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MODÈLES STOCHASTIQUES INTERAGISSANTS : SYNCHRONISATION ET RÉDUCTION À UN SYSTÈME DE PHASES

SYNCHRONIZATION AND PHASE REDUCTION IN INTERACTING STOCHASTIC SYSTEMS

Sous la direction de **Giambattista GIACOMIN**

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Introduction

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1.1 Modèle de Kuramoto

1.1.1 Modèle de Kuramoto et réduction à un système de phases

Les phénomènes de synchronisation sont omniprésents dans les sciences physiques et les sciences du vivant. Il est impossible de donner une liste exhaustive de leurs apparitions dans la littérature scientifique, mais l'on peut par exemple mentionner les émissions synchronisées de flashs lumineux par des populations de lucioles [17], les contractions synchronisées des cellules cardiaques [83], la synchronisation de réseaux de lasers [60], l'émission synchronisée de signaux électriques dans des réseaux de neurones [78]... Pour une description de nombreux autres exemples voir [87, 105].

Les modélisations de ces phénomènes font intervenir des systèmes dynamiques couplés complexes, généralement difficiles à étudier analytiquement. Dans le cas où chacun des systèmes admet un cycle limite (une courbe attractive), une simplification possible du modèle apparaît naturellement : la trajectoire de chaque système est à partir d'un certain temps proche de son cycle limite, et en projetant chaque dynamique sur le cycle associé (et donc en négligeant la distance séparant les trajectoires des cycles limites) on peut diminuer de façon radicale la dimension du modèle. Cette méthode a été pour la première fois mise en place par Kuramoto [62], qui a montré que cette approximation donne lieu, dans le cas où l'interaction entre les oscillateurs est faible et où les oscillateurs ont des cycles limites presque identiques, à un système universel de phases en interaction :

$$\dot{\theta}_j = \omega_j + \sum_{i=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \qquad i = 1 \dots N\,, \tag{1.1.1}$$

où ω_j est la fréquence naturelle de parcours du cycle limite pour le $j^{\text{ème}}$ oscillateur, et la fonction Γ_{ij} correspond à l'interaction induite par le $i^{\text{ème}}$ oscillateur sur le $j^{\text{ème}}$. Dans la suite nous allons étudier une version bruitée de ce système où l'interaction est de type champ moyen et sinusoïdale, et appelée modèle de Kuramoto. Ces simplifications ne donnent pas de sens physique particulier au modèle, mais facilitent sa résolution, l'objectif étant de parvenir à extraire les aspects fondamentaux des phénomènes de synchronisation et les mécanismes mis en jeu. Souvent le modèle général (1.1.1) est également appelé modèle de Kuramoto.

Dans le cadre des neurosciences, Ermentrout et Kopell [29] ont démontré de manière rigoureuse la validité d'une réduction à un système de phases : en se basant sur la théorie des sous-variétés normalement hyperboliques (voir la section 1.3.3 pour une présentation de cette notion), ils ont démontré qu'un modèle du type Hodgkin-Huxley tend dans une certaine échelle vers un modèle de phase sur le cercle unitaire, le Θ-*model*. Par la suite le comportement de systèmes de Θ-*models* mis en interaction a également été étudié [30, 76]. Ce modèle est également connu sous le nom de *Active Rotators* [101, 95], et sera étudié dans le chapitre 2.

La méthode de réduction à un système de phases a été principalement développée dans le cas déterministe. Dans le cas où le système initial d'oscillateurs est bruité, le choix des termes à négliger et la validité de la méthode ont été très discutés récemment (voir [113, 108] et les références s'y trouvant). Dans le chapitre 5 nous étudierons la validité de la réduction à un système de phases pour le problème de sortie de domaine induite par un bruit.

1.1.2 Présentation du modèle de Kuramoto

Le modèle de Kuramoto est défini par le système de N équations stochastiques couplées

$$\mathrm{d}\varphi_j^\omega(t) \;=\; \omega_j\,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N}\sin(\varphi_j^\omega(t)-\varphi_i^\omega(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t)\,, \tag{1.1.2}$$

où

- $\{B_j\}_{j=1\ldots N}$ est une famille de mouvements Browniens indépendants : d'un point de vue physique cette famille représente le bruit thermique ;
- $\{\omega_j\}_{j=1\ldots N}$ est une famille de variables aléatoires indépendantes identiquement distribuées de loi μ, qui sont les fréquences naturelles d'oscillation de chaque particule. Nous appellerons cette autre source de bruit le désordre. L'exposant ω est une notation abrégée signifiant que le modèle dépend de la suite $\{\omega_j\}_{j=1\ldots N}$;
- K et σ sont des paramètres positifs, mais l'on s'intéressera uniquement aux cas où ils sont strictement positifs. Nous nous référons à [1, 104] et aux références s'y trouvant pour une présentation du comportement du modèle dans le cas $\sigma = 0$.

Nous regardons les variables φ_j^ω comme des éléments de $\mathbb{S} := \mathbb{R}/2\pi\mathbb{R}$ (les positions d'oscillateurs distribuées sur un cercle). La positivité du paramètre K implique que le terme d'interaction dans (1.1.2) est un terme d'attraction. Ce terme pousse les particules à se rapprocher les unes des autres, alors que les deux termes de bruits les incitent à se comporter de manière individuelle.

En l'absence de désordre, le modèle de Kuramoto est très proche de la version 2-dimensionnelle du modèle de solution de polymères rigides avec interaction de type Maier-Saupe [27, 69]. Dans ce modèle les angles $\varphi_j(t)$ décrivent l'orientation de polymères dans le plan. Lorsque le milieu dans lequel évoluent les polymères est homogène, on néglige la dépendance de ces orientations en la position des polymères dans l'espace, et l'évolution de ces orientations est donnée par le système d'équation (1.1.2) sans désordre et où le $\sin(\cdot)$ intervenant dans le terme d'interaction est remplacé par un $\sin(2\,\cdot)$ (en effet on s'intéresse dans ce cas à l'orientation de barres, les angles φ_j sont à regarder modulo π).

En l'abscence de désordre, le modèle devient également un modèle classique de la mécanique statistique, et, comme nous l'expliquons dans la prochaine section, cette propriété est souvent cruciale dans notre argumentation.

Il est à noter que, comme les termes d'interaction et de bruit ne dépendent pas de la position des oscillateurs, le système (1.1.2) est invariant par rotation : si $\{\varphi_j^\omega(t)\}_{j=1\ldots N}$ est une solution de (1.1.2), alors $\{\varphi_j^\omega(t)+\psi\}_{j=1\ldots N}$ est également solution pour tout réel ψ.

1.2 Modèle de Kuramoto sans désordre

Dans cette partie nous étudions le modèle de Kuramoto (1.1.2) dans le cas où $\mu = \delta_0$. Dans ce cas l'exposant ω des variables φ_i^ω est inutile. L'équation (1.1.2) devient alors

$$\mathrm{d}\varphi_j(t) \;=\; -\frac{K}{N}\sum_{i=1}^{N}\sin(\varphi_j(t)-\varphi_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t)\,. \tag{1.2.1}$$

1.2.1 Réversibilité

Un aspect fondamental du modèle de Kuramoto, sur lequel repose ce travail, est sa réversibilité en l'absence de désordre. Définissons la mesure de Gibbs sur $\mathbb{S}^N$

$$\pi_{N,K}(\,\mathrm{d}\varphi) \;=\; \frac{1}{Z_{N,K}} \exp\left(-\frac{2K}{\sigma^2} H_N(\varphi)\right) \lambda_N(\,\mathrm{d}\varphi)\,, \tag{1.2.2}$$

où λ_N est la mesure uniforme sur $\mathbb{S}^N$, H_N est l'Hamiltonien

$$H_N(\varphi) \;=\; -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \cos(\varphi_i - \varphi_j)\,, \tag{1.2.3}$$

et

$$Z_{N,K} \;=\; \int_{\mathbb{S}^N} \exp\left(-\frac{2K}{\sigma^2} H_N(\varphi)\right) \lambda_N(\,\mathrm{d}\varphi)\,. \tag{1.2.4}$$

La mesure $\pi_{K,N}$ est celle du modèle classique de spins XY en champs moyen, aussi appelé modèle d'oscillateurs plans en champ moyen [102, 82]. La mesure $\pi_{N,K}$ est stationnaire et réversible pour (1.2.1). En effet (voir [9] pour plus de détails) le générateur $L_{K,N}$ du processus Markovien défini par (1.2.1) qui est défini pour toute fonction $f : \mathbb{S}^N \to \mathbb{R}$ par

$$L_{K,N} f \;=\; \frac{\sigma^2}{2}\sum_{i=1}^{N} \frac{\partial^2}{\partial \varphi_i^2} f(\varphi) - K \sum_{i=1}^{N} \frac{\partial}{\partial \varphi_i} H_N(\varphi) \frac{\partial}{\partial \varphi_i} f(\varphi) \tag{1.2.5}$$

satisfait la propriété de symétrie

$$\int_{\mathbb{S}^N} f L_{K,N} g \,\mathrm{d}\pi_{K,N} \;=\; \int_{\mathbb{S}^N} g L_{K,N} f \,\mathrm{d}\pi_{K,N}\,. \tag{1.2.6}$$

L'invariance par rotation de (1.1.2) implique l'invariance par rotation de la mesure stationnaire $\pi_{N,K}$:

$$\pi_{N,K}\Theta_\psi \;=\; \pi_{N,K}\,, \tag{1.2.7}$$

où pour tout $\psi \in \mathbb{S}$

$$\Theta_\psi(\varphi)_j \;=\; \varphi_j + \psi \qquad \text{pour tout} \qquad j = 1 \ldots N\,.$$

Bien sûr si le désordre est distribué suivant la loi $\mu = \delta_c$ où $c \neq 0$, (1.1.2) est réversible modulo la translation $\varphi_j^\omega(t) \mapsto \varphi_j^\omega(t) - ct$. Ce sont en fait les seuls cas où le modèle de Kuramoto est réversible (voir [9]). La perte de réversibilité par l'ajout de désordre non trivial rend le modèle beaucoup plus complexe à étudier, et représente un défi en soi.

1.2.2 Limite du nombre infini de particules en temps fini

La convergence du modèle (1.2.1) est un cas particulier de la théorie classique de convergence des modèles de diffusions à interaction non locale de type champ moyen et à coefficient lipschitziens [40, 71, 73, 107]. Pour une preuve dans notre cas particulier, voir [9]. Considérons la mesure empirique $(\nu_{N,t})_{t\in[0,T]}$ associée à (1.2.1), c'est à dire l'élément de $C([0,T], \mathcal{M}_1(\mathbb{S}))$ défini par

$$\nu_{N,t}(\,\mathrm{d}\theta) \;=\; \frac{1}{N}\sum_{j=1}^{N} \delta_{\varphi_j(t)}(\,\mathrm{d}\theta)\,. \tag{1.2.8}$$

Si $\nu_{N,0}$ converge dans $\mathcal{M}_1(\mathbb{S})$ vers une mesure p_0, alors pour tout $T > 0$ la mesure empirique $\nu_{N,t}$ converge dans l'espace $C([0,T], \mathcal{M}_1(\mathbb{S}))$ vers une limite déterministe, qui est l'unique solution de l'équation aux dérivées partielles

$$\partial_t p_t(\theta) \;=\; \frac{1}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta[p_t(\theta) J * p_t(\theta)]\,, \tag{1.2.9}$$

où $*$ est l'opérateur de convolution et $J(\theta) := -K\sin(\theta)$. Ce type d'équation est appelé équation de Fokker-Planck ou équation de McKean-Vlasov. Pour toute condition initiale $p_0 \in \mathcal{M}_1(\mathbb{S})$ il existe une unique solution p_t à (1.2.8), et cette solution est régulière, en fait $p_t \in C^\infty((0,T) \times \mathbb{S}, \mathbb{R})$ et $p_t > 0$ pour tout $t > 0$ (voir [9], la preuve reposant sur les propriétés régularisantes du noyau de la chaleur).

L'invariance par rotation de (1.2.1) implique l'invariance par rotation de la mesure empirique, qui subsiste à la limite. Ainsi, si p_t est solution de (1.2.9), $p_t(\cdot - \alpha)$ l'est également pour tout $\alpha \in \mathbb{R}$.

Ce résultat de convergence implique en particulier qu'il y a propagation du chaos dans ce modèle : si au temps $t = 0$ les phases $\varphi_i(0)$ sont distribuées indépendamment sur le cercle $\mathbb{S}$ ($\nu_{N,0}$ est une mesure produit), alors à la limite $N \to \infty$ les phases $\varphi_i(t)$ demeurent décorrélées. Plus précisément à la limite $N \to \infty$ la dynamique de chaque phase est donnée par le processus

$$\mathrm{d}\varphi_t \;=\; -J * p_t(\varphi_t)\,\mathrm{d}t + \mathrm{d}B_t\,, \tag{1.2.10}$$

où B est un mouvement Brownien et p_t la limite de la mesure empirique (i.e. la solution de (1.2.9)) : l'interaction ne subsiste qu'à travers une moyenne selon la loi p_t du processus limite. Le processus ψ_t donné par (1.2.10) est un processus de Markov non-linéaire : la loi p_t du processus apparaît dans son équation d'évolution.

1.2.3 Solutions stationnaires et synchronisation

Comme les solutions de (1.2.9) sont régulières, il en va de même pour les solutions stationnaires. Les solutions stationnaires de (1.2.9) sont les fonctions éléments de $C^\infty(\mathbb{S}, \mathbb{R})$ solutions de l'équation

$$\frac{1}{2}q''(\theta) - [qJ * q]'(\theta) \;=\; 0\,. \tag{1.2.11}$$

Remarquons que le terme $J * q(\theta)$ est en fait une combinaison linéaire des deux modes de Fourier du premier ordre de q :

$$J * q(\theta) = -K\sin(\theta)\int \cos(\theta')q(\theta')\,\mathrm{d}\theta' + K\cos(\theta)\int \sin(\theta')q(\theta')\,\mathrm{d}\theta'\,, \tag{1.2.12}$$

et rappelons que (1.2.9) est invariante par rotation, donc si q est stationnaire, $q(\cdot - \theta_0)$ est stationnaire pour tout θ_0. On peut donc réduire le problème (1.2.11) en ne recherchant que les solutions stationnaires q vérifiant $\int \sin(\theta')q(\theta')\,\mathrm{d}\theta' = 0$ et $r := \int \cos(\theta')q(\theta')\,\mathrm{d}\theta' \geqslant 0$. On voit facilement que les solutions stationnaires vérifiant ces deux propriétés sont les fonctions de la forme

$$q_0(\theta) \;=\; \frac{\exp(2Kr\cos(\theta))}{\int_{\mathbb{S}} \exp(2Kr\cos(\theta))\,\mathrm{d}\theta} \tag{1.2.13}$$

où r est une solution positive du problème de point fixe

$$r \;=\; \Psi(2Kr), \qquad \text{avec } \Psi(x) := \frac{I_1(x)}{I_0(x)}\,, \tag{1.2.14}$$

où I_0 et I_1 sont les fonctions de Bessel modifiées d'ordre 0 et 1 définies par

$$I_0(x) = \int_{\mathbb{S}} \exp(x\cos(\theta))\,\mathrm{d}\theta\,, \tag{1.2.15}$$

et

$$I_1(x) = \int_{\mathbb{S}} \cos(\theta)\exp(x\cos(\theta))\,\mathrm{d}\theta\,. \tag{1.2.16}$$

La fonction Ψ est strictement concave [82], le nombre de solutions à ce problème de point fixe dépend donc de la dérivée en 0 de la fonction $r \mapsto \Psi(2Kr)$. Comme Ψ vérifie $\Psi'(0) = 1/2$, il y a deux cas de figures, selon si $K \leqslant 1$ ou $K > 1$.

- Si $K \leqslant 1$ le seul point fixe est $r = 0$. La solution stationnaire correspondant est en fait la solution triviale $q_0(\theta) = 1/2\pi$.
- Si $K > 1$ la solution triviale est toujours stationnaire, mais il existe en plus une unique solution strictement positive $r > 0$ au problème de point fixe (1.2.14), associée à une solution stationnaire q_0 non triviale donnée par (1.2.13). Par invariance par rotation, on obtient en fait tout un cercle de solutions stationnaires

$$M := \{q_\psi,\, \psi \in \mathbb{S}\}\,, \tag{1.2.17}$$

où

$$q_\psi(.) := q_0(.-\psi)\,. \tag{1.2.18}$$

On parle de solutions synchronisées pour les solutions stationnaires q_ψ car contrairement à la solution triviale, elle définissent une répartition de particules qui n'est pas uniforme sur le cercle : les particules ont tendance à se concentrer autour de la phase ψ. Il y a donc une transition de phase pour le modèle de Kuramoto sans désordre à $K = 1$. Nous avons caractérisé cette transition de phase du point de vue dynamique, à l'aide des solutions stationnaires du modèle limite. Cette transition est à mettre en relation avec celle observée dans le modèle de spins XY (voir le résultat de [102], en se souvenant du résultat de concavité de [82]). Pour une étude des états stationnaires dans le cadre des modèles de polymères rigides en dimension 3, voir [20].

Dans la littérature portant sur la synchronisation, le modèle de Kuramoto est étudié à l'aide des degré de synchronisation $r_{N,t}$ et centre de synchronisation $\Psi_{N,t}$, définis par

$$r_{N,t}e^{i\Psi_{N,t}} = \frac{1}{N}\sum_{j=1}^{N} e^{i\varphi_j(t)} = \int_{\mathbb{S}} e^{i\theta}\nu_{N,t}(\,\mathrm{d}\theta)\,. \tag{1.2.19}$$

A la limite $N \to \infty$, dans le régime stationnaire, le degré de synchronisation correspond à la solution r du problème de point fixe (1.2.14), et si l'on se trouve dans le cas d'une solution stationnaire non triviale q_ψ, le centre de synchronisation correspond à la phase ψ paramétrant cette solution.

Intéressons nous maintenant à la stabilité des différentes solutions stationnaires que nous avons obtenues. L'opérateur d'évolution linéarisée autour d'une solution stationnaire q est défini par

$$-L_q u(\theta) := \frac{1}{2}u'' - [uJ * q + qJ * u]'\,, \tag{1.2.20}$$

et a pour domaine $\{u \in C^2(\mathbb{S},\mathbb{R}) : \int_{\mathbb{S}} u = 0\}$. Pour la solution stationnaire trivial $q(\theta) = 1/2\pi$, (1.2.20) se réduit à

$$-L_{1/2\pi}u = \frac{1}{2}u'' - \frac{1}{2\pi}[J * u]'\,. \tag{1.2.21}$$

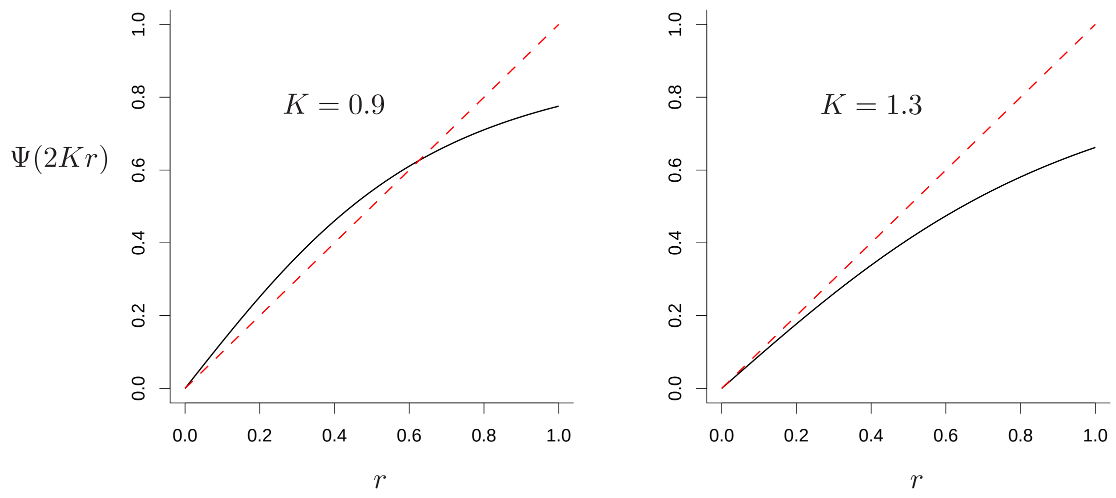


FIGURE 1.1. Comportement de $\Psi(2K\cdot)$ pour $K = 0.9$ et $K = 1.3$.

Cet opérateur a une écriture très simple en termes de coefficients de Fourier : pour une fonction u décomposée sur la base de Fourier comme suit

$$u(\theta) \;=\; \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)\,, \tag{1.2.22}$$

on obtient (dans le cas où la série qui suit converge)

$$L_{1/2\pi}u \;=\; \frac{1}{2}(1-K)a_1\cos(\theta) + \frac{1}{2}(1-K)b_1\sin(\theta) + \frac{1}{2}\sum_{n=2}^{\infty} n^2 a_n \cos(n\theta) + n^2 b_n \sin(n\theta)\,. \tag{1.2.23}$$

On voit facilement que pour $K < 1$ la solution $1/2\pi$ est linéairement stable, alors que pour $K > 1$ elle est linéairement instable sur le sous-espace des fonctions u vérifiant $\int_{\mathbb{S}} u(\theta)e^{i\theta}\,\mathrm{d}\theta = 0$.

Pour $K > 1$ et $q \in M$, l'écriture de l'opérateur L_q en Fourier n'est plus aussi efficace. Cependant l'on peut remarquer que L_q est symétrique dans un espace bien choisi [9], l'espace $H_{-1,1/q}$, un espace de Sobolev muni du poids $1/q$ défini comme suit : étant donnée une fonction régulière $k : \mathbb{S} \to (0,\infty)$ on définit l'espace de Hilbert H_k^{-1} comme la fermeture de l'espace des fonctions régulières de $\mathbb{S}$ vers $\mathbb{R}$ telles que $\int_{\mathbb{S}} u = 0$ muni de la norme (issue d'un produit scalaire) $\|u\|_{-1,k} := \sqrt{\int_{\mathbb{S}} k\mathcal{U}^2}$, où $\mathcal{U} = \mathcal{U}_w$ est la primitive de u vérifiant $\int_{\mathbb{S}} k\mathcal{U} = 0$. On note $\langle\cdot,\cdot\rangle_{-1,k}$ le produit scalaire associé. Pour une définition alternative de ces espaces faisant intervenir la notion d'espace pivot, voir [9]. Lorsque $k(\cdot) \equiv 1$ il s'agit de l'espace de Sobolev classique, que l'on note H^{-1}.

La fermeture de L_q dans $H^{-1}_{1/q}$ (que l'on note aussi L_q) est un opérateur auto-adjoint avec résolvante compacte, et donc a un spectre discret. Plus précisément ce spectre est inclus dans $[0,\infty)$, $L_q q' = 0$ et q' génère le noyau de L_q. L_q admet donc un trou spectral strictement positif que l'on dénotera λ_K (pour une preuve de ces résultats voir [9], où l'on peut de plus trouver une borne explicite pour le trou spectral λ_K). En particulier le cercle M est donc stable linéairement, ce qui entraîne sa stabilité locale (voir [43]).

L'étude des opérateur L_q permet de caractériser la stabilité locale des solutions stationnaires. Il est cependant possible (voir [43]) de donner une description complète de la dynamique :

– Si $K \leqslant 1$, la solution $1/2\pi$ est un attracteur global pour (1.2.9). Dans [43] cette stabilité est donnée en terme d'espaces de Gevrey. De plus l'existence d'un *central manifold*, c'est à dire une variété de dimension finie qui attire de manière exponentielle les trajectoires, est démontrée.
– Pour $K > 1$, pour toute condition initiale p_0 élément de

$$U = \left\{ p \in \mathcal{M}_1(\mathbb{S}) : \int_{\mathbb{S}} \exp(i\theta) p(\,\mathrm{d}\theta) = 0 \right\}, \tag{1.2.24}$$

la solution de (1.2.9) converge vers $1/2\pi$ (en effet dans ce cas l'EDP se réduit à l'équation de la chaleur, puisque le noyau J n'agit que sur les termes de Fourier du premier degré). Au contraire, si $p_0 \notin U$, p_t converge vers un élément de M.

Remarque 1.2.1. *Le modèle de Kuramoto est un modèle de phases issu d'une réduction. Sa limite $N \to \infty$ est un modèle infini dimensionnel qui est à priori très éloigné d'un modèle de phase. En réalité il peut lui aussi, en temps long, être décrit de manière efficace par une phase (voir la figure 1.2).*

Il est à noter que la stabilité locale de M n'implique pas que chaque élément q_ψ de M est stable. En effet une solution de (1.2.9) partant d'un voisinage d'un élément q_ψ de M peut ne pas converger vers q_ψ, mais vers un autre élément $q_{\psi'}$ de M proche de q_ψ.

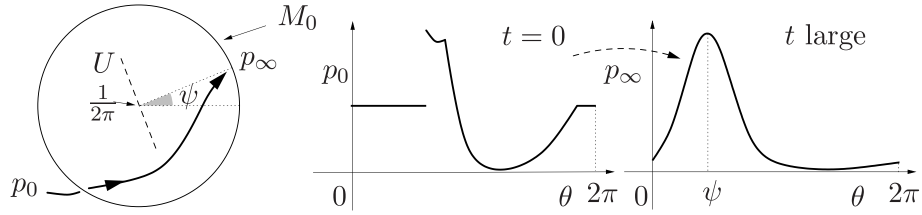


FIGURE 1.2. Comportement en temps long d'une solution de (1.2.9) avec condition initiale $p_0 \notin U$ lorsque $K > 1$.

Il est utile d'étudier les espaces d'interpolation de L_q, c'est-à-dire les espaces de Hilbert V_q^n associés aux normes $\|u\|_{V_q^n} = \|(1 + L_q)^{n/2} u\|_{-1,1/q}$. Plus particulièrement nous définiront la notion d'hyperbolicité normale dans le cadre donné par l'espace V_q^1. Nous montrons dans l'Appendice 2.5 que ces normes d'interpolations sont en fait équivalentes à des normes d'espaces de Sobolev : pour tout n entier $\|\cdot\|_{V_q^n}$ est équivalente à $\|\cdot\|_{H_{n-1}}$, où la norme $\|\cdot\|_{H_{n-1}}$ de l'espace H_{n-1} de Sobolev d'ordre $n-1$ peut être définie à l'aide de coefficients de Fourier de la façon suivante :

$$\|u\|_{H_k}^2 = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} m^{2k} u_m^2 , \tag{1.2.25}$$

où

$$u_m = \int_{\mathbb{S}} u b_m \qquad \text{et} \qquad b_m(\theta) = e^{im\theta/2\pi} . \tag{1.2.26}$$

En particulier la norme $\|.\|_{V_q^1}$ est équivalente à la norme $\|.\|_2$ de l'espace L^2. L'espace V_q^1 est donc homéomorphe à l'espace

$$L_0^2 = \left\{ u \in L^2 : \int u = 0 \right\} . \tag{1.2.27}$$

1.2.4 Comportement en temps long

Le résultat de convergence exposé dans la section 1.2.2 montre que pour un grand nombre d'oscillateurs le système (1.2.1) peut être approximé par l'EDP (1.2.9). La précision de cette approximation est donnée par les corrections dues à la taille finie du système. Les fluctuations de la mesure empirique $\nu_{N,t}$ sont de taille $1/\sqrt{N}$, et décrites à l'aide du processus $\sqrt{N}(\nu_{N,t}(\,\mathrm{d}\theta) - p_t(\theta)\,\mathrm{d}\theta)$. Hitsuda et Mitoma [52], puis Fernandez et Méléard [33] ont montré que ce processus converge vers une processus de type Ornstein-Uhlenbeck. Ce résultat a été démontré par Dai Pra et den Hollander [21] puis Eric Luçon [66] dans le cas de Kuramoto avec désordre (1.1.2), qui, comme on le verra dans la section 1.4, peut être approximé par une famille d'équations aux dérivées partielles indicée par le désordre (en particulier le résultat de [66] est un résultat *quenched*, i.e. à désordre fixé, alors que dans ces domaines les résultats sont souvent montrés en moyennant par rapport au désordre). Ces résultats sont démontrés avec un horizon de temps fini (sur un intervalle de temps $[0, T]$ avec T indépendant de N). Une question naturelle se pose : que se passe-t-il sur des temps plus longs, variant avec la taille N du système ? Ces petites déviations (de l'ordre de $1/\sqrt{N}$) peuvent-elles donner lieux en temps long à des phénomènes macroscopiques ? Sur quelle échelle de temps ces phénomènes apparaissent-ils ?

Cette question est très riche, et dépend fortement de la structure du système limite. Pour visualiser cette dépendance, plaçons nous dans un cadre simple. Considérons le système de $\mathbb{R}^n$ perturbé

$$\mathrm{d}X_t \;=\; -V'(X_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}B_t\,, \tag{1.2.28}$$

où V est un potentiel régulier et B_t un mouvement Brownien. Pour faire une analogie entre ce modèle simple et notre système d'oscillateurs en interaction, considérons des amplitudes de bruit d'ordre $\varepsilon = 1/\sqrt{N}$. Le comportement de (1.2.28) en temps long dépend des propriétés vérifiées par les minimums de V. Si V admet un minimum isolé, admettant un voisinage compact K inclus dans son domaine d'attraction, les trajectoires de (1.2.28) s'échappent de K sur des temps d'ordre $\exp(N)$. Ce type de phénomènes sera décrit plus en détail dans la section 1.5 en dimension finie, en se reposant en particulier sur le travail fondateur de Freidlin et Wentzell [34]. Pour des exemples de ce type de phénomènes en dimension infinie, voir par exemple [75]. Lorsque V comporte un point fixe instable, les trajectoires s'éloignent de ce point sur des temps d'ordre $\log(N)$ (voir par exemple le chapitre 5 de [90] pour des phénomènes de ce type). Le cas qui nous concerne tout particulièrement est celui où V admet tout une courbe stable de solutions stationnaires (c'est la cas de notre système d'oscillateurs lorsque $K > 1$). Dans ce cas, les erreurs dues à la taille finie du système peuvent se propager sans contrainte le long de la courbe. La déviation apparaît si l'on rééchelonne le temps d'un facteur proportionnel à N. Cette déviation macroscopique due aux fluctuations dans cette échelle de temps dans le système (1.2.1) avait déjà mise en évidence numériquement par Pikovsky et Ruffo dans [86] (voir la figure 1.3). Le fait que ces erreurs soient une somme d'erreurs indépendantes et identiquement distribuées entraîne l'apparition d'un mouvement Brownien dans l'échelle de temps adéquate.

Une étude du modèle de Kuramoto en temps long a déjà été effectuée mais dans un cadre différent : Collet et Dai Pra [19] ont étudié les fluctuations du modèle de Kuramoto avec désordre au cas critique. Dans ce cas il y a une unique solution stationnaire, mais l'évolution linéarisée au voisinage de cette solution est sans effet sur un sous-espace (pour plus de détails sur la transition de phase pour le modèle avec désordre, voir la section 1.4). Ils ont montré que dans ce cas l'échelle de temps adaptée est de rééchelonner d'un facteur proportionnel à $\sqrt{N}$. Le phénomène analysé dans [19] est celui des fluctuations non linéaires au cas critique.

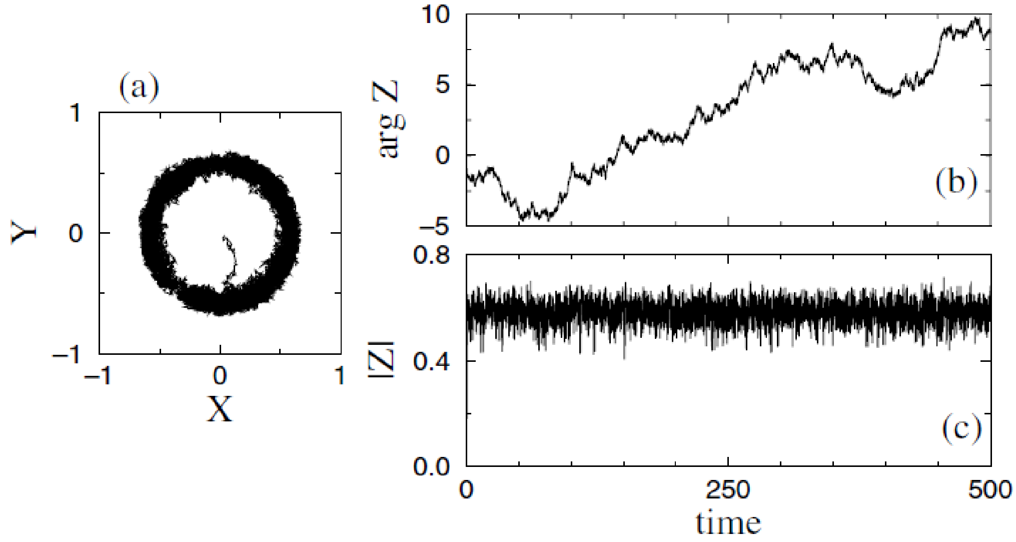


FIGURE 1.3. Simulations effectuées par Pikovsky et Ruffo (figure issue de [86]) dans le cas $N = 500$, $K = 2.5$ et $\sigma = 1/2$. $Z = X + iY$ correspond ici à la variable $r_{N,t}e^{i\Psi_{N,t}}$ introduite dans (1.2.19). Les graphiques a), b) et c) représentent respectivement $r_{N,t}e^{i\Psi_{N,t}}$, $\Psi_{N,t}$ et $r_{N,t}$.

Le type de phénomène auquel nous nous intéressons a déjà été établi dans le contexte des EDPS dans la limite de faible bruit. En particulier dans [15, 36] les auteurs étudient des équations stochastiques de type réaction diffusion munies de potentiels bistables, appelées modèle de Cahn-Allen stochastique et référencé comme *model A* dans la classification de Hohenberg et Halperin [53]. Pour Cahn-Allen stochastique, lorsque la condition initiale est proche du profil reliant les deux phases, la trajectoire de l'interface entre les deux phases est à faible bruit celle d'un mouvement Brownien. Ce type de résultat a été développé dans plusieurs directions, notamment par l'addition d'asymétries [13] qui induisent l'apparition d'une dérive dans la dynamique de l'interface, ou en restreignant le problème dans un domaine borné (la borne dépendant de la taille du bruit), ce qui induit un phénomène de répulsion au niveau du bord [7].

Dans le contexte des systèmes de particules en interaction ce type de phénomène a été étudié dans le cadre de particules Browniennes en dimension d soumises à une interaction locale induite par un potentiel pair [37]. Dans ce modèle des amas structurés de particules se forment, et ces amas se déplacent suivant un mouvement Brownien à basse température. Dans le cas particulier $d = 1$ la diffusion d'amas de particules a également été étudié [38].

Dans notre cas, par rapport à [37, 38], notre résultat est valide pour toute température (tout paramètre d'interaction $K > 1$), mais d'un autre côté l'interaction de notre modèle est plus facile à traiter, de type champ moyen. Même si le modèle que nous considérons est un système de particules, notre approche est proche de celle utilisée dans le cadre des EDPS [15]. La difficulté supplémentaire dans notre cas est que nous étudions une mesure empirique, ce qui nous oblige à nous placer dans un espace de Sobolev d'exposant négatif, et non dans un espace de fonctions continues comme c'est le cas dans [15, 36, 13, 7, 8]. De plus nous munissons ces espaces de Sobolev de poids, pour les adapter à dynamique linéarisée au voisinage de M (c'est-à-dire la structure donnée par les opérateurs L_q définis en (1.2.20)).

Le résultat suivant, qui correspond au théorème 4.1.1 du chapitre 4, a été établi en collaboration avec Lorenzo Bertini et Giambattista Giacomin [10].

Théorème 1.2.2. *Soient une constante τ_f et une mesure de probabilité $p_0 \in \mathcal{M}_1 \setminus U$. Si pour tout $\varepsilon > 0$*

$$\lim_{N\to\infty} \mathbb{P}\left(\|\nu_{N,0} - p_0\|_{-1} \leq \varepsilon\right) = 1\,, \tag{1.2.29}$$

alors il existe une constante ψ_0 qui ne dépend que de $p_0(\cdot)$ et, pour tout N, un processus continu $\{W_{N,\tau}\}_{\tau\geq 0}$ adapté à la filtration générée par la suite $\{W^j_{N\cdot}\}_{j=1,2,\ldots,N}$, tel que $W_{N,\cdot} \in C([0,\tau_f];\mathbb{R})$ converge en loi vers un mouvement Brownien standard et tel que pour tout $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{\tau\in[\varepsilon_N,\tau_f]} \left\|\nu_{N,\tau N} - q_{\psi_0 + D_K W_{N,\tau}}\right\|_{-1} \leq \varepsilon\right) = 1\,, \tag{1.2.30}$$

où $\varepsilon_N := C/N$, $C = C(K, p_0, \varepsilon) > 0$, et

$$D_K := \frac{1}{\sqrt{1 - (I_0(2Kr))^{-2}}}\,, \tag{1.2.31}$$

où I_0 est la fonction de Bessel modifiée d'ordre 0 *(voir* (1.2.15)*) et r la solution strictement positive du problème de point fixe* (1.2.14).

En utilisant la terminologie propre à la littérature portant sur la synchronisation (voir (1.2.19)), ce résultat devient :

Corolaire 1.2.3. *Sous les hypothèses du théorème 1.2.2 le processus $\Psi_{N,N\cdot} \in C([\varepsilon,\tau_f];\mathbb{S})$ converge, pour tout $\varepsilon \in (0,\tau_f]$, vers $(\psi_0 + D_K W_\cdot)\mathrm{mod}(2\pi)$.*

Remarque 1.2.4. *Ce théorème montre que pour N très grand mais fini le modèle peut être décrit de manière efficace, même sur des intervalles de temps très longs, par un modèle (stochastique) de phase.*

1.3 Active Rotators et excitabilité

Cette section introduit les résultats du chapitre 2, qui ont été établis en collaboration avec Giambattista Giacomin, Khashayar Packdaman et Xavier Pellegrin, et ont fait l'objet d'une publication dans le journal SIAM Journal on Mathematical Analysis [44].

1.3.1 Systèmes excitables bruités en interaction

Un système excitable est un système fournissant une réponse très variable suivant l'amplitude des stimulations qu'il subit. Plus précisément ce type de système est caractérisé par un état de repos stable et un état d'excitation. Lorsqu'il n'est pas stimulé le système reste dans son état de repos. Une petite perturbation n'induit qu'une réponse linéaire du système, qui revient rapidement à son état de repos stable. Cependant si l'amplitude de la perturbation qu'il subit dépasse un certain seuil, le système passe d'abord par son état d'excitation avant de revenir à l'état de repos, et ce retour se produit à travers une trajectoire complexe et non linéaire.

Un neurone est un exemple pertinent de système excitable : soumis à une perturbation suffisamment importante, il émet un signal électrique vers les neurones auxquels il est relié (cette émission correspond à la trajectoire non linéaire qu'il suit avant de retourner à son état de repos). Comme l'attestent de nombreuses expériences cette excitabilité est centrale

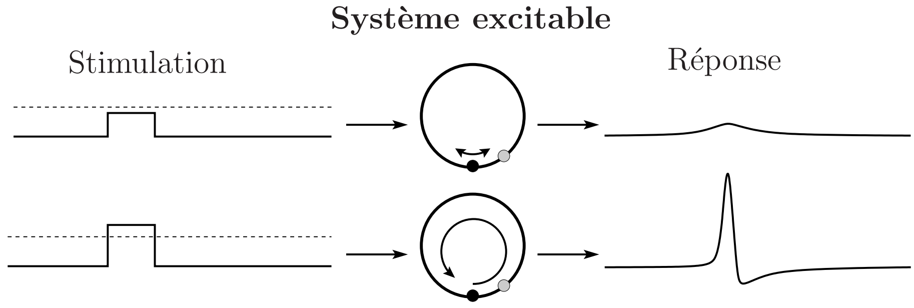


FIGURE 1.4. Réponse d'un système excitable suivant l'amplitude de la stimulation.

dans le mécanisme de transmission des informations des systèmes nerveux. Du point de vue de la modélisation, elle peut être caractérisée en terme de géométrie de diagrammes de phase de systèmes dynamiques [55].

Les systèmes excitables sont particulièrement sensibles au bruit. En effet un signal aléatoire peut contenir, de manière imprévisible, des sections ayant une amplitude dépassant le seuil d'excitabilité du système. Cette association entre aléa et non linéarité peut donner lieu à une grande variété de comportements, en particulier lorsque l'on considère des systèmes excitables bruités mis en interaction [64]. En particulier dans ce type de systèmes des oscillations synchronisées et régulières peuvent être générées par le bruit. C'est sur ce phénomène que nous allons porter notre attention.

Si des unités excitables isolées sont bruitées de manières indépendantes, chaque unité est excitée de manière indépendante et irrégulière. L'ajout d'interactions dans le système induit des cohérences dans ces excitations. Le système est synchronisé lorsque ces excitations coïncident presque parfaitement (la présence de bruit ne permet pas la synchronisation parfaite des différentes unités), le système global se comporte alors comme une seule unité excitable. Les excitations synchronisées peuvent se produire de façon irrégulière, mais un aspect surprenant des systèmes excitables bruités en interaction est que pour un certain choix d'amplitude de bruit et d'interaction, ces excitations se produisent de manière régulière. Cette régularité est optimale pour une certaine valeur non nulle de bruit, on dit alors que le système entre en résonance cohérente. Le rôle de ces deux phénomènes, synchronisation et résonance cohérente, dans la création d'oscillations régulières pour le système global a été mis en lumière dans l'analyse de modèles de neurones élémentaires [47, 84, 85, 91]. Le fait que ces oscillations puissent se produire uniquement sous l'influence du bruit (c'est-à-dire en l'absence de signal périodique extérieur, et d'unités comportant une dynamique périodique intrinsèque) leur donne un rôle centrale dans le système nerveux [59].

Le point marquant de ces oscillations synchronisées est sa généralité : elles ne semblent reposer ni sur des propriétés spécifiques des systèmes excitables, ni sur le type d'interaction mise en jeu. A notre connaissance une des premières mentions de ce phénomène est due à McGregor et Palasek, dans leur étude d'un système d'unités électroniques ayant un comportement analogue à celui d'un neurone (*neuromimes*) [67]. Depuis ce phénomène a été mis en évidence dans dans des modèles neuronaux classiques, comme celui de Hodgkin-Huxley [111], de FitzHugh-Nagumo [109], de Morris-Lecar [47], de Hindmarsh-Rose [112], ou d'autres modèles basés sur les propriétés biophysiques des neurones [59]. Cette énumération, qui n'a pas l'ambition d'être exhaustive, contient des modèles très différents, ayant des types d'interaction diverses : couplages par l'intermédiaire du bruit ou à travers les excitations, selon un réseau aléatoire ou en champ moyen.

La généralité de ces oscillations synchronisées motive l'étude de ce phénomène sur des modèles mathématiques simples, qui n'ont pas l'ambition de modéliser un phénomène biologique ou physique réel, mais de part leur simplicité de permettre l'étude précise des mécanismes mis en jeu. Le modèle que nous allons étudier est celui des *Active Rotators*, qui est une généralisation du modèle de Kuramoto (1.2.1).

Remarque 1.3.1. *Le fait que les systèmes synchronisés se comportent comme une seule unité excitable isolée va nous permettre à nouveau de décrire notre modèle à l'aide d'un modèle de phase. Cette réduction drastique (on décrit un modèle infini-dimensionnel à l'aide d'un modèle uni-dimensionnel) nous permet d'identifier efficacement les cas dans lesquels le modèle global a un comportement périodique.*

1.3.2 Le modèle des Active Rotators

Le modèle des *Active rotators* est une généralisation du modèle de Kuramoto : chaque particule est soumise, en plus de l'interaction et du bruit, à l'influence d'un potentiel V. Nous considérons des fonctions V de $\mathbb{R}$ régulière, avec V' 2π-périodique, mais nous n'imposons pas de condition de périodicité sur V. Plus précisément nous considérons le système d'équations différentielles stochastiques

$$\mathrm{d}\varphi_j(t) \;=\; -\delta V'(\varphi_j(t))\,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N}\sin(\varphi_j(t)-\varphi_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t)\,, \tag{1.3.1}$$

où δ est un paramètre réel. L'ajout de ce potentiel V entraîne la perte de l'invariance par rotation du modèle et, si V n'est pas lui-même périodique, la perte de la propriété de réversibilité. Comme dans le cas réversible, dans la limite $N \to \infty$, la mesure empirique associée à (1.3.1) converge vers la solution d'une EDP de type Fokker Planck, donnée par

$$\partial_t p_t^\delta(\theta) \;=\; \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta[p_t^\delta(\theta) J * p_t^\delta(\theta)] + \delta\partial_\theta[p_t^\delta(\theta)V'(\theta)]\,, \tag{1.3.2}$$

et qui conserve les même propriétés de régularité que dans le cas réversible si V est régulier.

Le potentiel V est choisi de telle manière que le système déterministe isolé (SDI) $\dot{\psi} = -V'(\psi)$ soit excitable. Nous étudierons plus particulièrement le cas où V est donné par le polynôme trigonométrique $V(\psi) = \psi - a\cos(\psi)$. Pour ce choix de potentiel le SDI comporte une bifurcation de type point selle en $|a| = 1$: si $|a| < 1$ la dynamique de le SDI est un mouvement périodique, alors que pour $|a| > 1$ le SDI comporte deux points fixes, un stable et l'autre instable.

Notre but est d'étudier le comportement du système dynamique induit par (1.3.2), et en particulier de rechercher les valeurs de paramètres pour lequel il admet des solutions périodiques. Comme nous allons le voir, le modèle des *Active Rotators* illustre bien le fait que des oscillations synchronisées peuvent être induites par le bruit (en l'absence de périodicité intrinsèque) : il est possible de trouver des valeurs de paramètres du système telles que le système global (1.3.2) ait un comportement périodique tandis que le SDI suit une dynamique de point fixe. Plus précisément, pour le choix de potentiel $V(\psi) = \psi - a\cos(\psi)$, il existe un intervalle $(1, a_0)$ tel que pour tout $a \in (1, a_0)$ et tout paramètre d'interaction K appartenant à un intervalle (K_-, K_+) (dépendant de a), (1.3.2) admet une solution périodique stable. Comme nous l'avons déjà fait remarquer, dans ce cas le SDI comporte deux points fixes. D'autre part, il est possible d'obtenir le phénomène inverse :

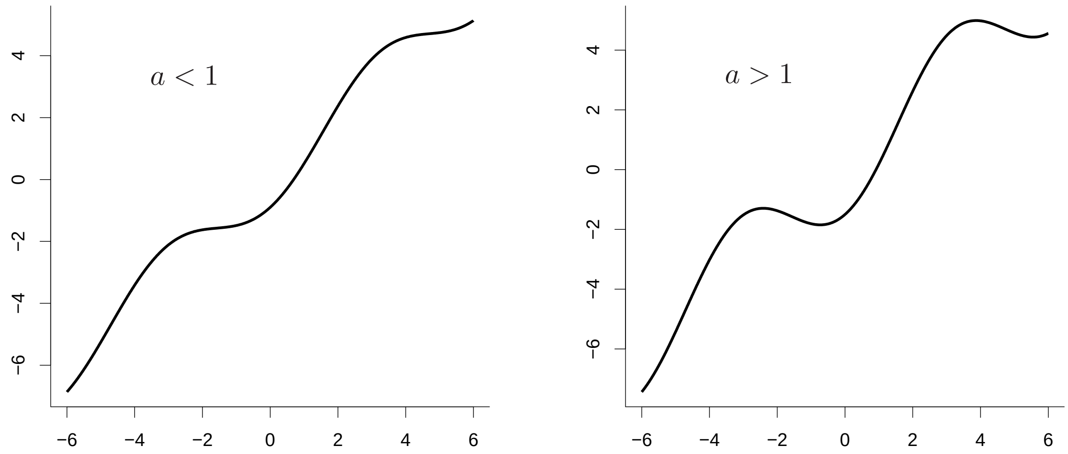


FIGURE 1.5. Potentiel $V(\psi) = \psi - a\cos(\psi)$ pour $a < 1$ et $a > 1$.

pour certaines valeurs de paramètres le SDI peut avoir un mouvement périodique alors que le système global a un comportement de point fixe.

Ces phénomènes étaient dans une certaine mesure déjà connus. En particulier, dans leur étude numérique des *Active Rotators*, Shinomoto et Kuramoto [101] font état d'oscillations couplées induites par le bruit. Ils établissent de plus numériquement le portrait de bifurcations de l'équation de type Fokker Planck (1.3.2). Cette analyse a ensuite été rendu plus précise dans [95]. Dans [63] Kurrer et Schulten ont approximé les solutions de l'équation de Fokker Planck (1.3.2) par des distributions gaussiennes. Ils peuvent ainsi obtenir des équations fermées pour la moyenne et la variance des solutions approximées, et donc d'itentifier les paramètres pour lesquelles la dynamique est périodique. Des travaux similaires été effectués, sur le modèle muni d'un nombre fini N de particules dans [48, 74], et pour d'autres modèles similaires dans [79, 80, 100]. Cette idée de recherche d'une équation fermée pour des grandeurs caractéristiques du système est également présente dans [49], ou He, Le Bris et Lelievre étudient un modèle de polymères rigide avec interaction de type Maier-Saupe et réorientation (due au gradient de vitesse du fluide dans lequel les polymères évoluent). En approximant leur modèle (en appliquant une fermeture dite de type Doi [27]), ils obtiennent une équations fermée pour le tenseur de conformation du système, qui dans certains cas admet une solution périodique.

D'un point de vue mathématique ces phénomènes sont seulement partiellement étudiés. Dans cet esprit de recherche de solutions périodiques pour des processus de Markov non linéaires, on peut citer les travaux [93, 96, 97, 98, 110]. Dans [96] Scheutzow considère un processus de Markov non-linéaire admettant une solution périodique, alors qu'en absence de bruit le système a un comportement de point fixe stable. Cependant le système qu'il considère est très particulier, dans le sens qu'il admet des solutions gaussiennes. L'étude de la périodicité de ces solutions particulières ne nécessite que l'étude de leur espérance et de leur variance, ce qui réduit le problème à un système de dimension 2. C'est également l'approche utilisée dans [110], où Touboul Hermann et Faugeras étudient la présence de solutions périodiques pour des processus de Markov non linéaires modélisant des neurones, et ayant la propriété de conserver le caractère gaussien des solutions. Cette propriété vient du fait que dans leur cas, le potentiel auquel est soumise chaque particule est linéaire. Dans notre cas, ce potentiel est non-linéaire, et le caractère gaussien des solutions n'est pas conservé par la dynamique. Dans [93] Rybko Shlosman et Vladimirov étudient un modèle de serveurs connectés en réseau qui se comporte de manière synchronisée et périodique dans la limite de volume infini lorsque le taux d'occupation des serveurs est suffisant.

Dans ce modèle l'interaction se produit à travers le taux d'occupation de chaque serveur. Dai Pra Fischer et Regoli démontrent l'apparition d'un comportement périodique dans la limite thermodynamique et à faible température dans un modèle de Curie-Weiss avec dissipation [22]. Dans leur cas c'est l'ajout de dissipation dans l'interaction qui entraîne l'apparition du comportement périodique.

Pour étudier le comportement du système dynamique associé à (1.3.2) nous utilisons une méthode de perturbation. Nous nous appuyons sur les propriétés structurelles de l'équation de Fokker Planck avec $\delta = 0$, c'est-à-dire (1.2.9), en particulier sur l'existence du cercle de solution stationnaire M et de sa stabilité linéaire (voir la section 1.2.3). Ce cercle M est en fait une sous-variété normalement hyperbolique stable, qui est une structure stable sous perturbation (voir les sections 1.3.3 et 1.3.4). Pour $\delta \neq 0$ suffisamment petit, (1.3.2) admet également une courbe fermée stable M^δ, qui est une déformation régulière de M. Cela montre que dans ce cas le système globale a le comportement d'un unique oscillateur isolé, dont la dynamique est celle de la dynamique restreinte sur M^δ. On peut ainsi réduire le système infini-dimensionnel (1.3.2) à un système de phases sur M_δ. L'étude cette dynamique restreinte (plus précisément on effectue un développement limité de cette dynamique) permet de déterminer l'existence d'une solution périodique pour (1.3.2).

1.3.3 Sous-variétés normalement hyperboliques stables

Dans cette section nous rappelons la notion de sous-variété normalement hyperbolique stable. Notre but est d'appliquer cette notion aux systèmes dynamique donnés par des équations du type (1.2.9) ou (1.3.2). Nous faisons le choix de nous placer dans la structure donnée par l'espace d'interpolation V_q^1 de L_q (voir section 1.2.3), qui comme nous l'avons vu a une topologie équivalente à celle de l'espace L_0^2. Nous étudions donc les solutions de (1.2.9) et (1.3.2) dans l'espace (rappelons que nous nous intéressons à des probabilités, et donc des solutions de masse égale à 1)

$$L_1^2 = \left\{ u \in L^2 : \int u = 1 \right\} . \tag{1.3.3}$$

Le résultat d'existence de solutions de la section 1.2.2 implique en particulier que pour tout condition initiale $u_0 \in L_1^2$, il existe une unique solution à (1.2.9) et (1.3.2) dans L_1^2.

De manière générale considérons un semi-groupe dans L_1^2 qui définit pour tout $u \in L_1^2$ une trajectoire $\{u_t\}_{t \geqslant 0}$ de condition initiale $u_0 = u$. On peut y associer un semi-groupe linéaire $\{\Phi(u,t)\}_{t \geqslant 0}$ dans L_1^2 satisfaisant $\partial_t \Phi(u,t)v = A(t)\Phi(u,t)v$ et $\Phi(u,0)v = v$, où $A(t)$ est l'opérateur linéaire obtenu en linéarisant l'évolution autour de la trajectoire u_t.

Une sous-variété normalement hyperbolique stable de caractéristiques λ_1, λ_2 et $C > 0$ ($0 \leqslant \lambda_1 < \lambda_2$) est un compact $M \subset L_1^2$ connexe invariant pour la dynamique tel qu'il existe pour tout $u \in M$ une projection $P^o(u)$ sur l'espace tangent de M en u (noté $\mathcal{T}_u M$) satisfaisant pour tout $v \in L_0^2$ les propriétés suivantes (rappelons que $\|\cdot\|_2$ désigne la norme de l'espace L^2) :

1. pour tout $t \geq 0$ on a

$$\Phi(u,t)P^o(u_0)v = P^o(u_t)\Phi(u,t)v \,, \tag{1.3.4}$$

2. on a

$$\|\Phi(u,t)P^o(u_0)v\|_2 \leq C \exp(\lambda_1 t)\|v\|_2 \,, \tag{1.3.5}$$

et, pour $P^s := 1 - P^o$, on a

$$\|\Phi(u,t)P^s(u_0)v\|_2 \,\leq\, C\exp(-\lambda_2 t)\|v\|_2\,, \tag{1.3.6}$$

pour tout $t \geq 0$;

3. il existe un prolongement sur les temps négatif de la trajectoire $\{u_t\}_{t\leq 0}$ et du semi-groupe linéaire $\{\Phi(u,t)P^o(u_0)v\}_{t\leq 0}$ et pour chacun de ces prolongements on a

$$\|\Phi(u,t)P^o(u_0)v\|_2 \,\leq\, C\exp(-\lambda_1 t)\|v\|_2\,, \tag{1.3.7}$$

pour tout $t \leq 0$.

Montrons que lorsque $K > 1$ le cercle de solutions stationnaires M de l'équation de Fokker Planck associée au modèle de Kuramoto réversible (1.2.11) est bien une sous-variété normalement hyperbolique stable. M est bien compact (un cercle), est C^∞, et est invariant pour la dynamique (1.2.11) puisque constitué de solutions stationnaires. Son espace tangent à chaque point q est donné par $\mathcal{T}_q M = \{aq' \,:\, a \in \mathbb{R}\}$. Considérons la projection P^o_q sur $\mathcal{T}_q M$ définie par (voir la section 1.2.3 pour la définition du produit scalaire $\langle\cdot,\cdot\rangle_{-1,1/q}$)

$$P^o_q v \,:=\, \frac{\langle v, q'\rangle_{-1,1/q}}{\langle q', q'\rangle_{-1,1/q}}\,. \tag{1.3.8}$$

Comme les points q sont stationnaires, le semi-groupe $\Phi(q,t)$ est en fait $\exp(-tL_q)$. La condition de commutativité 1) ci-dessus est donc vérifiée de façon triviale, et l'identité $L_q q' = 0$ implique que 2) et 4) sont vérifiés en prenant par exemple $\lambda_1 = 0$. D'autre part pour tout élément $v \in \mathcal{R}(L_q)$, l'existence du trou spectral λ_K pour L_q implique

$$\|v\|^2_{V^1_q} \,=\, \langle(1+L_q)v, v\rangle_{-1,1/q} \,\leqslant\, \left(1+\frac{1}{\lambda_K}\right)\|L_q^{1/2}v\|^2\,. \tag{1.3.9}$$

Ainsi, l'équivalence des normes $\|.\|_2$ et $\|.\|_{V^1_q}$ et le trou spectral entraînent la majoration pour tout $v \in L^2_0$

$$\begin{aligned}\|\Phi(q,t)P^s_q v\|_2 \,\leqslant\, C\|L_q^{1/2}\exp(-tL_q)P^s_q v\|_{-1,1/q} \,\leqslant\, C\exp(-\lambda_K t)\|L_q^{1/2}P^s_q v\|_{-1,1/q}\\ \leqslant\, C'\exp(-\lambda_K t)\|v\|_2\,,\end{aligned} \tag{1.3.10}$$

ce qui implique l'assertion 2) ci-dessus.

