either $u \prec v$ or $u \approx v$.

Write $x \overset{s}{\to} y$ if $x \underset{s}{\to} y$ but no weak path whose word is a splitting of s connects x to y. A *path* from x to y with word $u = s_1 \cdots s_n$, denoted by $x \overset{u}{\to} y$, is a weak path $x = x_0, \ldots, x_n = y$ such that $x_i \xrightarrow{s_{i+1}} x_{i+1}$ for $i = 0, \ldots, n-1$. Notice that a permutation of a path is again a path. If $x \underset{u}{\to} y$, then $x \overset{v}{\to} y$ for some *reduction* $v \preceq u$. Thus, we say that a word $v = s_1 \cdots s_n$ is *reduced* if there is no pair $i \neq j$ such that $s_i \subset s_j$ and s_i commutes with all letters in v between s_i and s_j. A path is *reduced* if its associated word is. A word u, resp. a path P, is reduced if and only if any permutation of u, resp. of P, is.

A dual quasi-building is *simply connected* if there are no non-trivial closed reduced paths, or equivalently, if the word of any closed path can be reduced to the trivial word 1. The dual quasi-building $\mathrm{B}^0(\Gamma)$ happens to be simply connected [BMZ14a, Theorem 3.16], since any reduced path between two elements must have singletons as letters. In a simply connected dual quasi-building X, given two elements x and y and reduced paths connecting x to y with words u and v, we have that $u \approx v$. Thus, we denote this reduced word connecting x to y, unique up to permutation, by $\mathrm{w}_X(x, y)$. Simple connectedness is an elementary property [BMZ14a, Theorem 3.26]. Indeed, given a weak path of length n between x and y of length n whose word is reduced, it suffices to require that there is no splitting of length n at each intermediate step to conclude that the reduction of the weak path cannot be trivial.

The theory of $\mathrm{B}^0(\Gamma)$ is then completely axiomatised by the collection of axioms PS_Γ stating that the universe is a simply connected dual quasi-building with associated Coxeter graph Γ such that the $\sim_\gamma$-class of every element is infinite. In order to show that PS_Γ is complete [BMZ14a, Theorem 4.12], we observe that every ω-saturated model can be obtained as an increasing chain of *nice* extensions. A non-empty subset A of a dual quasi-building X is *nice* if, whenever x and y in A are connected in X by a reduced path of word u, then so are they in A. Observe that a nice subset of a simply connected dual quasi-building is again simply connected. As a by-product of the completeness of PS_Γ, we deduce that it is ω-stable and that the quantifier-free type of a nice subset determines its type. Furthermore, the model $\mathrm{B}^0(\Gamma)$ is the prime model, since it is constructible and countable.

6.3 Equationality and Ampleness of Right-Angled Buildings

Work now inside a big sufficiently saturated model M of the theory PS_Γ, as a universal domain. Given a nice subset A of M and an element x, there is some y in A such that $u = \mathrm{w}(x, y)$ is not only $\prec$-minimal but $\prec$-smallest among all possible connections of x to some element in A. Such a point y is called a *base-point* of x over A, and $x \mathop{\smile\!\!\!\!\!\!\mid}_y A$ with respect to non-forking independence, since the theory is stable. The type of x over A is uniquely determined by y and u.

The canonical base of x over A is a certain collection of residues of any base-point of x over A, determined exclusively by $\mathrm{w}(x, y)$. This shows one of the advantages of considering Γ-spaces instead of dual quasi-buildings, for it grants elimination of imaginaries, since residues become now real elements. Forking in PS_Γ is trivial [BMZ14a, Corollary 7.27]: given three pairwise independent tuples, they form an independent set . Thus, no infinite group can be interpreted.

In order to show that the theory PS_Γ is equational (cf. Section 1.2), *non-splitting reductions* are required, that is, reduction of words where no splitting ever occurs. Up to permutation, such reductions are unique [BMZ14b, Corollary 5.3]. Thus, given two reduced words u and v, we denote by $[u \cdot v]$ the non-splitting reduct of $u \cdot v$. The collection of reduced words equipped with the partial order $\preceq$ and the above operation is hence an ordered monomial [BMZ14b, Lemma 5.29]. Furthermore, if u and v are reduced words, then $u \preceq [u \cdot v]$.

