GALOIS COHOMOLOGY OF FIELDS WITH A DIMENSION

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ABSTRACT. We consider fields K with an abstract notion of dimension as stated by Pillay and Poizat in their paper of 1995. We prove that, for every finite extension L of K and for every finite Galois extension L_1 of L, the Brauer group $\mathrm{Br}(L_1/L)$ is finite. Moreover, given an algebraic group G defined over L, we have that $\mathrm{H}^1(L_1/L, G)$ is finite.

1. Introduction and preliminaries

The existence of a rank induces some primitive concept of *dimension* on definable sets. In [6], basic requirements for the well-behaviour of a dimension were considered and in particular, fields equipped with a dimension such that, given a

definable set X and a definable equivalence relation E on X, there is only a finite number of classes of dimension equal to the dimension of X. These fields were called *surgical* (from the french *chirurgical*). Examples of these fields are finite fields, totally transcendental fields and *o*-minimal fields. It was proven in the aforementioned article that surgical fields are perfect and have bounded Galois group: i.e.for each n in $\mathbb{N}$, there a e only finitely many non-isomorphic field extensions of degree n.

In [7], the above treatment was also applied to the case of supersimple fields in some of the proofs. By a supersimple field we mean a definable field in some sufficiently saturated model of a theory which is supersimple. In [3], supersimple fields were later studied and some results about their cohomological behaviour was exhibited. We realized that the proofs of [7] and [3] can be generalized to this setting if we are purely interested in finiteness of certain cohomological groups.
We will work inside a fixed sufficiently saturated structure $\mathcal{M}$. We remark that by definable we mean usually definable in $\mathcal{M}^{\mathrm{eq}}$ (other people will call this *interpretable*). Neve theless, for simplicity, we will use the word *definable* in a more general etting. We also remarkark that *interpretable* means interpretab e in the structurrem maybe with parameters. The structure $\mathcal{M}$ is emphsurgical if there is some poset such that we can assign (in a way that is invariant under definable automorphisms of $\mathcal{M}$) to each efinable set U an elemmaent $\dim(U)$ of the aforemarkentioned po et satisfying the following conditions: rem

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- If U and V are both definable in $\mathcal{M}$, and there is a finite partition of U into definable subsets $U_1, \ldots, U_n$ such that each U_i can be mapped to V via some definable (in $\mathcal{M}$) finite-to-one function, then $\dim(U) \leq \dim(V)$.
- If U is a definable set and E is a definable equivalence relation on U, then there are only finitely many equivalence classes of dimension $\dim(U)$.

By a *surgical* field K, we mean a field K that is definable in $\mathcal{M}$. Moreover, we fix an algebraic closure for K, and denote it by $\overline{K}$.
Given a perfect field K, we consider a *central simple algebra* A over K: a finite dimensional K-algebra whose center is K and with no non-trivial two-sided ideals. Any such algebra A is a matrix algebra $M_m(A)$ over some finite dimensional division ring whose center is K (note that m may vary in $\mathbb{N}$). The *Brauer group* $\mathrm{Br}(K)$ classifies the classes of central simple algebras over K modulo K-isomorphism of their respective matrix algebras. The trivial elemmaent in $\mathrm{Br}(K)$ corresponds to the class of K. Given a finite Galois extension L/K and a central simple algebra A over K, we obtain a central simple algebra A' over L by setting $A' = A \otimes L$. Hence, we obtain a map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ and we denote by $\mathrm{Br}(L/K)$ its kernel.
An abstract variety V defined over K is *Severi-Brauer* if V is rationally isomorphic to $\mathbb{P}^n$ over $\overline{K}$. Equivalently, V is a Severi-Brauer variety over K if V and $\mathbb{P}^n$ are rationally isomorphic over K', where K'/K is a finite algebraic extension. The set of classes of Severi-Brauer varieties over K modulo rational isomorphism over K is in bijection with $\mathrm{Br}(K)$. A Severi-Brauer variety V corresponds to the trivial elemmaent in $\mathrm{Br}(K)$ if V has a K-rational point.
With the above notation, given an algebraic group G over K, we say that an abstract variety E over K is a *principal homogeneous space* (denoted by PHS) for G if E is non-empty and G acts strictly transitively on it (that is, for each x and y in E, there is a unique $g \in G$ such that $y = gx$). Two PHS's E and E' for G defined over K are *isomorphic* if there is a rational G-isomorphism $\phi\colon E \to E'$ defined over K (i.e.for any g in G and $e \in E$ $\phi(ge) = g\phi(e)$ and likewise for ϕ^{-1}). We denote by $\mathrm{H}^1(K, G)$ the set of classes under rational G-isomorphism of $PHS's$ for G defined over K. Again, considering K as a subfield of L, we obtain a map $\mathrm{H}^1(K, G) \to \mathrm{H}^1(L, G)$ and we denote its kernel by $\mathrm{H}^1(L/K, G)$.
We will prove the following:

Theorem 1.1. *Let K be a surgical field. For each finite algebraic extension L of K and for each finite Galois extension L_1 of L, the Brauer group $\mathrm{Br}(L_1/L)$ is finite. Moreover, $\mathrm{H}^1(L_1/L, G)$ is finite for any algebraic group G defined over L.*

The proof of the above theorem uses an induction argument on the degree of the extension, reducing it to the cyclic case, for which there is a particular description of the above objects as quotient sets. By our hypotheses, we need only show that each class in the quotient has at least the dimension of the ambient set, and hence, there are only finitely many. In order to do so, we reduce it to the case of abelian varieties (in the case of H^1), and exhibit a finite-to-one map using the p-torsion points of the abelian variety, which is a finite set.
We should remark that, due to the weakness of our hypotheses, we cannot expect to obtain the same results as in [7], where in fact, triviality of the Brauer group was shown. Note that $\mathrm{Br}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. Hence, triviality cannot hold in this more general context.

2. Galois Cohomology

In this section we give some details about the nature of the cohomology groups that we will use. The reader is referred to [9] and [10] for a more detailed exposition about Galois Cohomology over perfect fields.
Let L/K be a Galois extension. The Galois group $\mathrm{Gal}(L/K)$ is the projective limit of all Galois groups $\mathrm{Gal}(L_i/K)$, where $L/L_i/K$ is a Galois subextension with L_i/K finite. Hence, $\mathrm{Gal}(L/K)$ is a *profinite* group (i.e. a compact, Hausdorff and totally disconnected topological group). Any closed subgroup of a profinite group is again profinite.
A *supernatural number* is a formal product $\prod_{p\,\text{prime}} p^{n_p}$, where $n_p \in \mathbb{Z}^{\geq 0}$ or $n_p = \infty$. In a natural way we define the product, l.c.m. and the g.c.d. of a collection of supernatural numbers.
Let G be a profinite group and $H \leq G$ a closed subgroup. We define the *index* of H in G by

$$(G:H) = \text{l.c.m.}\{(G/U : H/(H\cap U)) \mid U \leq G \text{ open of finite index}\}$$

By the *order* of G we mean $(G:1)$. Let p be a prime number. The subgroup H is a *pro-p-subgroup* of G if its order is a p^{th}-power (as a supernatural number). Equivalently, H is a pro-p-subgroup if it is a projective limit of finite p-groups. The group H is a *Sylow* p-subgroup if it is a pro-p-group whose index in G is coprime to p. As in the case of finite groups, there are Sylow subgroups for each prime dividing the order of G and moreover, any two are conjugate in G.
Let us now return to the case of a Galois extension L/K and let A be an abelian algebraic group defined over K endowed with the discrete topology. There is an action of $\mathrm{Gal}(L/K)$ on $A(L)$. Given a subgroup $U \leq \mathrm{Gal}(L/K)$, we define $A^U = \{x \in A(L) \mid \forall g \in U\; gx = x\}$.
Setting $G = \mathrm{Gal}(L/K)$, we denote by $C^n(G,A)$ the set of continuous functions from G^n to $A(L)$ (with $C^0(G,A) = A$). We define the *coboundary maps* $\{\delta_n\}_{n\in\mathbb{Z}^{\geq 0}}$ as follows:

$$\begin{array}{ccccc} C^0(G,A) & \xrightarrow{\delta_0} & C^1(G,A) & & \\ a & \longrightarrow & \delta_0(a) & :G \longrightarrow & A(L) \\ & & & g \longrightarrow & ga - a \end{array}$$

$$\begin{array}{ccccl} C^n(G,A) & \xrightarrow{\delta_n} & C^{n+1}(G,A) & & \\ f & \longrightarrow & \delta_n(f): & G^{n+1} \longrightarrow & A(L) \\ & & & (g_1,\dots,g_{n+1}) \longrightarrow & g_1\cdot f(g_2,\dots,g_{n+1}) \\ & & & & + \sum_{i=1}^{n} (-1)^i f(g_1,\dots,g_{i-1}\cdot g_{i+1},\dots,g_{n+1}) \\ & & & & +(-1)^{n+1} f(g_1,\dots,g_n) \end{array}$$

Note that $\delta_{n+1}\circ\delta_n = 0$. The family $C^\bullet(G,A) = \{C^n(G,A),\delta_n\}_{n\in\mathbb{Z}^{\geq 0}}$ is called a complex. We define the n^{th}-*cohomology group* of the complex $C^\bullet(G,A)$ as $\mathrm{H}^n(G,A) = \mathrm{Ker}(\delta_n)/\mathrm{Im}(\delta_{n-1})$, for $n \geq 1$ ($\mathrm{H}^0(G,A) = A^G = A(K)$), and call it the n^{th}-cohomology group of G with coefficients in A. Elemmaents of $\mathrm{H}^n(G,A)$

in the same coset are called *cohomologous*. Elemmaents of $\mathrm{Ker}(\delta_n)$ are called *n-cocycles*. We write $\mathrm{H}^n(L/K, A) = \mathrm{H}^n(\mathrm{Gal}(L/K), A)$.
For each $n \geq 1$, the group $\mathrm{H}^n(G, A)$ is torsion. If $H = \mathrm{Gal}(L/K')$ is a closed subgroup of G, then H acts on $A(L)$. Hence, $\mathrm{H}^n(H, A)$ is defined. The following holds:

Lemma 2.1. *Let G, A and H as above. If H is normal, the following exact sequences hold:*

$$0 \to \mathrm{H}^1(K'/K, A) \to \mathrm{H}^1(L/K, A) \to \mathrm{H}^1(L/K', A)$$

$$0 \to \mathrm{H}^2(K'/K, A) \to \mathrm{H}^2(L/K, A) \to \mathrm{H}^2(L/K', A)$$

If B is another commutative algebraic group defined over K containing A, the quotient group $C = B/A$ is again an algebraic group defined over K and we obtain the following exact sequence:

$$0 \to A(K) \to B(K) \to C(K) \to \mathrm{H}^1(L/K, A) \to \mathrm{H}^1(L/K, B) \to \mathrm{H}^1(L/K, C)$$

Suppose now that A is an algebraic group, not necessarily commutative. In this case, only the zeroth and first cohomology groups can be defined naturally, similarly as done above. Note that the set $\mathrm{H}^1(G, A)$ need no longer have a composition law. It is a pointed set (i.e.it has a distinguished elemmaent, the class of the unit cocycle, called the *neutral elemmaent*). Hence, the notion of an exact sequence can be extended to this setting (that is, the image of a map is equal to the inverse image of the neutral elemmaent). In fact, by means of *twisting* principal homogeneous spaces, an exact sequence gives again information about the equivalence relation that a map determines. Suppose A and B are algebraic groups defined over K and $u\colon A \to B$ is a rational homomorphism defined over K. It induces a map $v\colon \mathrm{H}^1(L/K, A) \to \mathrm{H}^1(L/K, B)$ and by twisting, we transform each fiber of v into a kernel, so that they occur in exact sequences.