Nous définirons dans la section 1.4.4 la notion de sous-variété hyperbolique dans des espaces plus grands, adaptés à l'évolution du modèle de Kuramoto avec désordre (1.1.2), faisant intervenir la loi du désordre μ. Dans la section 1.5.3, nous verrons que pour des systèmes dynamiques de $\mathbb{R}^n$, l'hyperbolicité normale peut être caractérisée par des exposants de Lyapunov, et ne nécessite pas la définition d'une projection particulière P^o_q sur l'espace tangent, toutes les normes étant équivalentes dans $\mathbb{R}^n$.

1.3.4 Problème général et persistance des sous-variétés normalement hyperboliques stables

Dans cette section nous énonçons le théorème classique de persistance des sous-variétés normalement hyperboliques stables. Ce théorème a été démontré dans le cas de la dimension finie par Fenichel [32], et en dimension infinie par Hirsch Pugh et Shub [51], Bates Lu et Zeng [5], Sell et You [99]. Nous nous basons sur le théorème de persistance de [99], qui est adapté à l'étude des équations aux dérivées partielles, en portant une plus grande attention à la dépendance en la taille de la perturbation. Le théorème que nous énonçons, qui correspond au théorème 2.2.1 du Chapitre 2, est valable dans un contexte plus général que celui des *Active Rotators*. Plus précisément nous considérons les modèles donnés par les équation aux dérivées partielles du type

$$\partial_t p_t^\delta(\theta) \;=\; \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta[p_t^\delta(\theta) J * p_t^\delta(\theta)] + \delta G[p_t^\delta](\theta)\,, \tag{1.3.11}$$

où G est une application de L^2_1 dans H^{-1}, et telle qu'il existe un $\eta > 0$ tel que $G \in C^1(N^\eta, H_1)$ où N^η est le voisinage de M des point $p \in L^2_1$ situés à distance L^2 inférieure à η de M, et telle que la différentielle DG est uniformément bornée sur ce voisinage N^η. Ces propriétés sont vérifiées par l'application G des *Active Rotators* $p \mapsto (pV)'$, mais également par exemple par des applications G du type

$$G[p](\theta) \;=\; \partial_\theta[p(\theta)\widetilde{J} * p(\theta)]\,, \tag{1.3.12}$$

où $\widetilde{J} \in L^\infty$, ou

$$G[p](\theta) \;=\; \partial_\theta \left[p(\theta)\int h(\theta,\theta')p(\theta')\right]\,, \tag{1.3.13}$$

où $h \in L^\infty$.

Théorème 1.3.2. *Pour tout $K > 1$ il existe $\delta_0 > 0$ tel que si $\delta \in [0, \delta_0]$ il existe une sous-variété normalement hyperbolique stable M_δ de L^2_1 pour l'équation* (1.3.11)*. De plus on peut définir M_δ de la façon suivante :*

$$M_\delta \;=\; \{q_\psi + \phi_\delta\,(q_\psi) :\, \psi \in \mathbb{S}\}\,, \tag{1.3.14}$$

où l'application $\phi_\delta \in C^1(M, L^2_0)$ vérifie les propriétés suivantes :
- *$\phi_\delta(q) \in \mathcal{R}(L_q)$;*
- *Il existe $C > 0$ tel que* $\sup_\psi(\|\phi_\delta\,(q_\psi)\,\|_2 + \|\partial_\psi\phi_\delta(q_\psi)\|_2) \leq C\delta$.

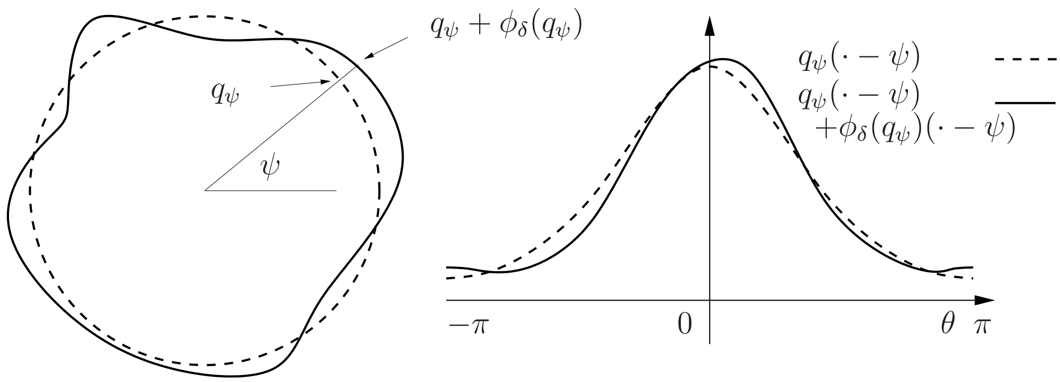


FIGURE 1.6. La courbe M est modifiée de manière régulière par la perturbation δG en une courbe M_δ invariante pour (1.3.11).

1.3.5 Dynamique de phases sur M_δ

Le théorème 1.3.2 montre que pour tout $K > 1$ et δ suffisamment petit, (1.3.11) admet une courbe stable M_δ, La caractérisation de M_δ (1.3.14) montre de plus que la dynamique restreinte sur M_δ peut être décrite par l'évolution d'une phase ψ sur $\mathbb{S}$. Ce phénomène illustre le caractère synchronisé de la dynamique. L'équation satisfaite par cette phase n'est pas explicite, mais il est possible d'en effectuer un développement limité, exposé dans le théorème suivant. Dans ce théorème nous effectuons également un développement limité de l'application ϕ qui définit la distance séparant les courbes M et M_δ (voir (1.3.14)). Cette approximation est utile dans la suite, en particulier pour justifier du fait que la dynamique sur $\mathbb{S}$ donnée par l'approximation (1.3.15) est proche de la dynamique réelle sur M_δ. Ce théorème correspond au théorème 2.2.2 du chapitre 2.

Théorème 1.3.3. *Pour tout $\delta \in [0, \delta_0]$ l'application $t \mapsto \psi_t^\delta$ est C^1 et*

$$\dot{\psi}_t^\delta + \delta \frac{\left\langle G[q_{\psi_t^\delta}], q'_{\psi_t^\delta} \right\rangle_{-1,1/q_{\psi_t^\delta}}}{\langle q', q' \rangle_{-1,1/q}} = O(\delta^2)\,, \tag{1.3.15}$$

où le $O(\delta^2)$ est uniforme en temps. De plus si l'on note n_ψ l'unique solution de

$$L_{q_\psi} n_\psi = G[q_\psi] - \frac{\langle G[q_\psi], q'_\psi \rangle_{-1,1/q_\psi}}{\langle q', q' \rangle_{-1,1/q}} q'_\psi \qquad \text{et} \qquad \langle n_\psi, q'_\psi \rangle_{-1,1/q_\psi} = 0\,, \tag{1.3.16}$$

on a

$$\sup_\psi \|\phi_\delta(q_\psi) - \delta n_\psi\|_{H_1} = O(\delta^2)\,. \tag{1.3.17}$$

Remarquons que l'invariance par rotation du modèle (1.2.9) implique que le produit scalaire $\langle q', q' \rangle_{-1,1/q}$ ne dépend pas de l'élément $q \in M$ choisi. Dans le cas où G est donné par un potentiel, c'est-à-dire $G[p] = (pV')'$, le caractère impair de la vitesse en fonction de δ implique que le $O(\delta^2)$ de l'équation (1.3.15) est en fait un $O(\delta^3)$.

Notre but est de montrer que la dynamique sur $\mathbb{S}$ donnée par les termes d'ordre 1 de (1.3.15), c'est-à-dire donnée par l'équation

$$\dot{\psi}_t^\delta = -\delta \frac{\left\langle G[q_{\psi_t^\delta}], q'_{\psi_t^\delta} \right\rangle_{-1,1/q_{\psi_t^\delta}}}{\langle q', q' \rangle_{-1,1/q}}\,, \tag{1.3.18}$$

est une bonne approximation de la dynamique de phase sur M_δ donnée par (1.3.11), et en particulier qu'elle conserve ses propriétés topologiques.

Nous définissons une fonction $f \in C^1(\mathbb{S}, \mathbb{R})$ comme générique ou hyperbolique si elle comporte un nombre fini de zéros et que ces zéros sont simples, c'est-à-dire que si $f(\psi) = 0$, alors $f'(\psi) \neq 0$. Le sous-ensemble des fonctions génériques est ouvert et dense dans $C^1(\mathbb{S}, \mathbb{R})$. Si deux fonctions génériques f et g sont suffisamment proches, ($\|f - g\|_{C^1} \leqslant \varepsilon$ pour un ε suffisamment petit, dépendant de f et g), alors les systèmes dynamiques associés à $\dot{\psi} = -f(\psi)$ et $\dot{\varphi} = -g(\varphi)$ sont topologiquement équivalents. Cela signifie que si $\psi(\psi_0, .)$ et $\varphi(\varphi_0, .)$ sont les flots associés à ces systèmes dynamiques, c'est-à-dire vérifiant $\partial_t \psi(\psi_0, t) = -f(\psi(\psi_0, t))$ et $\psi(\psi_0, 0) = \psi_0$, $\partial_t \varphi(\varphi_0, t) = -g(\varphi(\varphi_0, t))$ et $\varphi(\varphi_0, 0) = \varphi_0$, alors il existe un homéomorphisme $h : \mathbb{S} \to \mathbb{S}$ tel que les ensembles $\{h(\psi(\psi_0, t) : t \in \mathbb{R}\}$ et $\{\varphi(h(\psi_0), t) : t \in \mathbb{R}\}$ coïncident. De plus cet homéomorphisme h conserve le sens de la dynamique : pour tout ψ_0 il existe $T_0 > 0$ tel que pour $t \in [0, T_0)$ et $|s| \in [0, T_0)$, si $\psi(\psi_0, t) \neq \psi_0$ et $h(\psi(\psi_0, t)) = \varphi(h(\psi_0), s)$, alors $s > 0$.

Dans notre cas, si l'on définit

$$f(\psi) = \frac{\langle G[q_\psi], q'_\psi \rangle_{-1,1/q_\psi}}{\langle q', q' \rangle_{-1,1/q}}, \tag{1.3.19}$$

alors la dynamique de phase sur M_δ donnée par (1.3.11) renormalisée en temps par un facteur δ^{-1} peut s'écrire

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi^\delta_{t/\delta} = -f(\psi^\delta_{t/\delta}) + \frac{1}{\delta} R_\delta(\psi^\delta_{t/\delta}), \tag{1.3.20}$$

où

$$R_\delta(\psi) := \frac{\left\langle [\phi_\delta(q_\psi) J * \phi_\delta(q_\psi)]' + \delta \left(G\left[q_\psi + \phi_\delta(q_\psi)\right] - G\left[q_\psi\right] \right) \,,\, q'_\psi \right\rangle_{-1,1/q_\psi}}{\langle q', q' \rangle_{-1,1/q}}. \tag{1.3.21}$$

La régularité de G et ϕ_δ assure le caractère C^1 de f et R_ψ. Un contrôle suffisant du reste R_δ permet donc de montrer que la dynamique de phase sur M_δ donnée par (1.3.11) et la dynamique induite par f sont à une renormalisation en temps près topologiquement équivalente. Le théorème 1.3.3 assure déjà que $\|R_\delta\|_\infty = O(\delta^2)$. Le théorème suivant, qui correspond au théorème 2.2.3 du chapitre 2 assure un contrôle suffisant de la dérivée de R_ψ. Les deux dynamiques sont donc topologiquement équivalentes.

Théorème 1.3.4. *Il existe $\delta \mapsto \ell(\delta)$, avec $\ell(\delta) = o(1)$ lorsque $\delta \searrow 0$, tel que*

$$\sup_{\psi \in \mathbb{S}} |R'_\delta(\psi)| \leq \delta\, \ell(\delta). \tag{1.3.22}$$

1.3.6 Périodicité induite par le bruit

Dans cette section nous étudions, lorsque $K > 1$ et δ est suffisamment petit, la dynamique induite par (1.3.11) sur la courbe M_δ dans le cas particulier $G[p] = \partial_\theta(pV')$ où $V(\psi) = \psi - a\cos(\psi)$. Rappelons que dans ce cas l'équation (1.3.11) correspond à la limite $N \to \infty$ du système de N particules en interaction (1.3.1). Dans ce cas l'on peut obtenir une formule explicite pour la fonction f donnée par (1.3.19). Commençons par calculer explicitement le produit scalaire $\langle q', q' \rangle_{-1,1/q}$. Pour cela nous avons besoin de connaître la primitive $\mathcal{Q}$ de q' vérifiant $\int \mathcal{Q}/q = 0$. Celle-ci est de la forme $q + c$ et un calcul direct donne $c = -1/2\pi I_0^2(2Kr)$, où I_0 est la fonction de Bessel modifiée d'ordre 0 (voir (1.2.15)), et r est la solution strictement positive du problème de point fixe (1.2.14).

Remarquons ensuite que pour le calcul du produit scalaire $\langle u, v \rangle_{-1,1/q}$ de deux fonctions u et v, si l'on connaît la primitive $\mathcal{U}$ de u vérifiant $\int \mathcal{U}/q = 0$, alors, puisque pour toute constante α on a $\int \alpha\mathcal{U}/q = 0$, n'importe qu'elle primitive de v suffit. Pour toute primitive $\mathcal{V}$ de v on a bien $\langle u, v \rangle_{-1,1/q} = \int \mathcal{U}\mathcal{V}/q$. Nous pouvons maintenant calculer le produit scalaire $\langle q', q' \rangle_{-1,1/q}$:

$$\langle q', q' \rangle_{-1,1/q} = \int \frac{1}{q} q \left(q - \frac{1}{2\pi I_0^2(2Kr)} \right) = 1 - \frac{1}{I_0^2(2Kr)}. \tag{1.3.23}$$

Procédons de la même façons pour calculer $\langle G[q_\psi], q'_\psi \rangle_{-1,1/q_\psi}$. On a

$$\begin{aligned} \langle G[q_\psi], q'_\psi \rangle_{-1,1/q_\psi} &= \int_{\mathbb{S}} \frac{1}{q_\psi(\theta)} q_\psi(\theta)(1 + a\sin\theta) \left(q_\psi(\theta) - \frac{1}{2\pi I_0^2(2Kr)} \right) \mathrm{d}\theta \\ &= 1 - \frac{1}{I_0^2(2Kr)} + a \int_{\mathbb{S}} q_\psi(\theta) \sin\theta \, \mathrm{d}\theta. \end{aligned} \tag{1.3.24}$$

En utilisant la formule trigonométrique $\sin\theta = \cos\psi \sin(\theta - \psi) - \sin\psi \cos(\theta - \psi)$ et en se rappelant de la définition de q_ψ (voir (1.2.13) et (1.2.18)), on obtient

$$\langle G[q_\psi], q'_\psi \rangle_{-1,1/q_\psi} = 1 - \frac{1}{I_0^2(2Kr)} + a\frac{I_1(2Kr)}{I_0(2Kr)} \sin\psi \,, \tag{1.3.25}$$

où I_1 est est la fonction de Bessel modifiée d'ordre 1 (voir (1.2.16)). Finalement on obtient la formule explicite pour f suivante :

$$f(\psi) = 1 + \frac{a}{a_c(K)} \sin(\psi) \,, \qquad \text{où}\, a_c(K) = \frac{I_0^2(2Kr) - 1}{I_0(2Kr) I_1(2Kr)} \,. \tag{1.3.26}$$

On peut donc déduire que dans ce cas particulier le système (1.3.11) admet une dynamique périodique sur M_δ si et seulement si $a < a_c(K)$ (rappelons que le système déterministe isolé donné par $\dot{\psi} = -V'(\psi)$ et $V'(\theta) = 1 + a\sin(\theta)$ a une dynamique périodique si et seulement si $a < 1$). La figure 1.7 donne la forme de la fonction $K \mapsto a_c(K)$, et l'on peut faire les observations suivantes (qui sont asymptotiques dans le sens que pour chaque valeur de K on se place dans le cas où δ est suffisamment petit pour que M_δ existe et pour que la dynamique approchée (1.3.18) soit topologiquement équivalente à la dynamique réelle sur M_δ) :

- La fonction $a_c(K)$ atteint un maximum $a_{\max}$ qui vérifie en particulier $a_{\max} > 1$. Si $a > a_{\max}$ la dynamique sur M_δ a deux points fixes, et est similaire à celle du système déterministe isolé.
- Si $a \in (1, a_{\max})$ le problème $a_c(K) = a$ a deux solutions $K_-(a) < K_+(a)$, et pour $K \in (K_-(a), K_+(a))$ la dynamique sur M_δ est périodique, alors que dans ce cas la dynamique du système déterministe isolé est une dynamique de points fixes.
- Si $a < 1$ le problème $a_c(K) = a$ a une seule solution $K(a)$. La dynamique sur M_δ est périodique si $K > K(a)$ et de type point fixe si $K < K(a)$, alors que celle du système déterministe isolé est dans les deux cas périodique.

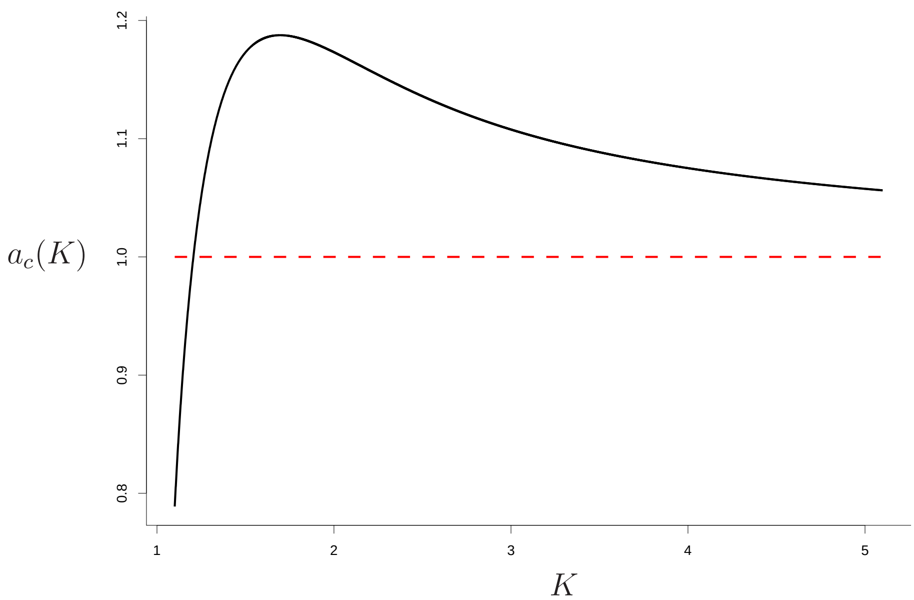


FIGURE 1.7. Tracé de la fonction a_c en fonction de K.

En particulier la deuxième observation montre qu'avec un bon choix de paramètres du modèle, le système de particules (1.3.11) peut à la limite avoir un comportement non trivial très différent de celui d'un oscillateur isolé déterministe subissant l'influence du même potentiel. L'action combinée de bruit et d'interaction peut donc être génératrice de dynamiques complexes.

Dans la cas où la dynamique sur M_δ est périodique l'on peut obtenir une approximation de la période $T_\delta(a, K)$ correspondante :

$$T_\delta(a,K) \,=\, \frac{\tau(a,K)}{\delta} + O(\delta) \qquad \text{où} \qquad \tau(a,K) \,:=\, \frac{2\pi}{\sqrt{1-(a/a_c(K))^2}}\,. \tag{1.3.27}$$

1.4 Modèle de Kuramoto désordonné

Dans cette section nous nous intéressons au modèle de Kuramoto avec désordre (1.1.2). La présence de désordre retire le caractère réversible du modèle, qui est ainsi plus complexe à étudier. Les idées que nous allons suivre dans cette étude sont de la même nature que celles utilisées pour le modèles des *Active Rotators* : nous allons nous placer dans la limite de faible désordre, et ainsi considérer le désordre comme une perturbation du modèle réversible (1.2.1), en s'appuyant sur la structure hyperbolique de ce modèle. Ce cadre nous permet de réduire la dynamique à une dynamique de phases, et d'établir l'existence de solutions périodiques (avec peut-être vitesse nulle). Dans le cas où le désordre est symétrique (cas pour lequel il existe un cercle de solutions stationnaires lorsque K est assez grand) nous étudions dans ce cadre l'évolution linéarisée autour du cercle de solutions stationnaires. Ce travail a été effectué en collaboration avec Giambattista Giacomin et Eric Luçon [42].

1.4.1 Limite du nombre infini de particules

Afin d'étudier la limite $N \to \infty$ du modèle (1.1.2), il est utile de considérer la mesure empirique $(\nu^\omega_{N,t})_{t\in[0,T]} \in C([0,T], \mathcal{M}_1(\mathbb{S}\times\mathbb{R}))$ définie par

$$\nu^\omega_{N,t} \,=\, \sum_{j=1}^N \delta_{\varphi^\omega_j(t),\omega_j}\,. \tag{1.4.1}$$

L'exposant ω désigne la réalisation du désordre $\omega_1, \ldots \omega_N$, et met en évidence la dépendance de la mesure empirique en cette réalisation. De manière similaire au cas réversible, lorsque la condition initiale $\nu_{N,0}$ converge en loi vers une mesure p_0, la mesure empirique converge vers l'unique solution du système d'équations aux dérivées partielles

$$\partial_t p_t(\theta,\omega) \,=\, \frac{1}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta\Big[p_t(\theta,\omega)(\langle J * p_t\rangle_\mu(\theta)+\omega)\Big]\,. \tag{1.4.2}$$

$\langle\cdot\rangle_\mu$ désigne l'intégration par rapport à μ, donc $\langle J{*}u\rangle_\mu(\theta) = \int_{\mathbb{R}}\int_{\mathbb{S}} J(\varphi)u(\theta{-}\varphi,\omega)\,\mathrm{d}\varphi\mu(\,\mathrm{d}\omega)$. Pour une preuve de ce résultat de convergence, voir [21, 66]. En particulier dans [66] ce résultat est obtenu sous l'hypothèse $\int|\omega|\mu(\,\mathrm{d}\omega) < \infty$ et pour toute μ-presque toute réalisation du désordre $\omega_1, \omega_2 \ldots$. Il est à noté que dans (1.4.2), ω désigne un élément de $\mathbb{R}$ inclus dans le support de μ, alors que dans la notation $\nu^\omega_{t,N}$ il désigne la suite $\omega_1, \ldots \omega_N$, mais ce conflit dans les notation ne posera pas de problème dans la suite.

Comme dans le cas réversible, les solutions de (1.4.2) sont régulières (voir l'appendice 3.A du chapitre 3). Plus précisément pour toute condition initiale $p_0 \in \mathcal{M}_1(\mathbb{S}\times\mathbb{R})$, il existe une unique solution $(p_t)_{t\in[0,T]}$ à (1.4.2) qui de plus est strictement positive sur $(0,T]$ et C^∞ en (t,θ).

1.4.2 Désordre symétrique et solutions stationnaires

Lorsque le désordre est symétrique, de façon similaire au cas réversible, les solutions stationnaires de (1.4.2) peuvent être formulées de façon semi-explicite (voir [94, 54]). Ces solutions stationnaire s'écrivent, à une rotation de type $\widetilde{q}_0(\theta-\theta_0,\omega)$ près,

$$\widetilde{q}_0(\theta,\omega) := \frac{S(\theta,\omega,2Kr)}{Z(\omega,2Kr)}, \tag{1.4.3}$$

avec

$$S(\theta,\omega,x) = e^{G(\theta,\omega,x)}\left[(1-e^{4\pi\omega})\int_0^\theta e^{-G(u,\omega,x)}\,\mathrm{d}u + e^{4\pi\omega}\int_0^{2\pi} e^{-G(u,\omega,x)}\,\mathrm{d}u\right], \tag{1.4.4}$$

$$G(u,y,x) = x\cos(u)+2yu, \tag{1.4.5}$$

$$Z(\omega,x) = \int_{\mathbb{S}} S(\theta,\omega,x)\,\mathrm{d}\theta, \tag{1.4.6}$$

et où $r=r(K)$ est solution du problème de point fixe

$$r=\Psi^\mu(2Kr), \qquad \text{avec} \qquad \Psi^\mu(x) = \int_{\mathbb{R}} \frac{\int_{\mathbb{S}}\cos(\theta)S(\theta,\omega,x)\,\mathrm{d}\theta}{Z(\omega,x)}\mu(\,\mathrm{d}\omega). \tag{1.4.7}$$

$r=0$ est toujours solution du problème de point fixe, ce qui signifie que la solution triviale $q(\theta,\omega)=\frac{1}{2\pi}$ est toujours stationnaire. Il s'agit de la seule solution lorsque K est inférieur à une valeur K_c dépendant du désordre μ, qui est majorée par $\widetilde{K}$ défini comme suit :

$$\widetilde{K} := \left(\int_{\mathbb{R}} \frac{\mu(\,\mathrm{d}\omega)}{1+4\omega^2}\right)^{-1}. \tag{1.4.8}$$

Dans le cas $K<K_c$ la solution triviale est stable (voir [103]). Lorsque $K>K_c$, il existe au moins une solution $r>0$ au problème de point fixe, qui est associée, par invariance du modèle par rotation, à un cercle de solutions stationnaires synchronisées. Contrairement au cas réversible, il n'y a pas forcément unicité de solution $r>0$ au problème de point fixes, et les cercles de solutions stationnaires associées, ne sont pas toujours stables. Cette unicité est attendue dans le cas où μ est uni-modale [54, 21], mais non prouvée.

1.4.3 Asymptotique de faible désordre

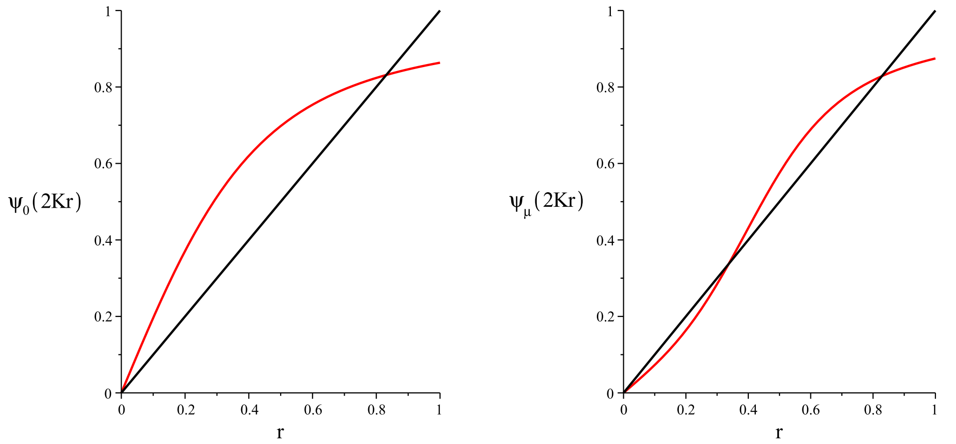
On peut limiter l'étude aux mesure de désordre μ de moyenne $m_\mu := \int_{\mathbb{R}}\omega\mu(\,\mathrm{d}\omega)$ nulle. En effet l'invariance par rotation du modèle permet de toujours se ramener à un désordre centré, en effectuant les translations $\varphi_j^\omega(t)-m_\mu t$ pour $j=1\ldots N$.

Notre but est de nous placer dans un cadre de désordre faible, afin de voir ce désordre comme une perturbation du modèle réversible. Fixons une mesure μ centrée à support dans $[-1,1]$, et considérons maintenant le modèle de Kuramoto dont les fréquences naturelles sont pondérées par un paramètre δ, c'est-à-dire le modèle donné par le système d'équations

$$\mathrm{d}\varphi_j^\omega(t) = \delta\omega_j\,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^N \sin(\varphi_j^\omega(t)-\varphi_i^\omega(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t). \tag{1.4.9}$$

Dans ce cadre, la mesure empirique (1.4.1) converge vers la solution de la famille d'équations aux dérivées partielles

$$\partial_t p_t^\delta(\theta,\omega) = \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta,\omega) - \partial_\theta\Big[p_t^\delta(\theta,\omega)(\langle J * p_t^\delta\rangle_\mu(\theta)+\delta\omega)\Big], \tag{1.4.10}$$



(a) $\Psi^\mu(2K\cdot)$ avec $\mu = \delta_0$, $K = 2$.

(b) $\Psi^\mu(2K\cdot)$ avec $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$, $K = 3.5$.

FIGURE 1.8. Tracé de la fonction $\Psi^\mu(2K\cdot)$ pour deux choix de K et μ. $\Psi^{\delta_0}(\cdot)$ est strictement concave (figure 1.8a) mais cette concavité peut disparaître même pour des choix simples de désordre μ (figure 1.8b), et dans ce cas il peut y avoir plusieurs points fixes non triviaux, chacun correspondant à un cercle de solutions stationnaires.

et dans le cas où la mesure μ est symétrique, le problème de point fixe (1.4.7) devient

$$r_\delta = \Psi^\mu_\delta(2Kr_\delta)\,, \qquad \text{avec} \qquad \Psi^\mu_\delta(x) = \int_{\mathbb{R}} \frac{\int_{\mathbb{S}} \cos(\theta) S(\theta, \delta\omega, x)\, \mathrm{d}\theta}{Z(\delta\omega, x)} \mu(\,\mathrm{d}\omega)\,. \tag{1.4.11}$$

Lorsque δ est suffisamment petit, Ψ^μ_δ est une perturbation régulière de Ψ (voir (1.2.14)), et conserve ainsi le caractère strictement concave de Ψ. Plus précisément, pour tout intervalle du type $[0, K_{max}]$, il existe un $\delta_1 = \delta_1(K_{max}) > 0$ tel que pour tout $\delta < \delta_1$ et $0 < K < K_{max}$, $x \mapsto \Psi^\mu_\delta(2Kx)$ reste strictement concave sur $[0, 1]$ (voir le lemme 3.2.3). Il existe donc dans ce cas une unique solution $r_\delta > 0$ au problème de point fixe (1.4.11), et donc un unique cercle de solutions stationnaires synchronisées pour (1.4.10). Dans la section 1.4.7 nous étudierons la question de la stabilité de ce cercle.

Lorsque μ n'est pas symétrique, notre but est de montrer l'existence d'une solution périodique stable (avec possibilité de vitesse nulle). L'idée principale que nous suivons est de considérer (1.4.10) comme une perturbation de l'équation avec $\delta = 0$

$$\partial_t p^0_t(\theta, \omega) = \frac{1}{2}\partial^2_\theta p^0_t(\theta, \omega) - \partial_\theta \Big[p^0_t(\theta, \omega)(\langle J * p^0_t \rangle_\mu(\theta)) \Big]\,, \tag{1.4.12}$$

que nous appelons EDP avec désordre figé. En effet dans ce modèle chaque oscillateur se voit attribuer une fréquence naturelle selon la loi μ, mais cette fréquence n'a pas de rôle dans la dynamique, qui est celle du modèle de Kuramoto réversible. Les solutions stationnaires de (1.4.12) sont donc celles du modèle réversible, définies à une rotation près par $q_0(\theta, \omega) = q_0(\theta)$ où $q_0(\theta)$ vérifie (1.2.13), et pour $K > 1$ il existe donc un cercle de solutions stationnaires synchronisées

$$M_0 = \{q_\psi : \psi \in \mathbb{S}\}\,, \tag{1.4.13}$$

où pour tout $\psi \in \mathbb{S}$ $q_\psi(\theta, \omega) = q_\psi(\theta)$ (voir (1.2.18) pour la définition de $q_\psi(\cdot)$). Comme $q_\psi(\theta, \omega)$ ne dépend pas de ω, on pourra confondre dans la suite $q_\psi(\theta, \omega)$ et $q_\psi(\theta)$. La stabilité linéaire de M_0 est déterminée par les opérateurs linéaires A_q qui pour tout $q \in M_0$ sont définis par

$$-A_q u(\theta, \omega) = \frac{1}{2}\partial^2_\theta u(\theta, \omega) - \partial_\theta \Big[q(\theta)\langle J * u \rangle_\mu + u(\theta, \omega) J * q(\theta) \Big]\,, \tag{1.4.14}$$

et qui ont pour domaine

$$\mathcal{D}(A) := \Big\{ u : (\mathbb{S}\times\mathbb{R}) \to \mathbb{R} :\ u(\cdot,\omega) \in C^2(\mathbb{S},\mathbb{R}) \text{ et } \int_{\mathbb{S}} u(\theta,\omega)\,\mathrm{d}\theta = 0 \quad \mu\text{-p.p.}, \\ \text{et } \int_{\mathbb{R}} \|u(\cdot,\omega)\|^2_{C^2(\mathbb{S},\mathbb{R})}\mu(\,\mathrm{d}\omega) < \infty \Big\}. \quad (1.4.15)$$

Comme la dynamique de (1.4.12) est celle de (1.2.9), avec néanmoins une répartition des particules suivant leur fréquences naturelles (fréquences qui, rappelons le, n'ont pas d'effet sur la dynamique), il est raisonnable de penser que les propriétés de l'opérateur A_q sont similaires à celles de l'opérateur L_q (voir (1.2.20)).

En effet la fermeture de l'opérateur A_q dans un espace de Sobolev à poids bien choisi est auto-adjointe. Plus précisément, définissons pour toute fonction régulière strictement positive $k : \mathbb{S} \to \mathbb{R}$ l'espace $H^{-1}_{k,\mu}$ défini par la fermeture de l'espace $\mathcal{D}(A)$ pour la norme $\|\cdot\|_{-1,k,\mu}$ issue du produit scalaire

$$\langle u, v\rangle_{-1,k,\mu} = \int_{\mathbb{R}}\int_{\mathbb{S}} k(\theta)\mathcal{U}(\theta,\omega)\mathcal{V}(\theta,\omega)\,\mathrm{d}\theta\mu(\,\mathrm{d}\omega)\,, \quad (1.4.16)$$

où ω p.s. $\mathcal{U}(\cdot,\omega)$ et $\mathcal{V}(\cdot,\omega)$ sont les primitives de u et v satisfaisant $\int_{\mathbb{S}} k(\theta)\mathcal{U}(\theta,\omega)\,\mathrm{d}\theta = 0$ et $\int_{\mathbb{S}} k(\theta)\mathcal{V}(\theta,\omega)\,\mathrm{d}\theta = 0$. Remarquons que

$$\|u\|^2_{-1,k_1,\mu} \leqslant \frac{\|k_1\|_\infty}{\|k_2\|_\infty}\|u\|_{-1,k_2,\mu}\,, \quad (1.4.17)$$

et donc que toutes les normes ainsi introduites sont équivalentes. Lorsque $\mu = \delta_0$ il s'agit exactement des espaces et normes définis dans la section 1.2.3. Dans le cas particulier $k \equiv 1$ nous utiliserons les notations H^{-1}_μ et $\|u\|_{-1,\mu}$ (à ne pas confondre avec la norme $\|u\|_{-1,1/q}$ de la section 1.2.3, qui elle correspond à la norme $\|u\|_{-1,1/q,\delta_0}$). Nous allons utiliser ces normes pondérées avec $k = 1/q$ où $q \in M_0$. Comme pour tout $q \in M_0$, $q(\theta,\omega) = q(\theta)$, on a en fait l'égalité

$$\langle \partial_\theta q, \partial_\theta q\rangle_{-1,1/q,\mu} = \langle q', q'\rangle_{-1,1/q}\,, \quad (1.4.18)$$

où dans le terme de droite on effectue un abus de notation en oubliant la dépendance en ω de q. De manière similaire au cas sans désordre, l'invariance par rotation du modèle avec désordre figé implique que ce produit scalaire ne dépend pas de l'élément $q \in M_0$ choisi.

L'opérateur A_q vérifie la propriété suivante, qui correspond à la proposition 3.2.1 du chapitre 3 :

Proposition 1.4.1. *A_q est essentiellement auto-adjoint dans $H^{-1}_{1/q,\mu}$. De plus son spectre est discret, inclus dans $[0,\infty)$, 0 est une valeur propre simple, dont l'espace propre associé est engendré par $\partial_\theta q$. Il existe de plus un trou spectral $\widetilde{\lambda}_K$ pour A_q, donné par la distance séparant 0 de sa plus petite valeur propre strictement positive.*

La preuve de cette propriété repose sur la preuve du résultat équivalent établi pour L_q dans [9]. En particulier les estimations obtenues sur le trou spectral $\widetilde{\lambda}_K$ de A_q sont les mêmes que celles obtenues dans [9] pour le trou spectral λ_K de L_q.

La proposition 1.4.1 permet de voir M_0 comme une sous-variété normalement hyperbolique, de la même façon que les propriétés de L_q permettaient de voir M comme une telle sous-variété dans la section 1.3.3, mais il faut cette fois-ci considérer des espaces plus grand, pour prendre en compte le désordre.

1.4.4 Hyperbolicité normale

Nous nous plaçons dans l'espace d'interpolation de A_q d'ordre 1, défini par la norme $\|(1+A)^{1/2}u\|_{-1,1/q,\mu}$. Cet espace d'interpolation est homéomorphe à l'espace

$$X_\mu^0 = \left\{ u \in L^2(\lambda \otimes \mu) : \int_{\mathbb{S}} u(\theta,\omega)\,\mathrm{d}\theta = 0 \qquad \omega \text{ p.s.} \right\}, \tag{1.4.19}$$

où λ est la mesure de Lebesgue sur $\mathbb{S}$. Ceci est dû à l'équivalence des normes $\|(1+L_q)^{1/2}u\|_{-1,1/q}$ et $\|u\|_2$ (voir section 1.2.3), qui est conservée sous l'ajout de désordre à travers l'équivalence des normes $\|(1+A)^{1/2}u\|_{-1,1/q,\mu}$ et $\|u\|_{2,\mu}$, où $\|u\|_{2,\mu}$ désigne la norme de l'espace $L^2(\lambda \otimes \mu)$. Comme nous nous intéressons à l'évolution de familles de probabilités indicées par ω, définissons également l'espace

$$X_\mu^1 = \left\{ u \in L^2(\lambda \otimes \mu) : \int_{\mathbb{S}} u(\theta,\omega)\,\mathrm{d}\theta = 1 \qquad \omega \text{ p.s.} \right\}. \tag{1.4.20}$$

Nous définissons une sous-variété normalement hyperbolique stable dans l'espace X_μ^1 de manière similaire à la définition dans L_1^2 (voir section 1.3.3).

Considérons un semi-groupe dans X_μ^1 qui définit pour tout $u \in X_\mu^1$ une trajectoire $\{u_t\}_{t \geqslant 0}$ de condition initiale $u_0 = u$. On peut y associer un semi-groupe linéaire $\{\Phi(u,t)\}_{t \geqslant 0}$ dans X_μ^1 satisfaisant $\partial_t \Phi(u,t)v = B(t)\Phi(u,t)v$ et $\Phi(u,0)v = v$, où $B(t)$ est l'opérateur linéaire obtenu en linéarisant l'évolution autour de la trajectoire u_t.

Une sous-variété normalement hyperbolique stable de caractéristiques λ_1, λ_2 et $C > 0$ ($0 \leqslant \lambda_1 < \lambda_2$) est un compact $M \subset X_\mu^1$ connexe invariant pour la dynamique tel qu'il existe pour tout $u \in M$ une projection $P^o(u)$ sur l'espace tangent de M en u satisfaisant pour tout $v \in X_\mu^0$ les propriétés suivantes :

1. pour tout $t \geq 0$ on a

$$\Phi(u,t)P^o(u_0)v \;=\; P^o(u_t)\Phi(u,t)v\,, \tag{1.4.21}$$

2. on a

$$\|\Phi(u,t)P^o(u_0)v\|_{2,\mu} \;\leq\; C\exp(\lambda_1 t)\|v\|_{2,\mu}\,, \tag{1.4.22}$$

   et, pour $P^s := 1 - P^o$, on a

$$\|\Phi(u,t)P^s(u_0)v\|_{2,\mu} \;\leq\; C\exp(-\lambda_2 t)\|v\|_{2,\mu}\,, \tag{1.4.23}$$

   pour tout $t \geq 0$;

3. il existe un prolongement sur les temps négatifs de la trajectoire $\{u_t\}_{t\leq 0}$ et du semi-groupe linéaire $\{\Phi(u,t)P^o(u_0)v\}_{t\leq 0}$ et pour chacun de ces prolongements on a

$$\|\Phi(u,t)P^o(u_0)v\|_{2,\mu} \;\leq\; C\exp(-\lambda_1 t)\|v\|_{2,\mu}\,, \tag{1.4.24}$$

   pour tout $t \leq 0$.

De manière similaire au cas sans désordre, il est clair que M_0 est une sous-variété normalement hyperbolique stable pour l'évolution (1.4.12), avec pour tout $p \in M_0$ la projection P_q^o définie par

$$P_q^o u \;=\; \frac{\langle u, q'\rangle_{-1,1/q,\mu}}{\langle q', q'\rangle_{-1,1/q}} q'\,. \tag{1.4.25}$$

Le théorème de persistance des sous-variétés normalement hyperboliques stables s'exprime dans ce cadre de la manière suivante (qui correspond au théorème 3.3.1 du chapitre 3) :

Théorème 1.4.2. *Pour tout $K > 1$ il existe $\delta_0 > 0$ tel que si $\delta \in [0, \delta_0]$ il existe un sous-variété normalement hyperbolique stable M_δ de X^1_μ pour l'équation* (1.4.10)*. De plus on peut définir M_δ de la façon suivante :*

$$M_\delta = \{q_\psi + \phi_\delta(q_\psi) : \psi \in \mathbb{S}\} , \tag{1.4.26}$$

où l'application $\phi_\delta \in C^1(M_0, X^0_\mu)$ vérifie les propriétés suivantes :
- *$\phi_\delta(q) \in \mathcal{R}(A_q)$;*
- *il existe $C > 0$ tel que $\sup_\psi(\|\phi_\delta(q_\psi)\|_{2,\mu} + \|\partial_\psi\phi_\delta(q_\psi)\|_{2,\mu}) \leq C\delta$.*

Remarque 1.4.3. *Ce théorème est valable pour des perturbations du modèle avec désordre figé* (1.4.12) *plus générales que* (1.4.10)*, plus précisément du type*

$$\partial_t p^\delta_t(\theta,\omega) = \frac{1}{2}\partial^2_\theta p^\delta_t(\theta,\omega) - \partial_\theta\Big[p^\delta_t(\theta,\omega)(\langle J * p^\delta_t\rangle_\mu(\theta))\Big] + \delta G[p^\delta_t](\theta,\omega) , \tag{1.4.27}$$

où G est un application C^1 d'un voisinage de M_0 dans X^1_μ vers H^{-1}_μ. C'est bien sûr le cas du modèle de Kuramoto avec désordre $G[p] = -\omega\partial_\theta p(\theta,\omega)$, mais également le cas du modèle des Actives Rotators avec désordre donné par $G[p](\theta,\omega) = \partial_\theta(\partial_\theta V(\theta,\omega)p(\theta,\omega))$, où $V(\cdot,\omega)$ est une fonction régulière ω p.s.. La preuve du théorème est la même que celle du cas sans désordre du chapitre 2.

1.4.5 Solutions périodiques

La preuve du théorème 1.4.2 implique de plus que pour tout $\delta \leqslant \delta_0$, M_δ est l'unique sous-variété normalement hyperbolique pour (1.4.10) située au voisinage de M_0. Cette unicité et l'invariance par rotation de (1.4.10) impliquent que M_δ dans ce cas est en fait un cercle, donné par les translations selon θ de $q_0 + \phi_\delta(q_0)$. Notons $\widetilde{q}_0 = q_0 + \phi_\delta(q_0)$. Il existe donc $c_\mu(\delta)$ tel que $\widetilde{q}_0(\theta - c_\mu(\delta)t, \omega)$ est solution de (1.4.10). Si $c_\mu(\delta) = 0$ M_0 est constitué de solutions stationnaires, alors que dans le cas contraire il existe un solution périodique non triviale.

Une méthode de perturbation permet d'obtenir un développement limité de $c_\mu(\delta)$. Ce résultat correspond au théorème 3.2.2 du chapitre 3.

Théorème 1.4.4. *On a*

$$c_\mu(\delta) = \delta^3 \frac{\big\langle \omega\partial_\theta n^{(2)}, q_0'\big\rangle_{-1,1/q_0,\mu}}{\langle q', q'\rangle_{-1,1/q}} + O(\delta^5) , \tag{1.4.28}$$

où $n^{(2)}$ est l'unique solution de

$$A_{q_0} n^{(2)} = -\omega\partial_\theta n^{(1)} \quad \text{et} \quad \big\langle n^{(2)}, q_0'\big\rangle_{-1,1/q_0,\mu} = 0 , \tag{1.4.29}$$

et $n^{(1)}$ est l'unique solution de

$$A_{q_0} n^{(1)} = -\omega\, q_0' \quad \text{et} \quad \big\langle n^{(1)}, q_0'\big\rangle_{-1,1/q_0,\mu} = 0 . \tag{1.4.30}$$

1.4.6 Retour sur les Actives Rotators

Nous nous intéressons dans cette section au modèle des *Active Rotators* avec désordre, c'est à dire au modèle donné par le système

$$\mathrm{d}\varphi_j^\omega(t) \,=\, -\delta\partial_\theta V(\varphi_j^\omega(t),\omega_j)\,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^N \sin(\varphi_j^\omega(t)-\varphi_i^\omega(t))\,\mathrm{d}t + \mathrm{d}B_j(t)\,, \qquad (1.4.31)$$

où ω p.s. $V(\cdot,\omega)$ est une fonction régulière. Comme indiqué dans la remarque 1.4.3, l'équation limite

$$\partial_t p_t^\delta(\theta,\omega) \,=\, \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta,\omega) - \partial_\theta\Big[p_t^\delta(\theta,\omega)(\langle J * p_t^\delta\rangle_\mu(\theta))\Big] + \delta\partial_\theta\Big[p_t^\delta(\theta,\omega)\partial_\theta V(\theta,\omega)\Big] \quad (1.4.32)$$

admet une courbe stable M_δ, paramétrée par une phase ψ par l'identité $M_\delta = \{q_\psi + \phi_\delta(q_\psi),\ \psi\in\mathbb{S}\}$. Une méthode de perturbation similaire à celle employée dans le cas sans désordre permet d'obtenir une approximation de la vitesse de parcours de M_δ :

$$\dot\psi_t^\delta \,=\, \delta\frac{\Big\langle \partial_\theta[q_{\psi_t^\delta}\partial_\theta V], q'_{\psi_t^\delta}\Big\rangle_{-1,1/q_{\psi_t^\delta},\mu}}{\langle q',q'\rangle_{-1,1/q}} + O(\delta^3)\,. \qquad (1.4.33)$$

Comme q_ψ n'a pas de dépendance en ω, l'intégration suivant ω peut être effectuée en premier, ce qui implique que le premier ordre de la vitesse est le même que celui donné par le modèle des *Active Rotators* sans désordre associé au potentiel $\int_{\mathbb{R}} V(\cdot,\omega)\mu(\mathrm{d}\omega)$, c'est-à-dire

$$\dot\psi_t^\delta \,=\, \delta\frac{\Big\langle \Big[q_{\psi_t^\delta}\left(\int_{\mathbb{R}}V(\cdot,\omega)\mu(\mathrm{d}\omega)\right)'\Big]', q'_{\psi_t^\delta}\Big\rangle_{-1,1/q_{\psi_t^\delta}}}{\langle q',q'\rangle_{-1,1/q}} + O(\delta^3)\,. \qquad (1.4.34)$$

1.4.7 Stabilité linéaire dans le cas du désordre symétrique

Revenons maintenant au modèle de Kuramoto dans le cas où le désordre est symétrique. Le théorème 1.4.2 implique que pour tout $K>1$, lorsque δ est assez petit, il existe un cercle M_δ stable pour (1.4.10). Nous avons déjà vu dans la section 1.4.3 que dans ce cas il s'agit en fait de solutions stationnaires. Plus précisément, on peut écrire

$$M_\delta \,=\, \{(\theta,\omega)\mapsto \widetilde{q}_\psi(\theta,\delta\omega):\ \psi\in\mathbb{S}\}\,, \qquad (1.4.35)$$

où $\widetilde{q}_\psi(\theta,\delta\omega)=\widetilde{q}_0(\theta-\psi,\delta\omega)$, et $\widetilde{q}_0$ est donné par (1.4.3). Le facteur δ dans la dépendance en ω viens de la renormalisation effectuée pour se placer dans l'asymptotique de faible désordre. Définissons l'opérateur d'évolution linéarisée $L^\omega_{\widetilde{q}_\psi}$ autour du point $\widetilde{q}_\psi$

$$-L^\omega_{\widetilde{q}_\psi}u \,=\, \frac{1}{2}\partial_\theta^2 u(\theta,\omega) - \partial_\theta\Big[u(\theta,\omega)\Big(\langle J*\widetilde{q}_\psi\rangle_\mu(\theta)+\delta\omega\Big) + \widetilde{q}_\psi(\theta,\delta\omega)\langle J*u\rangle_\mu(\theta)\Big]\,, \quad (1.4.36)$$

avec comme domaine $\mathcal{D}(A)$ (voir (1.4.15)). Le théorème suivant énonce des propriétés spectrales de $L^\omega_{\widetilde{q}_0}$, et correspond au théorème 3.2.5 du chapitre 3. En fait l'on obtient des estimations explicites, données dans la section 3.6, en particulier dans la proposition 3.6.11.

Théorème 1.4.5. *L'opérateur* $L^{\omega}_{\widetilde{q}_0}$ *vérifie les propriétés suivantes :* 0 *est une valeur propre simple pour* $L^{\omega}_{\widetilde{q}_0}$*, associée au sous-espace engendré par* $(\theta,\omega)\mapsto \partial_\theta \widetilde{q}_0(\theta,\delta\omega)$*. De plus, pour tout* $K>1$*,* $\rho\in(0,1)$*,* $\alpha\in(0,\pi/2)$*, il existe* $\delta_2=\delta_2(K,\rho,\alpha)$ *tel que pour out* $0\leqslant\delta\leqslant\delta_2$*, les propriétés suivantes sont vérifiées :*

- L^{ω}_{q} *admet une fermeture qui a le même domaine que l'extension auto-adjointe de* A *;*
- *Le spectre de* $L^{\omega}_{\widetilde{q}_0}$ *est contenu dans le cône* C_α *ayant pour sommet* 0 *et pour angle* α

$$C_\alpha := \left\{\lambda\in\mathbb{C}\,;\, -\frac{\pi}{2}+\alpha \leqslant \arg(\lambda)\leqslant \frac{\pi}{2}-\alpha\right\} \subseteq \{z\in\mathbf{C}\,;\,\Re(z)\leqslant 0\}\ ; \qquad (1.4.37)$$

- *Il existe* $\alpha'\in(0,\frac{\pi}{2})$ *tel que* $-L^{\omega}_{\widetilde{q}_0}$ *est le générateur infinitésimal d'un semi-groupe analytique défini sur le secteur* $\{\lambda\in\mathbb{C},\,|\arg(\lambda)|<\alpha'\}$ *;*
- *La distance séparant* 0 *et le reste du spectre est strictement positive et est au moins égale à* $\rho\widetilde{\lambda_K}$*, où* $\widetilde{\lambda_K}$ *est le trou spectral de* A_{q_0}*.*

1.5 Problème de sortie de domaine et réduction à un système de phases

Dans cette partie nous étudions la validité de la réduction à un système de phases dans le cadre du problème de sortie de domaine bruité. Nous montrons que cette méthode est valide en dimension finie lorsque le système est proche d'un système réversible. Ce travail correspond à la prépublication [89].

1.5.1 Problème de sortie de domaine

Considérons un système dynamique de $\mathbb{R}^n$ donné par (dans cette partie nous utilisons la notation $f[\cdot]$ pour les applications définies sur $\mathbb{R}^n$)

$$\mathrm{d}X_t \,=\, F[X_t]\,\mathrm{d}t\,, \qquad (1.5.1)$$

comportant un point fixe stable A de domaine d'attraction D, et considérons le modèle bruité comme suit :

$$\mathrm{d}X_t \,=\, F[X_t]\,\mathrm{d}t+\sqrt{\varepsilon}\,\mathrm{d}B_t\,, \qquad (1.5.2)$$

où B_t est un mouvement Brownien de $\mathbb{R}^n$. Avec l'ajout de ce bruit, les trajectoires issues du voisinage de A ne convergent plus vers A, mais fluctuent autour. Le point A devient métastable. Au bout d'un temps suffisamment long elles peuvent parvenir à s'échapper du domaine d'attraction de A. Les questions naturelles qui apparaissant alors sont où, quand et de quelle manière ces trajectoires parviennent à s'échapper du domaine D. Nous nous intéressons à ces questions dans la limite $\varepsilon\to 0$.

Ce type de questions a donné lieu à de nombreux travaux et publications. Freidlin et Wentzell ont les premiers effectué une étude rigoureuse du problème, en démontrant qu'il est relié aux grandes déviations du système stochastique (1.5.2), et plus précisément au quasipotentiel correspondant [34]. Pour tout domaine connexe K et tous points P_1 et P_2 de K, le quasipotentiel $W_K(P_1,P_2)$ est défini par

$$W_K(P_1,P_2) \,=\, \inf\{I_T^{P_1}(Y) : Y\in C([T,0],K),\, T<0,\, Y_T=P_1,\, Y_0=P_2\}\,, \qquad (1.5.3)$$

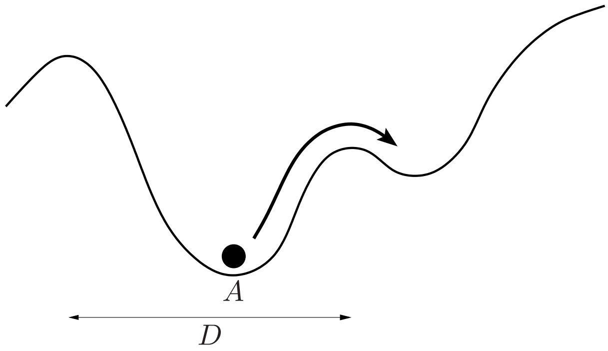


FIGURE 1.9. L'étude de la métastabilité de A est l'étude des échappements des trajectoires du domaine d'attraction de A induites par le bruit.

où I est la fonction de taux du problème de grandes déviations de (1.5.2), c'est-à-dire

$$I_T^x(Y) = \begin{cases} \frac{1}{2}\int_T^0 \left\| \dot{Y}_t - F[Y_t] \right\|^2 \mathrm{d}t & \text{si } Y \text{ est absolument continu} \\ & \text{et } Y_T = x\,, \\ +\infty & \text{sinon}\,, \end{cases} \tag{1.5.4}$$

où $\|\cdot\|$ est la norme associée au produit scalaire $\langle\cdot,\cdot\rangle$ de $\mathbb{R}^n$. Freidlin et Wentzell [34] ont démontré que si K est un voisinage compact à bords réguliers de A inclus dans D les points d'échappement de K des trajectoires de (1.5.2) sont situés (avec probabilité tendant vers 1 lorsque ε tend vers 0) dans le voisinage des points B du bord de K vérifiant $W_K(A,B) = \inf_{E\in\partial K} W_K(A,E)$. Dans un certain sens, le quasipotentiel $W_K(A,\cdot)$ représente le *prix* à payer pour atteindre B depuis A, les trajectoires les plus probables étant celles qui tendent à minimiser ce coût. Un argument de compacité et la continuité de $W_K(A,\cdot)$ (voir [34]) impliquent qu'il existe au moins un point $B \in \partial K$ réalisant le minimum $\inf_{E\in\partial K} W_K(A,E)$.

Freidlin et Wentzell ont également montré que pour toute condition initiale $x \in K$, les temps de sortie τ^ε vérifient

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_x \tau^\varepsilon = W_K(A,B)\,. \tag{1.5.5}$$

Après renormalisation ces temps de sorties suivent à la limite une loi exponentielle [23, 70, 75]. Cette perte de mémoire vient du fait qu'après chaque tentative manquée de sortie du domaine, les trajectoires de (1.5.2) retournent dans un voisinage du point fixe A. Dans la limite de bruit faible, la forme de la trajectoire d'une tentative échouée n'influe donc pas sur la tentative suivante.

Les résultats de [34] ont pu être généralisés, en particulier en affaiblissant les hypothèses sur le compact K. Ceci permet l'étude de sortie de domaine par des points selle, et donc l'étude des passages de trajectoires entre les domaines d'attraction de différents points fixes stables [24, 39, 75].

Lorsque la dynamique de (1.5.2) est réversible (i.e. $F = -\nabla V$ où V est régulier), le quasipotentiel est proportionnel au potentiel duquel est issue la dynamique : si B appartient au domaine d'attraction de A on a

$$W_K(A,B) = 2(V(B) - V(A))\,. \tag{1.5.6}$$

Dans ce cas particulier des méthodes analytiques (en particulier la théorie du potentiel [12, 6]) permettent d'aller plus loin dans l'étude de la métastabilité de A, et de montrer que le facteur précédent $e^{W_K(A,B)/\varepsilon}$ dans (1.5.5) suit la loi de Eyring-Kramer [31, 61]. Dans le cas où la dynamique de (1.5.2) est uni-dimensionnelle, celle-ci peut toujours être considérée comme réversible. En effet le problème de sortie de domaine ne dépend de la fonction F que sur le domaine borné K, et l'on peut toujours prolonger F en dehors de K de sorte que sur $\mathbb{R}$ F vérifie $F = -\nabla V$, avec V régulière.