A particular case of non-splitting happens when the letter t is *properly left-absorbed*, resp. *properly right-absorbed*, by the word $s_1 \cdots s_n$, that is, the letter t is properly contained in some s_i and commutes with $s_1 \cdots s_{i-1}$, resp. with $s_{i+1} \cdots s_n$. The notion of *left-absorbed*, resp. *right-absorbed* is defined similarly, but it may be that $t = s_i$. There is a *Symmetric Decomposition* [BMZ14b, Corollary 5.23] of two reduced words Given u and v, as follows:

$$u = u_1 \cdot u' \cdot w \qquad\qquad w \cdot v' \cdot v_1 = v,$$

such that:

(a). w is a commuting word,

(b). u' is properly left-absorbed by v_1,

(c). v' is properly right-absorbed by u_1,

(d). u', w and v' pairwise commute,

(e). $u_1 \cdot w \cdot v_1$ is reduced.

Furthermore,

$$[u \cdot v] = u_1 \cdot w \cdot v_1.$$

The above decomposition result, used all throughout [BMZ14b] and [BMZ14a], has many consequences. In particular, it allows to prove, given reduced words $u \preceq v$, the existence of a reduced word w, unique up to permutation, such that for every reduced word x,

$$[x \cdot u] \preceq v \Leftrightarrow x \preceq w.$$

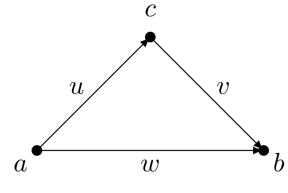
We denote the above word by v/u. Notice that $v/u \preceq v$.

Let us now describe which of the formulae form our collection of equations. Let $\mathrm{P}_u(x, y)$ state that, between x and y there is a weak path with word u. The formula $\mathrm{P}_u(x, y)$ is an equation. Indeed, the existence of the quotient v/u as above and an easy compactness argument imply that, given elements a and b in M with reduced path P connecting a to b, and reduced words u_1 and u_2 such that neither $\mathrm{P}_{u_1}(X, a)$ nor $\mathrm{P}_{u_2}(X, b)$ imply the other, the conjunction $\mathrm{P}_{u_1}(X, a) \wedge \mathrm{P}_{u_2}(X, b)$ is equivalent to

$$\bigvee_{i=1}^{n} \mathrm{P}_{w_i}(X, c_i)$$

for elements $c_1, \ldots, c_n$ occurring in some permutation of P and reduced words $w_1, \ldots, w_n \prec u_1, u_2$. Since the relation $\prec$ is well-founded, we conclude that each formula $\mathrm{P}_u(x, y)$ is an equation [BMZ14a, Corollary 7.6], as desired.

The theory PS_Γ is equational once we show that the type of n elements $x_1, \ldots, x_n$ is uniquely determined by the collection of words $\mathrm{w}(x_i, x_j)$ connecting each pair, for $i \neq j$ [BMZ14a, Theorem 7.24]. This is not a trivial consideration, though by induction, it suffices to prove it for three elements a, b and c with reduced words u, v and w, as in the following diagram:



By [BMZ14a, Proposition 7.21], there is a decomposition of the form:

$$u \approx u_1 \cdot \alpha^{-1} \cdot \epsilon^{-1},$$
$$v \approx \epsilon \cdot \beta \cdot v_1,$$
$$w \approx u_1 \cdot x \cdot v_1,$$

such that α, β and x pairwise commute, the word x is properly right-absorbed by ϵ, the word α is properly left-absorbed by v_1, and β is right-absorbed by u_1. The words u_1, v_1, ϵ, x, α and β are furthermore unique, up to permutation. There is some element d occuring in some permutation of the path $P : a \to b$, such that $\mathrm{w}(c, d) \approx \epsilon \cdot \alpha \cdot \beta$, and the scaffold consisting of the base set P together with the path from c to d is nice, so its type is determined by P and $\mathrm{w}(c, d)$, as desired.

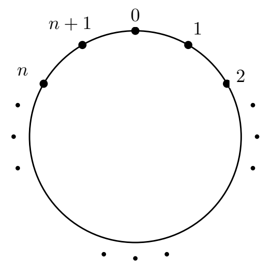
Let us now conclude this chapter by providing lower and upper bounds, explicitly computable in terms of the underlying Coxeter graph, on the ample degree of the theory PS_Γ. If Γ has at least one edge, define its *minimal valency* as the minimum of the valencies of the non-isolated vertices of Γ. A subgraph $Y \subset \Gamma$ is *full* if, whenever two vertices in Y are adjacent in Γ, then so are they in Y.

If the graph Γ has no edges, the theory PS_Γ is biinterpretable with the theory of an infinite set M partitioned into $|\Gamma|$ many infinite sets, which is a trivial theory of Morley rank 1 (and degree $|\Gamma|$) and 1-based. Otherwise, we have the following result.