Proposition 2.2. *Let A, B be algebraic groups over K such that $A \triangleleft B$, and denote B/A by C. There is a map $\delta\colon C(K) \to \mathrm{H}^1(L/K, A)$ such that the following sequence is exact:*

$$0 \to A(K) \to B(K) \to C(K) \xrightarrow{\delta} \mathrm{H}^1(L/K, A) \to \mathrm{H}^1(L/K, B) \to \mathrm{H}^1(L/K, C)$$

As stated in the introduction, $\mathrm{H}^1(L/K, A)$ is in bijection with the set of classes of isomorphism of $PHS's$ for A. Moreover, $\mathrm{Br}(L/K) = \mathrm{H}^2(L/K, L^*)$. We now state a well-known result about the cohomology of the additive and multiplicative group of a field:

Proposition 2.3. (Hilbert 90) *For any perfect field K and for any Galois extension L/K, the following hold:*

- $\mathrm{H}^1(L/K, L^*) = 0$.
- $\mathrm{H}^n(L/K, L^+) = 0$ *for each n in $\mathbb{Z}^{\geq 1}$.*

Suppose now A is a commutative algebraic group over K and the extension L/K is finite cyclic of order n. Choose a generator σ for $\mathrm{Gal}(L/K)$. In this case, we can define the following maps on A:

$$\begin{array}{ccccc} A(L) & \xrightarrow{D} & A(L) & \xrightarrow{N} & A(K) \\ Q & \longrightarrow & D(Q) = Q^\sigma - Q & & \\ & & P & \longrightarrow & N(P) = \sum_{i=0}^{n-1} P^{\sigma^i} \end{array}$$

With notation as above, we have:

Proposition 2.4. (see Section VIII.4 in [9])

- $\mathrm{H}^1(L/K, A) = \mathrm{Ker}(N)/\mathrm{Im}(D)$ *and,*
- $\mathrm{H}^2(L/K, A) = A(K)/\mathrm{Im}(N)$

We now state a result regarding the cohomological behaviour of fields with bounded absolute Galois group, that will be useful for the induction process in the proof of the theorem.

Proposition 2.5. (see Section III.4 in [10]) *Suppose K is a perfect field with bounded absolute Galois group* $\mathrm{Gal}(\overline{K}/K)$. *Given a finite algebraic group A defined over K, we have that* $\mathrm{H}^1(\overline{K}/K, A)$ *is finite. The same holds if A is a linear algebraic group defined over K.*

3. Results

As stated in the introduction, we work inside a sufficiently saturated surgical structure $\mathcal{M}$, and K denotes a definable field in $\mathcal{M}$. We also fix an algebraic closure $\overline{K}$ for K.
Let us first recall an obvious result:

Fact 3.1. $\dim(K^*) = \dim((K^*)^n)$ for each n in $\mathbb{N}$. Therefore, $(K^*)^n$ has finite index in K^*.

The theorem stated in the introduction refers to a finite algebraic extension L of K. Since such an extension is again interpretable in $\mathcal{M}$ via a basis for the extension L/K, and the surgical behaviour of $\mathcal{M}$ applies to any definable set, we may assume that $L = K$ is our base field, for simplicity of notation.