1.5.2 Présentation du modèle

Nous étudions le problème de sortie de domaine pour des modèles irréversibles, issus d'une perturbation régulière de modèles réversibles. Nous nous plaçons dans le cas où le modèle réversible admet une courbe stable de solutions stationnaires. Dans ce cas la persistance des sous-variétés normalement hyperboliques implique que si la perturbation est suffisamment petite, le modèle perturbé admet lui aussi une courbe invariante stable. Nous allons montrer que le problème de sortie de domaine associé à un point fixe appartenant à cette courbe invariante peut être approximé efficacement par le problème de sortie restreint sur la courbe.

Plus formellement nous nous intéressons au problème de sortie pour les systèmes dynamiques du type

$$\mathrm{d}X_t \;=\; (-\nabla V[X_t] + \delta G[X_t])\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t\,, \tag{1.5.7}$$

où $G \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ et $V \in C^4(\mathbb{R}^n, \mathbb{R}^n)$, et tels que le système dynamique non perturbé

$$\mathrm{d}X_t \;=\; -\nabla V[X_t] \tag{1.5.8}$$

admet une sous-variété compacte uni-dimensionnelle stable et formée de solutions stationnaires. Plus précisément nous supposons qu'il existe une courbe M, de régularité C^3 d'après le théorème d'inversion locale, sans croisements et vérifiant pour tout $X \in M$

$$\nabla V[X] \;=\; 0\,. \tag{1.5.9}$$

Par commodité nous supposons $V \equiv 0$ sur M. Nous supposons de plus l'existence d'un trou spectral pour l'évolution linéarisée au voisinage de M : pour tout vecteur v appartenant à l'espace tangent à M au point X et tout vecteur w orthogonal à u nous supposons que (en notant $H[X]$ la matrice Hessienne de V au point X)

$$H[X]v \;=\; 0\,, \tag{1.5.10}$$

et qu'il existe une constante strictement positive et indépendante du vecteur w telle que

$$\langle H[X]w, w\rangle \;\geqslant\; \lambda\|w\|^2\,. \tag{1.5.11}$$

Sous ces hypothèses M est une sous-variété normalement hyperbolique stable de $\mathbb{R}^n$. Nous rappelons dans la section suivante cette notion dans le cas de la dimension finie. La stabilité de ce type de structure sous perturbation nous permettra de justifier l'existence d'une courbe stable M_δ pour le système perturbé

$$\mathrm{d}X_t \;=\; (-\nabla V[X_t] + \delta G[X_t])\,\mathrm{d}t\,. \tag{1.5.12}$$

1.5.3 Hyperbolicité normale

Pour définir la notion de sous-variété normalement hyperbolique stable dans $\mathbb{R}^n$ nous suivons le formalisme introduit par Fénichel [32] (voir aussi [46]). Considérons un flot de $\mathbb{R}^n$ de classe C^r donné par

$$\dot{X} = F(X) \tag{1.5.13}$$

et supposons que ce flot admet une sous-variété compacte invariante M.

Comme dans le cas infini dimensionnel (voir la section 1.3.3), nous associons à chaque condition initiale $Q \in M$ le semi-groupe $\Phi(Q,t)$ d'évolution linéarisée défini par

$$\Phi(Q,0)u = u \tag{1.5.14}$$

pour tout $u \in \mathbb{R}^n$ et

$$\partial_t \Phi(Q,t) = DF(Q_t)\Phi(Q,t) \tag{1.5.15}$$

où Q_t est la trajectoire de (1.5.13) ayant pour condition initiale Q (et donc incluse dans le compact invariant M).

Comme en dimension finie toutes les normes sont équivalentes, la définition de sous-variété normalement hyperbolique stable ne nécessite pas la définition d'une projection sur l'espace tangent qui commute avec le semi-groupe $\Phi(Q,t)$ (voir (1.3.4)). On peut se contenter de considérer les espaces tangents T_Q à M et normaux N_Q en chaque point Q de M, ainsi que les projections orthogonales correspondantes P_Q^T et P_Q^N. Pour tout $Q \in M$ définissons également les exposants de Lyapunov généralisés

$$\nu(Q) := \inf\left\{a : \left(\frac{\|w\|}{\|P^N_{Q_t}\Phi(Q,t)w\|}\right)\Big/ a^{-t} \to 0 \quad \text{as} \quad t \downarrow -\infty \quad \forall w \in N_Q\right\}, \tag{1.5.16}$$

et, lorsque $\nu(Q) < 1$,

$$\sigma(Q) := \inf\left\{b : \frac{\|w\|^b/\|v\|}{\|P^N_{Q_t}\Phi(Q,t)w\|^b/\|P^T_{Q_t}\Phi(Q,t)v\|} \to 0 \quad \text{as} \quad t \downarrow -\infty \quad \forall v \in T_Q,\, w \in N_Q\right\}. \tag{1.5.17}$$

L'exposant ν quantifie la stabilité linéaire de M, alors que le coefficient σ compare les évolutions linéaires tangentielles et normales au voisinage de M. Plus σ est proche de 0, plus les trajectoires de (1.5.13) ont tendance à s'aplatir au voisinage de M. ν et σ sont des applications de régularité C^r [46], et admettent donc des maxima $\bar{\nu}(M)$ et $\bar{\sigma}(M)$ sur M.

M est une sous-variété normalement hyperbolique stable de $\mathbb{R}^n$ si $\bar{\nu}(M) < 1$ et $\bar{\sigma}(M) < 1$. Il est clair que dans notre problème (1.5.8), les hypothèses (1.5.10) et (1.5.11) impliquent $\bar{\nu}(M) \leqslant e^{-\lambda}$ et $\bar{\sigma}(M) = 0$, donc M est bien une sous-variété normalement hyperbolique stable.

Nous énonçons maintenant le résultat de persistance de notre courbe M sous perturbation. La preuve générale de persistance en dimension finie peut être trouvée dans [32, 46]. Pour se faire nous considérons une paramétrisation $\theta \mapsto q(\theta)$ de M définie sur $\mathbb{R}/L\mathbb{R}$ et vérifiant $\|q'(\theta)\| = 1$.

Théorème 1.5.1. *Si G est C^2, alors pour tout δ assez petit, il existe une application $\theta \mapsto \phi_\delta(\theta)$ de classe C^2 satisfaisant*

$$\langle \phi_\delta(\theta), q'(\theta)\rangle = 0\,, \tag{1.5.18}$$

$$\sup_{\theta\in\mathbb{R}/L\mathbb{R}} \{\|\phi_\delta(\theta)\|, \|\phi'_\delta(\theta)\|, \|\phi''_\delta(\theta)\|\} = O(\delta)\,, \tag{1.5.19}$$

et telle que

$$M_\delta = \{q(\theta)+\phi_\delta(\theta),\, \theta\in\mathbb{R}/L\mathbb{R}\} \tag{1.5.20}$$

est une sous-variété normalement hyperbolique stable pour (1.5.12).

1.5.4 Réduction à un système de phases

Considérons une paramétrisation $\{q_\delta(\varphi),\, \varphi\in\mathbb{R}/L_\delta\mathbb{R}\}$ de M^δ satisfaisant $\|q'_\delta(\varphi)\| = 1$ pour tout φ. La dynamique déterministe non perturbée (1.5.8) restreinte sur M_δ est donnée par la dynamique de phase

$$\mathrm{d}\varphi^\delta_t = b_\delta\left(\varphi^\delta_t\right)\,\mathrm{d}t\,, \tag{1.5.21}$$

où

$$b_\delta(\varphi) := \Big\langle -\nabla V[q_\delta(\varphi)] + \delta G[q_\delta(\varphi)], q'_\delta(\varphi)\Big\rangle\,. \tag{1.5.22}$$

La fonction b_δ est dans un certain sens (pour plus de détails voir les lemmes 5.2.6 et 5.2.7) une perturbation régulière de la fonction

$$b(\theta) = \langle G[q(\theta)], q'(\theta)\rangle\,, \tag{1.5.23}$$

où q est la paramétrisation de M introduite plus haut. La fonction b représente la projection de la dynamique perturbée sur M. Comme la perturbation que nous appliquons sur notre modèle réversible est régulière, les dynamiques induites par b sur M et b_δ sur M_δ sont conjuguées, et ont donc des propriétés topologiques similaires. Nous nous plaçons dans le cas où il existe un point fixe stable $\theta^0\in\mathbb{R}/L\mathbb{R}$ pour b, associé à un domaine d'attraction incluant l'intervalle $[\theta^0-\Delta_1, \theta^0+\Delta_2]$. Alors il existe une phase φ_{A^δ} (correspondant à un point $A^\delta = q_\delta(\varphi_{A^\delta})\in M_\delta$) stable pour b_δ et associée à un domaine d'attraction incluant également l'intervalle $[\varphi_{A^\delta}-\Delta_1, \varphi_{A^\delta}+\Delta_2]$ (voir la figure 1.10). De plus, comme M_δ est une courbe stable, A^δ est un point fixe stable pour (1.5.12).

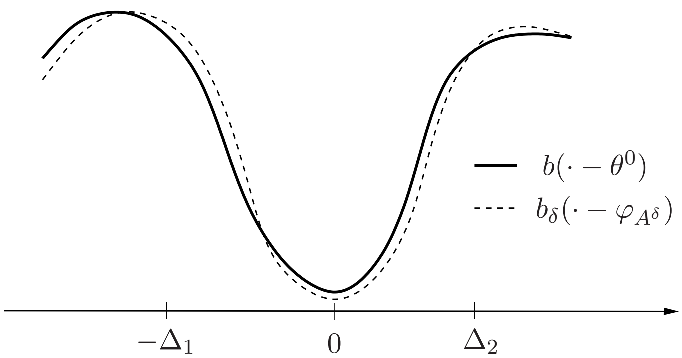


FIGURE 1.10. Pour δ assez petit, $[\varphi_{A^\delta}-\Delta_1, \varphi_{A^\delta}+\Delta_2]$ est inclus dans le domaine d'attraction de φ_{A^δ} pour b_δ.

Pour tout Z dans un voisinage de M_δ il existe un unique $q_\delta(\varphi)$ tel que $\|Z-q_\delta(\varphi)\| = \mathrm{dist}(Z, M^\delta)$. Notons $p_\delta(Z) := \varphi$ la phase de cette projection orthogonale et définissons le tube

$$U^\delta = \{Z\in\mathbb{R}^n,\, \mathrm{dist}(Z,M^\delta)\leqslant C_0\delta^{1/2},\, p_\delta(Z)\in[\varphi_{A^\delta}-\Delta_1, \varphi_{A^\delta}+\Delta_2]\}\,, \tag{1.5.24}$$

dépendant d'une constante C_0. Nous nous intéressons au quasipotentiel W_δ associé à (1.5.7) et au domaine U^δ, c'est-à-dire associé à la fonction de taux

$$I^x_{\delta,T}(Y) = \begin{cases} \frac{1}{2}\int_T^0 \|\dot{Y}_t + \nabla V[Y_t] - \delta G[Y_t]\|^2 \,\mathrm{d}t & \text{si } Y \text{ est absolument continu} \\ & \text{et } Y_T = x\,, \\ +\infty & \text{sinon}\,, \end{cases} \tag{1.5.25}$$

Nous nous intéressons plus particulièrement au minimum du quasipotentiel $W_\delta(A^\delta, \cdot)$ sur la frontière ∂U^δ (qui est atteint comme nous l'avons vu plus haut), ainsi qu'à la position des points où ce minimum est atteint. Comme il y a une différence d'échelle entre la longueur du tube et le diamètre de sa tranche (la longueur est d'ordre 1, la tranche d'ordre $\delta^{1/2}$), une trajectoire sortant du tube par une des deux extrémités, c'est-à-dire en un point B^δ satisfaisant $p_\delta(B^\delta) = \varphi_{A^\delta} - \Delta_1$ ou $p_\delta(B^\delta) = \varphi_{A^\delta} + \Delta_2$, reste très proche de la courbe M_δ. Notre but est de montrer que les points $B^\delta \in \partial U^\delta$ réalisant le minimum du quasipotentiel $W_\delta(A^\delta, \cdot)$ sur ∂U^δ appartiennent à ces extrémités.

La fonction de taux $I^{A^\delta}_{\delta,T}$ appliquée à une trajectoire du type $q_\delta(\varphi^\delta_t)$ partant du point A^δ se réduit au problème de phase

$$I^{A^\delta}_{\delta,T}(Y^\delta) = \int_T^0 \Big|\dot{\varphi}^\delta_t - \Big\langle -\nabla V[q_\delta(\varphi^\delta_t)] + \delta G[q_\delta(\varphi^\delta_t)], q'_\delta(\varphi^\delta_t)\Big\rangle\Big|^2 \,\mathrm{d}t\,. \tag{1.5.26}$$

Il s'agit de la fonction de taux que l'on obtient lorsque l'on considère la diffusion (unidimensionnelle)

$$\mathrm{d}\varphi^\delta_t = b_\delta(\varphi^\delta_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B^1_t\,, \tag{1.5.27}$$

où B^1 est un mouvement Brownien uni-dimensionnel. Notons W^{red}_δ le quasipotentiel associé à cette diffusion et au domaine $[\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$, c'est-à-dire le quasipotentiel défini pour tout $\varphi_1 \in \mathbb{R}/L_\delta\mathbb{R}$ et $\varphi_2 \in \mathbb{R}/L_\delta\mathbb{R}$ par

$$\begin{aligned} W^{red}_\delta(\varphi_1,\varphi_2) = \inf\Big\{ \int_T^0 |\dot{\varphi}_t - b_\delta(\varphi_t)|^2 \,\mathrm{d}t : \ & \varphi \in C([T,0],[\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]) \\ & \text{absolument continu, } T < 0,\ \varphi_T = \varphi_1,\ \varphi_0 = \varphi_2 \Big\}. \end{aligned} \tag{1.5.28}$$

Comme la dynamique réduite est uni-dimensionnelle et l'intervalle $[\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ est inclus dans le domaine d'attraction de φ_{A^δ}, le quasipotentiel $W^{red}_\delta(\varphi_1,\varphi_2)$ est en fait simplement donné par l'intégrale suivante :

$$W^{red}_\delta(\varphi_{A^\delta},\varphi) = \int_{\varphi_{A^\delta}}^{\varphi} b_\delta(\varphi')\,\mathrm{d}\varphi'\,. \tag{1.5.29}$$

Notre but est de montrer que pour l'étude du problème de sortie associé à un point fixe appartenant à M_δ, le quasipotentiel réduit W^{red}_δ est une bonne approximation du quasipotentiel W_δ. On a une majoration immédiate : comme W^{red}_δ est l'infimum de la fonction de taux pris sur le sous-ensemble des trajectoires incluses dans M_δ, on a

$$W_\delta(q_\delta(\varphi_1), q_\delta(\varphi_2)) \leqslant W^{red}_\delta(\varphi_1,\varphi_2)\,. \tag{1.5.30}$$

Pour montrer que W^{red}_δ est une bonne approximation de W_δ, il faut donc montrer que les trajectoires ne gagnent pas beaucoup à s'éloigner de la courbe M_δ. Pour cela nous utilisons des arguments de perturbation.

Théorème 1.5.2. *Il existe δ_0 et une constante C_0 tels que pour tout $\delta \leqslant \delta_0$, pour tout $B^\delta \in \partial U^\delta$ satisfaisant*

$$W_\delta(A^\delta, B^\delta) = \inf_{E \in \partial U^\delta} W_\delta(A^\delta, E)\,, \tag{1.5.31}$$

si l'on note $\varphi_{B^\delta} := p_\delta(B^\delta)$ alors φ_{B^δ} vérifie soit $\varphi_{B^\delta} = \varphi_{A^\delta} - \Delta_1$ soit $\varphi_{B^\delta} = \varphi_{A^\delta} + \Delta_2$, et de plus on a

$$\begin{aligned} W_\delta\left(A^\delta, B^\delta\right) &= W_\delta^{red}\Big(\varphi_{A^\delta}, \varphi_{B^\delta}\Big) + O\left(\delta^3 |\log \delta|^3\right) \\ &= \int_{\varphi_{A^\delta}}^{\varphi_{B^\delta}} b_\delta(\varphi)\,\mathrm{d}\varphi + O\left(\delta^3 |\log \delta|^3\right)\,. \end{aligned} \tag{1.5.32}$$

Ce théorème montre que le quasipotentiel peut être approximé de manière efficace pour les points réalisant le minimum du quasipotentiel $W_\delta(A^\delta, \cdot)$ dans le voisinage du tube U^δ. Il est naturel de penser que cette approximation est aussi valable pour les points situés sur la courbe M_δ, i.e les points B^δ du type $B^\delta = q_\delta(\varphi^\delta)$, mais qui ne vérifient pas nécessairement (1.5.31)). C'est le sujet du corollaire suivant, correspondant au corollaire 5.1.2 :

Corolaire 1.5.3. *Il existe δ_0 et une constante C_0 tels que pour tout $\delta \leqslant \delta_0$ et tout $\varphi^\delta \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ on a*

$$\begin{aligned} W_\delta\left(A^\delta, q_\delta\left(\varphi^\delta\right)\right) &= W_\delta^{red}\left(\varphi_{A^\delta}, \varphi^\delta\right) + O\left(\delta^3 |\log \delta|^3\right) \\ &= \int_{\varphi_{A^\delta}}^{\varphi^\delta} b_\delta(\varphi')\,\mathrm{d}\varphi' + O\left(\delta^3 |\log \delta|^3\right)\,. \end{aligned} \tag{1.5.33}$$

Ces résultats devraient être généralisables en dimension infinie, pour des systèmes issus de la perturbation d'un système réversible. Cette généralisation permettrait en particulier, dans le cas où la dynamique restreinte sur M_δ de (1.3.2) avec $V(\theta) = \theta + a\cos(\theta)$ est une dynamique de point fixe (voir la section 1.3.6), d'étudier les échappements de la mesure empirique associée au modèle (1.3.1) du domaine d'attraction de ce point fixe causés par les effets de taille finie du système.

Chapter 2

Synchronization and excitable systems

Contents

2.1 Introduction

2.1.1 Coupled excitable systems

There are diverse examples of threshold phenomena in natural systems. Dynamics of excitable systems, as exemplified by neuronal membranes (to which we restrict for sake of conciseness), constitute one of the common forms of threshold behavior. Excitable systems are characterized by their nonlinear response to perturbations. In the absence of inputs, they remain at a resting state. This state is locally stable in the sense that the system returns rapidly to it after small perturbations. However, for inputs beyond a critical range, the response of the system takes on a very different form, before regaining the resting state. In the phase portrait of the system, subthreshold responses correspond to monotonic returns to the stable equilibrium while suprathreshold ones appear as excursions that take the system transiently away from the stable equilibrium. Excitability is one of the key neuronal properties at the heart of signal processing and transmission in nervous systems. Motivated by their ubiquity and numerous experimental observations attesting to their functional importance, there has been a characterization of various forms of excitability in terms of the geometry of the phase portrait of dynamical systems [55].

Excitable systems are particularly sensitive to noise because such random signals contain consecutive sub and suprathreshold segments that occur in an unpredictable manner. The interplay between the nonlinearity inherent in the threshold mechanisms and the noise induced fluctuations can produce a large variety of dynamics in excitable systems, some of which are reviewed in [64]. In this paper, we consider one of these, namely, noise induced synchronous coherent oscillations in assemblies of coupled excitable systems.

Noisy excitable systems display irregular repetitive suprathreshold excursions henceforth referred to as firing. In ensembles of such units receiving independent noise, the firings of the units remain independent from one another as long as there are no interconnections between them. Coupling the units with one another introduces correlations between their firings. Synchrony is the extreme form of such correlations when the units fire almost simultaneously. However, synchronous firings can be irregular. One of the surprising effects of noise in assemblies of interacting excitable systems is that for some range of coupling strength and noise intensity, units fire synchronously and regularly. The wide occurrence of these noise induced coherent dynamics and their underlying mechanisms are well documented as explained below. Their putative functional role in nervous systems is to participate in rhythm generation in the absence of pacemaker units (see for instance [59]). Despite the large number of numerical explorations devoted to this phenomenon, it has not been analyzed from a mathematical standpoint. The purpose of the present paper is to deal with this aspect.

Two key elements are at play in the occurrence of noise induced regular synchronous firing in assemblies of interacting excitable units, one is that interacting excitable units act globally like a single excitable system at the population level, the other is that noise driven excitable systems undergo coherence resonance [88], a phenomenon whereby the firings of an excitable system have maximal regularity at an optimal non-zero noise intensity. How the combination of these two phenomena leads to noise induced regularly synchronous firing has been first highlighted in an analysis of networks of an elementary neuronal model [84, 85], see also [47, 91].

bursts of almost synchronous firing under the influence of noise.

The important point is the generality of this mechanisms. It neither relies on the refined properties of specific classes of excitable systems nor on the types of coupling. In fact, noisy assemblies of all common neuronal models, irrespective of the type of excitability, and whether coupled diffusively or through excitatory pulses or synapses, readily produce noise induced regular synchronous firing. To our knowledge, one of the earliest reports of this phenomenon goes back to the explorations of MacGregor and Palasek of randomly connected populations of neuromimes incorporating a large array of individual neuronal properties [67]. More recent examples include the description of the same phenomenon in common neuronal models such as the Hodgkin-Huxley [111], the FitzHugh-Nagumo [109], the Morris-Lecar [47], the Hindmarsch-Rose [112] and others implementing detailed biophysical properties [59]. In these references, besides differences in the models there are also differences in coupling and network architecture: in some the units are diffusively coupled, in others they are coupled through excitatory pulses; in some connectivity is all-to-all, while others deal with random networks. Our enumeration, which does not intend to be exhaustive, illustrates the ease with which assemblies of excitable units generate noise-induced synchronous regular activity, irrespective of model and network details.

The ubiquity of the phenomenon strongly supports investigating its key characteristics through the mathematical analysis of a minimal model that captures its essence. The model we consider is a general version of the so-called active rotator (AR) which is representative of the so-called class I excitable systems [55].

2.1.2 Active rotator models

The AR is a variant of the Kuramoto model for excitable-oscillatory systems that evolve on a unit circle [1]. Precisely, the AR model can be introduced via the stochastic equations

$$\mathrm{d}\psi_j(t) \;=\; -\delta V'\left(\psi_j(t)\right)\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N}\sin\left(\psi_j(t)-\psi_i(t)\right)\mathrm{d}t + \sigma\,\mathrm{d}w_j(t)\,, \tag{2.1.1}$$

where $j=1,\ldots,N$, N is a (large) integer, K, σ, and δ are non-negative constants, the w_j's are IID standard Brownian motions and V is a smooth function (in the applications the case in which V' is a trigonometric polynomial will play an important role, so me may as well think of this case). We look at ψ_j as an element of $\mathbb{S}:=\mathbb{R}/2\pi\mathbb{Z}$, that is ψ_j is a phase, and, of course we have to supply an initial condition for (2.1.1): for example we can take $\{\psi_j(0)\}_{j=1,\ldots,N}$ to be independent identically distributed random variables.

This set of equations defines a diffusion on $\mathbb{S}^N$ describing the evolution of N noisy interacting phases: note that since $K\geq 0$ the interaction has a tendency to *synchronize* the ψ_j's and let us stress from now that such an N-dimensional diffusion reduces for $\delta=0$ to a dynamics that is reversible with respect to the Gibbs measure with Hamiltonian given by $-\frac{K}{N}\sum_{i,j}\cos(\psi_i-\psi_j)$ and inverse temperature σ^{-2}. Such a Gibbs measure goes under the name of "mean field classical XY model": we refer to [9] for more details, but we point out that for $\delta>0$ (of course the case $\delta<0$ is absolutely analogous), unless V is a periodic function (which we do not assume: consider for example $V'(\psi)=1$), the dynamics is not reversible. Nevertheless, it is well known that the large N behavior of such a system can be described in terms of the Fokker-Planck or McKean-Vlasov PDE (the literature on this issue is very vast: see for example the references in [9]):

$$\partial_t p_t^\delta(\theta) \;=\; \frac{\sigma^2}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta\left[p_t^\delta(\theta)(J*p_t^\delta)(\theta)\right] + \delta\partial_\theta\left[p_t^\delta(\theta)V'(\theta)\right]\,, \tag{2.1.2}$$

where $J(\cdot):=-K\sin(\cdot)$ and $\theta\in\mathbb{S}$. To be precise, $p_t^\delta(\cdot)$ is a probability density and it captures the $N\to\infty$ limit of the empirical (probability) measure $\frac{1}{N}\sum_{j=1}^N\delta_{\psi_j(t)}(\mathrm{d}\theta)$, where δ_a is the Dirac delta measure on a. Actually, one can even describe with great accuracy (as $N\to\infty$) the dynamics of each unit system (in interaction!): it evolves following a non-local diffusion equation, called at times *non-linear diffusion*. The non-locality comes from the fact that ψ_j is subject not only to the force field V', but also to the field corresponding to the interaction with all other unit systems, and it all boils down to

$$\mathrm{d}\psi(t) \;=\; -\delta V'(\psi(t))\,\mathrm{d}t + (J*p_t^\delta)(\psi(t))\,\mathrm{d}t + \sigma\,\mathrm{d}w(t)\,, \tag{2.1.3}$$

with w a standard Brownian motion, and it turns out that the probability distribution of $\psi(t)$ is precisely p_t^δ if $\psi(0)$ has distribution p_0^δ.

In mathematical terms, the question that we want to tackle is: what is the relation between the simple deterministic one dimensional dynamics $\dot\psi=-V'(\psi)$ (Isolated Deterministic System: IDS) and the behavior of the associated N dimensional diffusion, for N large? The question is actually twofold. First, given a potential V for the IDS, what is the collective dynamic of the N large limit (2.1.2)? Conversely, what are the possible collective dynamics of (2.1.2)? In order to be more concrete let us ask the following sharper questions: is it possible that

– the IDS has only one stable point, for example if $V(\psi) = \psi - a\cos(\psi)$ for $a > 1$, but the $N \to \infty$ system exhibits stable periodic behavior, that is there is a stable periodic solution to (2.1.2)?
– the IDS has only periodic solutions, but the $N \to \infty$ system has stable stationary solutions?

The fact that the answer to these questions is positive is, to a certain extent, known. Notably, in their numerical investigations of the dynamics of coupled noisy ARs, Shinomoto and Kuramoto reported the existence of collective periodic oscillations, the same phenomenon we have referred to as noise induced regular synchronous activity [101]. They also performed numerical explorations of the transitions to and from this coherent state. The key ingredient in such analyses has been to consider the bifurcations of the associated Fokker-Planck equation (we anticipate that our results make rigorous some of their predictions, see Section 2.3). To clarify how noise generates such time-periodic global activity in coupled excitable ARs, Kurrer and Schulten approximated the solutions of the nonlinear Fokker-Planck equations by Gaussian distributions [63]. Under this assumption, they obtained closed ordinary differential equations for the mean and variance of the distribution and used the bifurcation diagram of these to investigate the regimes where the model generates periodic oscillations. Related work can be found for example in [48, 74], where finite N analysis has been performed, or in [79, 80, 100], where variants of the model have been considered.

However, from a mathematical viewpoint this phenomenon is only very partially understood. We are aware of the contributions [93, 96, 98, 110] that are somewhat close in spirit to what we are doing: these references deal with periodic behavior in nonlinear Markov processes and, more generally, with the effect of the noise on (mean field) interacting dynamical systems. We also deal with nonlinear Markov processes – the evolution equation (2.1.3) contains the law of the process itself – even if this aspect is not emphasized in the remainder of the paper. In particular, Scheutzow [96] provides examples of mean-field type systems in which periodic behavior arises in the $N \to \infty$ system, even if it is not present in absence of noise. The ingenious model set forth in [96] is however rather particular: for example the author plays with some stochastic differential equations of nonlinear Markov type that admit also Gaussian solutions and the analysis boils down to studying the behavior of the expectation and covariance of these solutions. This is close to the approach taken by Touboul, Hermann and Faugeras [110], who extensively exploit the *preservation of the Gaussian character* that holds for certain nonlinear Markov processes and they do so for models that aim at describing neural activity. We stress that in their approach the IDS dynamics is linear, while for us the nonlinearity of the IDS is a key feature. Rybko, Shlosman and Vladimirov in [93] study a connected network of servers that behaves in a periodic fashion in the infinite volume limit, when there are sufficiently many customers per server (*load per server*): in this regime there is an effective synchronization between servers and the *load per server* plays a role which is similar to the parameter K in our work, cf. (2.1.1).

2.1.3 Informal presentation of approach and results

The purpose of this work is to show that for general AR systems one can systematically (at least for some range of the parameters) and quantitatively exhibit the relation between the IDS and the infinite system. This is done by showing that the (infinite-dimensional!) AR system does behave like a one dimensional AR, and the latter can be throughly analyzed. We obtain such a drastic reduction of dimension by exploiting the fact that for the $\delta = 0$ case of (2.1.2) one can perform a rather detailed analysis (due to the fact that it is the grandient flow of a free energy functional [9]). In that case and when $K > K_c := \sigma^2$, stationary solutions of (2.1.2) are the constant $\frac{1}{2\pi}$ which is unstable, and a circle $M = \{q(\cdot - \theta_0) : \theta_0 \in \mathbb{S}\}$, which is a manifold of non-constant invariant solutions: these solutions describe the synchronized state of the oscillators that have a tendency to be close to θ_0. The function $q : \mathbb{S} \to (0, \infty)$ is explicitly known and one can show that M is stable. In fact it has been shown that M is stable in the sense that it is a *stable normally hyperbolic manifold* for the $\delta = 0$ evolution (See Section 2.2.1). A deep result in dynamical systems theory guarantees the robustness of normal hyperbolicity under suitable perturbations [51], see also [5, 99]: this means that, if $\delta > 0$ is not too large, there exists an invariant manifold M_δ which is stable and normally hyperbolic for the evolution (2.1.2), and M_δ is a *smooth deformation* of M. In particular, for small enough δ, M_δ is still a one dimensional manifold diffeomorphic to a circle, and the phase along this manifold plays the role of the natural phase $\psi \in \mathbb{S}$ of the IDS $\dot{\psi} = -V'(\psi)$. This makes clearly a direct link between the (one dimensional) IDS and the $N = \infty$ system (2.1.1), which is an infinite dimensional dynamical system.

The type of results that we obtain is well exemplified in the most basic of the active rotator models, namely the one in which we take $V(\psi) = \psi - a\cos(\psi)$ (without loss of generality: $a \geq 0$): note that, for $a < 1$, the IDS describes just a rotation on the circle, while for $a > 1$ the IDS has a stable point ($\psi = -\arcsin\left(\frac{1}{a}\right)$), to which it is driven, unless sitting on the unstable stationary point $\psi = \arcsin\left(\frac{1}{a}\right) + \pi$. Let us keep in mind that M_δ is close to M, which is a circle, so that also the dynamics on M_δ can be reduced to the dynamics of a phase (see Fig. 2.1). We are going to show in particular that

1. there exists (in fact, we give it explicitly) $a_0 > 1$ such that for $a \in (1, a_0)$ (so the IDS has a stable stationary point!) there exist $K_-, K_+ > 1$, with $K_- < K_+$ such that for $K \in (K_-, K_+)$, and $\delta > 0$ sufficiently small (2.1.2) has a stable periodic solution – a *pulsating wave* – which corresponds to the fact that the dynamics on M_δ is periodic. For $K \in (1, K_-)$ or $K > K_+$ instead the dynamics on the manifold M_δ has (only) one stable stationary point, so (2.1.2) has a stable stationary solution (like the IDS).
2. for every $a \in (0, 1)$, that is the IDS is rotating, one can find $K_0 > 1$ (sufficiently close to 1) such that whenever $K \in (1, K_0)$ for δ sufficiently small the dynamics on M_δ has (only) one stable stationary point.

Actually, these examples are just instances of results that we will establish for general potentials V. For example we will show that for any V such that V' changes sign (so the IDS has a stable point), for K large enough, the dynamics on the invariant curve stabilizes at an equilibrium for small δ. Or that for any V such that $V' > 0$ (so the IDS is rotating), but with nonzero first harmonic coefficient(s), for K close to 1 the dynamics on the invariant curve stabilizes at an equilibrium for small δ.

Finally, regarding the inverse problem, that is the range of possible dynamics, we show that given a noise and a coupling strength such that the $\delta = 0$ system exhibits synchronization, *any* (phase) dynamics can be produced on M_δ, for δ sufficiently small, by a suitable choice of the IDS dynamics (that is, of V) and the relation between these two dynamics is explicit.

Remark 2.1.1. The approach we propose and the results we obtain hold for arbitrary $K > K_c = \sigma^2$. This goes beyond the analytic approaches in the literature, mostly based on bifurcation analysis or on strong coupling/weak noise limits, see e.g. [1, 11, 95, 101, 100], and therefore restricted to K close to K_c or very large. On the other hand we wish to stress that, in spite of the fact that establishing synchronization is central to our analysis, we

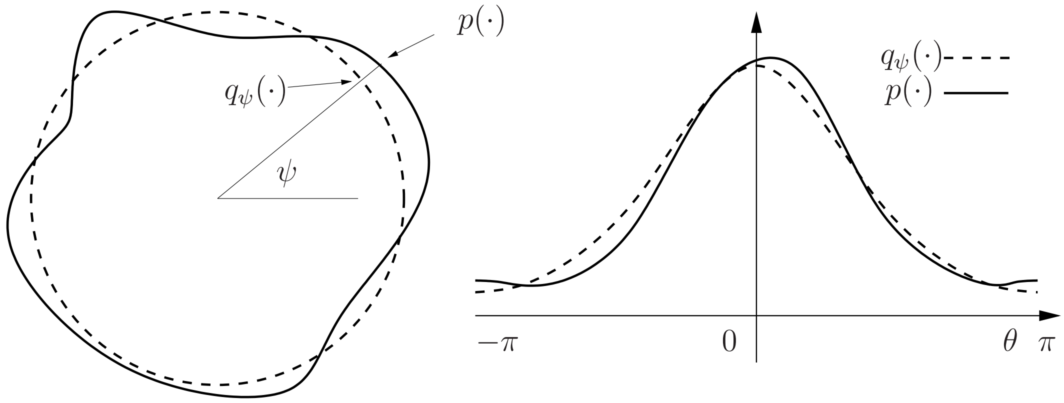


FIGURE 2.1. For δ sufficiently small the solutions of (2.1.2) with an initial condition in a L^2-neighborhood of M (in the figure on the left M is drawn by a dashed line), that is an initial condition close to a $q_\psi(\cdot) = q(\cdot + \psi)$, stay close to M for all times. In fact, they are attracted by a manifold M_δ (solid line, still on the left) that is a small (and smooth) deformation of M. For every function $q_\psi(\cdot)$ in M one associates only one function $p(\cdot)$ on M_δ. While the image on the right stresses the function viewpoint, the one on the left stresses the geometric viewpoint: M is a circle and it is hence parametrized just by one parameter (the phase ψ), but M_δ can also be reduced simply to ψ. The dynamics on the two manifolds is hence reduced to the dynamics of ψ, with the substantial difference that even if the dynamics for $\delta = 0$ is trivial in the sense that M is a manifold of stationary solutions of (2.1.2) with $\delta = 0$, for $\delta > 0$ the dynamics on M_δ can be non-trivial. As a matter of fact, we are going to show that by playing on the choice of $V(\cdot)$ *essentially any* phase dynamics can be observed on M_δ, and this for every K and σ such that $K > \sigma^2$.

do not focus on the synchronization transition: we just focus on the synchronized regime. The transition we are after is the one from a regime in which the system is asymptotically described by a stationary probability density to one in which the probability density has, asymptotically, a time periodic behavior. This is the type of phenomenon that Bonilla in [11] calls *non-equilibrium transition* and this terminology is particularly adapted because, as we pointed out in Section 2.1.2, for $\delta \neq 0$ we deal with a non-reversible dynamics and therefore, in a modern terminology (see e.g. [26]), with a non-equilibrium system. In the remainder of the paper we set $\sigma = 1$ because in the case $\delta = 0$ the true parameter of the model is K/σ^2 and we look for results for small values of δ (and K fixed). This simplification hides at first sight the role of the disorder, but in reality the results can be immediately transposed to the $\sigma \neq 1$ case by changing time by a factor σ^2. In this sense large K corresponds to small σ and the noise-induced character of the transition becomes apparent (see in particular § 2.3.1 and § 2.3.2).

Remark 2.1.2. Our approach can be generalized in several directions. In particular here we do not consider at all the important case in which *random natural frequencies* are present, like in the original Kuramoto synchronization model, or other cases in which random potentials are present [1]: this is taken up in [42].

2.2 Mathematical set-up and main results

2.2.1 On the reversible Kuramoto PDE

Let us first sum up a number of results about

$$\partial_t p_t^0(\theta) = \frac{1}{2}\partial_\theta^2 p_t^0(\theta) - \partial_\theta \left[p_t^0(\theta)(J * p_t^0)(\theta) \right] , \tag{2.2.1}$$

where $J(\theta) := -K\sin(\theta)$. Note that we have set $\sigma = 1$, see Remark 2.1.1. We start by introducing the weighted H_{-1} spaces that are going to play an important role in the sequel.

Given a positive smooth function $w : \mathbb{S} \to (0, \infty)$ we define the Hilbert space $H_{-1,w}$ as the closure of the set of smooth functions $\mathbb{S} \to \mathbb{R}$ such that $\int_{\mathbb{S}} u = 0$ with respect to the squared norm $\|u\|^2_{-1,w} := \int_{\mathbb{S}} w\mathcal{U}^2$, where $\mathcal{U} = \mathcal{U}_w$ is the primitive of u such that $\int_{\mathbb{S}} w\mathcal{U} = 0$. The alternative way to introduce such a space is in terms of *rigged Hilbert spaces* can be found in [9]. When $w(\cdot) \equiv 1$ we simply write H_{-1}. Let us remark immediately that

$$\|u\|^2_{-1,w_1} = \int_{\mathbb{S}} w_1 \left(\mathcal{U}_{w_2} - \frac{\int_{\mathbb{S}} w_1 \mathcal{U}_{w_2}}{\int_{\mathbb{S}} w_1} \right)^2 \le \int_{\mathbb{S}} w_1 \mathcal{U}^2_{w_2} \le \left\| \frac{w_1}{w_2} \right\|_\infty \|u\|^2_{-1,w_2}, \tag{2.2.2}$$

so that the the norms we have introduced are all equivalent. We will also use the affine space

$$\widetilde{H}_{-1} := \left\{ \frac{1}{2\pi} + u : u \in H_{-1} \right\}, \tag{2.2.3}$$

provided with the H_{-1} distance. The companion space $\widetilde{H}_1$, defined in the analogous way, will also appear later on.

Basic features and the stationary solutions of the reversible Kuramoto PDE

The reversible Kuramoto PDE has a number of features that we recall here. First of all, the reversible Kuramoto PDE has strong regularizing properties [43], so that we can safely talk about classical smooth solutions for all positive times, for example whenever the initial condition is in L^2. In particular, (2.2.1) defines an L^2-semigroup. Actually, the conservative character of the dynamics and the fact that we are dealing with probability distributions naturally lead to work on the affine space

$$L^2_1 := \{ f \in L^2 : \int f = 1 \}, \tag{2.2.4}$$

with the L^2 distance. One of the main feature of (2.2.1), directly inherited from being the limit of a reversible stochastic dynamics, is that it is the gradient flow of a free energy (which is therefore a Lyapunov functional for the evolution). These properties underlie what follows but we do not directly use them, and so we refer to [9], see also [43] for related results.

What plays a direct role in our analysis is the fact that all the stationary solutions of (2.2.1) can be written as

$$\frac{1}{Z} \exp(2Kr\cos(\cdot - \psi)), \tag{2.2.5}$$

where $\psi \in \mathbb{S}$ (this accounts for the rotation invariance of (2.2.1)), Z is the normalization constant (fixed by the requirement of working with probability densities) and $r \ge 0$ is a solution of the fixed point problem

$$r = \Psi(2Kr) \qquad \text{with} \quad \Psi(x) := \frac{\int_{\mathbb{S}} \cos(\theta)\exp(x\cos(\theta))\,\mathrm{d}\theta}{\int_{\mathbb{S}} \exp(x\cos(\theta))\,\mathrm{d}\theta}. \tag{2.2.6}$$

$\Psi(0) = 0$, so that $r = 0$ is a solution of the fixed point problem and $\frac{1}{2\pi}$ is a stationary solution. Moreover $\Psi(\cdot)$ is increasing and concave on the positive semi-axis, so that there exists at most one positive fixed point r and such a fixed point exists if and only if

$K > K_c = 1$ (see [9] and references therein). So for $K > 1$ (that we assume from now on) there is a manifold, in fact a curve, of stationary solution, besides the constant solution:

$$M := \{q_\psi(\cdot) = q_0(\cdot - \psi) : \psi \in \mathbb{S}\} \quad \text{with} \quad q_0(\theta) := \frac{\exp(2Kr\cos(\theta))}{\int_{\mathbb{S}} \exp(2Kr\cos(\theta))\, \mathrm{d}\theta}, \tag{2.2.7}$$

where $r = r(K)$ is the positive fixed point of (2.2.6). We will come back to the manifold structure of M, but we point out that the main result in [9] means that M is a *stable normally hyperbolic* manifold ([99, p. 494]: we are going to detail this just below): we stress that M is actually a manifold of stationary solutions and not only an invariant manifold. The key point is that if $p_t^0 = q \in M$ the linearized evolution operator

$$-L_q u := \frac{1}{2}u'' - [uJ * q + qJ * u]', \tag{2.2.8}$$

with domain $\{u \in C^2(\mathbb{S}, \mathbb{R}) : \int_{\mathbb{S}} u = 0\}$ is symmetric in $H_{-1,1/q}$ and its closure, that we still call L_q, is a self-adjoint operator operator with compact resolvent, hence the spectrum is discrete. Actually the spectrum is in $[0, \infty)$: $L_q q' = 0$ and q' generates the whole kernel of L_q: the spectral gap is therefore positive and it will be denoted λ_K (see [9] for a proof of all these facts and for an explicit lower bound on λ_K).

In such a framework it is useful to take advantage of some of the interpolation spaces associated to L_q. For us the (Hilbert) spaces V_q and V_q^2 with norms

$$\|v\|_{V_q} := \left\| \sqrt{1 + L_q}\, v \right\|_{-1,1/q} \quad \text{and} \quad \|v\|_{V_q^2} := \|(1 + L_q)\, v\|_{-1,1/q}, \tag{2.2.9}$$

will play an important role. In [9] it is shown that if $v \in L_0^2 := \{v \in L^2 : \int_{\mathbb{S}} v = 0\}$

$$c_K \|v\|_2 \leq \|v\|_{V_q} \leq c_K^{-1} \|v\|_2, \tag{2.2.10}$$

where here (and below) c_K denotes a suitable positive constant that depends only on K (it will not keep the same value through the text: in particular, in this case it is the same for every $q \in M$). Note that if $v \in \mathcal{R}(L_q)$, $\mathcal{R}(\cdot)$ denotes the range of $\cdot$, the spectral gap guarantees that

$$\|v\|_{V_q}^2 \leq \left(1 + \frac{1}{\lambda_K}\right) \left\| \sqrt{L_q}\, v \right\|_{-1,1/q}^2. \tag{2.2.11}$$

At this point it is also worth observing also that, by (2.2.2), there exists $c_K > 0$ such that for every $\psi_1, \psi_2 \in \mathbb{S}$ we have

$$c_K \|v\|_{-1,1/q_{\psi_1}} \leq \|v\|_{-1,1/q_{\psi_2}} \leq c_K^{-1} \|v\|_{-1,1/q_{\psi_1}}. \tag{2.2.12}$$

Of course, we have the analogous estimates in the case in which $1/q_{\psi_2}$, or $1/q_{\psi_1}$, is replaced by 1.

Stable normally hyperbolic manifolds

We now quickly review the notion of *stable normally hyperbolic manifold*, in the L_1^2 set-up, because it will play a central role in our results. For this we need a dynamics: what we have in mind is (2.2.20) but for the moment let us just think of an evolution semigroup in L_1^2 that gives rise to $\{u_t\}_{t \geq 0}$, with $u_0 = u$, to which we can associate a linear evolution semigroup $\{\Phi(u, t)\}_{t \geq 0}$ in L_0^2, satisfying $\partial_t \Phi(u, t)v = A(t)\Phi(u, t)v$ and $\Phi(u, 0)v = v$, where $A(t)$ is the operator obtained by linearizing the evolution around u_t.

For us a stable normally hyperbolic manifold $M \subset L_1^2$ (in reality we are interested only in 1-dimensional manifolds, that is curves, but at this stage this does not really play a role) of characteristics λ_1, λ_2 ($0 \le \lambda_1 < \lambda_2$) and $C > 0$ is a C^1 compact connected manifold which is invariant under the dynamics and for every $u \in M$ there exists a projection $P^o(u)$ on the tangent space of M at u, that is $\mathcal{R}(P^o(u)) =: T_u M$, which, for $v \in L^2_0$, satisfies the properties

1. for every $t \ge 0$ we have

$$\Phi(u,t)P^o(u_0)v \,=\, P^o(u_t)\Phi(u,t)v\,, \tag{2.2.13}$$

2. we have

$$\|\Phi(u,t)P^o(u_0)v\|_2 \,\le\, C\exp(\lambda_1 t)\|v\|_2\,, \tag{2.2.14}$$

and, for $P^s := 1 - P^o$, we have

$$\|\Phi(u,t)P^s(u_0)v\|_2 \,\le\, C\exp(-\lambda_2 t)\|v\|_2\,, \tag{2.2.15}$$

for every $t \ge 0$;

3. there exists a negative continuation of the dynamics $\{u_t\}_{t\le 0}$ and of the linearized semigroup $\{\Phi(u,t)P^o(u_0)v\}_{t\le 0}$ and for any such continuation we have

$$\|\Phi(u,t)P^o(u_0)v\|_2 \,\le\, C\exp(-\lambda_1 t)\|v\|_2\,, \tag{2.2.16}$$

for $t \le 0$.

As an example – for us a crucial example – let us show that M is a stable normally hyperbolic manifold for the L^2_1-semigroup associated to (2.2.1) (in Section 2.2.2 we give some details on this semigroup in the general case). As we have seen, M is an invariant manifold: it is in fact a set of stationary solutions, so that the dynamics has a (trivial) negative continuation, and it is easy to provide an explicit atlas, compatible with the L^2 topology, for which M is a C^∞ manifold and $T_q M_0 = \{aq' : a \in \mathbb{R}\} = \mathcal{R}(P^o_q)$. The projection P^o we choose is defined by

$$P^o(q)v = P^o_q v \,:=\, \frac{(v,q')_{-1,1/q}\, q'}{(q',q')_{-1,1/q}}\,, \tag{2.2.17}$$

and, since $L_q q' = 0$ for every $q \in M$, we see that λ_1 can be chosen equal to zero and any value $C \ge 1$ will do. Moreover if we set $v_t := \Phi(q,t)P^s_q v \in \mathcal{R}(P^s_q)$ then

$$\begin{aligned} \|v_t\|_2 \,\le\, c_K\|v_t\|_{V_q} \,\le\, c_K\sqrt{1+1/\lambda_K}\,\left\|\sqrt{L_q}\,v_t\right\|_{-1,1/q} \\ \le\, c_K\sqrt{1+1/\lambda_K}\exp\left(-\lambda_K t\right)\|\sqrt{L_q}\,v_0\|_{-1,1/q} \,\le\, c'_K\exp\left(-\lambda_K t\right)\|v\|_2\,, \end{aligned} \tag{2.2.18}$$

where we have used (2.2.12), then (2.2.11), then the spectral gap and finally (2.2.10). Therefore λ_2 can be chosen equal to λ_K, $C \ge c'_K$, and therefore M is a stable normally hyperbolic manifold in L^2 for the reversible Kuramoto evolution, with characteristics 0, λ_K and $C = \max(c'_K, 1)$.

For the sequel we observe also that $u \mapsto P^o_u$, a map from M to the bounded linear operators on L^2_0, is C^∞ as it can be easily verified by using for $v \in L^2_0$ the formula

$$\left(v, q'_\psi\right)_{-1,q_\psi} \,=\, \int_{\mathbb{S}} \mathcal{V} - \frac{\int_{\mathbb{S}} \mathcal{V}/q_\psi}{\int 1/q_0}\,, \tag{2.2.19}$$

where, like before, $\mathcal{V}(\theta) := \int_0^\theta v$, so that $\mathcal{V} : \mathbb{S} \to \mathbb{R}$ is (Hölder) continuous and $\psi \mapsto (v, q'_\psi)_{-1,q_\psi}$ is C^∞.

2.2.2 The full evolution equation

The type of limit evolution equations we are interested in can be cast into the form

$$\partial_t p_t^\delta(\theta) = \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta\left[p_t^\delta(\theta)(J * p_t^\delta)(\theta)\right] + \delta G[p_t^\delta](\theta)\,, \tag{2.2.20}$$

where $\delta \geq 0$ and for G we assume

1. $p \mapsto G[p]$ is a function from L_1^2 to H_{-1};
2. there exists $\eta > 0$ such that G is $C^1(L_1^2, H_{-1})$ for every p at L^2 distance at most η of M and the derivative DG is uniformly bounded (in the η-neighborhood of M that we consider).

Note that $p \mapsto (pJ * p)'$ is also in $C^1(L_1^2, H_{-1})$, in fact even in C^∞, so that the evolution equation can be cast in the abstract form $\partial p_t^\delta = Ap_t^\delta + F[p_t^\delta] + \delta G[p_t^\delta]$. A complete theory of this type of equations can be found in [99, Ch. 4], in particular for $p_0^\delta \in L_1^2$ such that $d_{L^2}(p_0^\delta, M) < \eta$ there exists of a unique mild solution in $C^0([0,T), L_0^2)$, for some $T > 0$.

Examples include:

1. the AR case, that is (2.1.2), with $G[p](\theta) = \partial_\theta[p(\theta)U(\theta)]$ and $\|U\|_\infty < \infty$;
2. the case of
$$G[p](\theta) = \partial_\theta[p(\theta)\widetilde{J} * p(\theta)] \tag{2.2.21}$$
with $\widetilde{J} \in L^\infty$;
3. the case of
$$G[p](\theta) = \partial_\theta[p(\theta)\int_{\mathbb{S}} h(\theta,\theta')p(\theta')]\,, \tag{2.2.22}$$
with $h \in L^\infty$, as well as generalizations like $\partial_\theta[p(\theta)\int_{\mathbb{S}} h(\theta,\theta',\theta'')p(\theta')p(\theta'')]$ and so on.

In all these examples actually one can prove global well-posedness for arbitrary initial condition in L_1^2. But the key point of our analysis is that if the initial condition is sufficiently close to M, then for δ smaller than a suitable constant, the solution will stay in a neighborhood of M for all times. More precisely, our approach is based on the following result, that is essentially contained in [99, Main Theorem, p. 495]. We say *essentially* because the result we need is more explicit for what concerns the various *small constants* that are involved: in Section 2.5 we detail this issue.

Theorem 2.2.1. *There exists $\delta_0 > 0$ such that if $\delta \in [0, \delta_0]$ there exists a stable normally hyperbolic manifold M_δ in L_1^2 for the perturbed equation* (2.2.20). *Moreover we can write*

$$M_\delta = \{q_\psi + \phi_\delta(q_\psi) : \psi \in \mathbb{S}\}\,, \tag{2.2.23}$$

for a suitable function $\phi_\delta \in C^1(M, L_0^2)$ with the properties that

- *$\phi_\delta(q) \in \mathcal{R}(L_q)$;*
- *there exists $C > 0$ such that* $\sup_\psi(\|\phi_\delta(q_\psi)\|_2 + \|\partial_\psi\phi_\delta(q_\psi)\|_2) \leq C\delta$.

We are now interested in the dynamics on M_δ, which is a curve and, given the mapping ϕ_δ, the position on the manifold is identified by the *phase* ψ_t^δ. A more detailed description demands information on $n_t^\delta := \phi_\delta(q_{\psi_t^\delta})$: of course $\psi_t^0 = \psi_0^0$ and $n_t^0 \equiv 0$ for every t.

We have the following:

Theorem 2.2.2. *For $\delta \in [0,\delta_0]$ we have that $t \mapsto \psi_t^\delta$ is C^1 and*

$$\dot{\psi}_t^\delta + \delta \frac{\left(G[q_{\psi_t^\delta}], q'_{\psi_t^\delta}\right)_{-1,1/q_{\psi_t^\delta}}}{(q', q')_{-1,1/q}} = O(\delta^2)\,, \tag{2.2.24}$$

with $O(\delta^2)$ uniform in t. Moreover if we call n_ψ the unique solution of

$$L_{q_\psi} n_\psi = G[q_\psi] - \frac{\left(G[q_\psi], q'_\psi\right)_{-1,1/q_\psi}}{(q', q')_{-1,1/q}} q'_\psi \quad \text{and} \quad \left(n_\psi, q'_\psi\right)_{-1,1/q_\psi} = 0\,, \tag{2.2.25}$$

we have

$$\sup_\psi \|\phi_\delta(q_\psi) - \delta n_\psi\|_{H_1} = O(\delta^2)\,. \tag{2.2.26}$$

A sharper control on the dynamics on M_δ can be obtained, under a slightly stronger assumption on the perturbation G: it all boils down to go beyond (2.2.24) and for this note that the left-hand side can be written as $R^\delta(\psi_t^\delta)$ where

$$R_\delta(\psi) := \frac{\left([\phi_\delta(q_\psi) J * \phi_\delta(q_\psi)]' + \delta \left(G\left[q_\psi + \phi_\delta(q_\psi)\right] - G\left[q_\psi\right]\right)\,,\, q'_\psi\right)_{-1,1/q_\psi}}{(q', q')_{-1,1/q}}\,. \tag{2.2.27}$$

It is clear that R_ψ is C^1, since ϕ^δ is C^1.

Theorem 2.2.3. *Under the same assumptions of the previous theorem and assuming in addition that DG (recall that $G \in C^1(L^2_1; H_{-1})$) is uniformly continuous in a L^2-neighborhood of M_0, we have that there exists $\delta \mapsto \ell(\delta)$, with $\ell(\delta) = o(1)$ as $\delta \searrow 0$, such that*

$$\sup_{\psi \in \mathbb{S}} |R'_\delta(\psi)| \le \delta\, \ell(\delta)\,. \tag{2.2.28}$$

2.3 Dynamics on M_δ: analysis of the active rotators case

Let us use the results of the previous section to tackle the questions we have raised in the introduction for the active rotators case and that, ultimately, boil down to: what is the relation between the Isolated Deterministic one dimensional System $\dot\psi = -V'(\psi)$ (IDS) and the behavior of the associated N dimensional diffusion, for N large? So we focus on (2.2.20) with $G[p] = (pV')'$ and regularity assumptions on V' are going to appear along the way. Theorem 2.2.1 tells us that if $\|V'\|_\infty < \infty$, at least when δ is small enough, the $N \to \infty$ limit system – ruled by (2.2.20) – is described by a dynamics on a one dimensional smooth and compact manifold M_δ equivalent to a circle and, via Theorem 2.2.2 and Theorem 2.2.3, we have a sharp control on this dynamics.

In order to be precise on this issue let us speed up time by $1/\delta$ in (2.2.24). If we keep just the leading terms we are dealing with the dynamics

$$\dot\psi = -f(\psi)\,, \tag{2.3.1}$$

where f is

$$f(\psi) := \frac{\left(G[q_\psi], q'_\psi\right)_{-1,1/q_\psi}}{(q', q')_{-1,1/q}}\,. \tag{2.3.2}$$

We say that $f \in C^1(\mathbb{S}, \mathbb{R})$ – not necessarily the f in (2.3.2) – is generic, or hyperbolic, if it has a finite number of zeroes on $\mathbb{S}$ and all of them are simple, i.e. for all ψ for which $f(\psi) = 0$, we have $f'(\psi) \neq 0$. Notice that the set of generic functions is open in $C^1(\mathbb{S}, \mathbb{R})$ and dense: if the C^1 distance of f and g is less than (a constant times) ϵ, we say that the dynamics generated by f and g are ϵ-close. Note that if ϵ is sufficiently small then the two dynamics are topologically equivalent. By this we mean that there exists a homeomorphism $h : \mathbb{S} \to \mathbb{S}$ such that $\{h(\psi(\psi_0, t)) : t \in \mathbb{R}\}$, where $\psi(\psi_0, \cdot)$ solves $\dot{\psi} = -f(\psi)$ and $\psi(\psi_0, 0) = \psi_0$, coincides with $\{(\phi(h(\psi_0), t) : t \in \mathbb{R}\}$, where $\phi(\phi_0, \cdot)$ solves $\dot{\phi} = -g(\phi)$ and $\phi(\phi_0, 0) = \phi_0$. Moreover we require that $h(\cdot)$ preserves the time orientation, that is for $a > 0$ sufficiently small and t, $|s| \in (0, a]$ we have that $\psi(\psi_0, t) \neq \psi_0$ and $h(\psi(\psi_0, t)) = \phi(h(\psi_0), s)$ imply $s > 0$.