Theorem 6.1. *[BMZ14a, Theorem 8.6] Let Γ be a Coxeter graph with at least one edge, and denote r its minimal valency and let n in $\mathbb{N}$ be maximal such that the graph $[0, n]$ fully embeds in Γ. The theory PS_Γ is n-ample but not $(|\Gamma| - r + 1)$-ample.*

Thus, the theory PS_Γ is not 1-based if and only if Γ contains at least one edge. The complete graph K_n has minimal valency $n - 1$, so the theories $\mathrm{PS}_{\mathrm{K}_n}$ are all CM-trivial, for every n.

Notice that these bounds are best possible, atttained either by the graph $[0, n]$ itself, whose associated theory is n-ample but not $(n + 1)$-ample (*cf.* [89, Theorem 3.3], [BMZ14b, Theorem 8.4]), or, as Evans and Wagner pointed out, by the graph consisting of $0, \ldots, n + 1$ arranged in a circular way:



which has valency 2, so its theory is n-ample but not $(n + 1)$-ample.

"This is the end.
Beautiful friend.

Of our elaborate plans, the end."

The Doors – The End

Perspectives

Future Research Directions

This last section contains some of the open problems and research directions, which arose from the work here presented, that we would like to pursue in the future, though the feasibility of some of them has not been thoroughly determined.

Given a geometric theory T, the expansion T^{ind} by adding a dense independent subset H was axiomatised and studied in [14]. In particular, if T is supersimple of SU rank 1, then so is T^{ind} of SU rank at most ω. Carmona showed [22] that ampleness is preserved betweend T and T^{ind}, for $n \geq 2$. If T is 1-based, the theory T^{ind} need not be. Take H a basis in a vector space V, and notice that, in the ω-stable group (V, H), the predicate H is not a boolean combination of cosets of subgroups. This marks a difference with respect to the theory of belles paires of a stable NFCP theory, for T_P is 1-based, whenever T is [11, Proposition 7.7].

Question / Problem A. *If T is stable NFCP and CM-trivial, then so is T_P? More generally, if T is stable NFCP and not n-ample, then so is T_P?*

In order to show that a simple theory is 1-based, notice that it suffices to show that it is *weakly* 1*-based* [13, Definition 2.3], that is, given a tuple a over a model M, there is some $a' \models \text{tp}(a/M)$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_M a$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}_a M$. The advantage of this formulation is that no canonical bases, and therefore no (hyper)-imaginaries, appear. We ignore any equivalent formulation of CM-triviality, where imaginary closure does not appear. The characterisation in T_P of non-forking independence stated before Lemma 4.1 holds exclusively for real sets, and unfortunately, the theory T_P of belles paires need not eliminate imaginaries modulo imaginaries from T, whenever T interprets an infinite group. A possible reformulation of n-ampleness similar to weakly 1-based could then be most helpful.

Work by Casanovas and Ziegler [24] implies that formulae in the theory T_P of belles paires of a stable NFCP theory with quantifier elimination are boolean combination of *bounded* formulae, that is, the quantified variables run over the predicate. Delon's language [30] for belles paires of algebraically closed field yields in particular that formulae in this theory are boolean combination of bounded existential formulae.

Question / Problem B. *Is there a natural expansion of the language which provides a description of formulae in the theory of belles paires of a stable NFCP theory ?*

The above relates to the following question:

Question / Problem C. *Let T be equational with NFCP. Is the theory T_P equational?*

Ongoing work with Ziegler suggests that the above holds for the theory of belles paires of algebraically fields. At the moment of writing, we have a complete proof in characteristic 0, which we hope to adapt to all characteristics. Unfortunately, our proof uses the expansion of differentially closed fields in characteristic 0, for which there is no equivalent in positive characteristic.

In belles paires of a stable NFCP 1-based theory, every $\mathrm{acl}^{\mathrm{eq}}$-closed set is P-independent. In particular, if T is 1-based strongly minimal with geometric elimination of quantifiers, the T_P-type of an infinite algebraically closed set is determined by its the quantifier-free type. Recall that 1-based theories are equational, which relates to the above question.

Question / Problem D. *Which geometrical conditions of T imply that (certain) structures are P-independent?*

Lascar and Junker [50] introduced a topology, the *indiscernible topology*, which agrees with Srour's topology on definable sets. They introduce an ordinal-valued invariant, called i_T, to mesure the complexity of this topology in a given theory. If T is stable, then $i_T \leq |T|^+$. If T is 1-based, then $i_T \leq 2$. They observe that for CM-trivial groups of finite Morley rank, the value i_T is finite. Carmona showed that the free pseudoplane has $i_T \leq 3$.

Question / Problem E. *If T is not n-ample, then is $i_T \leq n+1$? As a test: is $i_T \leq n+2$ for the free n-dimensional pseudospace?*

Lascar [61] and subsequently Evans, Ghadernezad and Tent [32] showed that, for a pure algebraically closed field or a differentially closed field in characteristic 0, given an unbounded strong automorphism, every strong automorphism can be written as a product of a fixed number of its conjugates. The proof uses the stationarity of strong types in a stable theory in a fundamental way, in order to show that a certain continuous map has non-meager image. If T is simple, stationarity no longer holds.