Theorem 3.2. *Let L/K be a finite Galois extension. Then* $\mathrm{Br}(L/K)$ *is finite.*

Proof. By 2.1, it is enough to prove the result for any Sylow subgroup of $\mathrm{Gal}(L/K)$. Since any p-group has a subgroup of index p, and again by 2.1, we reduce it to the case of L/K cyclic of degree p. It follows from 2.4 that $\mathrm{Br}(L/K) = K^*/\mathrm{Im}(N)$, where $N\colon L^* \to K^*$ is the norm map of the field extension. Recall that for any x in K^*, its norm is $N(x) = x^p$. Hence, $(K^*)^p \subset \mathrm{Im}(N) \subset K^*$, and it follows from 3.1 that the quotient $K^*/\mathrm{Im}(N)$ is finite. □

We now consider the case where A is an algebraic group defined over K, not necessarily commutative.

Theorem 3.3. *Let L/K be a finite Galois extension. The group* $\mathrm{H}^1(L/K, A)$ *is finite.*

Proof. Let A^0 denote the connected component of A (which is a connected algebraic group defined over K such that A/A^0 is a finite group). By 2.2, finiteness from $\mathrm{H}^1(L/K, A)$ will follow from finiteness of $\mathrm{H}^1(L/K, A^0)$ and $\mathrm{H}^1(L/K, A/A^0)$. Since $\mathrm{Gal}(L/K)$ is bounded, we have that $\mathrm{H}^1(L/K, A/A^0)$ is finite by 2.5. Hence, we need only consider the case where A is connected.
By Chevalley's theorem, we conclude that there exists a connected linear algebraic group T and an abelian variety B, both defined over K, such that the following exact sequence holds:

$$0 \to T \to A \to B \to 0$$

Again by 2.2, we need to consider $\mathrm{H}^1(L/K, T)$ and $\mathrm{H}^1(L/K, B)$. However, we have that $\mathrm{H}^1(L/K, T)$ is finite, since $\mathrm{Gal}(L/K)$ is bounded (see 2.5). Therefore, we may assume that A is an abelian variety defined over K. Hence, A is commutative. We can therefore reduce the proof, via a similar argument as in 3.2, to the case where L/K is cyclic of degree p.
We conclude by 2.4 that $\mathrm{H}^1(L/K, A) = \mathrm{Ker}(N)/\mathrm{Im}(D)$, where N and D are defined as in 2.4 via a generator σ of $\mathrm{Gal}(L/K)$. Choosing a basis of L over K and since A is definable in the field structure K, we have that both $\mathrm{Ker}(N)$ and $\mathrm{Im}(D)$ are definable in $\mathcal{M}$. Since $\mathcal{M}$ is surgical, it is enough to prove that each class in the quotient has dimension at least $\dim(\mathrm{Ker}(N))$. Let P be in $\mathrm{Ker}(N)$, and define the following map:

$$\begin{array}{ccc} \mathrm{Ker}(N) & \xrightarrow{\phi} & P/\mathrm{Im}(D) \\ Q & \longrightarrow & \phi(Q) = P + D(Q) \end{array}$$

The statement follows once we prove that ϕ is finite-to-one. This is clear: if Q and Q' in $\mathrm{Ker}(N)$ are such that $\phi(Q) = \phi(Q')$, we have that $D(Q - Q') = 0$. Hence, $Q - Q'$ is in $A(K)$. It follows that $0 = N(Q - Q') = [p](Q - Q')$, where $[p]$ denotes addition p times in the abelian variety. Therefore, $Q - Q'$ is in $A[p]$ (the p-torsion of A), which is a finite set. □

Remark 3.4. Inspecting the above proof, it follows that the whole argument goes through for the case $L = \overline{K}$ up to finiteness of $\mathrm{H}^1(\overline{K}/K, A)$, with A an abelian variety over K, that is, the induction argument. In general, we do not have tools enough to conclude finiteness for the absolute Galois group (considering the weakness of our assumptions, it would be too much to expect). Nontheless, it can be the case that, when specifying where the dimension comes from (i.e. o-minimal dimension, algebraic dimension, Shelah rank, etc), we are able to conclude stronger results.

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