Theorem 2.2.2 and Theorem 2.2.3 guarantee therefore that for δ sufficiently small the phase dynamics on the M_δ manifold speeded up by δ^{-1}

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi^\delta_{t/\delta} \;=\; -f(\psi^\delta_{t/\delta}) + \frac{1}{\delta}R_\delta(\psi^\delta_{t/\delta})\,, \tag{2.3.3}$$

is δ-close to the dynamics generated by $f(\cdot)$.

The layout of the remainder of this section is, first, to show that even if we fix $K > 1$, by playing on the choice of $V'(\cdot)$, one can generate *arbitrary generic phase dynamics on* M_δ. In this part we will make also more explicit the link between V' and f. Afterwards, we will work out in detail a few particular cases and expose some *a priori* surprising behaviors, notably that IDS with periodic behavior (active state) may lead to a $N \to \infty$ dynamics that settles down to a fixed point (quiescent state) or that IDS without periodic behavior may give origin to periodic $N \to \infty$ behaviors.

2.3.1 Noise and interaction induce arbitrary generic dynamics

It is practical and sufficient to work with $V'(\cdot)$ that is a trigonometric polynomial, that is

$$V'(\theta) \;=\; a_0 + \sum_{j=1}^{n} \left(a_j \cos(j\theta) + b_j \sin(j\theta)\right)\,. \tag{2.3.4}$$

Theorem 2.3.1. *For any generic dynamics on the circle $\dot{\psi}_t = -f(\psi_t)$ with $f \in C^1(\mathbb{S};\mathbb{R})$ and for any value of $K > 1$ there exists a trigonometric polynomial $V'(\cdot)$ (see Remark 2.3.2 for an explicit expression) such that for δ small enough, the phase dynamics on M_δ (2.3.3) is δ-close to $\dot{\psi} = -f'(\psi)$.*

Proof. Let f be a generic function in C^1. By the Stone-Weierstrass Theorem, for every $\varepsilon > 0$ there exists a trigonometric polynomial $P(\cdot)$ such that $\|f' - P\|_\infty \leq \varepsilon$. If c_0 is such that $\int_0^{2\pi}(P - c_0) = 0$ then, since $\int_0^{2\pi} f' = 0$, $|c_0| \leq \varepsilon$. Thus if we define the trigonometric polynomial $Q(\psi) := f(0) + \int_0^{\psi}(P(\theta) - c_0)\,\mathrm{d}\theta$ we have

$$\|Q - f\|_{C_1} \;=\; \|Q - f\|_\infty + \|P - c_0 - f'\|_\infty \;\leq\; (2\pi + 1)\|P - c_0 - f'\|_\infty \;\leq\; (4\pi + 2)\varepsilon\,, \tag{2.3.5}$$

so it suffices to consider functions f which are trigonometric polynomials:

$$f(\theta) \;=\; A_0 + \sum_{k=1}^{n} \left(A_k \cos(k\theta) + B_k \sin(k\theta)\right)\,. \tag{2.3.6}$$

Now we observe that if $V'(\cdot)$ is of the form (2.3.4) then a straightforward calculation gives

$$\frac{\left(G[q_\psi], q'_\psi\right)_{-1,1/q_\psi}}{\left(q', q'\right)_{-1,1/q}} = a_0 + \frac{I_0}{I_0^2 - 1} \sum_{k=1}^{n} \left(I_k a_k \cos(k\psi) + I_k b_k \sin(k\psi)\right) , \tag{2.3.7}$$

where

$$I_k = I_k(2Kr(K)) := \frac{1}{2\pi} \int_0^{2\pi} \cos(k\theta) e^{2Kr(K)\cos(\theta)} d\theta . \tag{2.3.8}$$

Therefore by making the choice $a_0 := A_0$ and for $k = 1, 2, \ldots, n$

$$a_k := \frac{I_0^2 - 1}{I_0 I_k} A_k \quad \text{and} \quad b_k := \frac{I_0^2 - 1}{I_0 I_k} B_k , \tag{2.3.9}$$

we obtain the function $V'(\cdot)$ we were after. □

Remark 2.3.2. The link between f and V' can be made more explicit. In fact from (2.3.8) and (2.3.9) and the fact that the Fourier series of q_0 is

$$q_0(\psi) = \frac{1}{2\pi I_0(2Kr)} e^{2Kr\cos(\psi)} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{j=1}^{+\infty} \frac{I_j(2Kr)}{I_0(2Kr)} \cos(j\psi) , \tag{2.3.10}$$

one directly extracts that

$$f = a_0 + \frac{I_0(2Kr(K))^2}{I_0(2Kr(K))^2 - 1} \left(q_0 * V' - a_0\right) = a_0 + D(K) q_0 * \left(V' - a_0\right) , \tag{2.3.11}$$

where we have set

$$D(K) := \frac{I_0^2(2Kr(K))}{(I_0^2(2Kr(K)) - 1)} , \tag{2.3.12}$$

and (2.3.11) can be applied also in the case in which f is not a trigonometric polynomial. It tells us that, for δ small, the *effective force* that drives the $N \to \infty$ system is, in a sense, obtained by smearing V' via the probability kernel q_0. To be precise, $V' - a_0$ is smeared and multiplied by $D(K)$, while the 0^{th} order Fourier coefficient is left unchanged. This is telling us that the effect of noise and interaction, to leading order, boil down to the size of $D(K)$ and to the smearing effect of the probability kernel $q_0(\cdot)$ (that depends on K too!).

While (2.3.11) is quite explicit, it is not always straightforward to read off it the qualitative properties of f. We start by analyzing the case of K very large and the case of K close to one, before moving to treating in detail some particular cases.

The $K \to \infty$ limit

It is straightforward to see that the probability density $q_0(\cdot)$ converges to the Dirac delta measure at the origin. Moreover $\lim_{K\to\infty} D(K) = 1$, since $\lim_{K\to\infty} r(K) = 1$ and $\lim_{x\to\infty} I_0(x) = \infty$. Therefore f and V' get closer and closer as K becomes large. More precisely one has that for every $s \in \mathbb{N}$ and every trigonometric polynomial $V'(\cdot)$ there is C such that

$$\|f - V'\|_{C^s} \le \frac{C}{\sqrt{K}} . \tag{2.3.13}$$

The proof can be obtained for example by using (2.3.9) that, with (2.3.12), tells us that A_j/a_j, as well as B_j/b_j, that is the ratio of the (non-vanishing) sine and cosine Fourier coefficients of f and V', is

$$D(K)\frac{I_j(2Kr(K))}{I_0(2Kr(K))}, \tag{2.3.14}$$

so that by using $(I_j(x)/I_0(x)) - 1 \overset{x\to\infty}{\sim} \frac{j^2}{2x}$ $(j = 1, 2, \ldots)$ and $\lim_{K\to\infty} D(K) = 1$ we readily obtain that the j^{th}-Fourier coefficients of $f(\cdot)$ are, to leading order, $j^2 a_j/(4K)$ and $j^2 b_j/(4K)$. Since we are just dealing with trigonometric polynomials and the estimate of the L^2 norm of arbitrary derivatives of $f - V'$, via Parseval formula, is straightforward, we get to (2.3.13). This means in particular that given a potential V such that V' has sign changes (so that the IDS has stable points), for any K large enough, the $N \to \infty$ system has stable stationary solutions, for δ small enough. We will encounter this phenomenon in the particular cases that we treat below.

The $K \searrow 1$ limit

This time we use $r(K) \overset{K\searrow 1}{\sim} \sqrt{2(K-1)}$ and we derive, first of all, that $D(K) \sim (4(K-1))^{-1}$, since $I_0(x) - 1 \overset{x\searrow 0}{\sim} x^2/4$. Once again we analyze the Fourier coefficients of f, via (2.3.14), and we use for $j = 1, 2, \ldots$

$$\frac{I_j(x)}{I_0(x)} \overset{x\searrow 0}{\sim} I_j(x) \sim \frac{x^j}{2^j j!}, \tag{2.3.15}$$

so that for $j = 1, 2, \ldots$

$$\frac{A_j}{a_j} = \frac{B_j}{b_j} \overset{K\searrow 1}{\sim} \frac{(K-1)^{-1+(j/2)}}{2^{2-(j/2)} j!}. \tag{2.3.16}$$

Notably, the first Fourier coefficients of f are enhanced with respect to the corresponding coefficients of V' by a factor that diverges like $(K-1)^{-1/2}$. The second Fourier coefficients of f are (asymptotically) just proportional to the ones of V', while higher coefficients in the $K \searrow 1$ limit are depressed passing from IDS to $N \to \infty$ behavior (recall that the 0^{th}-order coefficient is unchanged). A quantitative estimate in the spirit of (2.3.13) is easily established from these estimates.

What we retain from this $K \searrow 1$ analysis is that if the first Fourier coefficients are present, that is $|a_1| + |b_1| > 0$, then for K sufficiently close to one $f(\psi) = 0$ has two solutions and the dynamics will eventually settle to a fixed point (quiescent state). If instead $|a_1| + |b_1| = 0$, then it depends on the relative size of a_0 and a_2 or b_2 whether the system is in an activated or quiescent regime. But if also $a_2 = b_2 = 0$ (and $a_0 \neq 0$) then for K sufficiently close to one we have that $f(\psi)$ is close to a_0 and therefore $f(\psi) \neq 0$ for all ψ, so that the dynamics is periodic. Again, we will discuss in more detail these issues below, in specific examples.

Remark 2.3.3. The analysis for K large and close to one is helpful to get an idea on the relation between f and V', but the reader should keep in mind that the δ-closeness of the dynamics holds for fixed K, that is for $\delta < \delta_0(K)$. Quantitative estimates on how $\delta_0(K)$ behaves for extreme values of K is an interesting issue that we do not approach here.

2.3.2 Active rotators with $V(\theta) = \theta - a\cos(\theta)$

Without loss of generality we assume $a \geq 0$. Let us start the analysis by making a remark on the $a = 0$ case: the potential becomes just a straight line, and (2.2.20) reads

$$\partial_t p_t^\delta(\theta) = \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta) - \partial_\theta\left[p_t^\delta(\theta)(J * p_t^\delta)(\theta) - \delta p_t^\delta(\theta)\right]. \tag{2.3.17}$$

In this case $p_t^\delta(\theta - \delta t)$ solves (2.2.1), thus $M_\delta = M$ and the dynamics on M_δ is a rotation for all δ.

If $a > 0$ we exploit the analysis we have developed for Theorem 2.3.1 that tells us that the $N \to \infty$ phase dynamics is lead by the effective force

$$f(\psi) = -\left(1 + \frac{a}{a_c(K)} \sin(\psi)\right), \quad \text{with} \quad a_c(K) := \frac{I_0^2 - 1}{I_0 I_1}. \tag{2.3.18}$$

Therefore if $a < a_c(K)$, then the dynamic on M_δ is periodic for δ small enough (depending on K) and if $a > a_c(K)$, there are two fixed points. From this observation and the graph of $a_c(\cdot)$ (see Figure 2.2) we draw the following conclusions (see also Figure 2.3):

– Set $\hat{a}_c := \max_K a_c(K)(> 1)$. If $a > \hat{a}_c$ then for every K we have that $f(\theta) = 0$ has two solutions, so that the phase dynamics has two stationary hyperbolic point: one is stable and the other is unstable. In this case the dynamics of the IDS resembles to the phase dynamics of the $N \to \infty$ system.
– If $a \in (1, \hat{a}_c)$ then $a_c(K) = a$ has two solutions $K_-(a) < K_+(a)$ and for $K \in (K_-(a), K_+(a))$ we have $a < a_c(K)$, that is $f(\theta) < 0$ for every θ, and the motion is periodic: in this case the dynamics of the IDS, that has two fixed points, differs from the $N \to \infty$ phase dynamics. For $K > K_+(a)$ and for $K < K_-(a)$ instead the phase dynamics is driven to a (unique) stable fixed point (unless it starts from the unstable fixed point).
– If $a \leq 1$ instead $a_c(K) = a$ has only one solution $K(a)$ and the periodic behavior sets up for $K > K(a)$, otherwise $(K < K(a))$ the system eventually settles on a fixed point: this second case is another instance in which the dynamics of the IDS and the $N \to \infty$ system differ.

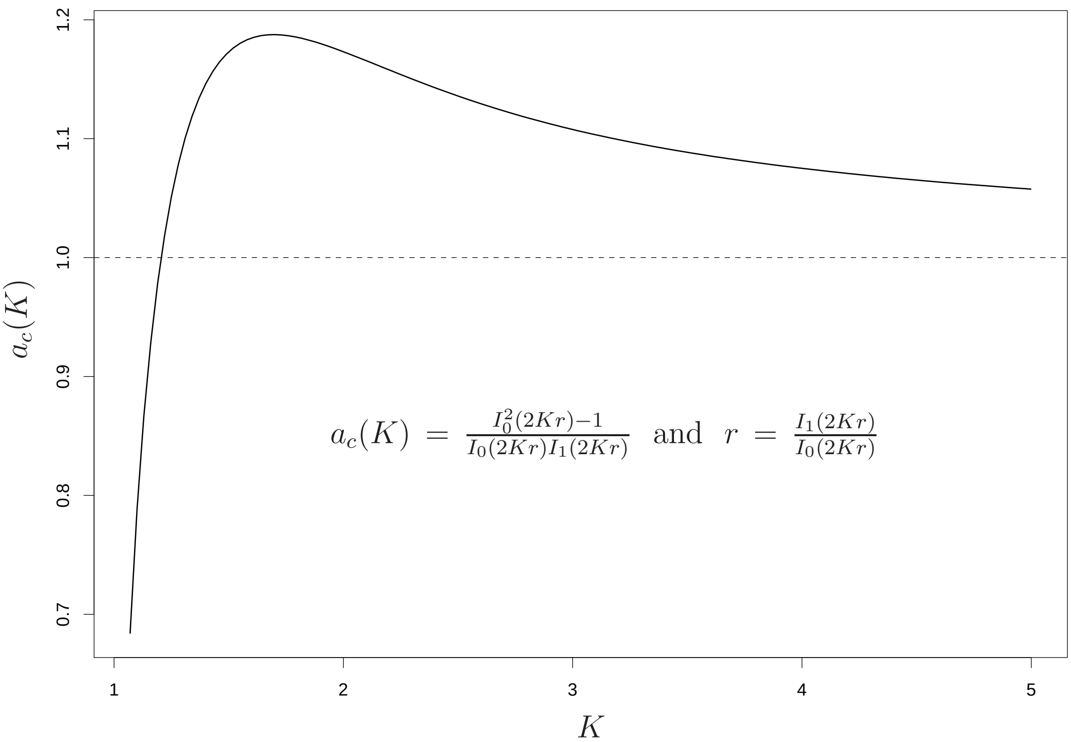


FIGURE 2.2. The graph of $a_c(\cdot)$. For $K \to \infty$ we have $a_c(K) = 1 + 1/(8K) + O(K^{-2})$, while for $K \searrow 1$ we have $a_c(K) \sim \sqrt{32(K-1)}$.

When the phase dynamics is periodic we can explicitly integrate the evolution equation (2.3.1) and compute the first order approximation the period $T_\delta(a, K)$ of the dynamics

δ	$T_\delta(1.1,2)$	$\tau(1.1,2)/\delta$
0.005	3615.59	3615.62
0.010	1807.79	1807.85
0.020	903.89	904.01
0.040	451.94	452.19
0.080	225.97	226.45
0.160	112.98	113.96
0.320	56.49	58.51
0.640	28.24	33.02

TABLE 2.1. We have simulated (2.1.2) with $V(\theta) = \theta - a\cos(\theta)$ for $a = 1.1$ and $K = 2$. In this case our estimates ensure the existence of periodic solutions for δ sufficiently small and the period given in (2.3.19) (in fact, $\tau := \tau(1.1,2) = 18.0779\ldots$). The simulation, that has been performed via Fourier decomposition (50 modes kept), gives $c = 0.333\ldots$, for the constant c such that $(\delta T_\delta(1.1,2)/\tau(1.1,2)) - 1 \sim c\delta^2$.

on M_δ:

$$T_\delta(a,K) = \frac{\tau(a,K)}{\delta} + O(1)\,, \quad \text{where} \quad \tau(a,K) := \frac{2\pi}{\sqrt{1-(a/a_c(K))^2}}\,. \tag{2.3.19}$$

Actually, it is possible to replace in this formula $O(1)$ with $O(\delta)$: in fact it is possible to show by induction that the phase speed on M_δ admits an expansion in (integer) powers of δ to any order (but with coefficients less explicit than the first order one), and it is easy to see that $\dot\psi_\delta$ is an odd function of δ. We have tested numerically this approximation and we report the result in Table 2.1.

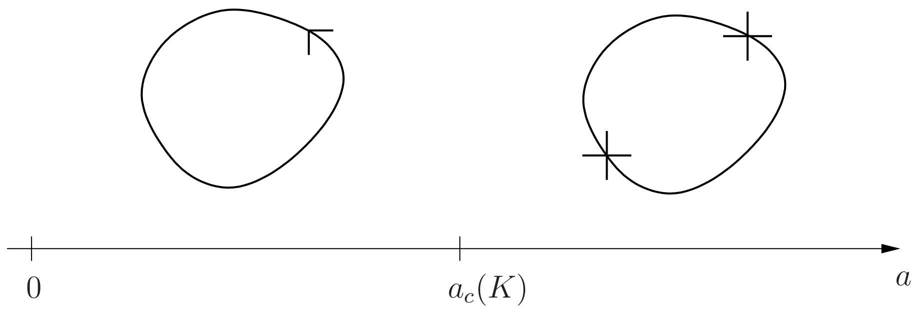


FIGURE 2.3. A sketch of the phase behavior for $V(\theta) = \theta - a\cos(\theta)$: for $a > K$ there are two fixed points, one attractive and one repulsive, while for $a < a_c(K)$ the force is bounded away from zero and the motion is periodic.

2.3.3 Active rotators with $V(\theta) = \theta - a\cos(j\theta)/j$, $j = 2, 3, \ldots$

In this case the $N \to \infty$ phase dynamics is lead by

$$f(\psi) = -\left(1 + a\frac{I_0 I_j}{I_0^2 - 1}\sin(j\psi)\right)\,, \tag{2.3.20}$$

and the behavior differs substantially from the $j = 1$ case (and the $j = 2$ case is different from the $j \geq 3$ case). In this case the crucial function is

$$a_{c,j}(K) := \frac{I_0^2 - 1}{I_0 I_j}\,. \tag{2.3.21}$$

Note that $a_{c,1} = a_c$. The criterion to have periodic behavior is, like for the $j = 1$ case, $a < a_{c,j}(K)$, while $a > a_{c,j}(K)$ leads to two fixed points. Figure 2.4 and its caption describes the (relatively surprising) phenomenology of the $j = 2$ and $j = 3$ cases (the case $j > 3$ is qualitatively the same as the case $j = 3$).

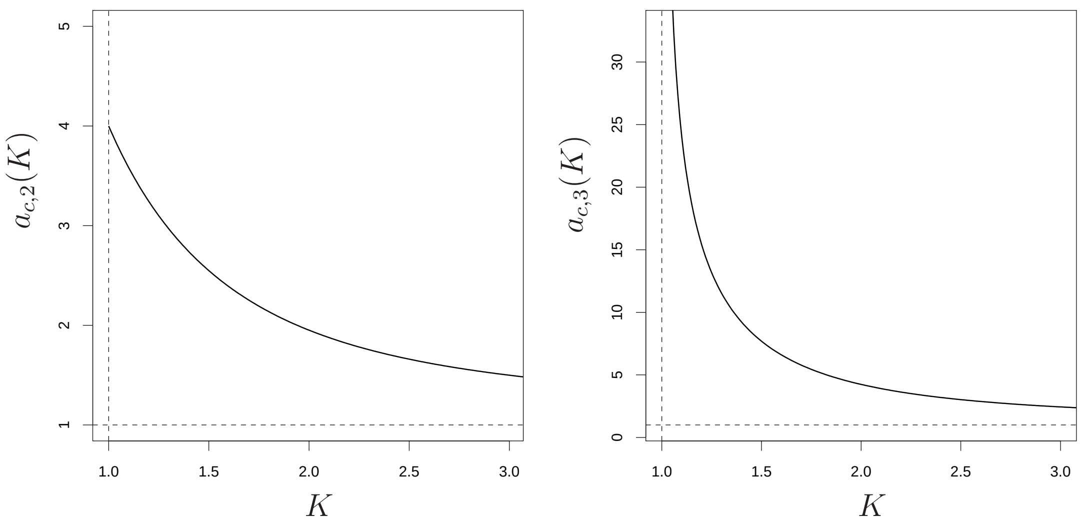


FIGURE 2.4. For $V(\theta) = \theta - a\cos(j\theta)/j$, $j \geq 2$, the $N \to \infty$ dynamics is always periodic for $a \leq 1$ (and δ sufficiently small), unlike the $j = 1$ case (recall that $a < a_{c,j}(K)$ corresponds to periodic motion, while $a > a_{c,j}(K)$ corresponds to two fixed points: see the text). Moreover, for $j = 2$ and $a > 4$ the dynamics has just two fixed points, but for $j \geq 3$ for arbitrarily large values of a one can observe periodic motion if K is sufficiently close to 1 (and, of course, δ sufficiently small).

Remark 2.3.4. Theorem 2.3.1 already tells us that one can produce arbitrary dynamics, so a very large variety of phenomena is observed. Here is a case that can be of some interest since it shows that playing on only one parameter one can produce three different dynamics (and the reader will directly infer how to induce arbitrarily many): if $V(\theta) = \theta - a(\cos(\theta) + \cos(2\theta))$ the $N \to \infty$ phase dynamics is lead by

$$f(\psi) \;=\; 1 + a\frac{I_0}{I_0^2 - 1}\left(I_1 \sin(\psi) + 2I_2 \sin(2\psi)\right) , \tag{2.3.22}$$

and in this case there can be two transitions as a varies. For example for $K = 2$ we have periodic behavior for $a < 0.600\ldots$, two fixed points if $a \in (0.600\ldots, 2.107\ldots)$ and four fixed points (of course two stable and two unstable ones) if $a > 2.107\ldots$.

2.4 Perturbation arguments

In this section we assume that $\delta \in (0, \delta_0]$ (cf. Theorem 2.2.1) and that we are on the invariant manifold M_δ of (2.2.20), that is $p_t^\delta \in M_\delta$ for every t. The result [99, Main Theorem, p. 495] actually contains also some estimates on the regularity of the semigroup on M_δ and notably that $t \mapsto p_t^\delta$ belongs to $C^0(\mathbb{R}; \widetilde{H}_1)$ and that it is (strongly) differentiable as a map from $\mathbb{R}$ to $\widetilde{H}_{-1}$. One directly sees that $\|u - v\|_{H_1} = \|u'' - v''\|_{-1}$, so that the right-hand side in (2.2.20) is $C^0(\mathbb{R}; H_{-1})$ and, in turn, $t \mapsto p_t^\delta$ is $C^1(\mathbb{R}, \widetilde{H}_{-1})$.

Since we are working in a neighborhood of M it is useful to introduce from now a parametrization of this region that will be particularly useful in the next section, but that we are going to use from now. The following facts are proven in Lemma 2.5.1: for every u in a sufficiently small H_{-1} neighborhood of M there exists a unique $q = v(u) \in M$ such that

$$(u - q, q')_{-1,1/q} \;=\; 0\,. \tag{2.4.1}$$

Furthemore $v \in C^1(\widetilde{H}_{-1}, \widetilde{H}_{-1})$ with differential

$$Dv(u) \,=\, P^o_{v(u)}\,. \tag{2.4.2}$$

Theorem 2.2.1 is telling us in particular that

$$v\left(q+\phi_\delta\left(q\right)\right) \,=\, q\,. \tag{2.4.3}$$

For the arguments that follow it is practical to use the notation introduced right after Theorem 2.2.1 and write

$$p^\delta_t \,=\, q_{\psi^\delta_t} + n^\delta_t\,, \tag{2.4.4}$$

where $q_{\psi^\delta_t} = v(p^\delta_t)$ and $n^\delta_t := \phi_\delta\left(q_{\psi^\delta_t}\right)$.

Proof of Theorem 2.2.2. Since the evolution on M_δ is $C^1(\mathbb{R}, \widetilde{H}_{-1})$, then $t \mapsto q_{\psi^\delta_t}$ is $C^1(\mathbb{R}, \widetilde{H}_{-1})$ too. This implies that, with f_1 and f_2 respectively sine and cosine, $\psi \mapsto \int_{\mathbb{S}} q_\psi(\theta) f_i(\theta) =: a_i(t)$ is C^1. Since $f_i(\psi_t) = a_i(t)/\sqrt{a_1^2(t)+a_2^2(t)}$, we see that $t \mapsto \psi^\delta_t$ is C^1. The fact that p^δ_t and $\psi^\delta_\cdot$ are C^1 directly implies that $t \mapsto n^\delta_t$ is $C^1(\mathbb{R}; H_{-1})$ (actually, since ϕ_δ is C^1 we have even $n^\delta_\cdot \in C^1(\mathbb{R}; L^2_0)$).

Notice furthermore that

$$-\dot{\psi}^\delta_t q'_{\psi^\delta_t} \,=\, P^o_{q_{\psi^\delta_t}} \partial_t p^\delta_t\,. \tag{2.4.5}$$

This follows by taking the time derivative of both sides of the equality $q_{\psi^\delta_t} = v(p^\delta_t)$ and by using (2.4.2).

Using (2.2.20) and the fact that q_ψ is a stationary solution of (2.2.1), we rewrite (2.4.5) as

$$-\dot{\psi}^\delta_t q'_{\psi^\delta_t} \,=\, P^o_{q_{\psi^\delta_t}}\left(-\partial_\theta\left[n^\delta_t(J*n^\delta_t)\right] + \delta G\left[q_{\psi^\delta_t}+n^\delta_t\right]\right)\,. \tag{2.4.6}$$

Recall that $\left\|n^\delta_t\right\|_2 \le C\delta$ (cf. Theorem 2.2.1): by

$$\|\left[n^\delta_t J * n^\delta_t\right]'\|_{-1} \,\le\, \|J\|_2 \|n^\delta_t\|_2^2 \,\le\, C^2\|J\|_2^2\delta^2\,, \tag{2.4.7}$$

and by the hypothesis on G that implies that

$$\left\|G\left[q_{\psi^\delta_t}+n^\delta_t\right] - G\left[q_{\psi^\delta_t}\right]\right\|_{-1} \,\le\, c_G C\delta\,, \tag{2.4.8}$$

from (2.4.6) we see that

$$\left\|\dot{\psi}^\delta_t q'_{\psi^\delta_t} + \delta G\left[q_{\psi^\delta_t}\right]\right\|_{-1} \le c\delta^2\,, \tag{2.4.9}$$

with c independent of t and of ψ^δ_0. To obtain (2.2.24) just take the $H_{-1,q_{\psi^\delta_t}}$ scalar product of $q'_{\psi^\delta_t}$ and the expression inside the norm in the left-hand side of (2.4.9).

For (2.2.26) rewrite (2.2.20) as

$$-\dot{\psi}^\delta_t q'_{\psi^\delta_t} - \partial_t n^\delta_t \,=\, -L_{q_{\psi^\delta_t}} n^\delta_t - \left[n^\delta_t J * n^\delta_t\right]' + \delta G\left[q_{\psi^\delta_t}+n^\delta_t\right]\,. \tag{2.4.10}$$

Note that for the second term on the left hand side we have

$$\|\partial_t n^\delta_{\psi^\delta_t}\|_{-1} \,\le\, c_K\|\partial_t n^\delta_{\psi^\delta_t}\|_2 \,\le\, c_K C\delta|\dot{\psi}^\delta_t|\,, \tag{2.4.11}$$

where we have use

$$\partial_t n^\delta_t \,=\, \dot{\psi}^\delta_t \partial_\psi \phi_\delta\left(q_\psi\right)\big|_{\psi=\psi^\delta_t}\,, \tag{2.4.12}$$

and the bound on the derivative of ϕ_δ given in Theorem 2.2.1.

Now plug (2.2.24) into (2.4.10) and use (2.4.7), (2.4.8) and (2.4.11) to obtain

$$\sup_{t,\psi_0^\delta} \left\| L_{q_{\psi_t^\delta}} n_{\psi_t^\delta} - \delta \left(G\left[q_{\psi_t^\delta}\right] - \frac{\left(G\left[q_{\psi_t^\delta}\right], q'_{\psi_t^\delta}\right)_{-1,1/q_{\psi_t^\delta}}}{(q',q')_{-1,1/q}} q'_{\psi_t^\delta} \right) \right\|_{-1} = O(\delta^2)\,. \tag{2.4.13}$$

Since ψ_0^δ can be chosen arbitrarily on $\mathbb{S}$, we can replace ψ_t^δ with ψ and take the supremum over ψ (and, by (2.2.2), we can freely switch between H_{-1} and $H_{-1,1/q_\psi}$ norms). Therefore (recall (2.2.25))

$$\sup_{\psi} \left\| L_{q_\psi}\left(n_\psi^\delta - \delta n_\psi\right) \right\|_{-1,1/q_\psi} = O(\delta^2)\,. \tag{2.4.14}$$

There result we are after, that is (2.2.26), follows from the equivalence of H_1 and V_q^2 (recall (2.2.9)) norms, which is proven in Appendix 2.A. □

Proof of Theorem 2.2.3. It is of course sufficient to estimate the numerator in the right-hand side of (2.2.27). It is the sum of two terms: the first one can be rewritten as

$$T_1(\psi) := \int_{\mathbb{S}} \phi_\delta(q_\psi) J * \phi_\delta(q_\psi) \left(1 - \frac{2\pi/q_\psi}{\int_{\mathbb{S}} 1/q}\right)\,, \tag{2.4.15}$$

and, by derivating and using the two L^2-estimates on $\phi_\delta(\cdot)$ and $D\phi^\delta(\cdot)$ in Theorem 2.2.1, it is straightforward to see that there exists $c > 0$ such that for $\delta \in [0, \delta_0]$

$$\sup_{\psi\in\mathbb{S}} |T_1'(\psi)| \le c\delta^2\,. \tag{2.4.16}$$

Let us turn to the second term, that is

$$T_2(\psi) = \int_{\mathbb{S}} \left(1 - \frac{2\pi/q_\psi(\theta)}{\int 1/q}\right) \int_0^\theta \left(G[q_\psi(\theta') + \phi^\delta(q_\psi(\theta'))] - G[q_\psi(\theta)]\right) d\theta' d\theta\,. \tag{2.4.17}$$

For this we write

$$H[y] = G[y + \phi^\delta(y)] - G[y]\,. \tag{2.4.18}$$

We have

$$DH[y] = DG[y + \phi^\delta(y)] - DG[y] + DG[y + \phi^\delta(y)] D\phi^\delta(y) \tag{2.4.19}$$

and thus, using the estimates of theorem 2.2.1 and the fact that DG is uniformly continuous on a neighborhood of M, we get that

$$\sup_{\psi\in\mathbb{S}} |T_2'(\psi)| \le l(\delta)\,. \tag{2.4.20}$$

with $l(\delta) = o(\delta)$ when $\delta \to 0$.

□

2.5 On the persistence of normally hyperbolic manifolds

In this section we prove theorem 2.2.1. The proof in a more general case can be found in [99] but we pay more attention on the relation between the various small parameters that enter the proof. We first give a lemma which defines a parametrisation in a neighbourhood of M using the scalar structure given by the operators L_q. The proof of this lemma is in [99, p. 501].

Lemma 2.5.1. *There exists a $\sigma > 0$ such that for all p in the neighborhood*

$$N_\sigma := \cup_{q\in M} B_{L^2}(q,\sigma), \tag{2.5.1}$$

of M there is one and only one $q = v(p) \in M$ such that $(p-q, q')_{-1,1/q} = 0$. Furthermore the mapping $p \mapsto v(p)$ is in $C^\infty(L^2_1, L^2_1)$, and

$$Dv(p) = P^o_{v(p)}. \tag{2.5.2}$$

Moreover, the analogous statement holds if N_σ is replaced by $\cup_{q\in M} B_{H_{-1}}(q,\sigma)$ and this time $p \mapsto v(p)$ is in $C^\infty(\widetilde{H}_{-1}, \widetilde{H}_{-1})$.

For the proof we look for conditions on δ in order to get a manifold, which is invariant for for (2.2.20), at distance ε from M: the condition in the end is going to be that δ needs to be smaller than a suitable constant times ε (and ε sufficiently small too), so that the invariant manifold is in a neighborhood of order δ of M. To simplify notations, we will write $F[u] = \partial_\theta(uJ * u)$, and (2.2.20) becomes:

$$\partial_t p_t = \frac{1}{2}\partial^2_\theta p_t - F[p_t] + \delta G[p_t]. \tag{2.5.3}$$

We will consider solutions with initial condition p_0 satisfying $\|p_0 - v(p_0)\|_2 \leqslant \varepsilon$. We need asumptions on ε and δ such that the solution stays in N_σ for a sufficiently long time. If q is in M, $w_t := p_t - q$ satisfies

$$w_t = e^{-tL_q} w_0 + \int_0^t e^{-(t-s)L_q}(F[w_s] + \delta G[q+w_s])\,\mathrm{d}s, \tag{2.5.4}$$

and we get

$$\|w_t\|_2 \leqslant \|w_0\|_2 + \int_0^t \|e^{-(t-s)L_q}\|_{\mathcal{L}(H_{-1},L_2)}(\|F[w_s]\|_{H_{-1}} + \delta\|G[q+w_s]\|_{H_{-1}})\,\mathrm{d}s. \tag{2.5.5}$$

Define

$$t_0 = \sup\{t \geqslant 0 :\ \|w_s\|_2 \leqslant \sigma \text{ for every } s \leqslant t\}. \tag{2.5.6}$$

Because of the continuity of w_t, $t_0 > 0$ if we suppose $\varepsilon < \sigma$. If $t \leqslant t_0$, using the spectral properties of L_q and the regularity of F and G, we get the bounds

$$\|e^{-(t-s)L_q}\|_{\mathcal{L}(H_{-1},L_2)} \leqslant C_L(1 + (t-s)^{-1/2}), \tag{2.5.7}$$

$$\|G[q+w_s]\|_{H_{-1}} \leqslant C_G(1 + \|w_s\|_2), \tag{2.5.8}$$

and

$$\|F[w_s]\|_{H_{-1}} \leqslant C_F\|w_s\|_2^2, \tag{2.5.9}$$

and thus for all $t_1 < t_0$

$$\|w_{t_1}\|_2 \leqslant (\varepsilon + C_G C_L(t_1 + 2\sqrt{t_1})\delta) + C_L(C_F\sigma + C_G\delta)\int_0^{t_1}\left(1 + \frac{1}{\sqrt{t_1 - s}}\right)\|w_s\|_2\,\mathrm{d}s. \tag{2.5.10}$$

We need the following lemma, that is a version of the Gronwall-Henry inequality

Lemma 2.5.2. *Let $t \mapsto y_t$ be a non-negative and continuous function on $[0,T)$ satisfying for all $t \in [0,T)$*

$$y_t \leqslant \eta_0 + \eta_1 \int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right) y_s \,\mathrm{d}s \,. \tag{2.5.11}$$

Then for all $t \in [0,T)$

$$y_t \leqslant 2\eta_0 e^{\alpha t}\,, \tag{2.5.12}$$

with $\alpha = 2\eta_1 + 4\eta_1^2 \left(\Gamma\left(\frac{1}{2}\right)\right)^2$ where $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}\,\mathrm{d}x$.

Proof of lemma 2.5.2 We consider the time

$$t^* = \sup\{t \geqslant 0, y_s \leqslant 2\eta_0 e^{\alpha s} \text{ for all } s \leqslant t\}\,. \tag{2.5.13}$$

We have to show that $t^* = T$. But if $t^* < T$, then

$$\begin{aligned} y_{t^*} &\leqslant \eta_0 \left(1 + 2\eta_1 \int_0^{t^*} \left(1 + \frac{1}{\sqrt{t^*-s}}\right) e^{\alpha s} ds\right) \\ &\leqslant \eta_0 \left(1 + \frac{2\eta_1}{\alpha}[e^{\alpha t^*} - 1] + \frac{2\eta_1}{\sqrt{\alpha}}\Gamma\left(\frac{1}{2}\right) e^{\alpha t^*}\right) < 2\eta_0 e^{\alpha t^*}\,, \end{aligned} \tag{2.5.14}$$

which contradicts $t^* < T$ since $y_.$ is continuous. □

Using Lemma 2.5.2 and (2.5.10) we get :

$$\|w_t\|_2 \leqslant C(t_1)(\delta + \varepsilon)\,, \tag{2.5.15}$$

where

$$C(t_1) = \max(1, C_G C_L(t_1 + 2\sqrt{t_1}))e^{\left(2\eta(\sigma,\delta) + 4\pi\eta(\sigma,\delta)^2\right)t_1}\,, \tag{2.5.16}$$

$$\eta(\sigma,\delta) = C_L(C_F\sigma + C_G\delta)\,. \tag{2.5.17}$$

For $T > 0$, if we choose ε and δ such that $C(2T)(\varepsilon + \delta) \leqslant \sigma$, then p_t lies in N_σ for $t \in [0; 2T]$. Take now T such that

$$C_{P^s} e^{-\lambda_1 T/2} \leqslant \frac{1}{16}\,, \tag{2.5.18}$$

$$e^{\lambda_1 T/2} \geqslant 4C_L\,, \tag{2.5.19}$$

where we recall that λ_1 is the spectral gap of L_q and we set

$$C_{P^s} = \max_{q \in M} \|P_q^s\|_{\mathcal{L}(L_0^2, L_0^2)}\,, \tag{2.5.20}$$

and P_q^s is a compact notation for $P^s(q)$ (defined just below (2.2.14)) and it is the orthogonal projection of the range of L_q (the scalar product is the one of $H_{-1,1/q}$). Define also

$$C_1 = C(2T), \quad C_2 = e^{\lambda_1 T/2} \text{ and } \varepsilon_0 = \frac{\sigma}{2C_1}\,. \tag{2.5.21}$$

For now we will take $\max\{\varepsilon, \delta\} \leqslant \varepsilon_0$, so that $p_t \in N_\sigma$ for $t \leq 2T$. We will use the following notations:

$$p_i := p(t, p_{i0})\,, \tag{2.5.22}$$

is the solution of (2.5.3) and

$$v_i := v_i(t, p_{i0}) := v(p_i)\,, \tag{2.5.23}$$

is given by Lemma 2.5.1. Moreover we set

$$n_i = p_i - v_i\,, \quad \Delta p := p_1 - p_2\,, \quad \Delta v := v_1 - v_2\,, \quad \Delta n := n_1 - n_2\,. \tag{2.5.24}$$

In the following lemma we compare the quantites we have just introduced with the initial conditions. It corresponds to Lemma 74.7 (page 507) in [99]. We remark that is in this lemma ε and δ play the same role and we stress that these are just preliminary estimates: some of them are going to be refined later on.

Lemma 2.5.3. *For all $\alpha > 0$, there exist $C_0 = C_0(T)$ and $\varepsilon_1 \leqslant \varepsilon_0$ such that if $\varepsilon \leqslant \varepsilon_1$ and $\delta \leqslant \varepsilon_1$ we have the following properties:*

1. *if $\|p_0 - v_0\|_2 \leqslant \varepsilon$ then for all $t \in [0, 2T]$*

$$\max\left(\|p(t,p_0) - v_0\|_2\,,\, \|v(t,p_0) - v_0\|_2\,,\, \frac{1}{2}\|n(t,p_0)\|_2 \right) \leqslant C_0(\varepsilon + \delta)\,; \tag{2.5.25}$$

2. *if $\|p_{i0} - v_{i0}\|_2 \leqslant \varepsilon$ and $\|\Delta v(0)\|_2 \leqslant \alpha\varepsilon$, then for all $t \in [0, 2T]$*

$$\max\left(\|\Delta p(t)\|_2\,,\, \|\Delta v(t)\|_2\,,\, \|\Delta n(t)\|_2\right) \leqslant C_2\|\Delta p(0)\|_2\,, \tag{2.5.26}$$

   with C_2 given in (2.5.21)*;*

3. *if $\|p_{i0} - v_{i0}\|_2 \leqslant \varepsilon$ and $\|\Delta p(0)\|_2 \leqslant 2\|\Delta v(0)\|_2$, then for all $t \in [0, 2T]$*

$$\frac{1}{2}\|\Delta v(0)\|_2 \leqslant \|\Delta v(t)\|_2 \leqslant \frac{3}{2}\|\Delta v(0)\|_2\,. \tag{2.5.27}$$

Proof of Lemma 2.5.3 For what concerns part (1) note that the first of the three inequalities in (2.5.25) is given above (see (2.5.15) with $t_0 = 2T$). The other inequalities come from the fact that the mapping $q \mapsto v(q)$ of Lemma 2.5.1 is Lipschitz, taking, if necessary, a bigger value for C_0.

For part (2)notice that, since $v_{20} \in M$, we can write the evolution in mild form around v_{20}, that is

$$\begin{aligned}\Delta p(t) &= e^{-tL_{v_{20}}}\Delta p(0)\\ &+ \int_0^t e^{-(t-s)L_{v_{20}}}\left(F[p_1(s) - v_{20}] - F[p_2(s) - v_{20}] + \delta(G[p_1(s)] - G[p_2(s)])\right) \mathrm{d}s\,,\end{aligned} \tag{2.5.28}$$

and thus

$$\|\Delta p(t)\|_2 \leqslant C_L\|\Delta p(0)\|_2 + C_L\Big(C_F(\alpha\varepsilon + C_0(\varepsilon+\delta) + C_G\delta\Big) \int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right) \|\Delta p(s)\|_2\, \mathrm{d}s\,. \tag{2.5.29}$$

Here we used the preceding point, (2.5.7) and the bounds

$$\|G[p_1(s)] - G[p_2(s)]\|_{-1} \leqslant C_G\|p_1(s) - p_2(s)\|_2\,, \tag{2.5.30}$$

$$\|F[p_1(s)] - F[p_2(s)]\|_{-1} \leqslant C_F(\alpha\varepsilon + C_0(\varepsilon + \delta))\|p_1(s) - p_2(s)\|_2\,, \tag{2.5.31}$$

(2.5.31) is obtained by applying the mean value inequality to F and DF and using the fact that $DF(0) = 0$: the constants C_G and C_F have a larger value than in (2.5.8) and (2.5.9). Applying Lemma 2.5.2 to (2.5.29), we obtain

$$\|\Delta p(t)\|_2 \leqslant 2C_L e^{\left(2\eta_1(\varepsilon,\delta) + 4\pi\eta_1(\varepsilon,\delta)^2\right)2T}\|\Delta p(0)\|_2 \tag{2.5.32}$$

with

$$\eta_1(\varepsilon,\delta) = C_L\Big(C_F(\alpha\varepsilon + 2C_0(\varepsilon+\delta)) + C_G\delta\Big), \tag{2.5.33}$$

Choose $\varepsilon_1 \leqslant \varepsilon_0$ such that (it is possible because of (2.5.19))

$$2C_L e^{\left(2\eta_1(\varepsilon_1,\varepsilon_1)+4\pi\eta_1(\varepsilon_1,\varepsilon_1)^2\right)2T} \leqslant e^{\lambda_1 T/2}. \tag{2.5.34}$$

The two other points come directly from the Lipschitz property of the mapping $q \mapsto v(q)$ taking, if necessary, a smaller value for ε_1.

For part (3) we prove first that for all $r > 0$, there exists $\varepsilon_2(r)$ such that for all $\varepsilon \leqslant \varepsilon_2(r)$ and $\delta \leqslant \varepsilon_2(r)$ we have for all $t \in [0, 2T]$

$$\frac{1}{2} \leqslant \frac{\|\Delta v(t)\|_2}{\|\Delta v(0)\|_2} \leqslant \frac{3}{2} \tag{2.5.35}$$

if $\|\Delta v(0)\|_2 \geqslant r$. In fact, in this case, using Lemma 2.5.1 :

$$\begin{aligned}\left|\frac{\|\Delta v(t)\|_2 - \|\Delta v(0)\|_2}{\|\Delta v(0)\|_2}\right| &\leqslant \frac{|\;\|\Delta v(t)\|_2 - \|\Delta v(0)\|_2\;|}{r}\\ &\leqslant \frac{\|\Delta v(t) - \Delta v(0)\|_2}{r}\\ &\leqslant \frac{\|v_1(t)-v_1(0)\|_2 + \|v_2(t)-v_2(0)\|_2}{r}\\ &\leqslant \frac{2C_0(\delta+\varepsilon)}{r}.\end{aligned} \tag{2.5.36}$$

We can choose $\varepsilon_2(r) = \min(\varepsilon_1, r/8C_0)$. Now it is sufficient to prove that, for $\|\Delta v(0)\|_2 \leqslant r_0$ with a certain r_0,

$$\|\Delta v(t) - \Delta v(0)\|_2 \leqslant \frac{1}{2}\|\Delta v(0)\|_2 \tag{2.5.37}$$

for all $t \in [0, 2T]$. Suppose that $\|\Delta v(0)\|_2 \leqslant r$ with $r \leqslant \alpha$. We use the following decomposition

$$\begin{aligned}\Delta v(t) - \Delta v(0) = &\Delta v(t) - P^o_{v_2}\Delta p(t) - \Delta v(0) + P^o_{v_{20}}\Delta p(0)\\ &+ (P^o_{v_2} - P^o_{v_{20}})\Delta p(t) + P^o_{v_{20}}(\Delta p(t) - \Delta p(0)).\end{aligned} \tag{2.5.38}$$

From Lemma 2.5.1 , part (2) and the hypothesis $\|\Delta p(0)\|_2 \leqslant 2\|\Delta v(0)\|_2$, we get

$$\begin{aligned}\|\Delta v(t) - P^o_{v_2}\Delta p(t)\|_2 &= \|v(p_1) - v(p_2) - Dv_{v_2}(p_1 - p_2)\|_2\\ &\leqslant 2C_v(r + 2C_0(\delta+\varepsilon))\|\Delta v(0)\|_2 \qquad (2.5.39)\\ \|\Delta v(0) - P^o_{v_{20}}\Delta p(0)\|_2 &= \|v(p_{10}) - v(p_{20}) - Dv_{v_{20}}(p_{10} - p_{20})\|_2\\ &\leqslant 4C_v r\|\Delta v(0)\|_2 \qquad (2.5.40)\end{aligned}$$

where C_v is the maximum of the second derivate of $q \mapsto v(q)$ in N_σ. Since P^o is C^1 on M, there exits L_{P^o} such that (By using part (1) and the hypothesis and defining L_{P^o} the maximum of the norme of DP on M, we get

$$\|(P^o_{v_2} - P^o_{v_{20}})\Delta p(t)\|_2 \leqslant L_{P^o}C_0(\delta+\varepsilon)\|\Delta v(0)\|_2 \tag{2.5.41}$$

For the last term, we write

$$\begin{aligned}\Delta p(t) - \Delta p(0) = (e^{-tL_{v_{20}}} - I)\Delta p(0) + \int_0^t e^{-(t-s)L_{v_{20}}}(F[p_{1s} - v_{20}] - F[p_{2s} - v_{20}]\\ + G[p_{1s}] - G[p_{2s}])ds.\end{aligned} \tag{2.5.42}$$

Notice that $P^o_{v20}(e^{-tL_{v20}} - I)\Delta p(0) = 0$. From (2.5.7), (2.5.31) and (2.5.30) it comes

$$\|P^o_{v_{20}}(\Delta p(t) - \Delta p(0))\|_2 \leqslant 4C_{P^o}C_LC_2\Big(C_F(r + 2C_0(\delta+\varepsilon)) + C_G\delta\Big)(T+\sqrt{2T})\|\Delta v(0)\|_2 \tag{2.5.43}$$

where C_{P^o} is the maximum of the norms $\|P^o_q\|_{\mathcal{L}(L^2_0,L^2_0)}$ for $q \in M$. In conclusion there exists a constant C_3 such that for all $t \in [0, 2T]$

$$\|\Delta v(t) - \Delta v(0)\|_2 \leqslant C_3(r+\varepsilon+\delta)\|\Delta v(0)\|_2\,. \tag{2.5.44}$$

To end the proof choose $r = r_0 := \min\left(\alpha, \frac{C_3}{3}\right)$ and reduce if nececessary the value of ε_1 to have $\varepsilon_1 \leqslant \min(\varepsilon_2(r_0), r_0)$.

□

We now move to the main body of the proof which is based on introducing a family of transformations of the manifold M by using the full dynamics and we aim at identifying the transformation that maps M to the manifold that is stable for the full dynamics and this is achieved by applying the Banach fixed point Theorem in a relevant space of functions.

Define the set $C(M, L^2_0)$ of continuous functions from M to L^2_0 provided with the norm

$$\|f\|_\infty = \sup\{\|f(v)\|_{L^2}\ ,\ v \in M\} \tag{2.5.45}$$

and consider the subset $\mathcal{F}(\varepsilon, l)$ of $C(M, L^2_0)$ of functions f satisfying :

1. $\|f\|_\infty \leqslant \varepsilon$
2. f is Lipschitz on M with Lipschitz constant $l \leqslant 1$
3. $(f(q), q')_{-1,1/q} = 0$ for all q in M

Notice that $\mathcal{F}(\varepsilon, l)$ is a complete subset of $C(M, L^2_0)$. We will now define a set of mappings $\{X_t\}_{t\in[T,2T]} : \mathcal{F}(\varepsilon, 1) \mapsto C(M, L^2_0)$ and show that

1. for all $\tau \in [T, 2T]$
$$X_\tau\left(\mathcal{F}(\varepsilon, 1)\right) \subset \mathcal{F}\left(\varepsilon, \frac{1}{4C_2}\right) \tag{2.5.46}$$
(recall that $C_2 = e^{\lambda_1 T/2}$ and thus $\frac{1}{4C_2} \leqslant 1$)
2. X_T is a contraction on $\mathcal{F}(\varepsilon, 1)$:
$$\|X_T(f_1) - X_T(f_2)\|_\infty \leqslant \frac{1}{2}\|f_1 - f_2\|_\infty \tag{2.5.47}$$
for all $f_1, f_2 \in \mathcal{F}(\varepsilon, 1)$.

Notice that the third point of (2.5.3) and an argument of connexion (see [99] page 513) show that for all $f \in \mathcal{F}(\varepsilon, 1)$ and $t \in [0, 2T]$, the mapping $q \mapsto g_{t,f}(q) := v(t, f(q))$ is a bijection of M. So we can define the mappings

$$\begin{array}{rccc} X_\tau(f)(u): & M & \to & L^2_0 \\ & u & \mapsto & n\left(\tau, (i_d + f)\circ g^{-1}_{\tau,f}(u)\right) \end{array} \tag{2.5.48}$$

It is easy to see that for all $\tau \in [T, 2T]$ and $f \in \mathcal{F}(\varepsilon, 1)$, $X_\tau(f)$ is the unique mapping satisfying for all $q \in M$

$$X_\tau(f)(v(\tau, p_0)) = n(\tau, p_0) = p(\tau, p_0) - v(\tau, p_0) = P^s_{v(\tau,p_0)}(p(\tau, p_0) - v(\tau, p_0)) \tag{2.5.49}$$

where $p_0 = q + f(q)$. We can see $X_t(f)$ as the *distance* (in the sense of (2.5.1)) of the trajectory p_t from M, starting at the time 0 at a *distance* f from M.

In the following, we will first prove that (2.5.46) and (2.5.47) imply that there exists an invariant manifold M_ε for (2.5.3) at distance ε of M. Then we will prove (2.5.46) and (2.5.47) in three lemmas, paying attention on the relations between the different parameters.

Suppose that the mappings X_τ satisfy (2.5.46) for $\tau \in, [T, 2T]$ and that X_T satisfies (2.5.47). Then X_T has a unique fixed point in $\mathcal{F}(\varepsilon, 1)$, which will be noted f_0. Define $\phi^\varepsilon = id + f_0$ on M and $M_\varepsilon = \phi_0(M)$. Since f_0 is a fixed point of X_T, if $p_0 \in M_0$, then $p_{kT} \in M_0$ for all $k \in \mathbb{N}$. Then to prove that M_ε is an invariant manifold of (2.5.3), it is sufficient to prove that for all $t \in (0, T)$, the functions f_t defined by $f_t = X_t(f_0)$ are equal to f_0. Using the property of semi-group and $X_T(f_0) = f_0$ it is easy to see that $f_t = X_{T+t}(f_0)$, and thus (2.5.46) implies that $f_t \in \mathcal{F}(\varepsilon, 1)$. But the same arguments show that f_t is a fixed point of X_T for all $t \in (0, T)$. In conclusion, M_ε is invariant for (2.5.3).

Now we prove (2.5.46) and (2.5.47) in the three following lemmas, which correspond to Lemmas 74.8, 74.9 and 74.10 in [99].

Lemma 2.5.4. *There exists a* $\varepsilon_3 \leqslant \varepsilon_1$ *such that if* $\varepsilon \leqslant \varepsilon_3$, *there exists a* $\delta_3(\varepsilon)$ *of the form* $\min(C\varepsilon, \varepsilon_3)$ *such that if* $\delta \leqslant \delta_3(\varepsilon)$, *we have for all* $\tau \in [T, 2T]$ *and* $f \in \mathcal{F}(\varepsilon, 1)$

$$\|X_\tau(f)\|_\infty \leqslant \varepsilon \tag{2.5.50}$$

Proof Let $v_0 \in M$, $p_0 = v_0 + f(v_0)$. We write (see (2.5.49))

$$X_\tau(f)(v(\tau)) = P^s_{v(\tau)}(p(\tau) - v_0) - P^s_{v(\tau)}(v(\tau) - v_0)\,. \tag{2.5.51}$$

The first term can be written as

$$P^s_{v(\tau)}(p(\tau) - v_0) = P^s_{v(\tau)}\left(e^{-\tau L_{v_0}}(p_0 - v_0) + \int_0^\tau F[p(s) - v_0] + G[p(s)]ds\right)\,. \tag{2.5.52}$$

Using the spectral gap, we bound the linear term

$$\|P^s_{v(\tau)} e^{-\tau L_{v_0}}(p_0 - v_0)\|_2 \leqslant C_{P^s} e^{-\lambda_1 \tau}\varepsilon \tag{2.5.53}$$

and the remaining term of (2.5.52) can be bounded in the same way as (2.5.43). Furthemore the second term of (2.5.51) is quadratic in ε and δ, using a Taylor argument as in (2.5.40). Finaly, we get

$$\|X_\tau(f)(v_\tau)\|_2 \leqslant C_4\Big((\delta + \varepsilon)^2 + \delta)\Big) + C_{P^s} e^{-\lambda_1 \tau}\varepsilon\,. \tag{2.5.54}$$

We supposed $C_{P^s} e^{-\lambda_1 T} \leqslant \frac{1}{16}$, thus we can choose

$$\varepsilon_3 = \min\left(\varepsilon_1, \frac{1}{12C_4}\right) \quad \text{and} \quad \delta_3(\varepsilon) = \min\left(\varepsilon_1, \varepsilon, \frac{1}{3C_4}\varepsilon\right)\,. \tag{2.5.55}$$

□

Lemma 2.5.5. *There exists* $\varepsilon_4 \leqslant \varepsilon_3$ *such that if* $\varepsilon \leqslant \varepsilon_4$, *there exists a* $\delta_4(\varepsilon)$ *of the form* $\min(C\varepsilon, \varepsilon_4)$ *such that if* $\delta \leqslant \delta_4(\varepsilon)$, *then for all* $f \in \mathcal{F}(\varepsilon, 1)$ *we have* $X_\tau(f) \in \mathcal{F}\left(\varepsilon, \frac{1}{4C_2}\right)$ *for all* $\tau \in [T, 2T]$.