Question / Problem F. *Describe the algebraic structure of the group of Lascar strong automorphisms of a generic difference field modulo the group generated by σ and Frobenius, if the characteristic is positive.*

Recall that the canonical base property CBP generalises 1-basedness. Chatzidakis [26] shows that, if a supersimple theory $T = T^{\mathrm{eq}}$ of finite SU rank has the CBP with respect to an invariant family Σ, then every algebraically closed set E contains a smallest algebraically closed set subset A such that the type of E/A is almost Σ-internal.

Question / Problem G. *Does the above hold for a supersimple CM-trivial theory (of finite* SU*-rank)? Do supersimple CM-trivial theories have the CBP?*

Hrushovski and Pillay studied groups in certain local fields [45, 46]. A Nash affine group definable in a real closed field is hence Nash isomorphic to the real points of an algebraic group. A remarkable example of a lovely pair of geometric structures is the theory of dense pairs of real closed fields, which is complete and has o-minimal open core [93].

Question / Problem H. *Describe definable groups in dense pairs of real closed fields, analogously to the theory of belles paires of algebraically closed fields in characteristic* 0*.*

Were the above problem solved successfully, then the study of interpretable groups in dense pairs of real closed fields seems the natural step. However, the issue of geometric elimination of imaginaries could represent an obstacle. Boxall showed [18] that imaginaries in the theory of dense pairs of real closed fields are interalgebraic with imaginaries coming from the associated belle paire of algebraically closed fields in characteristic 0. However, his proof is not explicit and does not provide natural geometric sorts, in contrast to Pillay's proof [72].

Question / Problem I. *Describe geometric sorts to have elimination of imaginaries in dense pairs of real closed fields.*

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Groupes définissables dans des expansions de théories stables

Ampleur et notions relatives

Résumé: Les travaux de recherche présentés dans ce document portent sur la hiérarchie ample, introduite par Pillay et Evans comme réformulation du Principe de la Trichotomie pour les ensembles fortement minimaux. Nous étudions la théorie de modèles d'immeubles à angles droits et résidues infinis en une expansion du langage naturelle des immeubles, pour donner des bornes explicites sur le degré d'ampleur, qui se calculent à partir du graphe de Coxeter associé. En particulier, la hiérarchie ample est stricte.
En outre, nous nous inspirons des divers résultats sur les groupes définissables dans certaines théories des corps munis d'opérateurs, fondamentales en théorie des modèles géométriques, pour donner une approche générale à l'étude des groupes définissables dans une théorie simple à partir de son ampleur relative à une (ou plusieurs) théorie stable de base, en présence d'un opérateur clôture modéré. Cette approche, qui s'applique aux corps différentiellement clos de caractéristique nulle et aux corps aux différences génériques en toute caractéristique ainsi qu'à tous les amalgames de Hrushovski connus, nous permet aussi d'établir une description des groupes définissables dans des corps colorés, introduits par Poizat, et en particulier dans le mauvais corps obtenu par collapse du corps vert. Les outils développés pour cette étude s'étendent aux groupes définissables et interprétable dans des belles paires de corps algébriquement clos. En outre, nous caractériserisons les automorphismes bornés de nombreuses théories des corps munis d'opérateurs.

Mots clés: Théorie des modèles Géométrique, Groupes, Ampleur.

Definable groups in expansions of stable theories

Abstract: We present research around the ample hierarchy, as introduced by Pillay and Evans to reformulate the Trichotomy Principle for strongly minimal theories. We study the model theory of right-angled buildings with infinite residues in an expansion of the natural language of buildings, in order to provide explicit bounds on their ample degree, computable in terms of the associated Coxeter graph. In particular, the ample hierarchy is strict.
We generalise various results on definable groups in several theories of fields with operators, such as differentially closed fields in characteristic 0 as well as generic difference fields in all characteristics, which play a crucial role in geometric model theory, to study type-definable groups in a simple theory according to its relative ampleness over some base stable reducts with respect to a tame closure operator. This approach applies to all known examples of Hrushovski's amalgams. We pursue further this analysis to definable groups in colored fields, as introduced by Poizat, and in particular the bad field obtained as a collapse of the green field. The tools we develop thereupon can be adapted to study definable groups in belles paires of algebraically closed fields, as well as to characterise bounded automorphisms in various theories of fields with operators.

Keywords: Geometric Model Theory, Groups, Ampleness.

Image en couverture : La configuration de groupe (d'après E. Hrushovski). Crédit image : Thomas Blossier.

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