Proof It is sufficient to prove that $X_\tau(f)$ is Lipschitz with Lipschitz constant $\frac{1}{4C_2}$ on all $M \cap B_2(q, \rho_0)$ with $\rho_0 = 8C_2\varepsilon$. Indeed in this case, if $\|q_1 - q_2\|_2 > \rho_0$, then

$$\|X_\tau(f)(q_1) - X_\tau(f)(q_2)\|_2 \leqslant \frac{2\varepsilon}{\rho_0}\|q_1 - q_2\|_2 \leqslant \frac{1}{4C_2}\|q_1 - q_2\|_2 . \tag{2.5.56}$$

Take $u_1, u_2 \in M$ such that $\|u_1 - u_2\|_2 \leqslant \rho_0$ and f with Lipschitz constant $l \leqslant 1$. There exists $v_{10}, v_{20} \in M$ such that $u_i = v(\tau, p_{i0})$ with $p_{i0} = v_{i0} + f(v_{i0})$. Our goal is to show that under the hypothesis

$$\frac{\|X_\tau(f)(u_1) - X_\tau(f)(u_2)\|_2}{\|u_1 - u_2\|_2} = \frac{\|X_\tau(f)(v_1(\tau)) - X_\tau(f)(v_2(\tau))\|_2}{\|v_1(\tau) - v_2(\tau)\|_2} = \frac{\|\Delta n(\tau)\|_2}{\|\Delta v(\tau)\|_2} \leqslant \frac{1}{4C_2} . \tag{2.5.57}$$

We use the decomposition

$$\begin{aligned}
\Delta n(\tau) &= e^{-\tau L_{v20}} P^s_{v20} \Delta n(0) + \Delta n(\tau) - e^{-\tau L_{v20}} P^s_{v20} \Delta n(0) \\
&= e^{-\tau L_{v20}} P^s_{v20} \Delta n(0) + \Delta p(\tau) - \Delta v(\tau) - e^{-\tau L_{v20}} P^s_{v20} \Delta p(0) + e^{-\tau L_{v20}} P^s_{v20} \Delta v(0) \\
&= e^{-\tau L_{v20}} P^s_{v20} \Delta n(0) + \Delta p(\tau) - P^o_{v20} \Delta p(t) + P^o_{v20} \Delta p(t) - \Delta v(\tau) - e^{-\tau L_{v20}} P^s_{v20} \Delta p(0) \\
&\quad + e^{-\tau L_{v20}} P^s_{v20} \Delta v(0) \\
&= \left[e^{-\tau L_{v20}} P^s_{v20} \Delta n(0)\right] + \left[(P^s_{v_2(\tau)} - P^s_{v20}) \Delta p(\tau)\right] + \left[P^s_{v20}(\Delta p(\tau) - e^{-\tau L_{v20}} \Delta p(0))\right] \\
&\quad + \left[e^{-\tau L_{v20}} P^s_{v20} \Delta v(0)\right] + \left[P^o_{v_2(\tau)} \Delta p(\tau) - \Delta v(\tau)\right] .
\end{aligned}$$

We bound the first term using the spectral gap of $L_{v_{20}}$, the second term using the smoothness of P^s and Lemma 2.5.3, and the third term in a similar way as (2.5.43). We use a Taylor decomposition for the two last terms, as in (2.5.40). Then we get (recall (2.5.20))

$$\|\Delta n(\tau)\|_2 \leqslant \Big(C_{P^s} e^{-\lambda_1 T} l + C_5(\rho_0 + \delta + \varepsilon)\Big)\|\Delta v(0)\|_2 . \tag{2.5.58}$$

Since f is Lipschitz with Lipschitz constant $l \leqslant 1$ we have $\|\Delta p(0)\|_2 \leqslant 2\|\Delta v(0)\|_2$. Then using the part (3) of Lemma 2.5.3 we deduce

$$\|\Delta v(0)\|_2 \leqslant 2\|\Delta v(\tau)\|_2 . \tag{2.5.59}$$

Furthemore we have chosen T such that $C_{P^s} e^{-\lambda_1 T/2} \leqslant \frac{1}{16}$, and thus $C_{P^s} e^{-\lambda_1 T} \leqslant \frac{1}{16C_2}$ (recall that $C_2 = e^{\lambda_1/2}$). We obtain

$$\frac{\|\Delta n(\tau)\|_2}{\|\Delta v(\tau)\|_2} \leqslant \frac{1}{8C_2} l + 2C_5((1 + 8C_2)\varepsilon + \delta) . \tag{2.5.60}$$

Finally choose

$$\varepsilon_4 = \min\left(\varepsilon_3, \frac{1}{32C_2C_5(1 + 4C_2)}\right) \quad \text{and} \quad \delta_4(\varepsilon) = \min\left(\varepsilon_4, \delta_3(\varepsilon)\right) , \tag{2.5.61}$$

and the proof is complete. □

Lemma 2.5.6. *There exists $\varepsilon_5 \leqslant \varepsilon_4$ such that if $\varepsilon \leqslant \varepsilon_5$, there exists a $\delta_5(\varepsilon)$ of the form $\min(C\varepsilon, \varepsilon_5)$ such that for all $f_i \in \mathcal{F}\left(\varepsilon, \frac{1}{4C_2}\right)$:*

$$\|X_T(f_1) - X_T(f_2)\|_\infty \leqslant \frac{1}{2}\|f_1 - f_2\|_\infty . \tag{2.5.62}$$

Proof This time take $v_{10} = v_{20} = v_0$ and $p_{i0} = v_0 + f_i(v_0)$. With the same decomposition as in Lemma 2.5.5 (with fewer terms, since $v_{10} = v_{20}$) we get

$$\|\Delta n(T)\|_2 \leqslant \Big(C_{P^s} e^{-\lambda_1 T} + C_6(\delta + \varepsilon)\Big) \|\Delta p(0)\|_2 \,. \tag{2.5.63}$$

We choose

$$\varepsilon_5 \;=\; \min\left(\varepsilon_4, \frac{1}{16C_6}\right) \quad \text{and} \quad \delta_5(\varepsilon) \;=\; \min\left(\varepsilon_5, \delta_4(\varepsilon)\right) \,, \tag{2.5.64}$$

and in this case we get

$$\|\Delta n(T)\|_2 \leqslant \frac{1}{4}\|f_1 - f_2\|_\infty \,. \tag{2.5.65}$$

Now notice that

$$\|(X_T(f_1) - X_T(f_2))(v_2(T))\|_2 \leqslant \|\Delta n(T)\|_2 + \|X_T(f_1)(v_1(T)) - X_T(f_1)(v_2(T))\|_2 \,, \tag{2.5.66}$$

and since $X_T(f_1)$ is Lipschitz with Lipschitz constant $\frac{1}{4C_2}$, we get, using Lemma 2.5.1

$$\|X_T(f_1)(v_1(T)) - X_T(f_1)(v_2(T))\|_2 \leqslant \frac{1}{4C_2}\|\Delta v(T)\|_2 \leqslant \frac{1}{4}\|f_1 - f_2\|_\infty \,. \tag{2.5.67}$$

□

Proof of Theorem 2.2.1. In these three lemmas, we see that if ε is small enough, we can take δ proportional to ε, thus adding a perturbation of type $\delta G[p_t]$ to (2.2.1) creates an invariant manifold M_δ situated at a distance $O(\delta)$ from M. It is proven in [99, (theorem 74.15, p. 531)] that the manifold M_δ is C^1 in L^2_1 and normally hyperbolic. Remark furthermore that $\left(\phi^\delta\right)^{-1}(p) = v(p)$ for all $p \in M_\delta$. So to prove that ϕ^δ is C^1, it suffices to prove that v satisfies the hypothesis of the local inverse theorem between manifolds, that is Dv is a bijection between the tangent spaces of de two manifolds. Since the manifold is of dimension one, this property is implied by the lipschitz property of ϕ^δ. Furthermore we can estimate the differential of ϕ^δ : (2.5.60) for ϕ^δ gives an inequality for the local Lipschitz constant l^δ of ϕ_δ on all neighborhoods $M \cup B_2(q, \rho_0)$ (ρ_0 is introduced right before (2.5.56)):

$$l^\delta \leqslant \frac{1}{8C_2} l^\delta + C_7 \delta \,, \tag{2.5.68}$$

and we get that for a $C_8 > 0$

$$l^\delta \leqslant C_8 \delta \,, \tag{2.5.69}$$

which yields the bound we claim on the differential of ϕ_δ. □

2.A On a norm equivalence

The goal is to prove that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_{V^2_q}$ are equivalent, with

$$-L_q u \;:=\; \frac{1}{2}u'' - [uJ * q + qJ * u]' \tag{2.A.1}$$

and

$$\|u\|_{V^2_q} \;:=\; \|(C + L_q)u\|_{-1,1/q} \tag{2.A.2}$$

with $C > 0$. Remark that by changing the constant C we get an equivalent norm. Since the norms $\|\cdot\|_{-1,1/q}$ and $\|\cdot\|_{-1}$ are equivalent, we will study $\|(C + L_q)u\|_{-1}$. We write

$$u(\theta) \;=\; \sum a_n e^{in\theta} \,. \tag{2.A.3}$$

$J * q$ is of the type $\alpha e^{i\theta} - \alpha e^{-i\theta}$, thus we can write

$$uJ * q(\theta) = \alpha \left(\sum a_n e^{i(n+1)\theta} - \sum a_n e^{i(n-1)\theta} \right) . \tag{2.A.4}$$

Furthermore

$$J * u(\theta) = -\frac{Ka_1}{2i} e^{i\theta} + \frac{Ka_{-1}}{2i} e^{-i\theta} . \tag{2.A.5}$$

So if we denote

$$q(\theta) = \sum c_n^q e^{i\theta} \tag{2.A.6}$$

then

$$qJ * u = -\frac{Ka_1}{2i} \sum c_n^q e^{(n+1)\theta} + \frac{Ka_{-1}}{2i} \sum c_n^q e^{(n-1)\theta} . \tag{2.A.7}$$

Consequently

$$\|(C + L_q)u\|_{-1} =$$
$$\sum (1+n^2)^{-1} \left| Ca_n + n^2 a_n - i\alpha n(a_{n-1} - a_{n+1}) + n\frac{Ka_1}{2} c_{n-1}^q - n\frac{Ka_{-1}}{2} c_{n+1}^q \right|^2 . \tag{2.A.8}$$

Suppose now that $u \in H_1$. It is easy to see that there exists $c > 0$ such that $\|u\|_{V_q^2} \leqslant c\|u\|_{H_1}$. Thus $\sum n^2 |a_n|^2 < \infty$ implies that $\|(C + L_q)u\|_{-1} < \infty$ and so $H_1 \subset V_q^2$. By expanding (2.A.8) and using Cauchy-Schwartz inequality we get

$$\|(C + L_q)u\|_{-1} \geqslant \sum (1+n^2)^{-1} \left(C^2 + n^4 + 2Cn^2 - \alpha_1 n^3 - \alpha_2 Cn \right) |a_n|^2 \tag{2.A.9}$$

where $\alpha_1, \alpha_2 \geqslant 0$ do not depend on u. It is clear that for C big enough (depending on α_1 and α_2) we have

$$\frac{C^2}{2} + n^4 - \alpha_1 n^3 \geqslant \frac{1}{2} n^4 \tag{2.A.10}$$

$$\frac{C^2}{2} + 2Cn^2 - \alpha_2 Cn \geqslant 0 \tag{2.A.11}$$

and thus $\|(C + L_q)u\|_{-1} \geqslant \frac{1}{4}\|u\|_{H_1}$. We have shown that there exist $c > 0$ such that for all $u \in H_1$,

$$c^{-1}\|u\|_{V_q^2} \leqslant \|u\|_{H_1} \leqslant c\|u\|_{V_q^2} . \tag{2.A.12}$$

But H_1 is dense in V_q^2 (consider the finite sums of fourier series). If $v \in V_q^2$, there exists a sequence v_n in H_1 such that $v_n \to v$ for the V_q^2 norm. Then v_n is a Cauchy sequence for the H_1 norm, and since H_1 is complete, $v \in H_1$. In conclusion V_q^2 and H_1 have the same elements.

Remark 2.A.1. *By replacing* $(1+n^2)^{-1}$ *by* $(1+n^2)^k$*, we can prove in the same way that* $\|(C + L_q)u\|_{H_k}$ *is equivalent to* $\|u\|_{H_{k+2}}$*. Thus* $\|u\|_{V_q^n} = \|(1 + L_q)^{n/2}u\|_{-1,1/q}$ *is equivalent to* $\|u\|_{H_{n-1}}$*.*

2.B Erratum

This chapter corresponds to an article written in collaboration with Giambattista Giacomin, Khashayar Pakdaman and Xavier Pellegrin and published in *SIAM Journal on Mathematical Analysis* [44]. Some mistakes were found after the publication of this article.

The differential of the projection given in Lemma 2.5.1 at a point p (equation (2.5.2)) is false, and also depends on the distance of p from the manifold M. The correct differential is given in the finite dimension case in Lemma 5.2.1. This implies some minor corrections in the proof of Theorem 2.2.2: (2.4.5) should be replaced by

$$-\dot{\psi}_t^\delta q'_{\psi_t^\delta} + \dot{\psi}_t^\delta \partial_\psi \varphi_\delta(q_\psi)|_{\psi=\psi_t^\delta} \,=\, \partial_t p_t^\delta\,,$$

where $\partial_\psi \varphi_\delta(q_\psi)|_{\psi=\psi_t^\delta}$ is of order δ, due to Theorem 2.2.1.

Chapter 3

Kuramoto model : the effect of disorder

Contents

3.1 Introduction

3.1.1 Collective phenomena in noisy coupled oscillators

Coupled oscillator models are omnipresent in the scientific literature because the emergence of coherent behavior in large families of interacting units that have a periodic behavior, that we generically call *oscillators*, is an extremely common phenomenon (crickets chirping, fireflies flashing, planets orbiting, neurons firing,...). It is impossible to properly account for the literature and the various models proposed for this kind of phenomena, but while a precise description of each of the different instances in which synchronization emerges demands specific, possibly very complex, models, the *Kuramoto model* has emerged as capturing some of the fundamental aspects of synchronization [1]. It can be introduced via the system of N stochastic differential equations

$$\mathrm{d}\varphi_j^\omega(t) = \omega_j \,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N} \sin(\varphi_j^\omega(t) - \varphi_i^\omega(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t)\,, \tag{3.1.1}$$

for $j = 1, \ldots, N$, where

1. $\{B_j\}_{j=1,\ldots,N}$ is a family of standard independent Brownian motions: in physical terms, this is a *thermal noise*;
2. $\{\omega_j\}_{j=1\cdots N}$ is a family of independent identically distributed random variables of law μ: they are are the *natural frequencies* of the oscillators and, in physical terms, they can be viewed as a *quenched disorder*;
3. K and σ are non-negative parameters, but one should think of them as positive parameters since the cases in which they vanish have only a marginal role in the what follows.

The variables φ_j^ω are meant to be angles (describing the position of rotators on the circle $\mathbb{S}$), so we focus on φ_j^ω mod 2π and (3.1.1) defines, once an initial condition is supplied, a diffusion process on $\mathbb{S}^N$. Note that if $\{\varphi_j^\omega(\cdot)\}_{j=1,\ldots,N}$ solves (3.1.1), also $\{\varphi_j^\omega(\cdot)+\varphi\}_{j=1,\ldots,N}$, with $\varphi \in \mathbb{S}$, is a solution: this is the rotation symmetry of the system that will repeatedly make surface in the remainder of the paper.

Some of the main features (3.1.1) are easily grasped: each oscillator rotates at its own speed, it is perturbed by independent noise and it interacts with all the other oscillators: the interaction tends to align the rotators. It may be helpful at this stage to point out that if $\mu = \delta_0$, that is the natural frequencies are just zero, then the dynamics is reversible with invariant probability measure that, up to normalization, is

$$\exp\left(\frac{K}{\sigma^2}\sum_{i,j=1}^{N} \cos\left(\varphi_i - \varphi_j\right)\right) \lambda_N(\,\mathrm{d}\varphi)\,, \tag{3.1.2}$$

where λ_N is the uniform measure on $\mathbb{S}^N$. The Gibbs measure in (3.1.2) is a well known statistical mechanics model – it is the classical XY spin mean field model or rotator mean field model – treated analytically in [102, 82] in the $N \to \infty$ limit. In particular, the model exhibits a phase transition at $K = K_c := 1/\sigma^2$, that is effectively a *synchronization transition*: in the $N \to \infty$ limit we have that for $K \le K_c$ the rotators become independent and uniformly distributed over $\mathbb{S}$, while for $K > K_c$ the limit measure is obtained by choosing a phase θ uniformly in $\mathbb{S}$ and by choosing the values of the phase of each oscillator by drawing it at random following a suitable distribution that concentrates around θ. However, in [9, Prop. 1.2], it is shown that, unless $\mu = \delta_0$, the model is not reversible (for μ almost surely all the realization of ω) and one effectively steps into the domain of non-equilibrium statistical mechanics.

Our approach actually relies on a sharp control of the reversible case and works when the system is not too far from reversibility, that is for weak disorder. Our approach actually applies well beyond (3.1.1): here we will treat explicitly the case ω_j is replaced by $U(\varphi_j^\omega, \omega_j)$, that is the natural frequency ω_j is replaced by a *natural dynamics* that can be substantially different from one oscillator to another. This model is a disordered version of the active rotator model considered for example in [101].

Since we will focus on $\sigma > 0$, from now on, for ease of exposition, we set $\sigma := 1$.

3.1.2 The Fokker-Planck or McKean-Vlasov limit

An efficient way to tackle (3.1.1) is to consider the empirical probability on $\mathbb{S} \times \mathbb{R}$

$$\nu^{\omega}_{N,t}(\,\mathrm{d}\theta,\,\mathrm{d}\omega) \,:=\, \frac{1}{N}\sum_{j=1}^{N}\delta_{(\varphi^{\omega}_j(t),\omega_j)}(\,\mathrm{d}\theta,\,\mathrm{d}\omega)\,. \tag{3.1.3}$$

In fact, in the $N \to \infty$ limit, the sequence of measures $\{\nu^{\omega}_{N,t}\}_{N=1,2,\ldots}$ converges to a limit measure whose density (with respect to $\lambda_1 \otimes \mu$) solves the nonlinear Fokker-Planck equation

$$\partial_t p_t(\theta,\omega) \,=\, \frac{1}{2}\Delta p_t(\theta,\omega) - \partial_\theta\Big(p_t(\theta,\omega)(\langle J * p_t\rangle_\mu(\theta) + \omega)\Big), \tag{3.1.4}$$

where $J(\theta) = -K\sin(\theta)$, $*$ denotes the convolution and $\langle\cdot\rangle_\mu$ is a notation for the integration with respect to μ, so $\langle J * u\rangle_\mu(\theta) = \int_{\mathbb{R}}\int_{\mathbb{S}} J(\varphi)u(\theta-\varphi,\omega)\,\mathrm{d}\varphi\mu(\,\mathrm{d}\omega)$ is the convolution of J and u, averaged with respect to the disorder. Here and throughout the whole paper Δ means ∂^2_ϑ. The Fokker-Planck PDE (3.1.4) appears repeatedly in the physics and biology literature, see e.g. [1, 94, 103], and a mathematical proof (and precise statement) of the result we just stated can be found in [21, 66]. Notably, in [66] the result is established under the assumption that $\int|\omega|\mu(\,\mathrm{d}\omega) < \infty$ and emphasis is put on the fact that the result holds for almost every realization of the disorder sequence $\{\omega_j\}_{j=1,2,\ldots}$. Let us point out that in (3.1.4) ω is a one dimensional real variable, while in (3.1.1) the superscript ω is a short for the whole sequence of natural frequencies. Since what follows is really about (3.1.4) this abuse of notation will be of limited impact.

In Appendix 3.A, we detail the fact that (3.1.4) generates an evolution semigroup in suitable spaces. Here we want to stress that (3.1.4) can be viewed as a family of coupled PDEs, one for each value of ω in the support of μ: $p_t(\cdot,\omega)$ is the distribution of phases in the population of oscillators with natural frequency ω.

3.1.3 About stationary solutions to (3.1.4)

Remarkably ([94], see also [54]), if μ is symmetric all the stationary solutions to (3.1.4) can be written in a semi-explicit way as $q(\theta+\theta_0,\omega)$ (θ_0 is an arbitrary constant that reflects the rotation symmetry) where

$$q(\theta,\omega) \,:=\, \frac{S(\theta,\omega,2Kr)}{Z(\omega,2Kr)}\,, \tag{3.1.5}$$

with

$$S(\theta,\omega,x) \,=\, e^{G(\theta,\omega,x)}\left[(1-e^{4\pi\omega})\int_0^\theta e^{-G(u,\omega,x)}\,\mathrm{d}u + e^{4\pi\omega}\int_0^{2\pi} e^{-G(u,\omega,x)}\,\mathrm{d}u\right], \tag{3.1.6}$$

and $G(u,y,x) = x\cos(u) + 2yu$, $Z(\omega,x) = \int_{\mathbb{S}} S(\theta,\omega,x)\,\mathrm{d}\theta$ is the normalization constant and $r \in [0,1]$ satisfies the fixed-point relation

$$r \,=\, \Psi^\mu(2Kr), \quad \text{where} \quad \Psi^\mu(x) \,:=\, \int_{\mathbb{R}} \frac{\int_{\mathbb{S}}\cos(\theta)S(\theta,\omega,x)\,\mathrm{d}\theta}{Z(\omega,x)}\mu(\,\mathrm{d}\omega)\,. \tag{3.1.7}$$

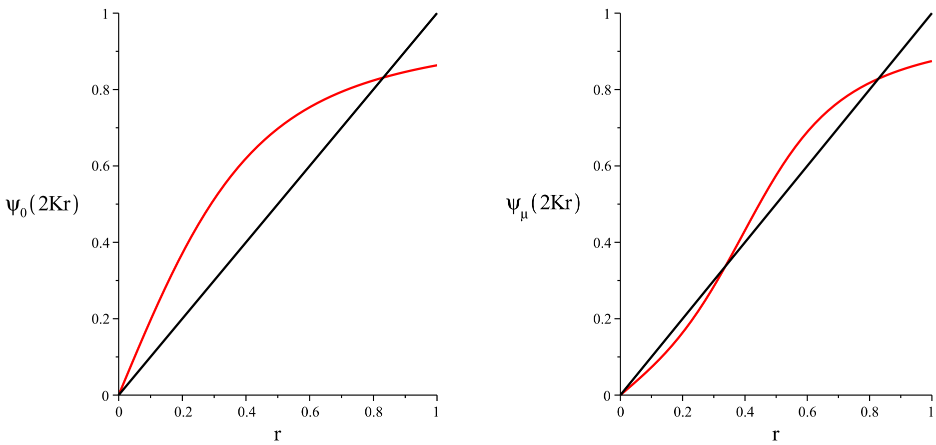
A series of remarks are in order:

1. $r = 0$ solves (3.1.7) and this corresponds to the fact that $q(\cdot) \equiv \frac{1}{2\pi}$ is a stationary solution. It is the only one as long as K does not exceed critical value K_c which is in any case not larger than

$$\widetilde{K} \,:=\, \left(\int_{\mathbb{R}} \frac{\mu(\,\mathrm{d}\omega)}{1+4\omega^2}\right)^{-1}\,, \tag{3.1.8}$$

as one can easily see by computing (see e.g. [54]) the derivative of $\Psi^\mu(2K\cdot)$ at the origin and noticing that is larger than one if and only if $K > \widetilde{K}$ and that $\Psi^\mu(\cdot) < 1$, see Figure 3.1.

2. When (3.1.7) admits a fixed point $r > 0$, and this is certainly the case if $K > \widetilde{K}$, a nontrivial stationary solution is present and in fact, by rotation symmetry, a circle of non-trivial stationary solutions. Such solutions correspond to a synchronization phenomenum, since the distribution of the phases is no longer trivial.
3. As explained in Figure 3.1 and its caption, in general there can be more than one fixed point $r > 0$: in absence of disorder there is only one positive fixed point (when it exists, that is for $K > 1$), but this fact is non-trivial even in this case (see below). Uniqueness is expected for μ which is unimodal, but this has not been established.
4. While the local stability of $\frac{1}{2\pi}$ is understood [103] and it holds only if $K \leq \widetilde{K}$, the stability properties of the non-trivial solutions are a more delicate issue.



(a) Function $\Psi^\mu(2K\cdot)$ for $\mu = \delta_0$, $K = 2$. (b) Function $\Psi^\mu(2K\cdot)$ for $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$, $K = 3.5$.

FIGURE 3.1. Plot of the fixed-point function $\Psi^\mu(2K\cdot)$ for two choices of K and μ. $\Psi^{\delta_0}(\cdot)$ is strictly concave with derivative at the origin equal to $1/2$ (Fig. 3.1a) but even for a simple instance of μ (Fig. 3.1b) concavity is lost and there are several non-trivial fixed-points, each of them corresponding to one circle of non trivial stationary solutions. Note that in the case of Fig. 3.1b, $K < \widetilde{K} = 5$ so that the phase transition is not given by the derivative of $\Psi^\mu(2K\cdot)$ at the origin.

3.1.4 An overview of the results we present

Here are two natural questions:

– What are the stability properties of the non-trivial stationary solutions?

– What happens if μ is not symmetric?

Our work addresses these two questions and provides complete answers for weak disorder. The precise set-up of our work is better understood if we remark from now that we can assume $m_\omega := \int \omega\mu(\mathrm{d}\omega) = 0$. In fact, if this is not the case we can map the model to a model with $m_\omega = 0$ by putting ourselves on the frame that rotates with speed m_ω, that is if we consider the diffusion $\{\varphi_j^\omega(t) - m_\omega t\}_{j=1,\ldots,N}$. So, we assume henceforth $m_\omega = 0$ and we rewrite the natural frequencies as $\delta\omega$, with δ a non-negative parameter. We assume moreover that

$$\mathrm{Supp}(\mu) \subseteq [-1, 1] \,. \tag{3.1.9}$$

In this set-up, (3.1.4) becomes

$$\partial_t p_t^\delta(\theta,\omega) \;=\; \frac{1}{2}\Delta p_t^\delta(\theta,\omega) - \partial_\theta\Big(p_t^\delta(\theta,\omega)(\langle J * p_t^\delta\rangle_\mu(\theta) + \delta\omega)\Big)\,. \tag{3.1.10}$$

Note that this leads to (obvious) changes to (3.1.5)-(3.1.7). We have introduced this parameterization because the results that we present are for small values of δ. In particular we are going to show that for any $K > 1$, there exists $\delta_0 > 0$ such that for $\delta \in [0, \delta_0]$

- there exists a solution $p_t^\delta(\theta,\omega)$ to (3.1.10) of the form $q(\theta - c_\mu(\delta)t)$, we show that $c_\mu(\delta) = O(\delta^3)$ and we actually give an expression for $\lim_{\delta\searrow 0} c_\mu(\delta)/\delta^3$: this is a rotating wave (or limit-cycle) for the dynamical system (3.1.10) and we establish its stability under perturbations;
- when μ is symmetric and $K > \widetilde{K}$ we show that there is, up to rotation symmetry, only one non-trivial solution and that it is (linearly and non-linearly) stable.

The results we obtain are based on the rather good understanding that we have of the case $\delta = 0$ that, as we have already explained, is reversible and the corresponding Fokker-Planck PDE is of gradient flow type (e.g. [77] and references therein). These properties have been exploited in [9] in order to extract a number of properties of the Fokker-Planck PDE (denoted from now on: reversible PDE)

$$\partial_t p_t(\theta) \;=\; \frac{1}{2}\Delta p_t(\theta) - \partial_\theta\Big(p_t(\theta)(J * p_t)(\theta)\Big)\,, \tag{3.1.11}$$

and notably the linear stability of the non-trivial stationary solutions. In fact one can find in [9] an analysis of the evolution operator linearized around the non-trivial stationary solutions. Some of the results in [9] are recalled in the next section, but they are not directly applicable because the $\delta = 0$ case that corresponds to what interests us is rather

$$\partial_t p_t(\theta,\omega) \;=\; \frac{1}{2}\Delta p_t(\theta,\omega) - \partial_\theta\Big(p_t(\theta,\omega)(\langle J * p_t\rangle_\mu(\theta)\Big)\,, \tag{3.1.12}$$

which we call *non-disordered PDE*. So the *natural frequencies* have no effective role beyond separating the various rotators into populations with given natural (ineffective) frequency that now are just labels. But in order to set-up a proper perturbation procedure we need to control (3.1.12) and, in particular, we need (and establish) a spectral gap inequality for the evolution (3.1.12) linearized around the non-trivial solutions.

This spectral analysis is going to be central both for the general and for the symmetric disorder case. In the general set-up we are going to exploit the *normally hyperbolic structure* [51, 99] of the manifold of stationary solutions of (3.1.12) and the robustness of such structures (like in [44]). In the case of symmetric μ we can get more precise results by ad hoc estimates, made possible by the explicit expressions (3.1.5)-(3.1.7), and use results in the general theory of operators [81] and perturbation theory of self-adjoint operators [57].

The normal hyperbolic manifold approach allows to treat cases that are substantially more general and notably the case of

$$\partial_t p_t(\theta,\omega) \;=\; \frac{1}{2}\Delta p_t(\theta,\omega) - \partial_\theta\Big(p_t(\theta,\omega)(\langle J * p_t\rangle_\mu(\theta) + \delta U(\theta,\omega))\Big)\,, \tag{3.1.13}$$

which is the large N limit of (3.1.1) with the term $\omega_j\,\mathrm{d}t$ replaced by $U(\varphi_j^\omega(t),\omega_j)\,\mathrm{d}t$, with $U \in C^1(\mathbb{S}\times\mathbb{R};\mathbb{R})$. In this case each oscillator has its own non-trivial dynamics which may be very different from the dynamics of other oscillators: consider for example

$$U(\varphi,\omega) \;=\; b + \omega + a\sin(\varphi)\,, \quad a, b \in \mathbb{R}\,, \tag{3.1.14}$$

and μ uniform over $[-1,1]$. For $a \in (-1,1)$ there are some *active rotators* [101, 44] that in absence of noise and interaction ($\sigma = K = 0$) rotate (this happens if $|b+\omega| > |a|$ and of course the direction of rotation depends on the sign of $b+\omega$) and others that instead are stuck at a fixed point (this happens if $|b+\omega| \le |a|$). Our approach allows us to establish that there is a synchronization regime for $K > 1$ and δ small and to describe the dynamics of the system in this regime. This is going to be detailed in Section 3.5.

The two questions raised at the beginning of this section have been already repeatedly approached but looking at synchronized solutions as bifurcation from incoherence. The results are hence for K close to the critical value corresponding to the breakdown of linear stability of $1/2\pi$: one can find a detailed review of the vast literature on this issue in [1, Sec. III]. Our results are instead for arbitrary $K > 1$, but δ smaller than $\delta_0(K)$ and of course $\delta_0(K)$ vanishes as K approaches 1.

3.2 Mathematical set-up and main results

3.2.1 The reversible and the non-disordered PDE

We first recall some results about the reversible PDE (3.1.11). The stationary solutions $q_0(\theta) = q(\theta, 0)$ are, up to rotation invariance, given by (3.1.5)-(3.1.7), but formulas get simpler, namely

$$q_0(\theta) = \frac{1}{Z_0(2Kr_0)} \exp(2Kr_0 \cos(\theta)), \tag{3.2.1}$$

where $Z_0(x) := Z(0,x)^{\frac{1}{2}}$ and this time we have the more explicit expression $Z_0(x) = \int_{\mathbb{S}} e^{x\cos(\theta)}\, \mathrm{d}\theta = 2\pi I_0(x)$ is the normalization constant and r_0 is a solution of the fixed-point problem

$$r_0 = \Psi_0(2Kr_0) \qquad \text{where} \qquad \Psi_0(x) := \frac{I_1(x)}{I_0(x)}, \tag{3.2.2}$$

where we used standard notations for the modified Bessel functions

$$I_i(x) = \frac{1}{2\pi} \int_{\mathbb{S}} (\cos(\theta))^i \exp(x\cos(\theta))\, \mathrm{d}\theta \qquad i = 0, 1. \tag{3.2.3}$$

The mapping Ψ_0 is increasing, concave (see [82]) and with derivative at 0 equal to $\frac{1}{2}$. Consequently if $K \leqslant 1$, $r_0 = 0$ is the unique solution of the fixed-point problem, and $q(\cdot) \equiv \frac{1}{2\pi}$ is the only stationary solution of (3.1.11). If $K > 1$, we get in addition a circle (because of the rotation invariance) of nontrivial stationary solutions

$$M_{\text{rev}} := \{q_{\psi,0}(\cdot) := q_0(\cdot - \psi) : \psi \in \mathbb{S}\} \qquad \text{with} \qquad q_0(\theta) := \frac{\exp(2Kr_0\cos(\theta))}{\int_{\mathbb{S}} \exp(2Kr_0\cos(\theta))} \tag{3.2.4}$$

where $r_0 = r_0(K)$ is the unique non trivial fixed-point (3.2.2).

Let us now focus on the non-disordered PDE (3.1.12) and let us insist on the fact that we are interested in solutions such that $\varphi_t^\delta(\cdot, \omega)$ is a probability density. Observe then that if $q(\theta,\omega)$ is a stationary solution of (3.1.12), we see (Appendix 3.A) that q is C^∞ with respect to θ and that $\langle q \rangle_\mu$ is a stationary solution for (3.1.11). So there exists $\psi \in \mathbb{S}$ such that $\langle q \rangle_\mu = q_\psi$ and a short computation leads to

$$\langle J * q \rangle_\mu(\theta) = -K\sin(\theta - \psi), \tag{3.2.5}$$

and, since $\int_{\mathbb{S}} q(\theta,\omega)\, \mathrm{d}\theta = 1$ for almost all ω, we obtain that $q(\cdot,\omega) = q_\psi(\cdot)$ for almost all ω. In conclusion, with some abuse of notation, we can say the stationary solutions of (3.1.11) and (3.1.12) are the same: of course in the second case the function space includes the dependence on ω, so we choose a different notation, that is M_0, for the corresponding circle of non-trivial stationary solutions.

An important issue for us is the stability of M_0 (for its existence we are assuming $K > 1$) and for this we denote for all $\psi \in \mathbb{S}$ by $A_{q_{\psi,0}}$ the linearized evolution operator of (3.1.12) around $q_{\psi,0}$

$$A_{q_{\psi,0}} u(\theta,\omega) := \frac{1}{2}\Delta u(\theta,\omega) - \partial_\theta \Big(q_{\psi,0}(\theta) \langle J * u \rangle_\mu(\theta) + u(\theta,\omega) J * q_{\psi,0}(\theta) \Big) \qquad (3.2.6)$$

with domain

$$\mathcal{D}(A) := \Big\{ u : (\mathbb{S}\times\mathbb{R}) \to \mathbb{R} : \, u(\cdot,\omega) \in C^2(\mathbb{S},\mathbb{R}) \text{ et } \int_{\mathbb{S}} u(\theta,\omega)\, \mathrm{d}\theta = 0 \quad \mu\text{-p.p.}, \\ \text{et } \int_{\mathbb{R}} \|u(\cdot,\omega)\|^2_{C^2(\mathbb{S},\mathbb{R})} \mu(\mathrm{d}\omega) < \infty \Big\}. \qquad (3.2.7)$$

By invariance by rotation, we only need to study the spectral properties of A_{q_0}, that will denote $A := A_{q_0}$ for simplicity. For any smooth positive function $k : \mathbb{S} \mapsto \mathbb{R}$, we introduce the Hilbert space $H^{-1}_{k,\mu}$ defined by the closure of $\mathcal{D}(A)$ for the norm $\|\cdot\|_{-1,k,\mu}$ associated with the scalar product

$$\langle u, v \rangle_{-1,k,\mu} := \int_{\mathbb{R}\times\mathbb{S}} \frac{\mathcal{U}(\theta,\omega)\mathcal{V}(\theta,\omega)}{k(\theta)} \mathrm{d}\theta \mu(\mathrm{d}\omega), \qquad (3.2.8)$$

where ω a.s., $\mathcal{U}(\cdot,\omega)$ is the primitive of $u(\cdot,\omega)$ such that $\int_{\mathbb{S}} \frac{\mathcal{U}(\theta,\omega)}{k(\theta)} \mathrm{d}\theta = 0$, and $\mathcal{V}(\cdot,\omega)$ is defined in the analogous fashion. Let us remark (see [44, Sec. 2]) immediately that

$$\|u\|^2_{-1,k_1,\mu} \leqslant \frac{\|k_2\|_\infty}{\|k_1\|_\infty} \|u\|^2_{-1,k_2,\mu}, \qquad (3.2.9)$$

so that all the norms we have introduced are equivalent. For the case $k(\cdot) \equiv 1$ we use the notations H^{-1}_μ and $\|\cdot\|_{-1,\mu}$. We will prove the following result, which is just technical, but it will be of help to understand our main results:

Proposition 3.2.1. *A is essentially self-adjoint in $H^{-1}_{q_0,\mu}$. Moreover the spectrum lies in $(-\infty, 0]$, 0 is a simple eigenvalue, with eigenspace spanned by $\partial_\theta q_0$, and there is a spectral gap, that is the distance λ_K between 0 and the rest of the spectrum is positive.*

The proof of this result builds on [9, Th. 1.8] that deals with the reversible case and the (lower) bound on the spectral gap λ_K that we obtain coincides with the quantity $\lambda(K)$ in [9, Th. 1.8] (this bound can be improved as explained in in [9, Sec. 2.5] and sharp estimates on the spectral gap can be obtained in the limit $K \searrow 1$ and $K \nearrow \infty$). For the reversible evolution, the linear operator L_{q_0} is defined by

$$L_{q_0} u(\theta) := \frac{1}{2}\Delta u(\theta) - \partial_\theta \Big(q_0(\theta) J * u(\theta) + u(\theta) J * q_0(\theta) \Big), \qquad (3.2.10)$$

with domain $D(L_{q_0})$ given by the $C^2(\mathbb{S},\mathbb{R})$ functions with zero integral.

3.2.2 Synchronization: the main result without symmetry assumption

Proposition 3.2.1 is a key ingredient for our main results and the functional space H^{-1} appears in it, but an important role is played also by $L^2(\lambda \otimes \mu)$, λ is the Haar measure on $\mathbb{S}$, whose norm is denoted by $\| \cdot \|_{2,\mu}$. For $C > 0$ and $M \subset L^2(\lambda \otimes \mu)$ we set $\mathcal{N}_{2,\mu}(M, C) := \{u : \text{there exists } v \in M \text{ such that } \|u - v\|_{2,\mu} \leq C\}$. In the statement below $q \in M_0$ is the element of the manifold such that $q(\cdot, \omega) = q_0(\cdot)$, cf. (3.2.1), with $r_0(K) > 0$ (hence $K > 1$).

Theorem 3.2.2. *For every $K > 1$ there exists $\delta_0 = \delta_0(K) > 0$ such that for $|\delta| \leq \delta_0$ there exists $\widetilde{q}_\delta \in L^2(\lambda \otimes \mu)$, satisfying $\|\widetilde{q}_\delta - q\|_{2,\mu} = O(\delta)$ and a value $c_\mu(\delta) \in \mathbb{R}$ such that if we set*

$$q_t^{(\psi)}(\theta, \omega) := \widetilde{q}_\delta(\theta - c_\mu(\delta)t - \psi), \tag{3.2.11}$$

then $q_t^{(0)}$ solves (3.1.10). Moreover

1. *the family of solutions $\{q_\cdot^{(\psi)}\}_\psi$ is stable in the sense that there exist two positive constants $\beta = \beta(K)$ and $C = C(K)$ such that if $p_0^\delta \in \mathcal{N}_{2,\mu}(M_0, \delta)$, and $\int_{\mathbb{S}} p_0^\delta(\theta, \omega)\, \mathrm{d}\theta = 0$ $\mu(\mathrm{d}\omega)$-a.s., then there exists $\psi_0 \in \mathbb{S}$ such that for all $t \geq 0$*

$$\|q_t^{(\psi_0)} - p_t^\delta\|_{2,\mu} \leq 2C \exp(-\beta t). \tag{3.2.12}$$

2. *we have*

$$c_\mu(\delta) = \delta^3 \frac{\langle \omega \partial_\theta n^{(2)}, \partial_\theta q_0 \rangle_{-1,q_0,\mu}}{\langle \partial_\theta q_0, \partial_\theta q_0 \rangle_{-1,q_0}} + O(\delta^5), \tag{3.2.13}$$

where $n^{(2)}$ is the unique solution of

$$An^{(2)} = \omega \partial_\theta n^{(1)} \quad \text{and} \quad \langle n^{(2)}, \partial_\theta q_0 \rangle_{-1,q_0,\mu} = 0, \tag{3.2.14}$$

and $n^{(1)}$ is the unique solution of

$$An^{(1)} = \omega\, \partial_\theta q_0 \quad \text{and} \quad \langle n^{(1)}, \partial_\theta q_0 \rangle_{-1,q_0,\mu} = 0. \tag{3.2.15}$$

In the proof of Theorem 3.2.2 one finds also further estimates, in particular (see (3.4.18)) that one has

$$\widetilde{q}_\delta = q_0 + \delta n^{(1)} + \delta^2 n^{(2)} + O_{L^2}(\delta^3). \tag{3.2.16}$$

Actually, see Remark 3.4.2, the argument of proof can be pushed farther to obtain arbitrarily many terms in development (3.2.16), as well as in

$$c_\mu(\delta) = c_3\delta^3 + c_5\delta^5 + \ldots. \tag{3.2.17}$$

In Table 3.1 we report a comparison between the $c_\mu(\delta)$ obtained by solving numerically (3.1.10) and by evaluating the leading order c_3, i.e. by using (3.2.13).

3.2.3 Symmetric disorder case

Let us focus on the case in which the distribution of the disorder μ is symmetric. In this case, at least for small disorder, Theorem 3.2.2 is just telling us that the leading order in the development for the speed $c_\mu(\delta)$ is zero: one can actually work harder and show that such a development yields zero terms to all orders. In reality in this case we already know,

δ	$K=2$	$K=1.5$	$K=1.1$
0.5	$-1.56300\cdot 10^{-2}$	$-8.59626\cdot 10^{-2}$	$-3.01064\cdot 10^{-1}$
0.1	$-1.23998\cdot 10^{-2}$	$-6.84835\cdot 10^{-2}$	$-2.72117\cdot 10^{-1}$
0.05	$-1.23072\cdot 10^{-2}$	$-6.79553\cdot 10^{-2}$	$-2.69460\cdot 10^{-1}$
0.01	$-1.22776\cdot 10^{-2}$	$-6.77921\cdot 10^{-2}$	$-2.68603\cdot 10^{-1}$
0.005	$-1.22767\cdot 10^{-2}$	$-6.77869\cdot 10^{-2}$	$-2.68576\cdot 10^{-1}$
c_3	$-1.22764\cdot 10^{-2}$	$-6.77851\cdot 10^{-2}$	$-2.68567\cdot 10^{-1}$

TABLE 3.1. For the case $\mu = p\delta_{1-p} + (1-p)\delta_{-p}$, $p=0.2$, we have computed (numerically) $c_\mu(\delta)/\delta^3$ for three values of K and five values of δ. In the last line we report the value $c_3 = \lim_{\delta\searrow 0} c_\mu(\delta)/\delta^3$ that one obtains by using (3.2.13).

see (3.1.5)-(3.1.7), that for K sufficiently large there is at least a non-trivial stationary profile, hence, by rotation symmetry, at least one whole circle of stationary solutions. Actually, we can show that for δ small there is just one circle, that we call M_δ, of non-trivial stationary solutions and this circle converges to M_0 as $\delta \searrow 0$ (in C^j, for every j) so the rotating solutions found in Theorem 3.2.2 must be the stationary solutions in M_δ.

In order to be precise about this issue, we point out that (3.1.5)-(3.1.7) are written for (3.1.4) while we work rather with (3.1.10). The changes are obvious, but we introduce a notation for the analog of (3.1.7):

$$r_\delta = \Psi^\mu_\delta(2Kr_\delta), \qquad \text{where, } \Psi^\mu_\delta(x) := \int_{\mathbb{R}} \frac{\int_{\mathbb{S}} \cos(\theta) S(\theta,\delta\omega,x)\,\mathrm{d}\theta}{Z(\delta\omega,x)} \mu(\,\mathrm{d}\omega)\,. \tag{3.2.18}$$

Lemma 3.2.3. *For all $K_{\max} > 0$, there exists $\delta_1 = \delta_1(K_{\max}) > 0$ such that, for all $\delta \leqslant \delta_1$ the function Ψ^μ_δ is strictly concave on $[0, 2K_{\max}]$. Therefore for $K \leqslant K_{\max}$ and $\delta \leqslant \delta_1$* (3.1.7) *has at most one positive solution $r_\delta = r_\delta(K,\mu)$. Moreover* $\lim_{\delta\searrow 0} r_\delta = r_0$.

We point out that in spite of the fact that Ψ^μ is explicit (cf. (3.2.2)), it is not so straightforward to show that it is concave. We show that Ψ^μ_δ remains strictly concave for a small δ via a perturbation argument. But the conjecture (see [54] and [21]) that Ψ^μ is strictly concave for unimodal distributions μ is still an open issue.

Remark 3.2.4. A direct computation shows that the derivative of Ψ^μ_δ at the origin is $1/(2\widetilde{K}_\delta)$, for $\widetilde{K}_\delta := \left(\int_{\mathbb{R}} \frac{\mu(\,\mathrm{d}\omega)}{1+4\delta^2\omega^2}\right)^{-1}$ (of course $\widetilde{K}_1$ coincides with $\widetilde{K}$, introduced in (3.1.8)). Under the hypothesis of Lemma 3.2.3, one therefore sees that there is a synchronization transition at $K = \widetilde{K}_\delta$ in the sense that for $K \le \widetilde{K}_\delta$ the only stationary solution is $\frac{1}{2\pi}$ while for $K_{\max} \geqslant K > \widetilde{K}_\delta$ also the manifold of non-trivial stationary solutions appears (and there is no other stationary solution).

Theorem 3.2.2 provides a stability statement for M_δ. This result can be sharpened and for this let us introduce the linear operator

$$L^\omega_q u(\theta,\omega) := \frac{1}{2}\Delta u(\theta,\omega) - \partial_\theta \left(u(\theta,\omega)\left(\langle J * q\rangle_\mu(\theta) + \delta\omega\right) + q(\theta,\delta\omega)\langle J * u\rangle_\mu(\theta)\right), \tag{3.2.19}$$

The domain $\mathcal{D}(L^\omega_q)$ of the operator L^ω_q is chosen to be the same as for A, cf. (3.2.7).

We place ourselves within the framework of Lemma 3.2.3, in the sense that δ is small enough to ensure the uniqueness of a non-trivial stationary solution (of course existence requires $K > \widetilde{K}_\delta$ and this is implied by $K > 1$ if δ is sufficiently small). We prove a number of properties of the linear operator (3.2.19), saying notably that it has a simple eigenvalue at zero and the rest of spectrum is at a positive distance from zero and it is in a cone in that lies in the negative complex half plane. We summarize in the next statement the qualitative features of our results on L_q^ω, but what we really prove are quantitative explicit estimates: the interested reader finds them in Section 3.6.

Theorem 3.2.5. *The operator L_q^ω has the following spectral properties: 0 is a simple eigenvalue for L_q^ω, with eigenspace spanned by $(\theta,\omega) \mapsto q'(\theta,\omega)$. Moreover, for all $K > 1$, $\rho \in (0,1)$, $\alpha \in (0,\pi/2)$, there exists $\delta_2 = \delta_2(K,\rho,\alpha)$ such that for all $0 \leqslant \delta \leqslant \delta_2$, the following is true:*

- *L_q^ω is closable and its closure has the same domain as the domain of the self-adjoint extension of A;*
- *The spectrum of L_q^ω lies in a cone C_α with vertex 0 and angle α*

$$C_\alpha := \left\{\lambda \in \mathbb{C}\,;\, \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha\right\} \subseteq \{z \in \mathbf{C}\,;\, \Re(z) \leqslant 0\}\ ; \qquad (3.2.20)$$

- *There exists $\alpha' \in (0, \frac{\pi}{2})$ such that L_q^ω is the infinitesimal generator of an analytic semi-group defined on a sector $\{\lambda \in \mathbb{C},\, |\arg(\lambda)| < \alpha'\}$;*
- *The distance between 0 and the rest of the spectrum is strictly positive and is at least equal to $\rho\lambda_K$, where λ_K is the spectral gap of the operator A introduced in Proposition 3.2.1.*

3.2.4 Organization of remainder of the paper

In Section 3.3 we introduce the notion of stable normally hyperbolic manifold, we recall its robustness properties, and show that M_0 is in this class of manifolds. The essential ingredient is Proposition 3.2.1 that, directly or indirectly, plays a role in each subsequent section. Section 3.3 is also devoted to the proof of Proposition 3.2.1. The proof of Theorem 3.2.2 is then completed in Section 3.4, that is mainly devoted to perturbation arguments. The case of the active rotators is treated in Section 3.5, while Section 3.6 deals with the case symmetric disorder distribution and, notably, with the proof of Theorem 3.2.5 and of a number of related quantitative estimates.

3.3 Hyperbolic structures and periodic solutions

In this section we present the arguments proving the existence of the periodic solution of Theorem 3.2.2. We rely on the fact that the circle of stationary solutions M_0 is a stable normally hyperbolic manifold, and on the robustness of this kind of structure : adding the perturbation term $-\delta\partial_\theta(p_t(\theta,\omega)\omega)$ in (3.1.12), this manifold M_0 is deformed into another manifold M_δ, and thanks to the rotation invariance of the problem, M_δ is a circle too. The spectral gap of operator A (Property 3.2.1) which induces the hyperbolic property of M_0 is proved at the end of this section.

3.3.1 Stable normally hyperbolic manifolds

We start by quickly reviewing the notion of of stable normally hyperbolic manifold (SNHM). The evolution of (3.1.10) will be studied in the space X_μ^1 defined by

$$X_\mu^1 := \left\{u \in L^2(\lambda \otimes \mu),\ \int_{\mathbb{S}} u(\theta,\omega)\,\mathrm{d}\theta = 1 \quad \omega \text{ a.s.}\right\} \qquad (3.3.1)$$

where λ denotes the Lebesgue measure on $\mathbb{S}$. This is made possible by the conservative character of the dynamics. The L^2-norm with respect to the measure $\lambda\otimes\mu$ will be denoted by $\|\cdot\|_{2,\mu}$. We will also use the space X^0_μ defined by

$$X^0_\mu := \left\{ u \in L^2(\lambda\otimes\mu),\ \int_{\mathbb{S}} u(\theta,\omega)\,\mathrm{d}\theta = 0 \quad \omega \text{ a.s.} \right\}. \tag{3.3.2}$$

To define a SNHM, we need a dynamics: we have in mind (3.1.10) but for the moment let us just think of an evolution semigroup in X^1_μ that gives rise to $\{u_t\}_{t\geq 0}$, with $u_0 = u$, to which we can associate a linear evolution semigroup $\{\Phi(u,t)\}_{t\geq 0}$ in X^0_μ, satisfying $\partial_t\Phi(u,t)v = L(t)\Phi(u,t)v$ and $\Phi(u,0)v = v$, where $L(t)$ is the operator obtained by linearizing the evolution around u_t.

For us a SNHM $M \subset X^1_\mu$ (in reality we are interested only in 1-dimensional manifolds, that is curves, but at this stage this does not really play a role) of characteristics λ_1, λ_2 $(0 \leq \lambda_1 < \lambda_2)$ and $C > 0$ is a C^1 compact connected manifold which is invariant under the dynamics and for every $u \in M$ there exists a projection $P^o(u)$ on the tangent space of M at u, that is $\mathcal{R}(P^o(u)) =: T_uM$, which, for $v \in L^2_0$, satisfies the following properties:

1. for every $t \geq 0$ we have
$$\Phi(u,t)P^o(u_0)v \;=\; P^o(u_t)\Phi(u,t)v\,, \tag{3.3.3}$$
2. we have
$$\|\Phi(u,t)P^o(u_0)v\|_{2,\mu} \;\leq\; C\exp(\lambda_1 t)\|v\|_{2,\mu}\,, \tag{3.3.4}$$
and, for $P^s := 1 - P^o$, we have
$$\|\Phi(u,t)P^s(u_0)v\|_{2,\mu} \;\leq\; C\exp(-\lambda_2 t)\|v\|_{2,\mu}\,, \tag{3.3.5}$$
for every $t \geq 0$;
3. there exists a negative continuation of the dynamics $\{u_t\}_{t\leq 0}$ and of the linearized semigroup $\{\Phi(u,t)P^o(u_0)v\}_{t\leq 0}$ and for any such continuation we have
$$\|\Phi(u,t)P^o(u_0)v\|_{2,\mu} \;\leq\; C\exp(-\lambda_1 t)\|v\|_{2,\mu}\,, \tag{3.3.6}$$
for $t \leq 0$.

3.3.2 M_0 is a SNHM

First of all: the dynamics on M_0 is trivial. For $q_\psi \in M_0$, the projection $P^o_{q_\psi}$ on the tangent space is the projection on the subspace spanned by q'_ψ:

$$P^o_{q_\psi}u \;=\; \frac{\left\langle u, q'_\psi\right\rangle_{-1,q_\psi,\mu}}{\left\langle q'_\psi, q'_\psi\right\rangle_{-1,q_\psi}} q'_\psi \tag{3.3.7}$$

and since the dynamic on the manifold is trivial, we are allowed to choose for the parameters $\lambda_1 = 0$ and $\lambda_2 = \lambda_K$ (where we recall that λ_K is given by Proposition 3.2.1).

We are in the same situation as in [44]. For a suitable perturbation and if δ is small enough, the circle M_0 is smoothly transformed into another SNHM M_δ, which is close to M_0. The proof is the same as in [44, Sec. 5], which, in turn builds on results in [99]): the spaces we are working in are more general since we have to deal with the disorder. Here suitable perturbation means being an element of $C^1(X^0_\mu, H^{-1}_\mu)$, but it is clearly the case for the perturbation $u \mapsto -\delta\,\omega\,\partial_\theta u$ when μ is of compact support. The following theorem works for all $C^1(X^0_\mu, H^{-1}_\mu)$ perturbations:

Theorem 3.3.1. *[44, Sec. 5] For every $K > 1$ there exists $\delta_0 > 0$ such that if $\delta \in [0, \delta_0]$ there exists a stable normally hyperbolic manifold M_δ in X^1_μ for the perturbed equation* (3.1.10)*. Moreover we can write*

$$M_\delta = \{q_\psi + \phi_\delta(q_\psi) : \psi \in \mathbb{S}\} , \tag{3.3.8}$$

for a suitable function $\phi_\delta \in C^1(M_0, X^0_\mu)$ with the properties that
– *$\phi_\delta(q) \in \mathcal{R}(A)$;*
– *there exists $C > 0$ such that $\sup_\psi(\|\phi_\delta(q_\psi)\|_{2,\mu} + \|\partial_\psi\phi_\delta(q_\psi)\|_{2,\mu}) \leq C\delta$.*

Remark 3.3.2. A byproduct of the proof in [44, Sec. 5] is also that M_δ is the unique invariant manifold in a $L^2(\lambda, \mu)$-neighborhood of M_0. So in the case of (3.1.10), thanks to the symmetry of the problem that tells us that any rotation of M_δ is still a invariant manifold, M_δ is in fact a circle, and that the dynamics on this circle is a traveling wave of constant (possibly zero) speed $c_\mu(\delta)$. So the invariant manifold we get for (3.1.10) is even C^∞. In this sense, when dealing with (3.1.10), we are using only part of the strength of Theorem 3.3.1. Of course this symmetry argument does not apply when dealing with (3.1.13).

Remark 3.3.3. Theorem 3.3.1 addresses the existence and the linear stability of the manifold M_δ. The non-linear stability statement in Theorem 3.2.2(1) follows from Theorem 3.3.1 combined with [50, Theorem 8.1.1], when the dynamics is periodic with non zero speed on M_δ. If M_δ is a manifold of stationary points, the argument for the non-linear stability follows by repeating the argument in [43, Th. 4.8], where the non-disordered case is treated.

We now prove Proposition 3.2.1 and thus that M_0 is a SNHM.

3.3.3 The spectral gap estimate (proof of Proposition 3.2.1)

We start by remarking that A is symmetric for the scalar product $\langle\cdot\,,\,\cdot\rangle_{-1,q_0,\mu}$ (recall (3.2.8)). In fact, for u and v in $\mathcal{D}(A)$, a short computation gives (in the following we use the notation $u'(\theta,\omega) = \partial_\theta u(\theta,\omega)$)

$$\begin{aligned}\langle v\,,\,Au\rangle_{-1,q_0,\mu} &= \int_{\mathbb{R}\times\mathbb{S}} \left[\frac{\mathcal{V}(\theta,\omega)}{q_0(\theta)}\left(\frac{u'(\theta,\omega)}{2} - u(\theta,\omega)J * q_0(\theta) - q_0(\theta)\langle J * u\rangle_\mu(\theta)\right)\right] \mathrm{d}\theta\,\mathrm{d}\mu \\ &= -\frac{1}{2}\int_{\mathbb{R}\times\mathbb{S}} \frac{u(\theta,\omega)v(\theta,\omega)}{q_0(\theta)}\,\mathrm{d}\theta\,\mathrm{d}\mu + \int_{\mathbb{R}}\int_{(\mathbb{S})^2} v(\theta,\omega)\widetilde{J} * u(\theta,\omega')\,\mathrm{d}\theta\,\mathrm{d}\mu\otimes\mu\,,\end{aligned} \tag{3.3.9}$$

where $\widetilde{J}(\theta) = K\cos(\theta)$. We now first prove an inequality for A that is stronger than the spectral gap inequality and then deduce that A is (essentially) self-adjoint. We define the two following scalar products, which were used for the non-disordered case in [9]:

$$\langle u\,,\,v\rangle_{-1,q_0} := \int_{\mathbb{S}} \frac{\mathcal{U}(\theta)\mathcal{V}(\theta)}{q_0(\theta)}\,\mathrm{d}\theta\,, \tag{3.3.10}$$

where $\mathcal{U}(\cdot)$ is the primitive of $u(\cdot)$ such that $\int_{\mathbb{S}} \frac{\mathcal{U}(\theta)}{q_0(\theta)}\,\mathrm{d}\theta = 0$ and

$$\langle u\,,\,v\rangle_{2,q_0} := \int_{\mathbb{S}} \frac{u(\theta)v(\theta)}{q_0(\theta)}\,\mathrm{d}\theta\,. \tag{3.3.11}$$

We denote the closures of $\mathcal{D}(L_0)$ for these scalar products respectively by $H^{-1}_{q_0}$ and $L^2_{q_0}$. In the disordered case, $L^2_{q_0}$ corresponds to the space $L^2_{q_0,\mu}$, which we define by the closure of $\mathcal{D}(A)$ with respect to the norm $\|\cdot\|_{2,q_0,\mu}$ associated with the scalar product

$$\langle u\,,\,v\rangle_{2,q_0,\mu} \;:=\; \int_{\mathbb{R}}\int_{\mathbb{S}} \frac{u(\theta,\omega)v(\theta,\omega)}{q_0(\theta)}\,\mathrm{d}\theta\,\mathrm{d}\mu\,. \tag{3.3.12}$$

The two Dirichlet forms for the disordered and non-disordered case are respectively

$$\mathcal{E}_\mu(u) \;=\; -\,\langle Au\,,\,u\rangle_{-1,q_0,\mu}\;, \tag{3.3.13}$$

and

$$\mathcal{E}(u) \;=\; -\,\langle L_{q_0}u\,,\,u\rangle_{-1,q_0}\;. \tag{3.3.14}$$

As in [9], we first prove a spectral gap type inequality that involves the scalar product $\langle\cdot\,,\,\cdot\rangle_{2,q_0}$. For this we introduce the projections on the line spanned by q_0' in the spaces $L^2_{q_0,\mu}$ and $L^2_{q_0}$

$$P_{2,q_0,\mu}u = \frac{\langle u\,,\,q_0'\rangle_{2,q_0,\mu}}{\langle q_0'\,,\,q_0'\rangle_{2,q_0}}q_0' \qquad \text{for all}\, u=u(\theta,\omega)\in L^2_{q_0,\mu}\,, \tag{3.3.15}$$

and

$$P_{2,q_0}u \;=\; \frac{\langle u\,,\,q_0'\rangle_{2,q_0}}{\langle q_0'\,,\,q_0'\rangle_{2,q_0}}q_0' \qquad \text{for all } u\in L^2_{q_0}\,. \tag{3.3.16}$$

Remark that since q_0' does not depend on ω,

$$\langle q_0'\,,\,q_0'\rangle_{2,q_0,\mu} \;=\; \langle q_0'\,,\,q_0'\rangle_{2,q_0} \qquad \text{and} \qquad \langle q_0'\,,\,q_0'\rangle_{-1,q_0,\mu} \;=\; \langle q_0'\,,\,q_0'\rangle_{-1,q_0}\;, \tag{3.3.17}$$

and that for all $u\in L^2_{q_0,\mu}$

$$P_{2,q_0,\mu}u \;=\; \langle P_{2,q_0}u\rangle_\mu \;=\; P_{2,q_0}\langle u\rangle_\mu\,. \tag{3.3.18}$$

Proposition 3.3.4. *For all $u\in L^2_{q_0,\mu}$ such that for almost every ω, $\int_{\mathbb{S}} u(\cdot,\omega)=0$*

$$\mathcal{E}_\mu(u) \;\geqslant\; c_K\,\langle u-P_{2,q_0,\mu}u\,,\,u-P_{2,q_0,\mu}u\rangle_{2,q_0,\mu}\;, \tag{3.3.19}$$

with

$$c_K \;=\; 1-K(1-r_0^2)\in(0,1/2)\,. \tag{3.3.20}$$

The proof of this proposition relies on the corresponding result for the non-disordered case.:

Proposition 3.3.5. *(see [9, Prop. 2.3]) For all $u\in L^2_{q_0}$ such that for almost every ω, $\int_{\mathbb{S}} u(\cdot,\omega)=0$*

$$\mathcal{E}(v) \;\geqslant\; c_K\,\langle u-P_{2,q_0}u\,,\,u-P_{2,q_0}u\rangle_{2,q_0}\;. \tag{3.3.21}$$

Proof of Proposition (3.3.4). The first step of the proof is to make the Dirichlet form of the non-disordered case appear in the the disordered case one, that is

$$\mathcal{E}_\mu(u) \;=\; \langle\mathcal{E}(u)\rangle_\mu + \int_{\mathbb{R}}\int_{(\mathbb{S})^2} u(\theta,\omega)\widetilde{J}*[u(\theta,\omega)-u(\theta,\omega')]\,\mathrm{d}\theta\,\mathrm{d}\mu\otimes\mu \tag{3.3.22}$$

$$= \langle\mathcal{E}(u)\rangle_\mu + \frac{1}{2}\int_{\mathbb{R}}\int_{(\mathbb{S})^2} [u(\theta,\omega)-u(\theta,\omega')]\widetilde{J}*[u(\theta,\omega)-u(\theta,\omega')]\,\mathrm{d}\theta\,\mathrm{d}\mu\otimes\mu\,, \tag{3.3.23}$$

and from Proposition (3.3.5) we see that

$$\langle \mathcal{E}(u) \rangle_\mu \geqslant c_K \langle u - P_{2,q_0} u \, , \, u - P_{2,q_0} u \rangle_{2,q_0} \, . \tag{3.3.24}$$

Now remark that if we define

$$v = u - P_{2,q_0,\mu} u \, , \tag{3.3.25}$$

using (3.3.18) we get

$$v - P_{2,q_0} v = u - P_{2,q_0} u \, , \tag{3.3.26}$$

and so

$$\langle \mathcal{E}(u) \rangle_\mu \geqslant c_K \langle v - P_{2,q_0} v \, , \, v - P_{2,q_0} v \rangle_{2,q_0,\mu} \, . \tag{3.3.27}$$

We now introduce an orthogonal decomposition of the space $L^2_{q_0}$ which is well adapted to the convolution with $\widetilde{J}$.

Lemma 3.3.6. *(See [9, Lemma 2.1].) We have the following decomposition*

$$L^2_{q_0} = F_0 \oplus^\perp F_{1/2} \oplus^\perp F_{K-1/2} \tag{3.3.28}$$

where

$$F_0 := \left\{ \theta \mapsto a_0 + \sum_{j \geqslant 2} a_j \cos(j\theta) + b_j \sin(j\theta) \, ; \, \sum_j a_j^2 + b_j^2 < \infty \right\} \tag{3.3.29}$$

and both $F_{1/2}$ and $F_{K-1/2}$ are one dimensional subspaces generated respectively by $\theta \mapsto \sin(\theta) q(\theta) \, (= -q_0'(\theta)/2Kr_0)$ and by $\theta \mapsto \cos(\theta) q_0(\theta)$. Moreover, when $u \in F_\lambda$, then

$$\widetilde{J} * u = \frac{\lambda}{q_0} u \, . \tag{3.3.30}$$

With the help of Lemma 3.3.6 we can find a lower bound for the last term in (3.3.23): choose α such that $P_{2,q_0} u = \alpha q_0'$, so that we can write

$$\mathcal{E}_\mu(u) \geqslant c_K \langle v - P_{2,q_0} v \, , \, v - P_{2,q_0} v \rangle_{2,q_0,\mu} + \frac{\langle q_0' \, , \, q_0' \rangle_{2,q_0}}{4} \int_{(\mathbb{S})^2} (\alpha(\omega) - \alpha(\omega'))^2 \, \mathrm{d}\mu \otimes \mu \, . \tag{3.3.31}$$

But if $P_{2,q_0} v = \beta q_0'$ (recall that $v = u - P_{2,q_0,\mu} u$), then since $P_{2,q_0,\mu} u$ is colinear to q_0', for almost all ω, ω'

$$\beta(\omega) - \beta(\omega') = \alpha(\omega) - \alpha(\omega') \tag{3.3.32}$$

and since v is orthogonal to q_0' (with respect to $\langle \cdot, \cdot \rangle_{2,q_0,\mu}$) we get

$$\int_{\mathbb{R}} \beta(\omega) \, \mathrm{d}\mu = 0 \, . \tag{3.3.33}$$

So (3.3.31) becomes

$$\mathcal{E}_\mu(u) \geqslant c_K \langle v - P_{2,q_0} v \, , \, v - P_{2,q_0} v \rangle_{2,q_0,\mu} + \frac{\langle q_0' \, , \, q_0' \rangle_{2,q_0}}{2} \int_{\mathbb{S}} \beta^2(\omega) \, \mathrm{d}\mu \, . \tag{3.3.34}$$

It is sufficient to compare this last minoration with the norm $\langle v \, , \, v \rangle_{2,q_0,\mu}$, and from Lemma 3.3.6 it comes

$$\langle v \, , \, v \rangle_{2,q_0,\mu} = \langle v - P_{2,q_0} v \, , \, v - P_{2,q_0} v \rangle_{2,q_0,\mu} + \langle q_0' \, , \, q_0' \rangle_{2,q_0} \int_{\mathbb{S}} \beta^2(\omega) \, \mathrm{d}\mu \, . \tag{3.3.35}$$

This completes the proof of Proposition 3.3.4. □

We now need two lemmas comparing the scalar products $\langle \cdot\, , \cdot \rangle_{2,q_0,\mu}$ and $\langle \cdot\, , \cdot \rangle_{-1,q_0,\mu}$. They correspond to Lemmas 2.4 and 2.5 in [9]. Their proofs are very similar to the proofs of the results corresponding results in [9] (to which we refer also for the explicit values of the constants C and c appearing below) and they use in particular the rigged Hilbert space representation of $H^{-1}_{q_0,\mu}$ (see [16, p.82]): namely, one can identify $H^{-1}_{q_0,\mu}$ as the dual space V' of the space V closure of $\mathcal{D}(A)$ with respect to the norm $\|u\|_V := \left(\int_{\mathbb{R}\times\mathbb{S}} v'(\theta,\omega)^2\, \mathrm{d}\theta\mu(\mathrm{d}\omega)\right)^{\frac{1}{2}}$. The pivot space H is the usual $L^2(\lambda\otimes\mu)$ (endowed with the Hilbert norm $\|u\|_{2,\mu} := \left(\int_{\mathbb{R}\times\mathbb{S}} u(\theta,\omega)^2\, \mathrm{d}\theta\mu(\mathrm{d}\omega)\right)^{\frac{1}{2}}$). In particular, one easily sees that the inclusion $V \subseteq H$ is dense. Consequently, one can define $T : H \to V'$ by setting $Tu(v) = \int_{\mathbb{R}\times\mathbb{S}} u(\theta,\omega)v(\theta,\omega)\, \mathrm{d}\theta\mu(\mathrm{d}\omega)$. One can prove that T continuously injects H into V' and that $T(H)$ is dense into V' so that one can identify $u \in H$ with $Tu \in V'$. Then for $u \in H$,

$$\|u\|_{V'} = \|Tu\|_{V'} = \sup_{v\in V} \frac{\int \mathcal{U}v'}{\|v\|_V} = \sqrt{\int \frac{\mathcal{U}^2}{q_0}}\,, \tag{3.3.36}$$

which enables us to identify $H^{-1}_{q_0,\mu}$ with V'.

We define the projection in $H^{-1}_{q_0,\mu}$:

$$P_{-1,q_0,\mu}u \;=\; \frac{\langle u\, ,\, q_0'\rangle_{-1,q_0,\mu}}{\langle q_0'\, ,\, q_0'\rangle_{-1,q_0}} q_0'\,. \tag{3.3.37}$$

Lemma 3.3.7. *For every $K > 1$ there exists a constant $C = C(K) > 0$ such that for $u \in L^2_\mu$ such that $\int_{\mathbb{S}} u = 0$ for almost every ω*

$$\begin{aligned}\langle u - P_{2,q_0,\mu}u\, ,\, u - P_{2,q_0,\mu}u\rangle_{2,q_0,\mu} \;&\geqslant e^{4Kr_0}C\,\langle u - P_{-1,q_0,\mu}u\, ,\, u - P_{-1,q_0,\mu}u\rangle_{2,q_0,\mu}\\ &\geqslant C\,\langle u - P_{-1,q_0,\mu}u\, ,\, u - p_{-1,q_0,\mu}u\rangle_{-1,q_0,\mu}\,.\end{aligned} \tag{3.3.38}$$

Lemma 3.3.8. *For every $K > 1$ there exists $c = c(K) > 0$ such that for $u \in L^2_\mu$ such that $\int_{\mathbb{S}} u = 0$ for almost every ω and*

$$\langle u\, ,\, u\rangle_{-1,q_0,\mu} \;\geqslant c\,\langle P_{2,q_0,\mu}u\, ,\, P_{2,q_0,\mu}u\rangle_{2,q_0,\mu}\,. \tag{3.3.39}$$

Proof of Proposition 3.2.1. Of course Proposition 3.3.4 and Lemma 3.3.7 imply directly the spectral gap inequality for the Dirichlet form:

$$\mathcal{E}(u) \;\geqslant c_K C\,\langle u - P_{-1,q_0,\mu}u\, ,\, u - P_{-1,q_0,\mu}u\rangle_{-1,q_0,\mu} \qquad \text{for all}\, u \in H^{-1}_{q_0,\mu}\,. \tag{3.3.40}$$

We now prove the self-adjoint property of A. It is sufficient to prove that the range of $1 - A$ is dense in H^{-1}_μ (see [16, p.113]). For $u, v \in D(A)$, we have

$$\begin{aligned}\langle v\, ,\, (1-A)u\rangle_{-1,q_0,\mu} \;=\; &-\int_{\mathbb{R}}\int_{\mathbb{S}} v(\theta,\omega)\left(\int_0^\theta \frac{\mathcal{U}}{q_0}\right) \mathrm{d}\theta\, \mathrm{d}\mu + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{S}} \frac{vu}{q_0}\, \mathrm{d}\theta\, \mathrm{d}\mu\\ &-\int_{\mathbb{R}}\int_{(\mathbb{S})^2} v(\theta,\omega)\widetilde{J} * u(\theta,\omega')\, \mathrm{d}\theta\, \mathrm{d}\mu\otimes\mu\,.\end{aligned} \tag{3.3.41}$$

The right side of this expression is still defined for $u, v \in L^2(\lambda\otimes\mu)$ (recall that λ denotes the Lebesgue measure on $\mathbb{S}$, and that we denote the usual scalar product on $L^2(\lambda\otimes\mu)$ by $\|\cdot\|_{2,\mu}$) and there exists $c > 0$ such that

$$\langle v\, ,\, (1-A)u\rangle_{-1,q_0,\mu} \;\leqslant c\|u\|_{2,\mu}\|v\|_{2,\mu}\,, \tag{3.3.42}$$

Furthermore from (3.3.40) and Lemma 3.3.8 we have

$$\langle u\,,\,(1-A)u\rangle_{-1,q_0,\mu} \;\geqslant\; \frac{1}{c}\|u\|_{2,\mu}^2\,. \tag{3.3.43}$$

So the bilinear form $(u,v)\mapsto\langle v\,,\,(1-A)u\rangle_{-1,q_0,\mu}$ is continuous and coercive on $H_\mu^{-1}\times H_\mu^{-1}$. If $f\in H_\mu^{-1}$, the linear form $v\mapsto\langle v\,,\,f\rangle_{-1,q_0,\mu}$ is continuous on $L^2(\lambda\otimes\mu)$, therefore from Lax-Milgram Theorem we get that there exists a unique $u\in L^2(\lambda\otimes\mu)$ such that for all $v\in L^2(\lambda\otimes\mu)$

$$\langle v\,,\,(1-A)u\rangle_{-1,q_0,\mu} \;=\; \langle v\,,\,f\rangle_{-1,q_0,\mu}\,. \tag{3.3.44}$$

Since

$$\langle v\,,\,f\rangle_{-1,q_0,\mu} \;=\; -\int_{\mathbb{R}}\int_{\mathbb{S}} v(\theta,\omega)\left(\int_0^\theta \frac{\mathcal{F}}{q_0}\right)\mathrm{d}\theta\,\mathrm{d}\mu\,, \tag{3.3.45}$$

from (3.3.41) we obtain that for almost θ and ω

$$-\int_0^\theta \frac{\mathcal{U}(\theta',\omega)}{q_0(\theta')}\,\mathrm{d}\theta' + \frac{u(\theta,\omega)}{2q_0(\theta)} - \int_{\mathbb{R}}\left(\widetilde{J}*u\right)(\theta,\omega)\,\mathrm{d}\mu \;=\; -\int_0^\theta \frac{\mathcal{F}(\theta',\omega)}{q_0(\theta')}\,\mathrm{d}\theta'\,. \tag{3.3.46}$$

So it is clear that if f is continuous with respect to θ, then u has a version C^2 with respect to θ. Thus $u\in D(A)$ and applying $\partial_\theta(q_0(\theta)\partial_\theta\cdot)$ to the both sides of this last expression, we get $(1-A)u=f$. Since this kind of functions f is dense in H_μ^{-1}, we can conclude that the range of $1-A$ is dense, and that A is essentially self-adjoint. This completes the proof of Proposition 3.2.1. □

3.4 Perturbation arguments (completion of the proof of Theorem 3.2.2)

In this section we complete the proof of Theorem 3.2.2. Essentially, this section is devoted to computing the expansion of the speed $c_\mu(\delta)$ in. We first recall a lemma that gives a useful parametrization in the neighborhood of M_0. The proof of this lemma is given in [99] , and it is used in the proof of Theorem 3.3.1 (see [44, 99]).

Lemma 3.4.1. *There exists a $\sigma>0$ such that for all p in the neighborhood*

$$N_\sigma \;:=\; \cup_{q\in M_0} B_{L^2(\lambda\otimes\mu)}(q,\sigma)\,, \tag{3.4.1}$$

of M_0 there is one and only one $q=v(p)\in M_0$ such that $\langle p-q\,,\,\partial_\theta q\rangle_{-1,q_0,\mu}=0$. Furthermore the mapping $p\mapsto v(p)$ is in $C^\infty(X_\mu^1,X_\mu^1)$.

Proof of Theorem 3.2.2. The existence and stability of a *rotating* solution $\widetilde{q}_\delta(\theta-\psi-c_\mu(\delta)t)$ of (3.1.10) (ψ is arbitrary) has been established in Section 3.3 for $\delta\leqslant\delta_0$, see Theorem 3.3.1 and the two remarks that follow it. We are left with proving Theorem 3.2.2(2).

Thanks to the invariance by rotation, we can define $\widetilde{q}_\delta$ such that $v(\widetilde{q}_\delta)=q_0$. Now if we denote

$$n_\delta \;:=\; \widetilde{q}_\delta - v\left(\widetilde{q}_\delta\right)\,, \tag{3.4.2}$$

then n_δ verifies $n_\delta=\phi_\delta(q_0)$ and (see Lemma 3.4.1)

$$\langle n_\delta\,,\,q_0'\rangle_{-1,q_0,\mu}=0 \tag{3.4.3}$$

$$\langle An_\delta\,,\,q_0'\rangle_{-1,q_0,\mu}=0\,. \tag{3.4.4}$$

Moreover the estimates we have on the mapping ϕ_δ in Theorem 3.3.1 give

$$\|n_\delta\|_{2,\mu} \leqslant C\delta\,, \tag{3.4.5}$$

$$\|\partial_\theta n_\delta\|_{2,\mu} \leqslant C\delta\,. \tag{3.4.6}$$

Taking the derivative with respect to t, at time $t=0$, we get (we recall the notation $p_t^{(\psi)}(\theta,\omega) = \widetilde{q}_\delta(\theta-\psi-c_\mu(\delta)t))$:

$$-c_\mu(\delta)\,(q_0' + \partial_\theta n_\delta) \;=\; \partial_t p_0^{(0)}\,. \tag{3.4.7}$$

So (3.1.10) at time $t=0$ becomes (recall that q_0 is a stationary solution of (3.1.12)) :

$$-c_\mu(\delta)\,(q_0' + \partial_\theta n_\delta) \;=\; A n_\delta - \partial_\theta\left[n_\delta \langle J * n_\delta\rangle_\mu\right] - \delta\omega q_0' - \delta\omega\partial_\theta n_\delta\,. \tag{3.4.8}$$

From (3.4.5) we deduce the bound

$$\|\partial_\theta\left[n_\delta\langle J*n_\delta\rangle_\mu\right]\|_{-1,\mu} \;\leqslant\; \|J\|_2 C^2\delta^2\,, \tag{3.4.9}$$

so by taking the $H^{-1}_{q_0,\mu}$ scalar product of q' in (3.4.8), using (3.4.4), (3.4.5), (3.4.6) and the fact that $\int_{\mathbb{R}} w\,\mathrm{d}\mu = 0$, we get that $c_\mu(\delta)$ is of order δ^2. This implies, using the same arguments, that

$$\|A n_\delta - \delta\omega q_0'\|_{-1,\mu} \;=\; O(\delta^2)\,. \tag{3.4.10}$$

So

$$\|A(n_\delta - \delta n^{(1)})\|_{-1,\mu} \;=\; O(\delta^2)\,, \tag{3.4.11}$$

and since $\|(1-A)^{(1/2)}u\|_{-1,\mu} \sim \|u\|_{2,\mu}$ (see (3.3.42) and (3.3.43)), we have in particular

$$\|n_\delta - \delta n^{(1)}\|_{2,\mu} \;=\; O(\delta^2)\,. \tag{3.4.12}$$

It allows us to make a second order expansion for $c_\mu(\delta)$: taking again the $H^{-1}_{q_0,\mu}$ scalar product of q_0' in (3.4.8), using the same bounds as for the first order expansion and (3.4.11), we get :

$$c_\mu(\delta) \;=\; \delta^2\frac{\left\langle \omega\partial_\theta n^{(1)} + n^{(1)}\langle J*n^{(1)}\rangle_\mu\,,\, q_0'\right\rangle_{-1,q_0,\mu}}{\langle q_0'\,,\,q_0'\rangle_{-1,q_0,\mu}} + O(\delta^3)\,. \tag{3.4.13}$$

Indeed, from (3.4.11), $\|\omega\partial_\theta(n_\delta-\delta n^{(1)})\|_{-1,\mu}$, $\|\partial_\theta[(n_\delta-\delta n^{(1)})\langle J*n^{(1)}\rangle_\mu]\|_{-1,\mu}$, $\|\partial_\theta[n^{(1)}\langle J*(n_\delta-\delta n^{(1)})\rangle_\mu]\|_{-1,\mu}$ are of order δ^2 and $\|\partial_\theta[(n_\delta-\delta n^{(1)})\langle J*(n_\delta-\delta n^{(1)})\rangle_\mu]\|_{-1,\mu}$ of order δ^4. Since $c_\mu(\delta)$ is odd with respect to δ, the second order term in (3.4.13) is equal to 0. It is possible to get this fact directly : we remark that $n^{(1)}$ satisfies :

$$L_{q_0}\int_{\mathbb{R}} n^{(1)}\,\mathrm{d}\mu \;=\; \int_{\mathbb{R}} A n^{(1)}\,\mathrm{d}\mu \;=\; \left(\int_{\mathbb{R}}\omega\,\mathrm{d}\mu\right) q_0' \;=\; 0\,, \tag{3.4.14}$$

$$\left\langle \int_{\mathbb{R}} n^{(1)}\,\mathrm{d}\mu\,,\, q_0'\right\rangle_{-1,q_0} \;=\; \langle n^{(1)}\,,\,q_0'\rangle_{-1,q_0,\mu} \;=\; 0\,. \tag{3.4.15}$$

So since L_{q_0} is bijective on the orthogonal of q_0' in $H^{-1}_{1/q}$ (see [9]), we have $\int_{\mathbb{R}} n^{(1)}\,\mathrm{d}\mu = 0$ and $\langle J*n^{(1)}\rangle\mu = 0$. On the other hand, since the operator A conserves the parity with respect to θ, $n^{(1)}$ is odd with respect to θ and thus

$$\langle\omega\partial_\theta n^{(1)}\,,\,q_0'\rangle_{-1,q_0,\mu} \;=\; \int_{\mathbb{S}}\int_{\mathbb{R}}\frac{\omega n^{(1)}}{q_0}\left(q_0 - \frac{1}{2\pi I_0^2(2Kr_0)}\right)\mathrm{d}\theta\,\mathrm{d}\mu \;=\; 0\,. \tag{3.4.16}$$

Now back to (3.4.8): since $c_\mu(\delta)$ is of order δ^3 and using $\int_{\mathbb{S}} n^{(1)}\,\mathrm{d}\mu = 0$, we get

$$\left\|A\left(n_\delta - \delta n^{(1)} - \delta^2 \omega \partial_\theta n^{(1)}\right)\right\|_{-1,\mu} = O(\delta^3)\,, \tag{3.4.17}$$

and thus

$$\|n_\delta - \delta n^{(1)} - \delta^2 n^{(2)}\|_{2,\mu} = O(\delta^3)\,. \tag{3.4.18}$$

This allows us this time to do a third order expansion in (3.4.8) :

$$c_\mu(\delta) = \delta^3 \frac{\langle \omega\partial_\theta n^{(2)}\,,\, q_0'\rangle_{-1,q_0,\mu}}{\langle q_0'\,,\, q_0'\rangle_{-1,q_0,\mu}} + O(\delta^4)\,. \tag{3.4.19}$$

This procedure may be repeated recursively at any order: we do not go through the details again, but we do report the result below (Remark 3.4.2) and we point out that the $O(\delta^4)$ (3.4.19) turns out to be $O(\delta^5)$, in agreement with the fact that $c_\mu(\delta)$ is odd in δ. □

Remark 3.4.2. As anticipated above, one can get arbitrarily many terms in the formal series $c_\mu(\delta) = \sum_{i=1,2,\ldots} c_{2i+1}\delta^{2i+1}$ and the remainder, when the series is stopped at $i = n$, is $O(\delta^{2i+3})$. In fact, by arguing like above, we have

$$c_5 = \frac{\left\langle \partial_\theta[n^{(2)}\langle J * n^{(3)}\rangle_\mu] + \partial_\theta[n^{(3)}\langle J * n^{(2)}\rangle_\mu] + w\partial_\theta n^{(4)}, q_0'\right\rangle_{-1,q_0,\mu}}{\langle q_0', q_0'\rangle_{-1,q_0}}\,, \tag{3.4.20}$$

where

$$An^{(3)} = \partial_\theta[n^{(1)}\langle J * n^{(2)}\rangle_\mu] + w\partial_\theta n^{(2)} - \frac{\left\langle w\partial_\theta n^{(2)}, q_0'\right\rangle_{-1,q_0,\mu}}{\langle q_0', q_0'\rangle_{-1,q_0}} q_0'\,, \tag{3.4.21}$$

and

$$An^{(4)} = \partial_\theta[n^{(2)}\langle J * n^{(2)}\rangle_\mu] + \partial_\theta[n^{(1)}\langle J * n^{(3)}\rangle_\mu] + w\partial_\theta n^{(3)} + c_3\partial_\theta n^{(1)}\,. \tag{3.4.22}$$

Actually, by induction we obtain

$$c_{2i+1} = \frac{\left\langle \sum_{k+l=2i+1, k\geqslant 2, l\geqslant 2} \partial_\theta[n^{(l)}\langle J * n^{(k)}\rangle_\mu] + w\partial_\theta n^{(2i)}, q_0'\right\rangle_{-1,q_0,\mu}}{\langle q_0', q_0'\rangle_{-1,q_0}}\,, \tag{3.4.23}$$

and

$$n^{(2i)} = \sum_{k+l=2i, k\geqslant 2, l\geqslant 2} \partial_\theta[n^{(l)}\langle J * n^{(k)}\rangle_\mu] + w\partial_\theta n^{(2i-1)} + \sum_{k+l=2i, k\geqslant 2, l\geqslant 2} c_k\partial_\theta n^{(l)}\,, \tag{3.4.24}$$

$$n^{(2i+1)} = \sum_{k+l=2i+1, k\geqslant 2, l\geqslant 2} \partial_\theta[n^{(l)}\langle J * n^{(k)}\rangle_\mu] + w\partial_\theta n^{(2i)} - c_{2i+1}q_0'\,. \tag{3.4.25}$$

Since this procedure yields also $n^{(j)}$ for arbitrary j, one can generalizes also (3.4.18) and, hence, (3.2.16).

3.5 Active rotators

In this section we deal with the equation (3.1.13) and we do it in a rather informal way, because on one hand a formal statement would be very close to Theorem 3.2.2 and, on the the other hand, the large scale behavior of disordered active rotators is qualitatively and quantitatively close to the non disordered case, treated in [44], in a way that we explain below.

First of all, from a technical viewpoint the main difference between (3.1.13) and (3.1.4) is that (3.1.13) is (in general) not rotation invariant, so the manifold $M_\delta = \{q_\psi + \phi(q_\psi)\}$ we get after perturbation is not necessarily a circle. Unlike Theorem 3.2.2, the motion on M_δ is not uniform, and we describe the behaviour on M_δ by the phase derivate $\dot{\psi}$. We follow the same procedure as in the previous section : if p_t^δ is a solution (3.1.13) belonging to M_δ, we define (see Lemma 3.4.1)

$$q_{\psi_t^\delta} \,=\, v(p_t^\delta)\,, \quad \text{and} \quad n_t^\delta \,=\, p_t^\delta - v(p_t^\delta)\,. \tag{3.5.1}$$

In this context, (3.4.8) becomes

$$-\dot{\psi}_t^\delta q'_{\psi_t^\delta} + \partial_t n_t^\delta \,=\, A^{\psi_t^\delta} n_t^\delta - \partial_\theta[n_t^\delta \langle J * n_t^\delta\rangle_\mu] - \delta U q'_\psi - \delta U \partial_\theta n_t^\delta\,, \tag{3.5.2}$$

where A^ψ is the rotation of the operator A

$$A^\psi u(\theta,\omega) \,:=\, \frac{1}{2}\Delta u(\theta,\omega) - \partial_\theta \Big(q_0(\theta-\psi)\langle J * u\rangle_\mu(\theta) + u(\theta,\omega) J * q_0(\theta-\psi)\Big)\,. \tag{3.5.3}$$

Note that we can reformulate the second term of the left hand side in (3.5.2):

$$\partial_t n_t^\delta \,=\, \dot{\psi}_t^\delta \partial_\psi \phi(q_\psi)|_{\psi=\psi_t^\delta}\,. \tag{3.5.4}$$

So, as in the previous section, using the estimates on the mapping ϕ given in Theorem 3.3.1, we get the bounds

$$\|n_t^\delta\|_{2,\mu} \,\leqslant\, C\delta\,, \quad \|\partial_t n_t^\delta\|_{2,\mu} \,\leqslant\, C\delta|\dot{\psi}_t^\delta| \ \text{ and } \ \|\partial_\theta[n_t^\delta\langle J * n_t^\delta\rangle_\mu]\|_{2,\mu} \,\leqslant\, \|J\|_2 C^2\delta\,, \tag{3.5.5}$$

and we deduce the first order expansion

$$\dot{\psi}_t^\delta \,=\, \delta \frac{\langle (U q_{\psi_t^\delta})', q'_{\psi_t^\delta}\rangle_{-1,q_{\psi_t^\delta},\mu}}{\langle q'_0, q'_0\rangle_{-1,q_0}} + O(\delta^2)\,. \tag{3.5.6}$$

Since $\dot{\psi}$ is odd in δ and the expansion can be pushed further in δ, this $O(\delta^2)$ is in reality a $O(\delta^3)$ and one can actually improve this result both in the direction of obtaining a regularity estimate on the $O(\delta^2)$ rest in (3.5.6) (like in [44, Th. 2.3]) and of going to higher orders (like in Remark 3.4.2).

However the evolution for small δ is dominated by the leading order and from (3.5.6) we can directly read that, to first order, the effect of the disorder is rather simple: in fact

$$\langle (U q_\psi)', q'_\psi\rangle_{-1,q_\psi,\mu} \,=\, \int_{\mathbb{R}}\int_{\mathbb{S}} U(\theta,\omega) q_\psi(\theta)\left(q_\psi(\theta) - c\right) \mathrm{d}\theta \mu(\mathrm{d}\omega)\,, \tag{3.5.7}$$

where c is such that $\int_{\mathbb{S}}(q_\psi - c) = 0$, that is $1/c = 2\pi(I_0(2Kr_0))^2$ (recall (3.2.1)-(3.2.3): this computation is analogous to (3.4.14)). Since the integrand depends on ω only via U, this integration can be performed first and the system behaves to leading order in δ as the non-disordered model with active rotator dynamics led by the deterministic force $\int_{\mathbb{R}} U(\cdot,\omega)\mu(\mathrm{d}\omega)$. The rich phenomenology connected to these models is worked out in [44, Sec. 3].

3.6 Symmetric case: stability of the stationary solutions

3.6.1 On the non-trivial stationary solutions (proof of Lemma 3.2.3)

We start by observing that in the case with no disorder the strict concavity of the fixed-point function Ψ_0 has been proven in [82, Lemma 4, p.315], in the apparently different context of classical XY-spin model (for a detailed discussion on the link with these models see [9]). We are going to obtain the concavity of Ψ_δ^μ for small δ via a perturbation argument, by relying on the result in [82].

Since Ψ_δ^μ is a smooth perturbation of Ψ_0, one expects that the strict concavity of Ψ_0 will be preserved to Ψ_δ^μ for small $\delta > 0$, namely $\sup_x (\Psi_\delta^\mu)''(x) < 0$. Nevertheless, an easy calculation shows that $\Psi_0''(0) = 0$; in that sense one has to treat the concavity in a neighborhood of 0 as a special case.

In what follows, we suppose that the coupling strength K is bounded above by a fixed constant $K_{\max}$. We first prove the statement on the concavity in a neighborhood of 0: there exist $\eta_0 > 0$, $\delta > 0$ such that for all μ such that $\mathrm{Supp}(\mu) \subseteq [-1, 1]$, Ψ_δ^μ is strictly concave on $[0, \eta_0]$.

Indeed, one easily shows (using that the function $x \mapsto \Psi_\mu^\delta(x)$ is odd) that we have the following Taylor's expansion:

$$(\Psi_\delta^\mu)''(x) = -\frac{3}{4} D^\delta(\mu) x + \epsilon(x)\,, \tag{3.6.1}$$

where $\epsilon(x) = o(x)$ as $x \to 0$ and where for fixed μ, we write

$$D^\delta(\mu) := \int_{\mathbb{R}} h(\delta\omega)\mu(\,\mathrm{d}\omega), \tag{3.6.2}$$

where

$$h(\omega) := \frac{1}{2(1+\omega^2)} - \frac{8\omega^2}{(1+4\omega^2)^2}. \tag{3.6.3}$$

Note that the $o(x)$ can be chosen independently of μ and δ. A closer look at the function h shows that there exists $\delta > 0$ such that for all μ with $\mathrm{Supp}(\mu) \subseteq [-1, 1]$, $D^\delta(\mu) > \frac{1}{4}$. If we choose $\eta_0 > 0$ such that $\frac{1}{\eta_0} \sup_{0 \leqslant x < \eta_0} |\epsilon(x)| < \frac{16}{3}$ then $(\Psi_\delta^\mu)''(x) < 0$ for all $0 < x < \eta_0$, which is the desired result.

We are now left with proving concavity away from 0: namely, we prove that for all $\eta > 0$, all $K_{\max}$, there exists $\delta_0 > 0$ such that for all $K \leqslant K_{\max}$, for all $0 < \delta < \delta_0$, for any measure μ such that $\mathrm{Supp}(\mu) \subseteq [-1, 1]$, Ψ_δ^μ is strictly concave on $[\eta, 2K_{\max}]$.

Indeed, using the strict concavity of Ψ_0 proved in [82], there exists a constant $\alpha > 0$ such that for all $x \in [\eta, 2K_{\max}]$, $\Psi_0''(x) < -\alpha < 0$. But then, it easy to see that

$$\sup_{0<\delta<\delta_0} \ \sup_{\mu,\ \mathrm{Supp}(\mu)\subseteq[-1,1]} \ \sup_{x\in[0,2K_{\max}]} \left|(\Psi_\delta^\mu)''(x) - \Psi_0''(x)\right| \overset{\delta_0 \searrow 0}{\to} 0. \tag{3.6.4}$$

If one chooses δ_0 such that the latter quantity is smaller than or equal to $\frac{\alpha}{2}$, the result follows. The proof of Lemma 3.2.3 is therefore complete. □

3.6.2 On the linear stability of non-trivial stationary solutions

We now prove Theorem 3.2.5 along with a number of explicit estimates.

Remark 3.6.1. Note that, since the whole operator L_q^ω is no longer self-adjoint nor symmetric, its spectrum need not be real. In that extent, one has to deal in this section with the complexified versions of the scalar products defined in Section 3.2, (3.2.8) and in Section 3.3, (3.3.12). Thus, we will assume for the rest of this section that we work with complex versions of these scalar products. The results concerning the operator A are obviously still valid, since A is symmetric and real.

We will also use the following standard notations: for an operator F, we will denote by $\rho(F)$ the set of all complex numbers λ for which $\lambda - F$ is invertible, and by $R(\lambda, F) := (\lambda - F)^{-1}$, $\lambda \in \rho(F)$ the resolvent of F. The spectrum of F will be denoted as $\sigma(F)$.

Decomposition of L_q^ω

In what follows, $K > 1$ and $r_0 = \Psi_0(2Kr_0) > 0$ are fixed.

In order to study the spectral properties of the operator L_q^ω for general distribution of disorder, we decompose L_q^ω in (3.2.19) into the sum of the self-adjoint operator A defined in (3.2.6) and a perturbation B which will be considered to be small w.r.t. A, namely:

$$Bu(\theta,\omega) := -\partial_\theta \left(u(\theta,\omega)\langle J * \varepsilon(q)\rangle_\mu + \varepsilon(q)(\theta,\omega,\delta)\langle J * u\rangle_\mu(\theta) + \delta\omega u(\theta,\omega)\right), \qquad (3.6.5)$$

where

$$\varepsilon(q) := (\theta,\omega,\delta) \mapsto q(\theta,\delta\omega) - q_0(\theta), \qquad (3.6.6)$$

is the difference between the stationary solution with disorder and the one without disorder.

Proposition 3.6.2. *The (extension of the) operator A is the infinitesimal generator of a strongly continuous semi-group of contractions $T_A(t)$ on $H^{-1}_{q_0,\mu}$.*

Moreover, for every $0 < \alpha < \frac{\pi}{2}$ this semigroup can be extended to an analytic semigroup $T_A(z)$ defined on $\Delta_\alpha := \{z \in \mathbb{C}\,;\, |\arg(z)| < \alpha\}$.

We recall here the result we use concerning analytic extensions of strongly continuous semigroups. Its proof can be found in [81, Th 5.2, p.61].

Proposition 3.6.3. *Let $T(t)$ a uniformly bounded strongly continuous semigroup, whose infinitesimal generator F is such that $0 \in \rho(F)$ and let $\alpha \in (0, \frac{\pi}{2})$. The following statements are equivalent:*

1. *$T(t)$ can be extended to an analytic semigroup in the sector $\Delta_\alpha = \{\lambda \in \mathbf{C}\,;\, |\arg(\lambda)| < \alpha\}$ and $\|T(z)\|$ is uniformly bounded in every closed sub-sector $\bar{\Delta}'_\alpha$, $\alpha' < \alpha$, of Δ_α,*
2. *There exists $M > 0$ such that*

$$\rho(F) \supset \Sigma = \left\{\lambda \in \mathbf{C}\,;\, |\arg(\lambda)| < \frac{\pi}{2} + \alpha\right\} \cup \{0\}, \qquad (3.6.7)$$

and

$$\|R(\lambda, F)\| \leqslant \frac{M}{|\lambda|}, \quad \lambda \in \Sigma, \lambda \neq 0\,. \qquad (3.6.8)$$

Proof of Proposition 3.6.2. The proof in Section 3.3, Theorem 3.2.1 of the self-adjointness of A shows that A satisfies the hypothesis of Lumer-Phillips Theorem (see [81, Th 4.3, p.14]): A is the infinitesimal generator of a C_0 semi-group of contractions denoted by $T_A(t)$.

The rest of the proof is devoted to show the existence of an analytic extension of this semigroup in a proper sector. We follow here the lines of the proof of Th 5.2, p. 61-62, in [81], but with explicit estimates on the resolvent, in order to quantify properly the appropriate size of the perturbation.

Let us first replace the operator A by a small perturbation: for all $\varepsilon > 0$, let $A_\varepsilon := A - \varepsilon$, so that 0 belongs to $\rho(A_\varepsilon)$. The operator A_ε has the following properties: as A, it generates a strongly continuous semigroup of operators (which is $T_{A,\varepsilon}(t) = T_A(t)e^{-\varepsilon t}$).

Since A is self-adjoint, it is easy to see that

$$\forall \lambda \in \mathbf{C} \setminus \mathbb{R}, \| R(\lambda, A_\varepsilon) \|_{-1,q_0,\mu} \leqslant \frac{1}{|\Im(\lambda)|}, \tag{3.6.9}$$

and since the spectrum of A is negative, for every $\lambda \in \mathbf{C}$ such that $\Re(\lambda) > 0$,

$$\| R(\lambda, A_\varepsilon) \|_{-1,q_0,\mu} \leqslant \frac{1}{|\lambda|}. \tag{3.6.10}$$

For any $\alpha \in (0, \frac{\pi}{2})$, let

$$\Sigma_\alpha := \left\{ \lambda \in \mathbb{C}\,;\, |\arg(\lambda)| < \frac{\pi}{2} + \alpha \right\}. \tag{3.6.11}$$

Let us prove that for $\lambda \in \Sigma_\alpha$,

$$\| R(\lambda, A_\varepsilon) \|_{-1,q_0,\mu} \leqslant \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}. \tag{3.6.12}$$

Note that (3.6.12) is clear from (3.6.9) and (3.6.10) when λ is such that $\Re(\lambda) \geqslant 0$.

Let us consider $\sigma > 0, \tau \in \mathbb{R}$ to be chosen appropriately later.

Let us write the following Taylor expansion for $R(\lambda, A_\varepsilon)$ around $\sigma + i\tau$ (at least well defined in a neighborhood of $\sigma + i\tau$ since $\sigma > 0$):

$$R(\lambda, A_\varepsilon) = \sum_{n=0}^{\infty} R(\sigma + i\tau, A_\varepsilon)^{n+1} ((\sigma + i\tau) - \lambda)^n. \tag{3.6.13}$$

From now, we fix $\lambda \in \Sigma_\alpha$ with $\Re(\lambda) < 0$. This series $R(\lambda, A_\varepsilon)$ is well defined in λ if one can choose σ, τ and $k \in (0,1)$ such that $\| R(\sigma + i\tau, A_\varepsilon) \|_{-1,q_0,\mu} |\lambda - (\sigma + i\tau)| \leqslant k < 1$. In particular, using (3.6.9), it suffices to have $|\lambda - (\sigma + i\tau)| \leqslant k|\tau|$ and since $\sigma > 0$ is arbitrary, it suffices to find $k \in (0,1)$ and τ with $|\lambda - i\tau| \leqslant k|\tau|$ to obtain the convergence of (3.6.13). For this $\lambda \in \Sigma_\alpha$ with $\Re(\lambda) < 0$, let us define λ' and τ as in Figure 3.2. Then, $|\lambda - i\tau| \leqslant |\lambda' - i\tau| = \sin(\alpha)|\tau|$ with $\sin(\alpha) \in (0,1)$. So the series converges for $\lambda \in \Sigma_\alpha$ and one has, using again (3.6.9),

$$\| R(\lambda, A_\varepsilon) \|_{-1,q_0,\mu} \leqslant \frac{1}{(1 - \sin(\alpha))|\tau|} \leqslant \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}. \tag{3.6.14}$$

The fact that $T_{A,\varepsilon}(t)$ can be extended to an analytic semigroup $T_{A,\varepsilon}(z)$ on the domain Δ_α is a simple application of (3.6.14) and Proposition 3.6.3, with $M := \frac{1}{1-\sin(\alpha)}$.

Let us then define $\widetilde{T_A}(z) := e^{\varepsilon z} T_{A,\varepsilon}(z)$, for $z \in \Delta_\alpha$ so that $\widetilde{T_A}$ is an analytic extension of T_A (an argument of analyticity shows that $\widetilde{T_A}$ does not depend on ε). □

Remark 3.6.4. Note that estimate (3.6.12) is also valid in the limit as $\varepsilon \to 0$: for all $\alpha \in (0, \frac{\pi}{2})$, $\lambda \in \Sigma_\alpha$,

$$\| R(\lambda, A) \|_{-1,q_0,\mu} \leqslant \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}. \tag{3.6.15}$$

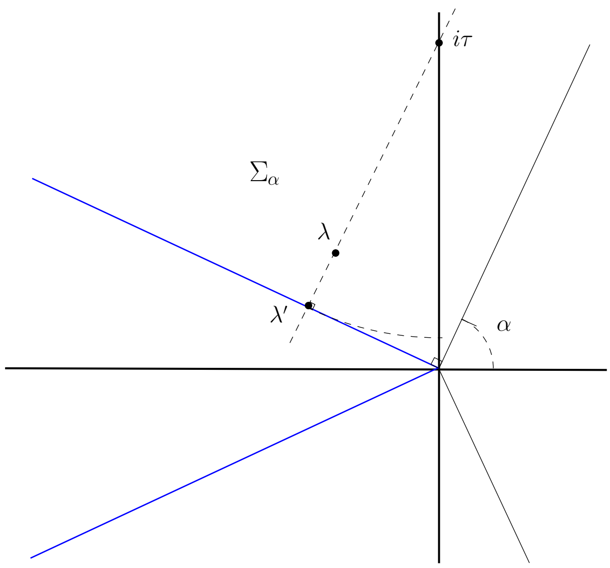


FIGURE 3.2. The set Σ_α.

Spectral properties of $L_q^\omega = A + B$

In this part, we show that if the perturbation B is small enough with respect to A, one has the same spectral properties for $L_q^\omega = A + B$ as for A. In this extent, we recall that μ is of compact support in $[-1, 1]$, and the disorder is rescaled by $\delta > 0$.

Proposition 3.6.5.
The operator B is A-bounded, in the sense that there exist explicit constants $a_{K,\delta}$ and $b_{K,\delta}$, depending on K and δ such that for all u in the domain of (the closure of) A

$$\| Bu \|_{-1,q_0,\mu} \leqslant a_{K,\delta} \| u \|_{-1,q_0,\mu} + b_{K,\delta} \| Au \|_{-1,q_0,\mu} . \tag{3.6.16}$$

Moreover, for fixed $K > 1$, $a_{K,\delta} = O(\delta)$ and $b_{K,\delta} = O(\delta)$, as $\delta \to 0$.

The latter proposition is based on the fact that the difference $\varepsilon(q)(\theta, \omega, \delta) = q(\theta, \delta\omega) - q_0(\theta)$ in (3.6.6) is small if the scale parameter δ tend to 0:

Lemma 3.6.6. *For $\delta > 0$, let us define*

$$\| \varepsilon(q) \|_\infty := \sup_{\substack{\theta\in\mathbb{S}, |\omega| \leqslant 1 \\ 0<u<\delta}} |\varepsilon(q)(\theta, \omega, u)| . \tag{3.6.17}$$

Then for all $K > 1$, $\| \varepsilon(q) \|_\infty = O(\delta)$, as $\delta \to 0$. More precisely, for $K > 1$, $\delta > 0$, the following inequality holds:

$$\| \varepsilon(q) \|_\infty \leqslant \varepsilon_{K,\delta} , \tag{3.6.18}$$

where the constant $\varepsilon_{K,\delta}$ can be chosen explicitly in terms of K and δ:

$$\varepsilon_{K,\delta} := \frac{\delta}{\pi} e^{8\pi\delta} \left(2 + 3e^{4\pi\delta}\right) e^{14K\bar{r}_\delta} \left(1 + 2\pi e^{2K\bar{r}_\delta}\right) , \tag{3.6.19}$$

where we recall that $\bar{r}_\delta = \max(r_0, r_\delta)$.

Proof of Lemma 3.6.6. Recall that the disordered stationary solution q (3.1.5) is given by

$$q(\theta, \delta\omega) := \frac{S(\theta, \delta\omega, 2Kr_\delta)}{Z(\delta\omega, 2Kr_\delta)}, \tag{3.6.20}$$

where $S(\theta,\omega,x)$ is defined in (3.1.6) and that the non-disordered one (3.2.1) is given by $q_0(\theta) = \frac{S(\theta,0,2Kr_0)}{Z(0,2Kr_0)} = \frac{e^{2Kr_0\cos(\theta)}}{\int_{\mathbb{S}} e^{2Kr_0\cos(\theta)}\,\mathrm{d}\theta}$. Since $q(\theta,\delta\omega) = q(-\theta,-\delta\omega)$, it suffices to consider the case $\delta\omega > 0$. A simple computation shows that

$$Z(\delta\omega, 2Kr_\delta) \geqslant 4\pi^2 e^{-4Kr_\delta} e^{-4\pi\delta}, \tag{3.6.21}$$

and that

$$|S(\theta,0)| \leqslant 2\pi e^{4Kr_0}. \tag{3.6.22}$$

Using $|q(\theta,\delta\omega) - q_0(\theta)| \leqslant \frac{1}{Z(\delta\omega)Z(0)}\left(Z(0)|S(\theta,\delta\omega) - S(\theta,0)| + |S(\theta,0)||Z(0) - Z(\delta\omega)|\right)$, one has to deal with, successively:

– for fixed $\theta \in \mathbb{S}$, $|S(\theta,\delta\omega) - S(\theta,0)| \leqslant \delta \cdot \sup_{|\omega|\leqslant 1} |\frac{\mathrm{d}}{\mathrm{d}\omega} S(\theta,\delta\omega)|$. A long calculation shows that the latter expression $|\frac{\mathrm{d}}{\mathrm{d}\omega} S(\theta,\delta\omega)|$ can be bounded above by $8\pi^2 e^{4Kr_\delta} e^{4\pi\delta}\left(2 + 3e^{4\pi\delta}\right)$, that is,

$$|S(\theta,\delta\omega) - S(\theta,0)| \leqslant \delta 8\pi^2 e^{4Kr_\delta} e^{4\pi\delta}\left(2 + 3e^{4\pi\delta}\right). \tag{3.6.23}$$

– Using $|Z(\delta\omega) - Z(0)| = \left|\int_{\mathbb{S}} (S(\theta,\delta\omega) - S(\theta,0))\,\mathrm{d}\theta\right|$ and (3.6.23), one has directly:

$$|Z(\delta\omega) - Z(0)| \leqslant \delta 16\pi^3 e^{4Kr_\delta} e^{4\pi\delta}\left(2 + 3e^{4\pi\delta}\right). \tag{3.6.24}$$

Putting together (3.6.21), (3.6.22), (3.6.23) and (3.6.24), one obtains the result. □

We are now in position to prove the A-boundedness of B:

Proof of Proposition 3.6.5. B is A-bounded: let us fix a u in the domain of the closure of A. Then we have $\| Bu \|_{-1,q_0,\mu} = \| \mathcal{B}u \|_{2,q_0,\mu}$, where $\mathcal{B}u$ is the appropriate primitive of Bu, namely:

$$\begin{aligned}\mathcal{B}u(\theta,\omega) := & -\left(u(\theta,\omega)\langle J * \varepsilon(q)\rangle_\mu + \varepsilon(q)(\theta,\omega,\delta)\langle J * u\rangle_\mu(\theta) + \delta\omega u(\theta,\omega)\right) \\ & + \left(\int_{\mathbb{S}} \frac{1}{q_0}\right)^{-1} \left(\int_{\mathbb{S}} \frac{u(\theta,\omega)\langle J * \varepsilon(q)\rangle_\mu + \varepsilon(q)(\theta,\omega,\delta)\langle J * u\rangle_\mu(\theta) + \delta\omega u(\theta,\omega)}{q_0(\theta)}\,\mathrm{d}\theta\right).\end{aligned} \tag{3.6.25}$$

One can easily shows that there exists a constant $c^{(1)}_{K,\delta}$, depending only on $K > 1$ and $\delta > 0$ such that:

$$\| Bu \|_{-1,q_0,\mu} \leqslant c^{(1)}_{K,\delta} \| u \|_{2,q_0,\mu}. \tag{3.6.26}$$

Indeed, an easy calculation shows that $|\langle J * \varepsilon(q)\rangle_\mu| \leqslant 4K \| \varepsilon(q) \|_\infty$ and that

$$\begin{aligned}|\langle J * u\rangle_\mu(\cdot)| &\leqslant K\left(\int_{\mathbb{S}} \sin(\cdot - \varphi)^2 q_0(\varphi)\,\mathrm{d}\varphi\right)^{\frac{1}{2}} \| u \|_{2,q_0,\mu} \\ &\leqslant K\left(\int_{\mathbb{S}} q_0(\varphi)\,\mathrm{d}\varphi\right)^{\frac{1}{2}} \| u \|_{2,q_0,\mu} = K \| u \|_{2,q_0,\mu}.\end{aligned} \tag{3.6.27}$$

So we have for all θ,ω (recall that Z_0 is the normalization constant in (3.2.1)):

$$\begin{aligned}|\mathcal{B}u(\theta,\omega)| \leqslant & \left(4K \| \varepsilon(q) \|_\infty + \delta|\omega|\right)|u| + 2K \| \varepsilon(q) \|_\infty \| u \|_{2,q_0,\mu} \\ & + Z_0^{-1}\left(4K \| \varepsilon(q) \|_\infty + \delta|\omega|\right)\left(\int_{\mathbb{S}} \frac{|u|^2}{q_0}\right)^{\frac{1}{2}}.\end{aligned} \tag{3.6.28}$$

Hence, inequality (3.6.26) is true for the following choice of $c^{(1)}_{K,\delta}$ (recall that $\varepsilon_{K,\delta}$ is defined in (3.6.19)):

$$c^{(1)}_{K,\delta} := \left(6\left(4K\varepsilon_{K,\delta} + \delta\right)^2 + 12K^2 Z_0^2 \varepsilon_{K,\delta}^2\right)^{\frac{1}{2}}. \tag{3.6.29}$$

Remark 3.6.7. Note that, thanks to Lemma 3.6.6, one has that $c^{(1)}_{K,\delta} = O(\delta)$ as $\delta \to 0$.

In order to complete the proof of the inequality (3.6.16), it suffices to prove that there exist constants $c^{(2)}_K$ and $c^{(3)}_K$, only depending on K such that, for all u:

$$\| u \|_{2,q_0,\mu} \leqslant c^{(2)}_K \| Au \|_{-1,q_0,\mu} + c^{(3)}_K \| u \|_{-1,q_0,\mu} \,. \tag{3.6.30}$$

The rest of this first of the proof is devoted to find explicit expressions of $c^{(2)}_K$ and $c^{(3)}_K$, and is based on an interpolation argument.

For all integer $n > 1$, one can compute the linear operator $f \mapsto f'$ in terms of a sum of two integral operators, namely:

$$f' = I_n(f'') + J_n(f)\,, \tag{3.6.31}$$

where $I_n : f \mapsto \int_0^{2\pi} i_n(\theta,\varphi) f(\varphi)\,\mathrm{d}\varphi$ (resp. $J_n : f \mapsto \int_0^{2\pi} j_n(\theta,\varphi) f(\varphi)\,\mathrm{d}\varphi$) is the integral operator whose kernel $i_n(\theta,\varphi)$ (resp. $j_n(\theta,\varphi)$) is defined by:

$$\begin{cases} i_n(\theta,\varphi) := \frac{\varphi^{n+1}}{2\pi\theta^n}\,, & j_n(\theta,\varphi) := -\frac{n(n+1)\varphi^{n-1}}{2\pi\theta^n}\,, \quad 0 \leqslant \varphi < \theta \leqslant 2\pi\,, \\ i_n(\theta,\varphi) := \frac{-(2\pi-\varphi)^{n+1}}{2\pi(2\pi-\theta)^n}\,, & j_n(\theta,\varphi) := \frac{n(n+1)(2\pi-\varphi)^{n-1}}{2\pi(2\pi-\theta)^n}\,, \quad 0 \leqslant \theta < \varphi \leqslant 2\pi\,. \end{cases} \tag{3.6.32}$$

Equality (3.6.31) can be easily verified by integrations by parts. Since,

$$\begin{cases} \int_0^{2\pi} |i_n(\theta,\varphi)|\,\mathrm{d}\varphi \leqslant \frac{2\pi}{n+2}, \quad \int_0^{2\pi} |i_n(\theta,\varphi)|\,\mathrm{d}\theta \leqslant \frac{2\pi}{n-1}\,, \\ \int_0^{2\pi} |j_n(\theta,\varphi)|\,\mathrm{d}\varphi \leqslant \frac{n+1}{\pi}, \quad \int_0^{2\pi} |j_n(\theta,\varphi)|\,\mathrm{d}\theta \leqslant \frac{n(n+1)}{\pi(n-1)}\,, \end{cases} \tag{3.6.33}$$

we see (cf. [57, p.143-144]) that I_n and J_n are bounded operators on $L^2(\mathbb{S})$, namely:

$$\|I_n\| \leqslant \frac{2\pi}{n-1}, \quad \|J_n\| \leqslant \frac{n(n+1)}{\pi(n-1)}\,. \tag{3.6.34}$$

So, applying relation (3.6.31) for $f = \mathcal{U}$ we get, for μ-almost every ω:

$$\left(\int_{\mathbb{S}} |u(\theta,\omega)|^2\,\mathrm{d}\theta\right)^{\frac12} \leqslant \frac{2\pi}{n-1}\left(\int_{\mathbb{S}} |u'(\theta,\omega)|^2\,\mathrm{d}\theta\right)^{\frac12} + \frac{n(n+1)}{\pi(n-1)}\left(\int_{\mathbb{S}} |\mathcal{U}(\theta,\omega)|^2\,\mathrm{d}\theta\right)^{\frac12}. \tag{3.6.35}$$

This gives

$$\| u \|_{2,\mu} \leqslant \frac{2\pi}{n-1} \| u' \|_{2,\mu} + \frac{n(n+1)}{\pi(n-1)} \| \mathcal{U} \|_{2,\mu}\,. \tag{3.6.36}$$

Since $\| \mathcal{U} \|_{2,q_0,\mu} = \| u \|_{-1,q_0,\mu}$, it only remains to control $\| u' \|_{2,q_0,\mu}$ with $\| Au \|_{-1,q_0,\mu}$: like for the beginning of this proof for the operator B, we have $\| Au \|_{-1,q_0,\mu} = \| \mathcal{A}u \|_{2,q_0,\mu}$, where $\mathcal{A}u$ is the appropriate primitive of Au:

$$\begin{aligned} \mathcal{A}u(\theta,\omega) := {} & \frac12 u'(\theta,\omega) - \left(u(\theta,\omega)(J*q_0) + q_0(\theta)\langle J*u\rangle_\mu(\theta)\right) \\ & + \left(\int_{\mathbb{S}} \frac{1}{q_0}\right)^{-1} \left(\int_{\mathbb{S}} \left\{\frac{u(\theta,\omega)(J*q_0)}{q_0(\theta)} + \frac12 u(\theta,\omega)\partial_\theta\left(\frac{1}{q_0(\theta)}\right)\right\}\mathrm{d}\theta\right). \end{aligned} \tag{3.6.37}$$

Using inequalities $|\langle J*u\rangle|_\mu(\cdot) \leqslant K\sqrt{\pi} \| u \|_{2,\mu}$, and $\int_{\mathbb{S}} \frac{|u(\cdot,\omega)|}{q_0} \leqslant Z_0^{\frac12} e^{Kr_0} \left(\int_{\mathbb{S}} |u(\cdot,\omega)^2|\right)^{\frac12}$, an easy calculation shows that:

$$|u'(\cdot,\omega)| \leqslant 2|\mathcal{A}u(\cdot,\omega)| + 2Kr_0|u(\cdot,\omega)| + 2\sqrt{\pi}Kq_0(\cdot) \| u \|_{2,\mu} + \frac{4Kr_0}{Z_0^{\frac12}} e^{Kr_0}\left(\int_{\mathbb{S}} |u(\cdot,\omega)^2|\right)^{\frac12}, \tag{3.6.38}$$

and thus,

$$\| u' \|_{2,\mu} \leqslant 4 \| \mathcal{A}u \|_{2,\mu} + 4K \left(r_0^2 + \pi Z_0^{-1} e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} \| u \|_{2,\mu} , \tag{3.6.39}$$

and by putting (3.6.36) and (3.6.39) together we obtain

$$\begin{aligned} \| u \|_{2,\mu} \leqslant\ & \frac{8\pi}{n-1} \| \mathcal{A}u \|_{2,\mu} + \frac{2\pi}{n-1} 4K \left(r_0^2 + \pi Z_0^{-1} e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} \| u \|_{2,\mu} \\ & + \frac{n(n+1)}{\pi(n-1)} \| u \|_{-1,q_0,\mu} . \end{aligned} \tag{3.6.40}$$

Let us choose the integer $n = \left\lfloor 16\pi K \left(r_0^2 + \pi Z_0^{-1} e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} + 1 \right\rfloor$ so that

$$\frac{2\pi}{n-1} 4K \left(r_0^2 + \pi Z_0^{-1} e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} \leqslant \frac{1}{2} . \tag{3.6.41}$$

In this case, we obtain:

$$\begin{aligned} \| u \|_{2,q_0,\mu} \leqslant\ & \frac{e^{2Kr_0}}{4K \left(r_0^2 + \pi Z_0^{-1} e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}}} \| Au \|_{-1,q_0,\mu} \\ & + \frac{e^{2Kr_0} \left(16K \left(r_0^2 + \pi Z_0^{-1} e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} + 3\right)^2}{16\pi^2 K \left(r_0^2 + \pi Z_0^{-1} e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}}} \| u \|_{-1,q_0,\mu} , \end{aligned} \tag{3.6.42}$$

which is precisely the inequality (3.6.30) we wanted to prove. Inequalities (3.6.26) and (3.6.30) give the result, for $a_{K,\delta} := c^{(1)}_{K,\delta} \cdot c^{(3)}_{K}$ and $b_{K,\delta} := c^{(1)}_{K,\delta} \cdot c^{(2)}_{K}$. □

Proposition 3.6.8. *For all $K > 1$, there exists $\delta_3(K) > 0$ such that for all $0 < \delta \leqslant \delta_3(K)$, the operator L_q^ω is closable. In that case, its closure has the same domain as the closure of A.*

Proof. Let us choose $\delta_3(K) > 0$ so that

$$b_{K,\delta_3(K)} < 1 \tag{3.6.43}$$

where $b_{K,\delta}$ is the constant introduced in (3.6.16), then, for all $0 < \delta \leqslant \delta_3(K)$, the operator B is A-bounded with A-bound strictly lower than 1. The result is then a consequence of Th. IV-1.1, p.190 in [57]. □

The spectrum of L_q^ω

We divide our study into two parts: the determination of the position of the spectrum within a sector and its position near 0.

Position of the spectrum away from 0

We prove mainly that the perturbed operator L_q^ω still generates an analytic semigroup of operators on an appropriate sector. An immediate corollary is the fact that the spectrum lies in a cone whose vertex is zero.

We know (Proposition 3.6.2) that for all $0 < \alpha < \frac{\pi}{2}$, A generates an analytic semigroup of operators on $\Delta_\alpha := \{\lambda \in \mathbf{C}\,;\, |\arg(\lambda)| < \alpha\}$.

Proposition 3.6.9. *For all $K > 1$, $0 < \alpha < \frac{\pi}{2}$ and $\varepsilon > 0$, there exists $\delta_4 > 0$ (depending on α, K and ε) such that for all $0 < \delta < \delta_4$, the spectrum of $L_q^\omega = A + B$ lies within $\Theta_{\varepsilon,\alpha} := \left\{\lambda \in \mathbf{C}\,;\, \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha\right\} \cup \{\lambda \in \mathbf{C}\,;\, |\lambda| \leqslant \varepsilon\}$. Moreover, there exists $\alpha' \in (0, \frac{\pi}{2})$ such that the operator L_q^ω still generates an analytic semigroup on $\Delta_{\alpha'}$.*

Proof of Proposition 3.6.9. Let $0 < \alpha < \frac{\pi}{2}$ be fixed. Following (3.6.16) and using (3.6.15), one can easily deduce an estimate on the bounded operator $BR(\lambda, A)$, for $\lambda \in \Sigma_\alpha$:

$$\begin{aligned} \| BR(\lambda, A)u \|_{-1,q_0,\mu} &\leqslant a_{K,\delta} \| R(\lambda, A)u \|_{-1,q_0,\mu} + b_{K,\delta} \| AR(\lambda, A)u \|_{-1,q_0,\mu} \\ &\leqslant a_{K,\delta} \frac{1}{(1-\sin(\alpha))|\lambda|} \| u \|_{-1,q_0,\mu} \\ &\quad + b_{K,\delta} \left(1 + \frac{1}{1-\sin(\alpha)}\right) \| u \|_{-1,q_0,\mu} \,. \end{aligned} \tag{3.6.44}$$

Let us fix $\varepsilon > 0$ and choose δ so that:

$$\max\left(4b_{K,\delta}\left(\frac{1}{1-\sin(\alpha)} + 1\right), \frac{4a_{K,\delta}}{(1-\sin(\alpha))\varepsilon}\right) \leqslant 1\,. \tag{3.6.45}$$

Then, for $\lambda \in \Sigma_\alpha$ such that $|\lambda| > \varepsilon \geqslant \frac{4a_{K,\delta}}{1-\sin(\alpha)}$, we have

$$\| BR(\lambda, A)u \|_{-1,q_0,\mu} \leqslant \frac{1}{2} \| u \|_{-1,q_0,\mu} \,. \tag{3.6.46}$$

In particular, $1 - BR(\lambda, A)$ is invertible with $\left\| (1 - BR(\lambda, A))^{-1} \right\|_{-1,q_0,\mu} \leqslant 2$. A direct calculation shows that

$$(\lambda - (A + B))^{-1} = R(\lambda, A)\,(1 - BR(\lambda, A))^{-1}\,. \tag{3.6.47}$$

One deduces the following estimates on the resolvent: for $\lambda \in \Sigma_\alpha$, $|\lambda| > \varepsilon$,

$$\left\| R(\lambda, L_q^\omega) \right\|_{-1,q_0,\mu} \leqslant \frac{2}{(1-\sin(\alpha))|\lambda|}\,. \tag{3.6.48}$$

Estimate (3.6.48) has two consequences: firstly, one deduces immediately that the spectrum $\sigma(L_q^\omega)$ of L_q^ω is contained in $\Theta_{\varepsilon,\alpha}$:

$$\sigma(L_q^\omega) \subseteq \left\{\lambda \in \mathbf{C}\,;\, \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha\right\} \cup \{\lambda \in \mathbf{C}\,;\, |\lambda| \leqslant \varepsilon\}\,. \tag{3.6.49}$$

Secondly, (3.6.48) entails that L_q^ω generates an analytic semigroup of operators on an appropriate sector. Indeed, if one denotes by $L_{q,\varepsilon}^\omega := L_q^\omega - \varepsilon$, one deduces from (3.6.49) that $0 \in \rho(L_{q,2\varepsilon}^\omega)$ and that for all $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ (in particular, $|\lambda| < |\lambda + 2\varepsilon|$)

$$\begin{aligned} \left\| R(\lambda, L_{q,2\varepsilon}^\omega) \right\|_{-1,q_0,\mu} = \left\| R(\lambda + 2\varepsilon, L_q^\omega) \right\|_{-1,q_0,\mu} &\leqslant \frac{2}{(1-\sin(\alpha))|\lambda + 2\varepsilon|}\,, \\ &\leqslant \frac{2}{(1-\sin(\alpha))|\lambda|}\,. \end{aligned} \tag{3.6.50}$$

Hence, using the same arguments of Taylor expansion as in the proof of Proposition 3.6.2 and applying Proposition 3.6.3, one easily sees that $L_{q,2\varepsilon}^\omega$ generates an analytic semigroup in a (a priori) smaller sector $\Delta_{\alpha'}$, where $\alpha' \in (0, \frac{\pi}{2})$ can be chosen as $\alpha' := \frac{1}{2}\arctan\left(\frac{1-\sin(\alpha)}{2}\right)$. But if $L_{q,2\varepsilon}^\omega$ generates an analytic semigroup, so does L_q^ω. ◻

Position of the spectrum near 0

Let us apply Proposition 3.6.9 for fixed $K > 1$, $\alpha \in (0, \frac{\pi}{2})$, $\rho \in (0,1)$ and $\varepsilon := \rho\lambda_K$, where we recall that λ_K is the spectral gap between the eigenvalue 0 for the non perturbed operator A and the rest of the spectrum $\sigma(A) \setminus \{0\}$. Let $\Theta^+_{\varepsilon,\alpha} := \{\lambda \in \Theta_{\varepsilon,\alpha}\,;\, \Re(\lambda) \geqslant 0\}$ be the subset of $\Theta_{\varepsilon,\alpha}$ which lies in the positive part of the complex plane (see Fig. 3.3). In order to show the linear stability, one has to make sure that one can choose a perturbation B small enough so that no eigenvalue of $A + B$ remains in the small set $\Theta^+_{\varepsilon,\alpha}$.

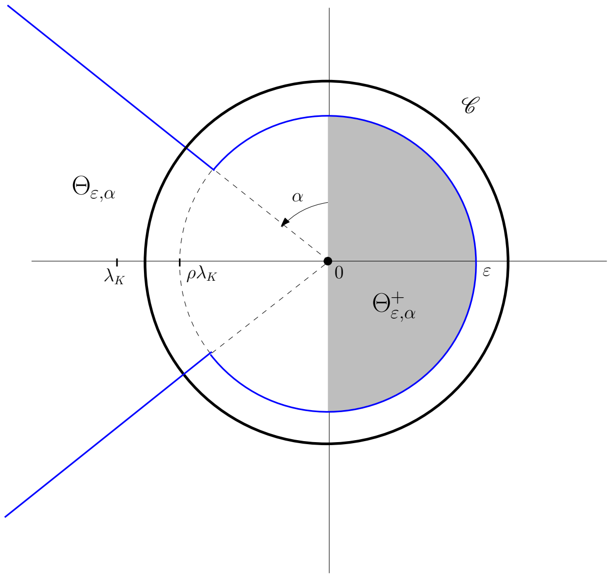


FIGURE 3.3. The set $\Theta_{\varepsilon,\alpha}$.

Since $\lambda_K > 0$, one can separate 0 from the rest of the spectrum of A by a circle $\mathscr{C}$ centered in 0 with radius $(\frac{\rho+1}{2})\lambda_K$. The appropriate choice of ε ensures that the interior of the disk delimited by $\mathscr{C}$ contains $\Theta^+_{\varepsilon,\alpha}$ (see Figure 3.3).

The main argument is the following: by construction of $\mathscr{C}$, 0 is the only eigenvalue (with multiplicity 1) of the non-perturbed operator A lying in the interior of $\mathscr{C}$. A principle of local continuity of eigenvalues shows that, while adding a sufficiently small perturbation B to A, the interior of $\mathscr{C}$ still contains exactly one eigenvalue (which is *a priori* close but not equal to 0) with the same multiplicity.

But we already know that for the perturbed operator $L^\omega_q = A + B$, 0 is always an eigenvalue (since $L^\omega_q q' = 0$). One can therefore conclude that, by uniqueness, 0 is the only element of the spectrum of L^ω_q within $\mathscr{C}$, and is an eigenvalue with multiplicity 1. In particular, there is no element of the spectrum in the positive part of the complex plane.

In order to quantify the appropriate size of the perturbation B, one has to have explicit estimates on the resolvent $R(\lambda, A)$ on the circle $\mathscr{C}$.

Lemma 3.6.10. *There exists some explicit constant* $c_{\mathscr{C}} = c_{\mathscr{C}}(K, \rho)$ *such that for all* $\lambda \in \mathscr{C}$,

$$\| R(\lambda, A) \|_{-1,q_0,\mu} \leqslant c_{\mathscr{C}}\,, \tag{3.6.51}$$

$$\| AR(\lambda, A) \|_{-1,q_0,\mu} \leqslant 1 + \left(\frac{1+\rho}{2}\right) \lambda_K \cdot c_{\mathscr{C}}\,. \tag{3.6.52}$$

One can choose $c_{\mathscr{C}}$ *as* $\frac{1}{\lambda_K} \max\left(\frac{2}{\rho+1}, \frac{2}{1-\rho}\right) := \frac{\ell(\rho)}{\lambda_K}$.

Proof of Lemma 3.6.10. Applying the spectral theorem (see [28, Th. 3, p.1192]) to the essentially self-adjoint operator A, there exists a spectral measure E vanishing on the complementary of the spectrum of A such that $A = \int_{\mathbb{R}} \lambda \, \mathrm{d}E(\lambda)$. In that extent, one has for any $\zeta \in \mathscr{C}$

$$R(\zeta, A) = \int_{\mathbb{R}} \frac{\mathrm{d}E(\lambda)}{\lambda - \zeta}. \tag{3.6.53}$$

In particular, for $\zeta \in \mathscr{C}$

$$\| R(\zeta, A) \|_{-1,q_0,\mu} \leqslant \sup_{\lambda \in \sigma(A)} \frac{1}{|\lambda - \zeta|} \leqslant \frac{\ell(\rho)}{\lambda_K}. \tag{3.6.54}$$

The estimation (3.6.52) is straightforward. □

We are now in position to apply our argument of local continuity of eigenvalues: Following [57, Th III-6.17, p.178], there exists a decomposition of the operator A according to $H^{-1}_{q_0,\mu} = H_0 \oplus H'$ (in the sense that $AH_0 \subset H_0$, $AH' \subset H'$ and $P\mathcal{D}(A) \subset \mathcal{D}(A)$, where P is the projection on H_0 along H') in such a way that A restricted to H_0 has spectrum $\{0\}$ and A restricted to H' has spectrum $\sigma(A) \smallsetminus \{0\}$.

Let us note that the dimension of H_0 is 1, since the characteristic space of A in the eigenvalue 0 is reduced to its kernel which is of dimension 1.

Then, applying [57, Th. IV-3.18, p.214], and using Proposition 3.6.5, we find that if one chooses $\delta > 0$, such that

$$\sup_{\lambda \in \mathscr{C}} \left(a_{K,\delta} \| R(\lambda, A) \|_{-1,q_0,\mu} + b_{K,\delta} \| AR(\lambda, A) \|_{-1,q_0,\mu} \right) < 1, \tag{3.6.55}$$

then the perturbed operator L_q^ω is likewise decomposed according to $H^{-1}_{q_0,\mu} = \widetilde{H}_0 \oplus \widetilde{H}'$, in such a way that $\dim(H_0) = \dim(\widetilde{H}_0) = 1$, and that the spectrum of L_q^ω is again separated in two parts by $\mathscr{C}$. But we already know that the characteristic space of the perturbed operator L_q^ω according to the eigenvalue 0 is, at least, of dimension 1 (since $L_q^\omega q' = 0$).

We can conclude, that for such an $\delta > 0$, 0 is the only eigenvalue in $\mathscr{C}$ and that $\dim(\widetilde{H}_0) = 1$.

Applying Lemma 3.6.10, we see that condition (3.6.55) is satisfied if we choose $\delta > 0$ so that:

$$a_{K,\delta} c_{\mathscr{C}} + b_{K,\delta} \left(1 + \left(\frac{1+\rho}{2} \right) \lambda_K c_{\mathscr{C}} \right) < 1. \tag{3.6.56}$$

In particular, in that case, the spectrum of L_q^ω is contained in

$$\left\{ \lambda \in \mathbb{C} \, ; \, \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha \right\} \subseteq \{ z \in \mathbf{C} \, ; \, \Re(z) \leqslant 0 \}. \tag{3.6.57}$$

Finally, the following proposition sums-up the sufficient conditions on δ for the conclusions of Theorem 3.2.5 to be satisfied:

Proposition 3.6.11. *Recall the definitions of $a_{K,\delta}$ and $b_{K,\delta}$ in Proposition 3.6.5. If $\delta > 0$ satisfies the following conditions*

$$\begin{aligned} b_{K,\delta} &\leqslant 1, \\ 4b_{K,\delta} \left(\frac{1}{1 - \sin(\alpha)} + 1 \right) &\leqslant 1, \\ \frac{4a_{K,\delta}}{\rho \lambda_K (1 - \sin(\alpha))} &\leqslant 1, \\ a_{K,\delta} \frac{\ell(\rho)}{\lambda_K} + b_{K,\delta} \left(1 + \left(\frac{1+\rho}{2} \right) \ell(\rho) \right) &< 1. \end{aligned} \tag{3.6.58}$$

the conclusions of Theorem 3.2.5 are true.

Proof. One has simply to sum-up conditions (3.6.43), (3.6.45) with $\varepsilon = \rho\lambda_K$ and (3.6.56). (3.6.59) can be obtained by (long) estimations on the coefficients $a_{K,\delta}$ and $b_{K,\delta}$. □

Remark 3.6.12. The conditions in Proposition 3.6.11 can be simplified. For example one can exhibit an explicit constant c such that if δ satisfies

$$\delta e^{12\pi\delta} \leqslant ce^{-20K\bar{r}_\delta} \max\left(1, \left(\frac{1-\sin(\alpha)}{2-\sin(\alpha)}\right), \frac{\rho\lambda_K(1-\sin(\alpha))e^{-4K\bar{r}_\delta}}{K^2}, \frac{\lambda_K}{K^2e^{4K\bar{r}_\delta}\ell(\rho) + \lambda_K\left(1+\left(\frac{1+\rho}{2}\right)\ell(\rho)\right)}\right) \tag{3.6.59}$$

the conditions in (3.6.58) are fulfilled. Explicit estimates on the spectral gap λ_K can be found in [9, Sec. 2.5].

3.A Regularity in the non-linear Fokker-Planck equation

The purpose of this section is to establish regularity properties of the solution of the non-linear equation (3.1.13) (where we fix $\delta = 1$ for simplicity). Note that this case also captures the situation where $U(\cdot,\omega) \equiv \omega$ (evolution (3.1.4)), as well as the situation where $U(\cdot,\cdot) \equiv 0$ (evolution (3.1.12)). In what follows we make the assumption that U is bounded and that for all $\omega \in \mathrm{Supp}(\mu)$, $\theta \mapsto U(\theta,\omega) \in C^\infty(\mathbb{S};\mathbb{R})$ with bounded derivatives.

The existence and uniqueness in $L^2(\lambda\otimes\omega)$ of a solution to (3.1.13) can be tackled using Banach fixed point arguments (see [99, Section 4.7]), but one can obtain more regularity from the theory of fundamental solutions of parabolic equations.

More precisely, it is usual to interpret Equation (3.1.13) as the strong formulation of the weak equation (where $\nu \in \mathcal{C}([0,T],\mathcal{M}_1(\mathbb{S}\times\mathbb{R}))$ and F is any bounded function on $\mathbb{S}\times\mathbb{R}$ with twice bounded derivatives w.r.t. θ):

$$\int_{\mathbb{R}\times\mathbb{S}} F(\theta,\omega)\nu_t(\,\mathrm{d}\theta,\,\mathrm{d}\omega) = \int_{\mathbb{R}\times\mathbb{S}} F(\theta,\omega)\nu_0(\,\mathrm{d}\theta,\,\mathrm{d}\omega) + \frac{1}{2}\int_0^t\int_{\mathbb{R}\times\mathbb{S}} F''(\theta,\omega)\nu_s(\,\mathrm{d}\theta,\,\mathrm{d}\omega)\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}\times\mathbb{S}} F'(\theta,\omega)\left(\int_{\mathbb{R}\times\mathbb{S}} J(\theta-\cdot)\,\mathrm{d}\nu_s + U(\theta,\omega)\right)\nu_s(\,\mathrm{d}\theta,\,\mathrm{d}\omega)\,\mathrm{d}s, \tag{3.A.1}$$

where the second marginal (w.r.t. to the disorder ω) of the initial condition $\nu_0(\,\mathrm{d}\theta,\,\mathrm{d}\omega)$ is $\mu(\,\mathrm{d}\omega)$ so that one can write

$$\nu_0(\,\mathrm{d}\theta,\,\mathrm{d}\omega) = \nu_0^\omega(\,\mathrm{d}\theta)\mu(\,\mathrm{d}\omega)\,, \tag{3.A.2}$$

where ν_0^ω is a probability measure on $\mathbb{S}$, for μ-a.e. ω.

As already mentioned, a proof of the existence of a solution on $[0,T]$ of (3.A.1) can be obtained from the almost-sure convergence of the empirical measure of the microscopic system [66]. One can also find a proof of uniqueness of such a solution relying on arguments introduced in [73].

The regularity result can be stated as follows:

Proposition 3.A.1. *For all probability measure* $\nu_0(\mathrm{d}\theta, \mathrm{d}\omega) = \nu_0^\omega(\mathrm{d}\theta)\mu(\mathrm{d}\omega)$ *on* $\mathbb{S}\times\mathbb{R}$, *for all* $T>0$, *there exists a unique solution* ν *to* (3.A.1) *in* $\mathcal{C}([0,T],\mathcal{M}_1(\mathbb{S}\times\mathbb{R}))$ *such that for all* $F\in\mathcal{C}(\mathbb{S}\times\mathbb{R})$,

$$\lim_{t\searrow 0}\int_{\mathbb{R}\times\mathbb{S}} F(\theta,\omega)\nu_t(\mathrm{d}\theta, \mathrm{d}\omega) = \int_{\mathbb{R}\times\mathbb{S}} F(\theta,\omega)\nu_0^\omega(\mathrm{d}\theta)\mu(\mathrm{d}\omega). \tag{3.A.3}$$

Moreover, for all $t>0$, ν_t *is absolutely continuous with respect to* $\lambda_1\otimes\mu$ *and for* μ*-a.e.* $\omega\in$ *Supp*(μ), *its density* $(t,\theta,\omega)\mapsto p_t(\theta,\omega)$ *is strictly positive on* $(0,T]\times\mathbb{S}$, *is* $\mathcal{C}^\infty$ *in* (t,θ) *and solves the Fokker-Planck equation* (3.1.13).

Proof of Proposition 3.A.1. Let us fix $T>0$, $\omega\in\mathrm{Supp}(\mu)$ and $t\mapsto\nu_t$ the unique solution in $\mathcal{C}([0,T],\mathcal{M}_1(\mathbb{S}\times\mathbb{R}))$ to (3.A.1). Let us define $R(t,\theta,\omega):=\int_{\mathbb{R}\times\mathbb{S}} J(\theta-\cdot)\,\mathrm{d}\nu_t + U(\theta,\omega)$ and consider the linear equation

$$\partial_t p_t(\theta,\omega) \;=\; \frac{1}{2}\Delta p_t(\theta,\omega) - \partial_\theta\Big(p_t(\theta,\omega)R(t,\theta,\omega)\Big)\,, \tag{3.A.4}$$

such that for μ-a.e. ω, for all $F\in\mathcal{C}(\mathbb{S})$,

$$\int_{\mathbb{S}} F(\theta)p_t(\theta,\omega)\,\mathrm{d}\theta \overset{t\searrow 0}{\longrightarrow} \int_{\mathbb{S}} F(\theta)\nu_0^\omega(\mathrm{d}\theta)\,. \tag{3.A.5}$$

For fixed $\omega\in\mathrm{Supp}(\mu)$, $R(\cdot,\cdot,\omega)$ is continuous in time and $\mathcal{C}^\infty$ in θ.

Suppose for a moment that we have found a weak solution $p_t(\theta,\omega)$ to (3.A.4)-(3.A.5) such that for μ-a.e. ω, $p_t(\cdot,\omega)$ is strictly positive on $(0,T]\times\mathbb{S}$. In particular for such a solution p, the quantity $\int_{\mathbb{S}} p_t(\theta,\omega)\,\mathrm{d}\theta$ is conserved for $t>0$, so that $p_t(\cdot,\omega)$ is indeed a probability density for all $t>0$. Then both probability measures $\nu_t(\mathrm{d}\theta, \mathrm{d}\omega)$ and $p_t(\theta,\omega)\,\mathrm{d}\theta\mu(\mathrm{d}\omega)$ solve

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{S}} F(\theta,\omega)\nu_t(\mathrm{d}\theta, \mathrm{d}\omega) = \int_{\mathbb{R}\times\mathbb{S}} F(\theta,\omega)\nu_0(\mathrm{d}\theta, \mathrm{d}\omega) + \frac{1}{2}\int_0^t\int_{\mathbb{R}\times\mathbb{S}} F''(\theta,\omega)\nu_s(\mathrm{d}\theta, \mathrm{d}\omega)\,\mathrm{d}s\\ + \int_0^t\int_{\mathbb{R}\times\mathbb{S}} F'(\theta,\omega)R(t,\theta,\omega)\nu_s(\mathrm{d}\theta, \mathrm{d}\omega)\,\mathrm{d}s.\end{aligned} \tag{3.A.6}$$

By [66] or [73, Lemma 10], uniqueness in (3.A.1) is precisely a consequence of uniqueness in (3.A.6). Hence, by uniqueness in (3.A.6), $\nu_t(\mathrm{d}\theta, \mathrm{d}\omega) = p_t(\theta,\omega)\,\mathrm{d}\theta\mu(\mathrm{d}\omega)$, which is the result. So it suffices to exhibit a weak solution $p_t(\theta,\omega)$ to (3.A.4) such that (3.A.5) is satisfied.

This fact can be deduced from standard results for uniform parabolic PDEs (see [3] and [35] for precise definitions). In particular, a usual result, which can be found in [3, §7 p.658], states that (3.A.4) admits a fundamental solution $\Gamma(\theta,t;\theta',s,\omega)$ $(t>s)$, which is bounded above and below (see [3, Th.7, p.661]):

$$\frac{1}{C\sqrt{t-s}}\exp\left(\frac{-C(\theta-\theta')^2}{\sqrt{t-s}}\right) \leqslant \Gamma(\theta,t;\theta',s,\omega) \leqslant \frac{C}{\sqrt{t-s}}\exp\left(\frac{-(\theta-\theta')^2}{C\sqrt{t-s}}\right). \tag{3.A.7}$$

Note that the constant $C>0$ only depends on T and the *structure* of the linear operator in (3.A.4) (see [3, Th.7, p.661] and [3, §1, p.615]). In particular, since $(\theta,\omega)\mapsto U(\theta,\omega)$ is bounded, this constant does not depend on ω.

Note that the proof given in [3] is done for $\theta\in\mathbb{R}$ but can be readily adapted to our case ($\theta\in\mathbb{S}$).

Moreover, thanks to Corollary 12.1, p.690 in [3], the following expression of $p_t(\theta,\omega)$

$$p_t(\theta,\omega) = \int_{\mathbb{S}} \Gamma(\theta,t;\theta',0,\omega)\nu_0^{\omega}(\,\mathrm{d}\theta') \tag{3.A.8}$$

defines a weak solution of (3.A.4) on $(0,T]\times\mathbb{S}$ (namely a weak solution on $(\tau,T]\times\mathbb{S}$, for all $0<\tau<T$) such that (3.A.5) is satisfied. The positivity and boundedness of $p_t(\cdot,\omega)$ for $t>0$ is an easy consequence of (3.A.7). The smoothness of $p_\cdot(\cdot,\omega)$ on $(0,T]\times\mathbb{S}$ can be derived by standard bootstrap methods. □

We focus now on the regularity of the solution $p_t(\theta,\omega)$ of (3.1.13) with respect to the disorder ω. We assume here that the initial condition ν_0 is such that for all $\omega\in\mathrm{Supp}(\mu)$, $\nu_0^{\omega}(\,\mathrm{d}\theta)$ is absolutely continuous with respect to the Lebesgue measure λ_1 on $\mathbb{S}$: there exists a positive integrable function $\gamma(\cdot,\omega)$ of integral 1 on $\mathbb{S}$ such that $\nu_0^{\omega}(\,\mathrm{d}\theta)=\gamma(\theta,\omega)\,\mathrm{d}\theta$. Then we have

Lemma 3.A.2 (Regularity w.r.t. the disorder)**.** *For every* $(t_0,\theta_0)\in(0,\infty)\times\mathbb{S}$*, for every* ω_0 *which is an accumulation point in* $\mathrm{Supp}(\mu)$ *such that the following holds*

$$\int_{\mathbb{S}} |\gamma(\theta,\omega)-\gamma(\theta,\omega_0)|\,\mathrm{d}\theta \to 0, \quad \textit{as } \omega\to\omega_0\,, \tag{3.A.9}$$

then the solution p *of* (3.1.13) *defined on* $(0,\infty)\times\mathbb{S}\times\mathrm{Supp}(\mu)$ *is continuous at the point* (t_0,θ_0,ω_0)*.*

Proof of Lemma 3.A.2. For any ω in the support of μ, let for all $t>0$, $\theta\in\mathbb{S}$

$$u(t,\theta,\omega) := p_t(\theta,\omega)-p_t(\theta,\omega_0), \tag{3.A.10}$$

where $(p_t(\cdot,\cdot))_{t\geqslant 0}$ is the unique solution of (3.A.4). It is easy to see that u is a strong solution to the following PDE

$$\partial_t u(t,\theta,\omega) - \left[\frac{1}{2}\Delta u(t,\theta) - \partial_\theta\left(u(t,\theta)R(t,\theta,\omega_0)\right)\right] = \mathcal{R}(t,\theta,\omega), \tag{3.A.11}$$

where $\mathcal{R}(t,\theta,\omega) := \partial_\theta\left[p_t(\theta,\omega)\left(R(t,\theta,\omega)-R(t,\theta,\omega_0)\right)\right]$ and with initial condition (since $\nu_0^{\omega}(\,\mathrm{d}\theta)=\gamma(\theta,\omega)\,\mathrm{d}\theta$ for all ω)

$$u(t,\theta,\omega)|_{t\searrow 0} = \gamma(\theta,\omega)-\gamma(\theta,\omega_0). \tag{3.A.12}$$

Then applying [35, Th. 12 p.25], $u(t,\theta,\omega)$ can be expressed as

$$u(t,\theta,\omega) = \int_{\mathbb{S}} \Gamma(\theta,t;\theta',0,\omega_0)(\gamma(\theta,\omega)-\gamma(\theta,\omega_0))\,\mathrm{d}\theta' - \int_0^t\int_{\mathbb{S}} \Gamma(\theta,t;\theta',s,\omega_0)\mathcal{R}(s,\theta',\omega)\,\mathrm{d}\theta'\,\mathrm{d}s. \tag{3.A.13}$$

For the first term of the RHS of (3.A.13), we have

$$\left|\int_{\mathbb{S}} \Gamma(\theta,t;\theta',0,\omega_0)(\gamma(\theta,\omega)-\gamma(\theta,\omega_0))\,\mathrm{d}\theta'\right| \leqslant \frac{C}{\sqrt{t}}\int_{\mathbb{S}}|\gamma(\theta,\omega)-\gamma(\theta,\omega_0)|\,\mathrm{d}\theta'\,, \tag{3.A.14}$$

which converges to 0, for fixed $t>0$, by hypothesis (3.A.12).

Secondly, it is easy to see from the definition (3.A.8) of the density p and the estimates (3.A.7) and [35, Th.9 p.263] concerning the fundamental solution Γ that both $p_t(\theta,\omega)$ and $\partial_\theta p_t(\theta,\omega)$ are bounded uniformly on $(t,\theta,\omega)\in[0,T]\times\mathbb{S}\times\mathrm{Supp}(\mu)$. In particular, a standard result shows that for fixed (t,θ), the second term of the RHS of (3.A.13) goes to 0 as $\omega\to\omega_0$. But then the joint continuity of p at (t_0,θ_0,ω_0) follows from (3.A.8) and uniform estimates on Γ (see [35, Th.9 p.263]). □

Chapter 4

Random long time behavior

Contents

4.1 Introduction

4.1.1 Overview

In a variety of instances partial differential equations are a faithful approximation – in fact, a law of large numbers – for particle systems in suitable limits. This is notably the case for stochastic interacting particle systems, for which the mathematical theory has gone very far [58]. The closeness between the particle system and PDE is typically proven in the limit of systems with a large number N of particles or for infinite systems under a space rescaling involving a large parameter N – for example a spin or particle system on $\mathbb{Z}^d$ and the lattice spacing scaled down to $\frac{1}{N}$ – and up to a time horizon which may depend on N. Of course the question of capturing the finite N corrections has been taken up too, and the related central limit theorems as well as large deviations principles have been established (see [58] and references therein). *Sizable* deviations from the law of large numbers, not just small fluctuations or rare events, can be observed beyond the time horizon for which the PDE behavior has been established and these phenomena can be very relevant.

The first examples that come to mind are the ones in which the PDE has multiple isolated stable stationary points: metastability phenomena happens on exponentially long time scales [75]. Deviations on substantially shorter time scales can also take place and this is the case for example of the noise induced escape from stationary unstable solutions, which is particularly relevant in plenty of situations: for example for the model in [90, Ch. 5] phase segregation originates from homogeneous initial data via this mechanism, on times proportional to the logarithm of the size of the system. The logarithmic factor is directly tied to the exponential instability of the stationary solution (see [90] for more literature on this phenomenon). Of course, the type of phenomena happen also in finite dimensional random dynamical systems, in the limit of small noise, but we restrict this quick discussion to infinite dimensional models and PDEs.

In the case on which we focus the deviations also happen on time scales substantially shorter than the exponential ones, but the mechanism of the phenomenon does not involve exponential instabilities. In the system we consider there are multiple stationary solutions, but they are not (or, at least, not all) isolated, and hence they are not stable in the standard sense. Deviations from the PDE behavior happen as a direct result of the cumulative effect of the fluctuations. More precisely, this phenomenon is due to the presence of whole stable manifold of stationary solutions: the deterministic limit dynamics has no dumping effect along the tangential direction to the manifold so, for the finite size system, the weak noise does have a macroscopic effect on a suitable time scale that depends on how large the system is. We review the mathematical literature on this type of phenomena in § 4.1.6, after stating our results.

Apart for the general interest on deviations from the PDE behavior, the model we consider – mean-field plane rotators – is a fundamental one in mathematical physics and, more generally, it is the basic model for synchronization phenomena. Our results provide a sharp description of the long time dynamics of this model for general initial data.

4.1.2 The model

Consider the set of ordinary stochastic differential equations

$$\mathrm{d}\varphi_t^{j,N} = \frac{1}{N}\sum_{i=1}^{N} J\left(\varphi_t^{j,N} - \varphi_t^{i,N}\right) \mathrm{d}t + \mathrm{d}W_t^j \,. \qquad (4.1.1)$$

with $j = 1, 2, \ldots, N$, $\{W_j\}_{j=1,2,\ldots}$ is an IID collection of standard Brownian motions and $J(\cdot) = -K\sin(\cdot)$. With abuse of notation, when writing $\varphi_t^{j,N}$ we will actually mean $\varphi_t^{j,N}\mathrm{mod}(2\pi)$ and for us (4.1.1), supplemented with an (arbitrary) initial condition, will give origin to a diffusion process on $\mathbb{S}^N$, where $\mathbb{S}$ is the circle $\mathbb{R}/(2\pi\mathbb{Z})$.

The choice of the interaction potential $J(\cdot)$ is such that the (unique) invariant probability of the system is

$$\pi_{N,K}(\mathrm{d}\varphi) \propto \exp\left(\frac{K}{N}\sum_{i,j=1}^{N}\cos(\varphi_i - \varphi_j)\right)\lambda_N(\mathrm{d}\varphi)\,, \qquad (4.1.2)$$

where λ_N is the uniform probability measure on $\mathbb{S}^N$. Moreover, the evolution is reversible with respect to $\pi_{N,K}$, which is the well known Gibbs measure associated to mean-field plane rotators (or classical XY model).

We are therefore considering the simplest Langevin dynamics of mean-field plane rotators and it is well known that such a model exhibits a phase transition, for $K > K_c := 1$, that breaks the continuum symmetry of the model (for a detailed mathematical physics literature we refer to [9]). The continuum symmetry of the model is evident both in the dynamics (4.1.1) and in the equilibrium measure (4.1.2): if $\{\varphi_t^{j,N}\}_{t\geq 0, j=1,\ldots,N}$ solves (4.1.1), so does $\{\varphi_t^{j,N} + c\}_{t\geq 0, j=1,\ldots,N}$, c an arbitrary constant, and $\pi_{N,K}\Theta_c^{-1} = \pi_{N,K}$, where Θ_c is the rotation by an angle c, that is $(\Theta_c\varphi)_j = \varphi_j + c$ for every j.

4.1.3 The $N \to \infty$ dynamics and the stationary states

The phase transition can be understood also taking a dynamical standpoint. Given the mean-field set up it turns out to be particularly convenient to consider the empirical measure

$$\mu_{N,t}(\,\mathrm{d}\theta) := \frac{1}{N}\sum_{j=1}^{N} \delta_{\varphi_t^{j,N}}(\,\mathrm{d}\theta)\,, \tag{4.1.3}$$

which is a probability on (the Borel subsets of) $\mathbb{S}$. It is well known, see [9] (for detailed treatment and original references), that if $\mu_{N,0}$ converges weakly for $N \to \infty$, then so does $\mu_{N,t}$ for every $t > 0$. Actually, the process itself $t \mapsto \{\mu_{N,t}\}$, seen as an element of $C^0([0,T], \mathcal{M}_1)$, where $T > 0$ and $\mathcal{M}_1$ is the space of probability measures on $\mathbb{S}$ equipped with the weak topology, converges to a non-random limit which is the process that concentrates on the unique solution of the non-local PDE ($*$ denotes the convolution)

$$\partial_t p_t(\theta) = \frac{1}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta\big((J * p_t)(\theta)p_t(\theta)\big), \tag{4.1.4}$$

with initial condition prescribed by the limit of $\{\mu_{N,0}\}_{N=1,2,\ldots}$ (by [43] this solution is smooth for $t > 0$). We insist on the fact that $p_t(\cdot)$ is a probability density: $\int_{\mathbb{S}} p_t(\theta)\,\mathrm{d}\theta = 1$. We will often commit the abuse of notation of writing $p(\theta)$ when $p \in \mathcal{M}_1$ and p has a density. Much in the same way, if $p(\cdot)$ is a probability density, p, or $p(\,\mathrm{d}\theta)$, is the probability measure.

It is worthwhile to point out that $(J * p)(\theta) = -\Re(\hat{p}_1)K\sin(\theta) + \Im(\hat{p}_1)K\cos(\theta)$ with $\hat{p}_1 := \int_{\mathbb{S}} p(\theta)\exp(i\theta)\,\mathrm{d}\theta$. This is to say that the nonlinearity enters only through the first Fourier coefficient of the solution, a peculiarity that allows to go rather far in the analysis of the model. Notably, starting from this observation one can easily (once again details and references are given in [9]) see that all the stationary solutions to (4.1.4), in the class of probability densities, can be written, up to a rotation, as

$$q(\theta) := \frac{\exp\left(2Kr\cos(\theta)\right)}{2\pi I_0(2Kr)}\,, \tag{4.1.5}$$

where $2\pi I_0(2Kr)$ is the normalization constant written in terms of the modified Bessel function of order zero ($I_j(x) = (2\pi)^{-1}\int_{\mathbb{S}}(\cos\theta)^j\exp(x\cos(\theta))\,\mathrm{d}\theta$, for $j = 0, 1$) and r is a non-negative solution of the fixed point equation $r = \Psi(2Kr)$, with $\Psi(x) = I_1(x)/I_0(x)$. Since $\Psi(\cdot) : [0,\infty) \to [0,1)$ is increasing, concave, $\Psi(0) = 0$ and $\Psi'(0) = 1/2$ we readily see that if (and only if) $K > 1$ there exists a non-trivial (i.e. non-constant) solution to (4.1.4). Let us not forget however that $\Psi(0) = 0$ implies that $r = 0$ is a solution and therefore the constant density $\frac{1}{2\pi}$ is a solution no matter what the value of K is. From now on we set $K > 1$ and choose $r = r(K)$, the unique positive solution of the fixed point equation, so that the probability density $q(\cdot)$ in (4.1.5) is non trivial and it achieves the unique maximum at 0 and the minimum at π. Note that the rotation invariance of the system immediately yields that there is a whole family of stationary solution:

$$M = \{q_\psi(\cdot) : q_\psi(\cdot) := q(\cdot - \psi) \text{ and } \psi \in \mathbb{S}\}\,, \tag{4.1.6}$$

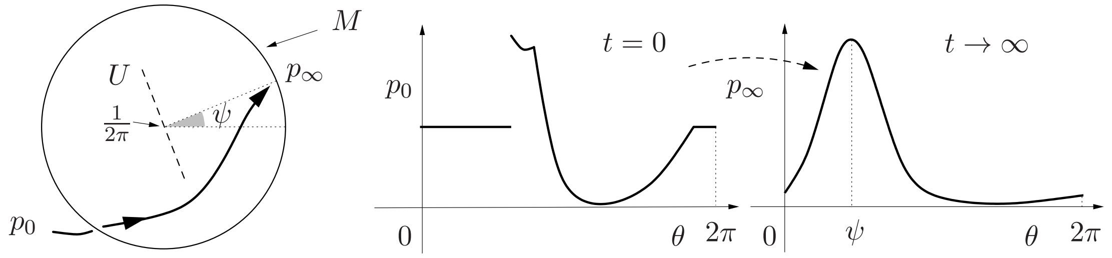


FIGURE 4.1. The limit evolution (4.1.4) instantaneously smoothens an arbitrary initial probability and, unless the Fourier decomposition such an initial condition has zero coefficients corresponding to the first harmonics (the hyperplane U), it drives it to a point p_∞ – a synchronized profile – on the invariant manifold M and of course it stays there for all times. This has been proven in [43], here we are interested in what happens for the finite size – N – system and we show that the PDE approximation is faithful up to times much shorter than N: on times proportional to N synchronization is kept and the center of synchronization ψ performs a Brownian motion on $\mathbb{S}$.

and, when $x \in \mathbb{R}$, $q_x(\cdot)$ of course means $q_{x\mathrm{mod}(2\pi)}(\cdot)$. M, which is more practically viewed as a manifold (in a suitable function space, see § 4.2.2 below), is invariant and stable for the evolution. The proper notion of stability is given in the context of *normally hyperbolic manifolds* (see [99] and references therein), but the full power of such a concept is not needed for the remainder. Nevertheless let us stress that in [43] one can find a complete analysis of the global dynamic phase diagram, notably the fact that unless $p_0(\cdot)$ belongs to the stable manifold U of the unstable solution $\frac{1}{2\pi}$ – the solution corresponding to $r = 0$ in (4.1.5) – $p_t(\cdot)$ converges (also in strong norms, controlling all the derivatives) to one of the points in M, see Figure 4.1. There is actually an explicit characterization of U:

$$U = \left\{ p \in \mathcal{M}_1 : \int_{\mathbb{S}} \exp(i\theta) p(\,\mathrm{d}\theta) = 0 \right\} . \tag{4.1.7}$$

As a matter of fact, it is easy to realize that if $p_0(\cdot) \in U$ then (4.1.4) reduces to the heat equation $\partial_t p_t(\theta) = \frac{1}{2}\partial_\theta^2 p_t(\theta)$ which of course relaxes to $\frac{1}{2\pi}$.

4.1.4 Random dynamics on M: the main result

In spite of the stability of M, $q_\psi(\cdot)$ itself is not stable, simply because if we start nearby, say from $q_{\psi'}$, the solution of (4.1.4) does not converge to $q_\psi(\cdot)$. The important point here is that the linearized evolution operator around $q(\cdot) \in M$ (q is an arbitrary element of M, not necessarily the one in (4.1.5): the phase ψ of q_ψ is explicit only when its absence may be misleading)

$$L_q u(\theta) := \frac{1}{2} u'' - \left[uJ * q + qJ * u \right]' , \tag{4.1.8}$$

with domain $\{u \in C^2(\mathbb{S}, \mathbb{R}) : \int_{\mathbb{S}} u = 0\}$ is symmetric in $H_{-1,1/q}$ – a weighted H_{-1} Hilbert space that we introduce in detail in Section 4.2.1 – and it has compact resolvent. Moreover the spectrum of L_q, which is of course discrete, lies in $(-\infty, 0]$ and the eigenvalue 0 has a one dimensional eigenspace, generated by q'. So q' is the only *neutral direction* and it corresponds precisely to the tangent space of M at $q(\cdot)$: all other directions, in function space, are contracted by the linear evolution and the nonlinear part of the evolution does not alter substantially this fact [43, 50].

Let us now step back and recall that our main concern is with the behavior of (4.1.1), with N large but finite, and not (4.1.4). In a sense the finite size, i.e. finite N, system is close to a suitable stochastic perturbation of (4.1.4): the type of stochastic PDE, with noise vanishing as $N \to \infty$, needs to be carefully *guessed* [41], keeping in particular in mind that we are dealing with a system with one conservation law. We will tackle directly (4.1.1), but the heuristic picture that one obtains by thinking of an SPDE with vanishing noise is of help. In fact the considerations we have just made on L_q suggest that if one starts the SPDE on M, the solution keeps very close to M, since the deterministic part of the dynamics is contractive in the orthogonal directions to M, but a (slow, since the noise is small) random motion on M arises because in the tangential direction the deterministic part of the dynamics is *neutral*. This is indeed what happens for the model we consider for N large. The difficulty that arises in dealing with the interacting diffusion system (4.1.1) is that one has to work with (4.1.3), which is not a function. Of course one can mollify it, but the evolution is naturally written and, to a certain extent, *closed* in terms of the empirical measure, and we do not believe that any significative simplification arises in proving our main statement for a mollified version. Working with the empirical measure imposes a clarification from now: as we explain in Section 4.2.1 and Appendix 4.A, if μ and $\nu \in \mathcal{M}_1$, then $\mu - \nu$ can be seen as an element of H_{-1} (or, as a matter of fact, also as an element of a weighted H_{-1} space).

Here is the main result that we prove (recall that $K > 1$):

Theorem 4.1.1. *Choose a positive constant τ_f and a probability $p_0 \in \mathcal{M}_1 \setminus U$. If for every $\varepsilon > 0$*

$$\lim_{N\to\infty} \mathbb{P}\left(\|\mu_{N,0} - p_0\|_{-1} \le \varepsilon\right) = 1\,, \tag{4.1.9}$$

then there exist a constant ψ_0 that depends only on $p_0(\cdot)$ and, for every N, a continuous process $\{W_{N,\tau}\}_{\tau\ge 0}$, adapted to the natural filtration of $\{W^j_{N\cdot}\}_{j=1,2,\ldots,N}$, such that $W_{N,\cdot} \in C^0([0,\tau_f];\mathbb{R})$ converges weakly to a standard Brownian motion and for every $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{\tau\in[\varepsilon_N,\tau_f]} \left\|\mu_{N,\tau N} - q_{\psi_0 + D_K W_{N,\tau}}\right\|_{-1} \le \varepsilon\right) = 1\,, \tag{4.1.10}$$

where $\varepsilon_N := C/N$, $C = C(K, p_0, \varepsilon) > 0$, and

$$D_K := \frac{1}{\sqrt{1 - (I_0(2Kr))^{-2}}}\,. \tag{4.1.11}$$

The result is saying that, unless one starts on the stable manifold of the unstable solution (see Remark 4.2.5 for what one expects if $p_0 \in U$), the empirical measure reaches very quickly a small neighborhood of the manifold M: this happens on a time scale of order one, as a consequence of the properties of the deterministic evolution law (4.1.4) (Figure 4.1), and, since we are looking at times of order N, this happens almost instantaneously. Actually, in spite of the fact that the result just addresses the limit of the empirical measure, the drift along M is due to fluctuations: the noise pushes the empirical measure away from M but the deterministic part of the dynamics *projects back* the trajectory to M and the net effect of the noise is a random shift – in fact, a rotation – along the manifold (this is taken up in more detail in the next section, where we give a complete heuristic version of the proof of Theorem 4.1.1).

Remark 4.1.2. *Without much effort, one can upgrade this result to much longer times: if we set $\tau_f(N) = N^a$ with an arbitrary $a > 1$, there exists an adapted process $W^a_{N,\tau}$ converging to a standard Brownian motion such that*

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{\tau\in[\varepsilon_N,\tau_f(N)]} \left\|\mu_{N,\tau N^a} - q_{\psi_0 + D_K N^{a-1} W^a_{N,\tau}}\right\|_{-1} \le \varepsilon\right) = 1\,. \tag{4.1.12}$$

This is due to the fact that our estimates ultimately rely on moment estimates, cf. Section 4.3. These estimates are obtained for arbitrary moments and we choose the moment sufficiently large to get uniformity for times $O(N)$, but working for times $O(N^a)$ would just require choosing larger moments. We have preferred to focus on the case $a = 1$; this is the natural scale, that is the scale in which the center of the probability density converges to a Brownian motion and not to an accelerated Brownian motion (this is really due to the fact that we work on $\mathbb{S}$ and marks a difference with [14, 7] where one can rescale the space variable).

4.1.5 The synchronization phenomena viewpoint

The model (4.1.1) we consider is actually a particular case of the Kuramoto synchronization model (the full Kuramoto model includes *quenched disorder* in terms of random constant speeds for the rotators, see [1, 9] and references therein). The mathematical physics literature and the more bio-physically oriented literature use somewhat different notations reflecting a slightly different viewpoint. In the synchronization literature one introduces the synchronization degree $\boldsymbol{r}_{N,t}$ and the synchronization center $\boldsymbol{\Psi}_{N,t}$ via

$$\boldsymbol{r}_{N,t} \exp(i\boldsymbol{\Psi}_{N,t}) := \frac{1}{N}\sum_{j=1}^{N} \exp(i\varphi_t^{j,N}) \left(= \int_{\mathbb{S}} \exp(i\theta)\mu_{N,t}(\mathrm{d}\theta)\right), \tag{4.1.13}$$

which clearly correspond to the parameters r and ψ that appear in the definition of M, but $\boldsymbol{r}_{N,t}$ and $\boldsymbol{\Psi}_{N,t}$ are defined for N finite and also far from M. Note that if (4.1.9) holds, then both $\boldsymbol{r}_{N,t}$ and $\boldsymbol{\Psi}_{N,t}$ converge in probability as $N \to \infty$ to the limits r and ψ, with $r\exp(i\psi) = \int_{\mathbb{S}} \exp(i\theta)p_0(\mathrm{d}\theta)$ and the assumption that $p_0 \notin U$ just means $r \neq 0$. Here is a straightforward consequence of Theorem 4.1.1:

Corollary 4.1.3. *Under the same hypotheses and definitions as in Theorem 4.1.1 we have that the stochastic process $\boldsymbol{\Psi}_{N,N\cdot} \in C^0([\varepsilon, \tau_f]; \mathbb{S})$ converges weakly, for every $\varepsilon \in (0, \tau_f]$, to $(\psi_0 + D_K W_\cdot)\mathrm{mod}(2\pi)$.*

It is tempting to prove such a result by looking directly at the evolution of $\boldsymbol{\Psi}_{N,t}$:

$$\begin{aligned}\mathrm{d}\boldsymbol{\Psi}_{N,t} = &\left(-K + \frac{1}{2N\boldsymbol{r}_{N,t}^2}\right)\frac{1}{N}\sum_{j=1}^{N}\sin(2(\varphi_t^{j,N} - \boldsymbol{\Psi}_{N,t}))\,\mathrm{d}t \\ &+ \frac{1}{\boldsymbol{r}_{N,t}N}\sum_{j=1}^{N}\cos(\varphi_t^{j,N} - \boldsymbol{\Psi}_{N,t})\,\mathrm{d}W_j(t)\,. \end{aligned}\tag{4.1.14}$$

But this clearly requires a control of the evolution of the empirical measure, so it does not seem that (4.1.14) could provide an alternative way to many of the estimates that we develop, namely convergence to a neighborhood of M and persistence of the proximity to M (see Section 4.3 and Section 4.5). On the other hand, it seems plausible that one could use (4.1.14) to develop an alternative approach to the dynamics on M, that is an alternative to Section 4.4. While this can be interesting in its own right, since the notion of synchronization center that we use in the proof and $\boldsymbol{\Psi}_{N,t}$ are almost identical (where they are both defined, that is close to M) we do not expect substantial simplifications.

4.1.6 A look at the literature and perspectives

Results related to our work have been obtained in the context of SPDE models with vanishing noise. In [15, 36] one dimensional stochastic reaction diffusion equations with bistable potential (also called *stochastic Cahn-Allen* or *model A*) are analyzed for initial data that are close to profiles that connect the two phases. It is shownthat the location of the phase boundary performs a Brownian motion. These results have been improved in a number of ways, notably to include *small asymmetries* that result in a drift for the arising diffusion process [13] and to deal with macroscopically finite volumes [7] (which introduce a repulsive effect approaching the boundary). Also the case of stochastic phase field equations has been considered [8].

For interacting particle systems results have been obtained for the zero temperature limit of d-dimensional Brownian particles interacting via local pair potentials in [37]: in this case the *frozen clusters* perform a Brownian motion and, in one dimension, also the merging of clusters is analyzed [38]. In this case the very small temperature is the small noise from which cluster diffusion originates. With respect to [37, 38], our results hold for any super-critical interaction, but of course our system is of mean field type. It is also interesting to observe that for the model in [37, 38] establishing the stability of the frozen clusters is the crucial issue, because the motion of the center of mass is a martingale, i.e. there is no drift. A substantial part of our work is in controlling that the drift of the center of synchronization vanishes (and controlling the drift is a substantial part also of [15, 36, 13, 7, 8]). This is directly related to the content of § 4.1.5.

As a matter of fact, in spite of the fact that our work deals directly with an interacting system, and not with an SPDE model, our approach is closer to the one in the SPDE literature. However, as we have already pointed out, a non negligible point is that we are forced to perform an analysis in distribution spaces, in fact Sobolev spaces with negative exponent, in contrast to the approach in the space of continuous functions in [15, 36, 13, 7, 8]. We point out that approaches to dynamical mean field type systems via Hilbert spaces of distribution has been already taken up in [33] but in our case the specific use of weighted Sobolev spaces is not only a technical tool, but it is intimately related to the geometry of the contractive invariant manifold M. In this sense and because of the iterative procedure we apply – originally introduced in [15] – our work is a natural development of [15, 7].

An important issue about our model that we have not stressed at all is that propagation of chaos holds (see e.g. [40]), in the sense that if the initial condition is given by a product measure, then this property is approximately preserved, at least for finite times. Recently much work has been done toward establishing quantitative estimates of chaos propagation (see for example the references in [18]). On the other hand, like for the model in [18], we know that, for our model, chaos propagation eventually breaks down: this is just because one can show by Large Deviations arguments that the empirical measure at equilibrium converges in law as $N \to \infty$ to the random probability density $q_X(\cdot)$, with X a uniform random variable on $\mathbb{S}$. But using Theorem 4.1.1 one can go much farther and show that chaos propagation breaks down at times proportional to N. From Theorem 4.1.1 one can actually extract also an accurate description of how the correlations build up due to the random motion on M.

It is natural to ask whether the type of results we have proven extend to the case in which random natural frequencies are present, that is to the disordered version of the model we consider that goes under the name of Kuramoto model. The question is natural because for the limit PDE [21, 66] there is a contractive manifold similar to M [42]. However the results in [65] suggest that a nontrivial dynamics on the contractive manifold is observed rather on times proportional to $\sqrt{N}$ and one expects a dynamics with a nontrivial random drift. The role of disorder in this type of models is not fully elucidated (see however [19] on the critical case) and the global long time dynamics represents a challenging issue.

The paper is organized as follows: we start off (Section 4.2) by introducing the precise mathematical set-up and a number of technical results. This will allow us to present quantitative heuristic arguments and sketch of poofs. In Section 4.3 we prove that if the system is close to M, it stays so for a long time. We then move on to analyzing the dynamics on M (Section 4.4) and it is here that we show that the drift is negligible. Section 4.5 provides the estimates that guarantee that we do approach M and in Section 4.6 we collect all these estimates and complete the proof our main result (Theorem 4.1.1).

4.2 More on the mathematical set-up and sketch of proofs

4.2.1 On the linearized evolution

We introduce the Hilbert space $H_{-1,1/q}$ or, more generally, the space $H_{-1,w}$ for a general weight $w \in C^1(\mathbb{S};(0,\infty))$ by using the rigged Hilbert space structure [16] with pivot space $\mathbb{L}^2_0 := \{u \in \mathbb{L}^2 : \int_{\mathbb{S}} u = 0\}$. In this way given an Hilbert space $V \subset \mathbb{L}^2_0$, V dense in $\mathbb{L}^2_0$, for which the canonical injection of V into $\mathbb{L}^2_0$ is continuous, one automatically obtains a representation of V' – the dual space – in terms of a third Hilbert space into which $\mathbb{L}^2_0$ is canonically and densely injected. If V is the closure of $\{u \in C^1(\mathbb{S};\mathbb{R}) : \int u = 0\}$ under the squared norm $\int_{\mathbb{S}}(u')^2/w$, that is $H_{1,1/w}$, the third Hilbert space is precisely $H_{-1,w}$. The duality between $H_{1,1/w}$ and $H_{-1,w}$ is denoted in principle by $\langle \cdot\,,\cdot \rangle_{H_{1,1/w},H_{-1,w}}$, but less cumbersome notations will be introduced when the duality is needed (for example, below we drop the subscripts).

It is not difficult to see that for $u, v \in H_{-1,w}$

$$(u,v)_{-1,w} = \int_{\mathbb{S}} w\mathcal{U}\mathcal{V}\,, \tag{4.2.1}$$

where $\mathcal{U}$, respectively $\mathcal{V}$, is the primitive of u (resp. v) such that $\int_{\mathbb{S}} w\mathcal{U} = 0$ (resp. $\int_{\mathbb{S}} w\mathcal{V} = 0$), see [9, § 2.2]. More precisely, $u \in H_{-1,w}$ if there exists $\mathcal{U} \in \mathbb{L}^2(\mathbb{S};\mathbb{R})$ such that $\int_{\mathbb{S}}\mathcal{U}w = 0$ and $\langle u, h\rangle = -\int_{\mathbb{S}}\mathcal{U}h'$ for every $h \in H_{1,1/w}$. One sees directly also that by changing w one produces equivalent $H_{1,w}$ norms [44, §2.1] so, when the geometry of the Hilbert space is not crucial, one can simply replace the weight by 1, and in this case we simply write H_{-1}. Occasionally we will need also H_{-2} which is introduced in an absolutely analogous way.

Remark 4.2.1. *One observation that is of help in estimating weighted H_{-1} norms is that computing the norm of u requires access to $\mathcal{U}$: in practice if one identifies a primitive $\widetilde{\mathcal{U}}$ of u, then $\|u\|^2_{-1,w} \le \int_{\mathbb{S}}\widetilde{\mathcal{U}}^2 w$. This is just because $\widetilde{\mathcal{U}} = \mathcal{U} + c$ for some $c \in \mathbb{R}$ and $\int_{\mathbb{S}}\widetilde{\mathcal{U}}^2 w = \int_{\mathbb{S}}\mathcal{U}^2 w + c^2\int_{\mathbb{S}} w$.*

The reason for introducing weighted H_{-1} spaces is because, as one can readily verify, L_q, given in (4.1.8), is symmetric in $H_{-1,1/q}$. A deeper analysis (cf. [9]) shows that L_q is essentially self-adjoint, with compact resolvent. The spectrum of $-L_q$ lies in $[0,\infty)$, there is an eigenvalue $\lambda_0 = 0$ with one dimensional eigenspace generated by q'. We therefore denote the set of eigenvalues of $-L_q$ as $\{\lambda_0, \lambda_1, \ldots\}$, with $\lambda_1 > 0$ and $\lambda_{j+1} \geq \lambda_j$ for $j = 1, 2, \ldots$. The set of eigenfunctions is denoted by $\{e_j\}_{j=0,1,\ldots}$ and let us point out that it is straightforward to see that $e_j \in C^\infty(\mathbb{S}; \mathbb{R})$. Moreover, if $u \in C^2(\mathbb{S}; \mathbb{R})$ is even (respectively, odd), then $L_q u$ is even (respectively, odd): the notion of parity is of course the one obtained by observing that $u \in C^2(\mathbb{S}; \mathbb{R})$ can be extended to a periodic function in $C^2(\mathbb{R}; \mathbb{R})$. This implies that one can choose $\{e_j\}_{j=0,1,\ldots}$ with e_j that is either even or odd, and we will do so.

Remark 4.2.2. *By rotation symmetry the eigenvalues do not depend on the choice of $q(\cdot) \in M$, but the eigenfunctions do depend on it, even if in a rather trivial way: the eigenfunction of L_{q_ψ} and $L_{q_{\psi'}}$ just differ by a rotation of $\psi' - \psi$. We will often need to be precise about the choice of $q(\cdot)$ and for this it is worthwhile to introduce the notations*

$$L_\psi := L_{q_\psi} \quad \text{and} \quad -L_\psi e_{\psi,j} = \lambda_j e_{\psi,j}\,. \tag{4.2.2}$$

The eigenfunctions are normalized in $H_{-1,1/q_\psi}$.

Remark 4.2.3. *Some expressions involving weighted H_{-1} norms can be worked out explicitly. For example a recurrent expression in what follows is $(u, q')_{1,1/q}$, for $u \in H_{-1}$ and $q \in M$. If $\mathcal{U}$ is the primitive of u such that $\int_\mathbb{S} \mathcal{U}/q = 0$, then we have $(u, q')_{1,1/q} = \int_\mathbb{S} \mathcal{U}(q - c)/q = \int_\mathbb{S} \mathcal{U}$, where c is uniquely defined by $\int_\mathbb{S}(q - c)/q = 0$, but of course the explicit value of c is not used in the final expression. In practice however it may be more straightforward to use an arbitrary primitive $\widetilde{\mathcal{U}}$ of u (i.e. $\int_\mathbb{S} \widetilde{\mathcal{U}}/q$ is not necessarily zero) for which we have*

$$(u, q')_{1,1/q} = \int_\mathbb{S} \widetilde{\mathcal{U}} \left(1 - \frac{c}{q}\right). \tag{4.2.3}$$

Since now c appears, let us make it explicit:

$$c = \frac{2\pi}{\int_\mathbb{S} 1/q} = \frac{1}{2\pi I_0^2(2Kr)}\,. \tag{4.2.4}$$

4.2.2 About the manifold M

As we have anticipated, we look at the set of stationary solutions M, defined in (4.2.2), as a manifold. For this we introduce

$$\widetilde{H}_{-1} := \left\{\mu : \mu - \frac{1}{2\pi} \in H_{-1}\right\}, \tag{4.2.5}$$

which is a metric space equipped with the distance inherited from H_{-1}, that is $\mathrm{dist}(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{-1}$. We have $M \subset \widetilde{H}_{-1}$ and M can be viewed as a smooth one dimensional manifold in $\widetilde{H}_{-1}$. The tangent space at $q \in M$ is $q'\mathbb{R}$ and for every $u \in H_{-1}$ we define the projection P^o_q on this tangent space as $P^o_q u = (u, q')_{-1,1/q} q'/(q', q')_{-1,1/q}$. The following result is proven in [99, p. 501] (see also [44, Lemma 5.1]):

Lemma 4.2.4. *There exists $\sigma > 0$ such that for all $p \in N_\sigma$ with*

$$N_\sigma := \cup_{q\in M} \left\{\mu \in \widetilde{H}_{-1} : \|\mu - q\|_{-1} < \sigma\right\}, \tag{4.2.6}$$

there is one and only one $q =: v(\mu) \in M$ such that $(\mu - q, q')_{-1,1/q} = 0$. Furthermore, the mapping $\mu \mapsto v(\mu)$ is in $C^\infty(\widetilde{H}_{-1}, \widetilde{H}_{-1})$, and (with D the Fréchet derivative)

$$Dv(\mu) = P^o_{v(\mu)}\,. \tag{4.2.7}$$

Note that the empirical (probability) measure $\mu_{N,t}$ that describes our system at time t is in $\widetilde{H}_{-1}$ (see Appendix 4.A) and Lemma 4.2.4 guarantees in particular that as soon as it is sufficiently close to M there is a well defined projection $v(\mu_{N,t})$ on the manifold. Since the manifold is isomorphic to $\mathbb{S}$ it is practical to introduce, for $\mu \in \widetilde{H}_{-1}$, also $\mathtt{p}(\mu) \in \mathbb{S}$, uniquely defined by $v(\mu) = q_{\mathtt{p}(\mu)}$. It is immediate to see that the projection $\mathtt{p}$ is $C^\infty(\widetilde{H}_{-1}, \mathbb{S})$.

4.2.3 A quantitative heuristic analysis: the diffusion coefficient

The proof of Theorem 4.1.1 is naturally split into two parts: the approach to M and the motion on M. The approach to M is based on the properties of the PDE (4.1.4): in [43] it is shown, using the gradient flow structure of (4.1.4), that if the initial condition is not on the stable manifold U (see (4.1.7)) of the unstable stationary solution $\frac{1}{2\pi}$, then the solution converges for time going to infinity to one of the probability densities $q = q_\psi \in M$ (of course ψ is a function of the initial condition), so given a neighborhood of q_ψ after a finite time (how large it depends only on the initial condition), it gets to the chosen neighborhood: due to the regularizing properties of the PDE, such a neighborhood can be even in a topology that controls all the derivatives [43], but here there is no point to use a strong topology, since at the level of interacting diffusions we deal with a measure (that we inject into H_{-1}). And in fact we have to estimate the distance between the empirical measure and the solution to (4.1.4) – controlling thus the effect of the noise – but this type of estimates on finite time intervals is standard. However here there is a subtle point: the result we are after is a matter of fluctuations and it will not come as a surprise that the empirical measure approaches M but does not reach it (of course: M just contains smooth functions, and $\mu_{N,t}$ is not a function), but it will stay in a $N^{-1/2}$-neighborhood (measured in the H_{-1} norm). How long will it take to reach such a neighborhood? The approach to M is actually exponential and driven by the spectral gap (λ_1) of the linearized evolution operator (at least close to M). Therefore in order to enter such a $N^{-1/2}$-neighborhood a time proportional to $\log N$ appears to be needed, as the quick observation that $\exp(-\lambda_1 t) = O(N^{-1/2})$ for $t \geq \log N/(2\lambda_1)$ suggests. The proofs on this stage of the evolution are in Section 4.5: here we just stress that

1. controlling the effect of the noise on the system on times $O(\log N)$ is in any case sensibly easier than controlling it on times of order N, which is our final aim;
2. on times of order N it is no longer a matter of showing that the empirical measure stays close to the solution of the PDE: on such a time scale the noise takes over and the finite N system, which has a non-trivial (random) dynamics, substantially deviates from the behavior of the solution to the PDE, which just converges to one of the stationary profiles.

Let us therefore assume that the empirical measure is in a $N^{-1/2}$-neighborhood of a given $q = q_\psi$. It is reasonable to assume that the dominating part of the dynamics close to q is captured by the operator L_q and we want to understand the action of the semigroup generated by L_q on the noise that stirs the system, on long times. Note that we cannot choose arbitrarily long times, in particular not times proportional to N right away, because in view of the result we are after, we expect that on such a time scale the empirical measure of the system is no longer in a neighborhood of q, but close to $q_{\psi'}$ for a $\psi' \neq \psi$. We will actually choose some intermediate time scale $N^{1/10}$ as we will see in § 4.2.4 and Remark 4.2.6, that guarantees that working with L_q makes sense, i.e. that the projection of the empirical measure on M is still sufficiently close to q. The point is that the effect of the noise on intermediate times is very different in the tangential direction and the orthogonal directions to M, simply because in the orthogonal direction there is a damping, that is absent in the tangential direction. So on intermediate times the the leading term in the evolution of the empirical measure turns out to be the projection of the evolution on the tangential direction, that is $(q', \mu_{N,t} - q)_{-1,1/q}/\|q'\|_{-1,1/q}$. One can now use Remark 4.2.3 to obtain

$$(q', \mu_{N,t} - q)_{-1,1/q} = -\int_{\mathbb{S}} \mathcal{K}(\theta)\, (\mu_{N,t}(\,\mathrm{d}\theta) - q(\theta)\,\mathrm{d}\theta)\ , \tag{4.2.8}$$

with $\mathcal{K}$ a primitive of $1 - c/q$ (c given in Remark 4.2.3). By applying Itô's formula we see that the term in (4.2.8) can be written as the sum of a drift term and of a martingale term. It is not difficult to see that to leading order the drift term is zero (a more attentive analysis shows that one has to show that the next order correction does not give a contribution, but we come back to this below). The quadratic variation of the martingale term instead turns out to be equal to t/N times

$$\int_{\mathbb{S}} (\mathcal{K}'(\theta))^2 q(\theta)\,\mathrm{d}\theta = 1 - \frac{(2\pi)^2}{\int_{\mathbb{S}} 1/q} = \|q'\|^2_{-1,1/q}\,. \tag{4.2.9}$$

Since $q_{\psi+\varepsilon} = q_\psi - \varepsilon q'_\psi + \cdots$ (note that q'_ψ is not normalized), (4.2.9) suggests that the diffusion coefficient D_K in Theorem 4.1.1 is $\|q'\|^{-1}_{-1,1/q}$, which coincides with (4.1.11).

To make this procedure work one has to carefully put together the analysis on the intermediate time scale, by setting up an adequate iterative scheme. Several delicate issues arise and one of the challenging points is precisely to control that the drift can be neglected. In fact the first order expansion of the projection that we have used

$$\mathtt{p}\,(q_\psi + h) = \psi - \frac{(h, q')_{-1,1/q}}{(q', q')_{-1,1/q}} + O(\|h\|^2_{-1})\,, \tag{4.2.10}$$

is not accurate enough and one has to go to the next order, see Lemma 4.A.5. This is due to the fact that the random contribution, which in principle appears as first order, fluctuates and generates a cancellation, so in the end the term is of second order.

Remark 4.2.5. *It is natural to expect that Theorem 4.1.1 holds true also when $p_0 \in U$ and this is just because the evolution is attracted to $\frac{1}{2\pi}$ and then the noise will cause an escape from this unstable profile after a time $\propto \log N$, since the exponential instability will make the fluctuations grow exponentially with a rate which is just given by the linearized dynamics (linearized around $\frac{1}{2\pi}$ of course). Arguments in this spirit can be found for example in [90, Ch. 5], see [4] and references therein for the finite dimensional counterpart. However*

1. *this is not so straightforward because it requires a good control on the dynamics on and around the heteroclinic orbits linking $\frac{1}{2\pi}$ to M_0 [43, Section 5];*
2. *the statement would require more details about the initial condition: the simple convergence to a point on U is largely non sufficient (the fluctuations of the initial conditions now matter!);*
3. *in general the initial phase ψ_0 on M is certainly going to be random: if the initial condition is rotation invariant (at least in law), like if $\{\varphi_0^{j,N}\}_{j=1,\dots,N}$ are IID variables uniformly distributed on $\mathbb{S}$ or if $\varphi_0^{j,N} = 2\pi j/N$, one expects ψ_0 to be uniformly distributed on $\mathbb{S}$. Note however that uniform distribution of ψ_0 is definitely not expected in the general case and asymmetries in the initial condition should affect the distribution of ψ_0.*

4.2.4 The iterative scheme

As we have explained in § 4.2.3, the analysis close to M requires an iterative procedure, which we introduce here. We assume that at $t=0$ the system is already close to M, while in practice this will happen after some time: in Section 4.6 we explain how to put together the results on the early stage of the evolution and the analysis close to M, that we start here. So, for $\mu_0 = \mu_{N,0} = \frac{1}{N}\sum_{j=1}^N \delta_{\varphi_0^{j,N}}$ such that $\mathrm{dist}(\mu_0, M) \le \sigma$ (here and below $\mathrm{dist}(\cdot,\cdot)$ is the distance built with the norm of H_{-1}), by Lemma 4.2.4 we can define $\psi_0 = \mathtt{p}(\mu_0)$. Applying the Itô formula to $\nu_t = \mu_t - q_{\psi_0}$ ($\mu_t = \mu_{N,t}$), we see that

$$\nu_t \,=\, e^{-tL_{\psi_0}}\nu_0 - \int_0^t e^{-(t-s)L_{\psi_0}}\partial_\theta[\nu_s J * \nu_s]\,\mathrm{d}s + Z_t\,, \tag{4.2.11}$$

where

$$Z_t \,=\, \frac{1}{N}\sum_{j=1}^N \int_0^t \partial_{\theta'}\mathcal{G}_{t-s}^{\psi_0}\left(\theta, \varphi_s^{j,N}\right)\,\mathrm{d}W_s^j\,, \tag{4.2.12}$$

and $\mathcal{G}_s^{\psi_0}(\theta,\theta')$ is the kernel of $e^{-sL_{\psi_0}}$ in $\mathbb{L}^2$. The evolution equation (4.2.11) and the noise term (4.2.12) have a meaning in H_{-1}, as well as the recentered empirical measures ν_t, and it is in this sense that we will use them: we detail this in Appendix 4.A, where one finds also an explicit expression and some basic facts about the kernel $\mathcal{G}_s^{\psi_0}(\theta,\theta')$. We have started here an abuse of notation that will be persistent through the text: $\partial_{\theta'}\mathcal{G}_{t-s}^{\psi_0}\left(\theta, \varphi_s^{j,N}\right)$ stands for $\partial_{\theta'}\mathcal{G}_{t-s}^{\psi_0}(\theta,\theta')\,|_{\theta'=\varphi_s^{j,N}}$.

Equations (4.2.11)–(4.2.12) are useful tools as long as we can properly define the phase associated to the empirical measure of the system and that this phase is close to ψ_0: in view of the result we want to prove, this is expected to be true for a long time, but it is certainly expected to fail for times of the order of N, since on this timescale the phase does change of an amount that does not vanish as N becomes large.

The idea is therefore to divide the evolution of the particle system up to a final time proportional to N into $n = n_N \overset{N\to\infty}{\longrightarrow} \infty$ time intervals $[T_i, T_{i+1}]$, where $T_i = iT$ and $T = T(N)$ is chosen close to a fractional power of N (see Remark 4.2.6). Moreover i runs from 1 up to $n = n_N$ so that $n_N T(N) = T_{n_N}$ and $\lim_N T_{n_N}/N$ is equal to a positive constant (the τ_f of Theorem 4.1.1). If the empirical measure μ_t stays close to the manifold M, we can define the projections of μ_{T_k} and successively update the reentering phase at all times T_k. The point then will be essentially to show that the process given by these phases, on the time scale $\propto N$, converges to a Brownian motion.

More formally, we construct the following iterative scheme: we choose

$$\sigma = \sigma_N := \lceil N^{2\zeta}\sqrt{T/N}\rceil \overset{N\to\infty}{\longrightarrow} 0\,, \tag{4.2.13}$$

for a suitable $\zeta > 0$ (see Remark 4.2.6), we set $\tau_{\sigma_N}^0 = 0$ and for $k = 1, 2, \ldots$ we define

$$\psi_{k-1} \,:=\, \mathtt{p}(\mu_{T_{k-1}})\,, \tag{4.2.14}$$

if $\mathrm{dist}(\mu_{T_{k-1}}, M) \le \sigma_N$ and

$$\tau_{\sigma_N}^k = \tau_{\sigma_N}^{k-1}\mathbf{1}_{\{\tau_{\sigma_N}^{k-1} < T_{k-1}\}} + \inf\{s \in [T_{k-1}, T_k] :\, \|\mu_s - q_{\psi_{k-1}}\|_{-1} > \sigma_N\}\mathbf{1}_{\{\tau_{\sigma_N}^{k-1} \geqslant T_{k-1}\}}\,. \tag{4.2.15}$$

Then we set

$$\nu_t^k := \mu_t - q_{\psi_{k-1}}\,, \tag{4.2.16}$$

for $t \in [T_{k-1}, T_k]$ and $t \le \tau^k_{\sigma_N}$, and otherwise $\nu^k_t := \nu^k_{\tau^k_{\sigma_N}}$ for every $t \ge \tau^k_{\sigma_N}$ (of course $\tau^k_{\sigma_N}$ can be smaller than T_{k-1} and, in this case, the definition becomes redundant). Therefore the ν process we have just defined solves for $t \in [T_{k-1}, T_k]$

$$\nu^k_t = \mathbf{1}_{\{\tau^k_{\sigma_N} < T_{k-1}\}} \nu^k_{T_{k-1}} + \mathbf{1}_{\{\tau^k_{\sigma_N} \geqslant T_{k-1}\}} \times \\ \left(e^{-(t\wedge\tau^k_{\sigma_N} - T_{k-1})L_{\psi_{k-1}}} \nu^k_{T_{k-1}} - \int_{T_{k-1}}^{t\wedge\tau^k_{\sigma_N}} e^{-(t\wedge\tau^k_{\sigma_N} - s)L_{\psi_{k-1}}} \partial_\theta[\nu^k_s J * \nu^k_s]\, \mathrm{d}s + Z^k_{t\wedge\tau^k_{\sigma_N}} \right), \quad (4.2.17)$$

where

$$Z^k_t = \frac{1}{N}\sum_{j=1}^N \int_{T_{k-1}}^t \partial_{\theta'} \mathcal{G}^{\psi_{k-1}}_{t-s}\left(\theta, \varphi^{j,N}_s\right) \mathrm{d}W^j_s\,. \quad (4.2.18)$$

Once again, we refer to Appendix 4.A for the precise meaning of (4.2.17) and (4.2.18).

Remark 4.2.6. *For the remainder of the paper we choose $T(N) \sim N^{1/10}$ and $\zeta \le 1/100$. The two exponents do not have any particular meaning: a look at the argument shows that the exponent for $T(N)$ has in any case to be chosen smaller than $1/2$, but then a number of technical estimates enter the game and we have settled for a value $1/10$ without trying to get the optimal value that comes out of the method we use.*

4.3 A priori estimates: persistence of proximity to M

The aim of this section is to prove that, if we are (say, at time zero) sufficiently close to M, we stay close to M for times $O(N)$. The arguments in this section justify the choice of the proximity parameter σ_N that we have made in the iterative scheme. We first prove some estimates on the size of the noise term and then we will give the estimates on the empirical measure.

4.3.1 Noise estimates

We define the event

$$B^N = \left\{ \sup_{1\leqslant k\leqslant n-1} \sup_{t\in[T_k, T_{k+1}]} \left\|Z^k_t\right\|_{-1} \leqslant \sqrt{\frac{T}{N}} N^\zeta \right\} \\ \bigcap \left\{ \sup_{1\leqslant k\leqslant n-1} \sup_{t\in[T_k, T_{k+1}]} \left\|Z^{k,\perp}_t\right\|_{-1} \leqslant \frac{1}{\sqrt{N}} N^\zeta \right\}, \quad (4.3.1)$$

where $Z^{k,\perp}_t$ is defined precisely like Z^k_t, see (4.2.18), except for the replacement of $\mathcal{G}^\psi_{t-s}(\cdot,\cdot)$ with $\mathcal{G}^\psi_{t-s}(\theta,\theta') - e_{\psi_{k-1},0}(\theta) f_{\psi_{k-1},0}(\theta')$ (see (4.3.3)).

Lemma 4.3.1. $\lim_{N\to\infty} \mathbb{P}\left(B^N\right) = 1$.

Proof. In order to perform the estimates we introduce and work with approximated versions of Z^k_t and $Z^{k,\perp}_t$ (see Lemma 4.A.4). Define for $T_{k-1} < t' < t$

$$Z^k_{t,t'} = \frac{1}{N}\sum_{j=1}^N \int_{T_{k-1}}^{t'} \partial_{\theta'} \mathcal{G}^{\psi_{k-1}}_{t-s}(\theta, \varphi^{j,N}_s)\, \mathrm{d}W^j_s\,. \quad (4.3.2)$$

The kernel $\mathcal{G}_\cdot^{\psi_{k-1}}$ in this case is (cf. Appendix 4.A)

$$\mathcal{G}_s^{\psi_{k-1}}(\theta,\theta') = \sum_{l=0}^{\infty} e^{-s\lambda_l} e_{\psi_{k-1},l}(\theta) f_{\psi_{k-1},l}(\theta') , \tag{4.3.3}$$

where λ_l are the ordered eigenvalues of $-L_{\psi_{k-1}}$, $e_{\psi_{k-1},l}$ are the associated eigenfunctions of unit norm in $H_{-1,1/q_{\psi_{k-1}}}$, cf. Remark 4.2.2, and $f_{\psi_{k-1},l}$ are the eigenfunctions of $L^*_{\psi_{k-1}}$, the adjoint in $\mathbb{L}^2$ (see Appendix 4.A).

Very much in in the same way we define

$$Z^{k,\perp}_{t,t'} = \frac{1}{N}\sum_{j=1}^{N}\int_{T_{k-1}}^{t'} \partial_{\theta'} \mathcal{G}_{t-s}^{\psi_{k-1},\perp}(\theta,\varphi_s^{j,N})\, \mathrm{d}W_s^j , \tag{4.3.4}$$

with

$$\mathcal{G}_s^{\psi_{k-1},\perp}(\theta,\theta') = \sum_{l=1}^{\infty} e^{-s\lambda_l} e_{\psi_{k-1},l}(\theta) f_{\psi_{k-1},l}(\theta') . \tag{4.3.5}$$

We decompose for $T_{k-1} < s' < s < t$ and $s' < t' < t$

$$\begin{aligned} Z^k_{t,t'} - Z^k_{s,s'} = \frac{1}{N}\sum_{j=1}^{N}\int_{T_{k-1}}^{s'} \Big(\partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{t-u}(\theta,\varphi_u^{j,N}) - \partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{s-u}(\theta,\varphi_u^{j,N})\Big)\, \mathrm{d}W_u^j \\ + \frac{1}{N}\sum_{j=1}^{N}\int_{s'}^{t'} \partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{t-u}(\theta,\varphi_u^{j,N})\, \mathrm{d}W_u^j , \end{aligned} \tag{4.3.6}$$

and an absolutely analogous formula holds for $Z^{k,\perp}$: in fact the bounds for Z^k and $Z^{k,\perp}$ are obtained with the same technique even if the results are slightly different due to the presence of the zero eigenvalue in Z^k. Moreover we apply $\|a+b\|^2 \leqslant 2(\|a\|^2+\|b\|^2)$, so that we can estimate the two terms in the right-hand side of (4.3.6) separately.

And we start with the second term of the right-hand side in (4.3.6): by the orthogonality properties of the eigenvectors we obtain

$$\begin{aligned} &\left\| \frac{1}{N}\sum_{j=1}^{N}\int_{s'}^{t'} \partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{t-u}(\cdot,\varphi_u^{j,N})\, \mathrm{d}W_u^j \right\|^2_{-1,1/q} \\ &\quad = \frac{1}{N^2}\sum_{l=0}^{\infty}\sum_{j,j'=1}^{N}\int_{s'}^{t'}\int_{s'}^{t'} e^{-(2t-u-u')\lambda_l} f'_{\psi_{k-1},l}(\varphi_u^{j,N}) f'_{\psi_{k-1},l}(\varphi_{u'}^{j',N})\, \mathrm{d}W_u^j\, \mathrm{d}W_{u'}^{j'} , \end{aligned} \tag{4.3.7}$$

and by taking the expectation

$$\begin{aligned} &\mathbb{E}\left[\left\| \frac{1}{N}\sum_{j=1}^{N}\int_{s'}^{t'} \partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{t-u}(\cdot,\varphi_u^{j,N})\, \mathrm{d}W_u^j \right\|^2_{-1,1/q}\right] = \\ &\qquad \frac{1}{N^2}\sum_{l=0}^{\infty}\sum_{j=1}^{N}\int_{s'}^{t'} e^{-2(t-u)\lambda_l}\mathbb{E}\left[(f'_{\psi_{k-1},l}(\varphi_u^{j,N}))^2\right] \mathrm{d}u . \end{aligned} \tag{4.3.8}$$

By Corollary 4.B.6 there exists a constant C_1 such that

$$\mathbb{E}\left[\left\| \frac{1}{N}\sum_{j=1}^{N}\int_{s'}^{t'} \partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{t-u}(\cdot,\varphi_u^{j,N})\, \mathrm{d}W_u^j \right\|^2_{-1,1/q}\right] \leqslant \frac{C_1}{N}\sum_{l=0}^{\infty}\int_{s'}^{t'} e^{-2(t-u)\lambda_l}\, \mathrm{d}u . \tag{4.3.9}$$

Proposition 4.B.4, Remark 4.B.3, leads us to

$$\sum_{l=0}^{\infty}\int_{s'}^{t'} e^{-2(t-u)\lambda_l}\,\mathrm{d}u \leqslant \sum_{l=0}^{\infty}\int_{s'}^{t'} e^{-(t-u)\frac{l^2}{C}}\,\mathrm{d}u \leqslant C\sum_{l=0}^{\infty}\frac{1}{l^2}\left(1-e^{-(t'-s')\frac{l^2}{C}}\right), \tag{4.3.10}$$

where the addend with $l=0$ (times C) has to be read as $t'-s'$. The right-most term in (4.3.10) for $t'-s'\geq 1$ can be bounded by $C(t'-s')+C\sum_{l=1}^{\infty}1/l^2\leq 3C(t'-s')$. Instead for $t'-s'<1$ we decompose the same term and then estimate as follows:

$$\begin{aligned} C\sum_{l=0}^{\llcorner (t'-s')^{-1/2}\lrcorner}\frac{1}{l^2}\left(1-e^{-(t'-s')\frac{l^2}{C}}\right)+C\sum_{l=\llcorner (t'-s')^{-1/2}\lrcorner+1}^{\infty}\frac{1}{l^2}\left(1-e^{-(t'-s')\frac{l^2}{C}}\right)\\ \leqslant \sum_{l=0}^{\llcorner (t'-s')^{-1/2}\lrcorner}(t'-s')+\sum_{l=\llcorner (t'-s')^{-1/2}\lrcorner+1}^{\infty}\frac{C}{l^2}\leq (3+2C)\sqrt{t'-s'}\,, \end{aligned} \tag{4.3.11}$$

where for the first term we have used $(1-\exp(-a))\leq a$, for $a\geq 0$. Therefore we have proven that there exists C such that for every k and every s, s', t, t' such that $T_{k-1}<s'<s<t$ and $s'<t'<t$ we have

$$\mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^{N}\int_{s'}^{t'}\partial_{\theta'}\mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot,\varphi_u^{j,N})\,\mathrm{d}W_u^j\right\|_{-1,1/q}^2\right]\leqslant \frac{Ch_1(t'-s')}{N}. \tag{4.3.12}$$

with

$$h_1(u) := u^{1/2}\mathbf{1}_{[0,1)}(u)+u\mathbf{1}_{[1,\infty)}\,. \tag{4.3.13}$$

We can do better in the case of $\mathcal{G}^{\psi_{k-1},\perp}$, for which a direct inspection of the argument we have just presented shows that the linearly growing term in the estimate can be avoided (since the term $l=0$ is no longer there) and the net result is

$$\mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^{N}\int_{s'}^{t'}\partial_{\theta'}\mathcal{G}_{t-u}^{\psi_{k-1},\perp}(\cdot,\varphi_u^{j,N})\,\mathrm{d}W_u^j\right\|_{-1,1/q}^2\right]\leqslant \frac{Ch_2(t'-s')}{N}. \tag{4.3.14}$$

with h_2 defined as

$$h_2(u) = u^{1/2}\mathbf{1}_{[0,1)}(u)+\mathbf{1}_{[1,\infty)}\,. \tag{4.3.15}$$

For what concerns the first term in the right-hand side of (4.3.6), we have

$$\begin{aligned} \mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^{N}\int_{T_{k-1}}^{s'}\left(\partial_{\theta'}\mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot,\varphi_u^{j,N})-\partial_{\theta'}\mathcal{G}_{s-u}^{\psi_{k-1}}(\cdot,\varphi_u^{j,N})\right)\mathrm{d}W_u^j\right\|_{-1,1/q}^2\right]\\ =\frac{1}{N^2}\sum_{l=1}^{\infty}\sum_{j=1}^{N}\int_{T_{k-1}}^{s'}\left(e^{-(t-u)\lambda_l}-e^{-(s-u)\lambda_l}\right)^2\mathbb{E}\left[(f'_{\psi_{k-1},l}(\varphi_u^{j,N}))^2\right]\mathrm{d}u\,, \end{aligned} \tag{4.3.16}$$

and, by proceeding like for (4.3.9), we see that the expression in (4.3.16) is bounded by

$$\frac{C_1}{N}\sum_{l=1}^{\infty}\frac{\left(1-e^{-\lambda_l(t-s')}\right)^2}{\lambda_l}\leq\frac{C_1C}{N}\sum_{l=1}^{\infty}\frac{\left(e^{-(t-s')\frac{l^2}{C}}-1\right)^2}{l^2}. \tag{4.3.17}$$

This last term is estimated once again by separating the two cases of $t-s'$ small and large. The net result is that there exists $C>0$ such that for every k, every s, s' and t such that $T_k<s'<s<t$ we have

$$\mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^{N}\int_{T_{k-1}}^{s'}\left(\partial_{\theta'}\mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot,\varphi_u^{j,N})-\partial_{\theta'}\mathcal{G}_{s-u}^{\psi_{k-1}}(\cdot,\varphi_u^{j,N})\right)\mathrm{d}W_u^j\right\|_{-1,1/q}^2\right]\leqslant Ch_2(t-s')\,. \tag{4.3.18}$$

In order to complete the proof of Lemma 4.3.1 quadratic estimates do not suffice: we need to generalize (4.3.12), (4.3.14) and(4.3.18) to larger exponents. We actually need estimates on moments of order $2m$, with m finite, but sufficiently large, so to apply the standard Kolmogorov Lemma type estimates and get uniform bounds. We are going to use

$$\|a+b\|^m\leqslant m(\|a\|^m+\|b\|^m)\,, \tag{4.3.19}$$

but actually we will not track the m dependence of the constants. We aim at showing that the expectation of the moments of order $2m$ of the quantities we are interested in are bounded by the m^{th} power of the estimate we found in the quadratic case, times an m-dependent constant.

For $m=1,2,\ldots$, the m^{th}–power of the expression in (4.3.7) gives

$$\begin{aligned}&\left\|\frac{1}{N}\sum_{j=1}^{N}\int_{s'}^{t'}\partial_{\theta'}\mathcal{G}_{t-u}^{\psi_{k-1}}(\theta,\varphi_u^{j,N})\,\mathrm{d}W_u^j\right\|_{-1,1/q}^{2m}\\&\quad=\frac{1}{N^{2m}}\sum_{l_1,\ldots,l_m=0}^{\infty}\ \sum_{j_1,j_1',\ldots,j_m,j_m'=1}^{N}F_{l_1}^{j_1}(t,s',t')F_{l_1}^{j_1'}(t,s',t')\cdots F_{l_m}^{j_m}(t,s',t')F_{l_m}^{j_m'}(t,s',t')\,,\end{aligned} \tag{4.3.20}$$

in which we have introduced the random variables

$$F_l^j(t,s',t')=\int_{s'}^{t'}e^{-\lambda_l(t-u)}f'_{\psi_{k-1},l}(\varphi_u^{j,N})\,\mathrm{d}W_u^j\,. \tag{4.3.21}$$

We now take the expectation of both terms in (4.3.20) and all the terms in the sum that do not include an even number of each Brownian motion vanish. The number of non-zero terms in the expectation can thus be bounded by $(2m)!N^m$. Applying the Itô formula to each of these non-zero terms, we get at most $(2m)!/(2^m m!)$ terms (the number of possibilities classifying $2m$ elements in couples) of the type $I_1\cdots I_m$, where

$$I_k=I_k(l_1,l_2)=\int_{s'}^{t'}e^{-(\lambda_{l_1}+\lambda_{l_2})(t-u)}\mathbb{E}\left[f'_{\psi_{k-1},l_1}(\varphi_u^{j,N})f'_{\psi_{k-1},l_2}(\varphi_u^{j,N})\right]\mathrm{d}u\,. \tag{4.3.22}$$

We now observe that

$$|I_k(l_1,l_2)|\leq\sqrt{I_k(l_1,l_1)I_k(l_2,l_2)}\,, \tag{4.3.23}$$

and for each index index l_i in the first sum in the right-hand side of (4.3.20) gives rise either directly to a term $I_k(l_i)$ (for this it is needed that the two terms share the Brownian motion), or in the arising products the terms $I_k(l_i,l_{i'})$ are associated with a term of the type $I_k(l_i,l_{i''})$. Therefore the expression obtained after applying Itô formula can be bounded by a sum of terms of the type $\hat{I}_1\cdots\hat{I}_m$, with

$$\hat{I}_k=I_k(l,l)=\int_{s'}^{t'}e^{-2\lambda_l(t-u)}\mathbb{E}\left[(f'_{\psi_{k-1},l}(\varphi_u^{j,N}))^2\right]\mathrm{d}u\,. \tag{4.3.24}$$

Therefore we are facing the same estimates that we have encountered in the quadratic case, see (4.3.8) and (4.3.9), except of course for combinatorial contribution. In the end we obtain that there exists $C = C_m$ such that

$$\mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N})\, \mathrm{d}W_u^j\right\|_{-1,1/q}^{2m}\right] \leqslant C\frac{h_1^m(t'-s')}{N^m}, \tag{4.3.25}$$

$$\mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1},\perp}(\cdot, \varphi_u^{j,N})\, \mathrm{d}W_u^j\right\|_{-1,1/q}^{2m}\right] \leqslant C\frac{h_2^m(t'-s')}{N^m}. \tag{4.3.26}$$

In a similar way

$$\begin{aligned}\left\|\frac{1}{N}\sum_{j=1}^N \int_{T_{k-1}}^{s'} \left(\partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) - \partial_{\theta'} \mathcal{G}_{s-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N})\right) \mathrm{d}W_u^j\right\|_{-1,1/q}^{2m} \\ = \frac{1}{N^{2m}} \sum_{l_1,\ldots,l_m=0}^{\infty} \sum_{j_1,j_1',\ldots,j_m,j_m'=1}^{N} G_{l_1}^{j_1}(s,t,s')G_{l_1}^{j_1'}(s,t,s')\cdots G_{l_m}^{j_m}(s,t,s')G_{l_m}^{j_m'}(s,t,s')\end{aligned} \tag{4.3.27}$$

with

$$G_l^j(s,t,s') = \int_{T_{k-1}}^{s'} \left(e^{-\lambda_l(t-u)} - e^{-\lambda_l(s-u)}\right) f'_{\psi_{k-1},l}(\varphi_u^{j,N})\, \mathrm{d}W_u^j. \tag{4.3.28}$$

We reduce the problem as above to the study of products of integral terms $J_1 \cdots J_k$ with

$$\begin{aligned}J_k = \int_{T_{k-1}}^{s'} \left(e^{-\lambda_{l_1}(t-u)} - e^{-\lambda_{l_1}(s-u)}\right)\left(e^{-\lambda_{l_2}(t-u)} - e^{-\lambda_{l_2}(s-u)}\right) \times \\ \mathbb{E}\left[f'_{\psi_{k-1},l_1}(\varphi_u^{j,N}) f'_{\psi_{k-1},l_2}(\varphi_u^{j,N})\right] \mathrm{d}u,\end{aligned} \tag{4.3.29}$$

and then, like before, in terms of products of *diagonal* terms of the type

$$\hat{J}_k = \int_{T_{k-1}}^{s'} \left(e^{-\lambda_l(t-u)} - e^{-\lambda_l(s-u)}\right)^2 \mathbb{E}\left[(f'_{\psi_{k-1},l}(\varphi_u^{j,N}))^2\right] \mathrm{d}u. \tag{4.3.30}$$

Again we are reduced to the estimating terms that have already appeared in the quadratic case, see (4.3.16), so we obtain that there exists $C = C_m$ such that

$$\begin{aligned}\mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^N \int_{T_{k-1}}^{s'} \left(\partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) - \partial_{\theta'} \mathcal{G}_{s-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N})\right) \mathrm{d}W_u^j\right\|_{-1,1/q}^{2m}\right] \\ \leqslant C\frac{h_2^m(t-s)}{N^m}.\end{aligned} \tag{4.3.31}$$

We now let $t' \nearrow t$ and $s' \nearrow s$ and by applying Fatou's Lemma and Lemma 4.A.4, from(4.3.6), (4.3.25), (4.3.26) and (4.3.31) we get

$$\mathbb{E}\left[\left\|Z_t^k - Z_s^k\right\|_{-1}^{2m}\right] \leqslant C\frac{h_1^m(t-s)}{N^m}, \tag{4.3.32}$$

and

$$\mathbb{E}\left[\left\|Z_t^{k,\perp} - Z_s^{k,\perp}\right\|_{-1}^{2m}\right] \leqslant C\frac{h_2^m(t-s)}{N^m}. \tag{4.3.33}$$

The fact that we are allowed to drop the weight in the H_{-1} norm is of course due to the norm equivalence.

We are now in good condition to apply the Garsia-Rodemich-Rumsey Lemma [106]:

Lemma 4.3.2. *Let p and Ψ be continuous, strictly increasing functions on $(0,\infty)$ such that $p(0) = \Psi(0) = 0$ and $\lim_{t\nearrow\infty}\Psi(t) = \infty$. Given $T > 0$ and ϕ continuous on $(0,T)$ and taking its values in a Banach space $(E, \|.\|)$, if*

$$\int_0^T \int_0^T \Psi\left(\frac{\|\phi(t)-\phi(s)\|}{p(|t-s|)}\right) \mathrm{d}s\,\mathrm{d}t \leqslant B < \infty , \tag{4.3.34}$$

then for $0 \leqslant s \leqslant t \leqslant T$:

$$\|\phi(t)-\phi(s)\| \leqslant 8 \int_0^{t-s} \Psi^{-1}\left(\frac{4B}{u^2}\right) p(\,\mathrm{d}u)\,. \tag{4.3.35}$$

We apply Lemma 4.3.2 with

$$\phi(t) = Z^k_{t-T_{k-1}}, \quad p(u) = u^{\frac{2+\zeta}{2m}} \text{ and } \Psi(u) = u^{2m}, \tag{4.3.36}$$

and $\zeta = 1/100$ (Remark 4.2.6). With these choices we can find an explicit constant $C = C(m,\zeta)$ such that

$$\|Z^k_t - Z^k_s\|^{2m}_{-1} \le C(t-s)^{\zeta} B\,, \tag{4.3.37}$$

for every s and t such that $T_{k-1} \leqslant s < t \leqslant T_k$ and B is a positive random variable such that

$$\mathbb{E}[B] \le \frac{C}{N^m}\int_0^T\int_0^T \frac{h_1^m(|t-s|)}{|t-s|^{2+\zeta}}\,\mathrm{d}s\,\mathrm{d}t\,, \tag{4.3.38}$$

where C is the constant in (4.3.32). For $m > 4$ the function $t \mapsto h_1^m(t)/t^{2+\zeta}$, defined for $t > 0$, is increasing (and it tends to zero for $t \searrow 0$). So $\mathbb{E}[B]$ is bounded by $CN^{-m}h_1^m(T)/T^{\zeta}$ and therefore

$$\mathbb{E}\left[\sup_{T_{k-1}\leqslant s<t\leqslant T_k} \frac{\|Z^k_t - Z^k_s\|^{2m}_{-1}}{|t-s|^{\zeta}}\right] \leqslant C\frac{T^{m-\zeta}}{N^m}\,, \tag{4.3.39}$$

which leads to

$$\mathbb{P}\left[\sup_{T_{k-1}\leqslant t\leqslant T_k} \|Z^k_t\|_{-1} \geqslant \sqrt{\frac{T}{N}}N^{\zeta}\right] \leqslant C\frac{1}{N^{m\zeta}}\,. \tag{4.3.40}$$

Then, (recall $n = n_N = \frac{N}{T}$) we deduce

$$\mathbb{P}\left[\sup_{1\leqslant k\leqslant n}\sup_{T_{k-1}\leqslant t\leqslant T_k} \|Z^k_t\|_{-1} \geqslant \sqrt{\frac{T}{N}}N^{\zeta}\right] \leqslant C\frac{1}{TN^{m\zeta-1}}\,, \tag{4.3.41}$$

where the right hand side tends to 0 when m is chosen sufficiently large. A similar argument gives for $Z^{k,\perp}_t$

$$\mathbb{P}\left[\sup_{1\leqslant k\leqslant n}\sup_{T_{k-1}\leqslant t\leqslant T_k} \|Z^{k,\perp}_t\|_{-1} \geqslant \frac{1}{\sqrt{N}}N^{\zeta}\right] \leqslant C\frac{T^{\zeta-1}}{N^{m\zeta-1}}\,. \tag{4.3.42}$$

□

We now give the main result of the section:

Proposition 4.3.3. *If* $\|\nu_0^1\|_{-1} \leqslant \frac{N^{2\zeta}}{\sqrt{N}}$ *and if the event* B^N *defined in* (4.3.1) *is realized (then, with probability approaching* 1 *as* $N \to \infty$*) we have*

$$\sup_{1\leqslant k\leqslant n} \sup_{t\in[T_{k-1},T_k]} \|\nu_t^k\|_{-1} \leqslant \sqrt{\frac{T}{N}} N^{2\zeta}\,, \tag{4.3.43}$$

and

$$\max_{1\leqslant k\leqslant n} \left\|\nu_{T_{k-1}}^k\right\|_{-1} \leqslant \frac{N^{2\zeta}}{\sqrt{N}}\,. \tag{4.3.44}$$

Proof. From (4.2.17) and Lemma 4.A.2, we get,for all $k = 1\dots n$ and $t \in [T_{k-1}, T_k]$

$$\|\nu_t^k\|_{-1} \leqslant Ce^{-\lambda_1(t-T_{k-1})}\|\nu_{T_{k-1}}^k\|_{-1} + C\int_{T_{k-1}}^t \left(1+\frac{1}{\sqrt{t-s}}\right)\|\nu_s^k\|_{-1}^2\,\mathrm{d}s + \|Z_t^k\|_{-1}\,. \tag{4.3.45}$$

The constant C in front of the first term of the right hand side above would be equal to 1 if we were using the $\|.\|_{-1,1/q_{\psi_{k-1}}}$ norm. Let us assume that $\|\nu_{T_{k-1}}^k\|_{-1} \leq N^{2\zeta}/\sqrt{N}$, since we are working in B^N we obtain

$$\|\nu_t^k\|_{-1} \leqslant Ce^{-\lambda_1(t-T_{k-1})}\frac{N^{2\zeta}}{\sqrt{N}} + C(T+\sqrt{T})\sup_{T_{k-1}\leqslant s\leqslant t}\|\nu_s^k\|_{-1}^2 + \frac{\sqrt{T}}{\sqrt{N}}N^{\zeta}\,. \tag{4.3.46}$$

Therefore we readily see that if we define

$$t^* = \sup\left\{t \in [T_{k-1}, T_k] :\ \|\nu_t^k\|_{-1} > \frac{\sqrt{T}}{\sqrt{N}}N^{2\zeta}\right\}\,, \tag{4.3.47}$$

we have that for $t \leq t^*$

$$\|\nu_t^k\|_{-1} \leq CN^{2\zeta-\frac{1}{2}} + 2CT^2N^{4\zeta-1} + \sqrt{T}N^{\zeta-\frac{1}{2}}\,. \tag{4.3.48}$$

Therefore since $\lim_N T^3N^{-1+4\zeta} = 0$ (see Remark 4.2.6), for N large enough, we have $t^* = T_k$ and (4.3.43) is reduced to proving n $\|\nu_{T_{k-1}}^k\|_{-1} \leq N^{2\zeta}/\sqrt{N}$ for $k = 1, 2, \dots, n$. This holds for $k = 1$: we are now going to show by induction (4.3.44) and therefore that the assumption propagates from k to $k + 1$.

To prove the bound on $\nu_{T_k}^{k+1}$, assuming the bound on $\nu_{T_{k-1}}^k$, we use the smoothness of the manifold M. Since we are working in B^N, $\tau_\sigma^k = T_k$ and we have

$$\begin{aligned}
\nu_{T_k}^{k+1} &= q_{\psi_{k-1}} + \nu_{T_k}^k - q_{\psi_k} \\
&= P_{\psi_k}^\perp\left[q_{\psi_{k-1}} + \nu_{T_k}^k - q_{\psi_k}\right] \\
&= \left(P_{\psi_k}^\perp - P_{\psi_{k-1}}^\perp\right)\left[q_{\psi_{k-1}} + \nu_{T_k}^k - q_{\psi_k}\right] + P_{\psi_{k-1}}^\perp\left[q_{\psi_{k-1}} - q_{\psi_k}\right] + P_{\psi_{k-1}}^\perp \nu_{T_k}^k\,,
\end{aligned} \tag{4.3.49}$$

Since the mapping $\psi \mapsto P_\psi^\perp$ is smooth on the compact M, we have (cf. § 4.2.2)

$$\left\|P_{\psi_k}^\perp - P_{\psi_{k-1}}^\perp\right\|_{\mathcal{L}(H_{-1},H_{-1})} \leqslant C\,|\psi_k - \psi_{k-1}|\,, \tag{4.3.50}$$

and the identities

$$\psi_k - \psi_{k-1} = \mathtt{p}(\mu_{T_k}) - \mathtt{p}(\mu_{T_{k-1}})\,, \tag{4.3.51}$$

and

$$\mu_{T_k} - \mu_{T_{k-1}} = \nu_{T_k}^k - \nu_{T_{k-1}}^k\,, \tag{4.3.52}$$

combined with the smoothness of $\mathtt{p}$, lead to (using (4.3.43))

$$\left\| P^{\perp}_{\psi_k} - P^{\perp}_{\psi_{k-1}} \right\|_{\mathcal{L}(H_{-1},H_{-1})} \leqslant C\frac{\sqrt{T}}{\sqrt{N}}N^{2\zeta}. \tag{4.3.53}$$

On the other hand, the smoothness of q_ψ with respect to ψ, (4.3.51) and (4.3.52) imply

$$\left\| q_{\psi_{k-1}} + \nu^k_{T_k} - q_{\psi_k} \right\|_{-1} \leqslant C\left(\|\nu^k_{T_{k-1}}\|_{-1} + \|\nu^k_{T_k}\|_{-1} \right), \tag{4.3.54}$$

so the first term in the last line of (4.3.49) is of order $\frac{T}{N}N^{4\zeta}$, which is much smaller than $\frac{N^{2\zeta}}{\sqrt{N}}$ nor $N \to \infty$, since $\lim_N TN^{2\zeta-\frac{1}{2}} = 0$ (see Remark 4.2.6). Moreover, Lemma 4.2.4 implies

$$\left\| P^{\perp}_{\psi_{k-1}} \left[q_{\psi_{k-1}} - q_{\psi_k} \right] \right\|_{-1} = \left\| P^{\perp}_{\psi_{k-1}} \left[v(\mu_{T_{k-1}}) - v(\mu_{T_k}) \right] \right\|_{-1} \leqslant C \left\| \mu_{T_{k-1}} - \mu_{T_k} \right\|_{-1}, \tag{4.3.55}$$

so the second term in the last line of (4.3.49) is also of order $\frac{T}{N}N^{4\zeta}$. Finally, projecting (4.2.17) on Range $\left(L_{q_{\psi_{k-1}}}\right)$ and by using again Lemma 4.A.2, we get

$$\left\| P^{\perp}_{\psi_{k-1}} \nu^k_t \right\|_{-1} \leqslant Ce^{-\lambda_1(t-T_{k-1})}\|\nu^k_{T_{k-1}}\|_{-1} + C\int_{T_{k-1}}^t \left(1 + \frac{1}{\sqrt{t-s}}\right) \|\nu^k_s\|^2_{-1}\,\mathrm{d}s + \|Z^{k,\perp}_t\|_{-1}, \tag{4.3.56}$$

which, since $\lim_N T^4N^{1-5\zeta} = 0$ (see Remark 4.2.6), leads for N large enough to

$$\left\| P^{\perp}_{\psi_{k-1}} \nu^k_{T_k} \right\|_{-1} \leqslant \frac{N^{3\zeta/2}}{\sqrt{N}}. \tag{4.3.57}$$

This takes care of the third term in the last line of (4.3.49) and by collecting the three estimates we obtain (4.3.44) and the proof is complete. □

4.4 The effective dynamics on the tangent space

The following result states that each rotation increment of our discretization scheme is well approximated by the projection the dynamical noise on the tangent space.

Proposition 4.4.1. *We have the first order approximation in probability: for every $\varepsilon > 0$*

$$\mathbb{P}\left(\left| \sum_{k=1}^n (\psi_k - \psi_{k-1}) - \sum_{k=1}^n \frac{(Z^k_{T_k}, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q',q')_{-1,1/q}} \right| \leq \varepsilon \right) = 1. \tag{4.4.1}$$

Proof. Lemma 4.A.5 and Proposition 4.3.3 give (assuming that B^N is realized: we will do this through all the proof)

$$\begin{aligned}\psi_k - \psi_{k-1} = &-\frac{(\nu^k_{T_k}, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q',q')_{-1,1/q}} \\ &- \frac{1}{2\pi I_0^2(2Kr)} \frac{(\nu^k_{T_k}, (\log q_{\psi_{k-1}})'')_{-1,1/q_{\psi_{k-1}}}}{(q',q')_{-1,1/q}} \frac{(\nu^k_{T_k}, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q',q')_{-1,1/q}} + o\left(\frac{T}{N}\right).\end{aligned} \tag{4.4.2}$$

Since $\log(q_{\psi_{k-1}})''$ is in $R(L_{q_{\psi_{k-1}}})$, we have

$$(\nu^k_{T_k}, (\log q_{\psi_{k-1}})'')_{-1,1/q_{\psi_{k-1}}} = ((\nu^k_{T_k})^\perp, (\log q_{\psi_{k-1}})'')_{-1,1/q_{\psi_{k-1}}}, \tag{4.4.3}$$

and thus using again Proposition 4.3.3 we get for the second term of the right-hand side

$$\left\| \frac{1}{2\pi I_0^2(2Kr)} \frac{(\nu^k_{T_k}, (\log q_{\psi_{k-1}})'')_{-1,1/q_{\psi_{k-1}}}}{(q',q')_{-1,1/q}} \frac{(\nu^k_{T_k}, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q',q')_{-1,1/q}} \right\|_{-1} \leqslant C \frac{1}{\sqrt{N}} N^{2\zeta} \frac{\sqrt{T}}{\sqrt{N}} N^{2\zeta}, \tag{4.4.4}$$

and hence it is $o(T/N)$, since $\lim_N N^{4\zeta}/\sqrt{T} = 0$ (see Remark 4.2.6). So only the component on the tangent space of M at the point ψ_{k-1} is of order T/N:

$$\psi_k - \psi_{k-1} = -\frac{(\nu^k_{T_k}, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q',q')_{-1,1/q}} + o\left(\frac{T}{N}\right). \tag{4.4.5}$$

We now decompose this tangent term. Our goal is to show that the projection of the noise Z_{T_k} is the only term that gives a non negligible contribution when N goes to infinity. However, a direct domination of the remainder – the nonlinear part of the evolution equation (4.2.17) – using the a priori bound $\|\nu^k_t\|_{-1} \leqslant \frac{\sqrt{T}}{\sqrt{N}} N^{2\zeta}$ is not sufficient. In fact

$$\left| \left(\int_{T_{k-1}}^{T_k} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta [\nu^k_s J * \nu^k_s]\, \mathrm{d}s, q'_{\psi_{k-1}} \right)_{-1,1/q_{\psi_{k-1}}} \right| \leqslant \frac{T^2}{N} N^{4\zeta}. \tag{4.4.6}$$

In order to improve this estimate the strategy is to we re-inject (4.2.17) into the projection $(I_k(T_k), q_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}$, where

$$I_k(t) = \mathbf{1}_{\{\tau^k_{\sigma N} \geqslant T_{k-1}\}} \int_{T_{k-1}}^{t \wedge \tau^k_{\sigma N}} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta [\nu^k_s J * \nu^k_s]\, \mathrm{d}s, \tag{4.4.7}$$

and this leads to a rather long expression

$$\sum_{k=1}^n (\psi_k - \psi_{k-1}) = \sum_{k=1}^n \frac{(Z^k_{T_k}, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q',q')_{-1,1/q}} + \sum_{k=1}^n \sum_{i=1}^9 A_{k,i} + o(1), \tag{4.4.8}$$

with

$$
\begin{aligned}
A_{k,1} &= \mathbf{1}_E \times \\
&\left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} \nu^k_{T_{k-1}} J * \left(e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} \nu^k_{T_{k-1}}\right)\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star \\
A_{k,2} &= \mathbf{1}_E \left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[I_k(s) J * I_k(s)\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star \\
A_{k,3} &= \mathbf{1}_E \left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[Z^k_s J * Z^k_s\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star \\
A_{k,4} &= \mathbf{1}_E \left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} \nu^k_{T_{k-1}} J * I_k(s)\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star \\
A_{k,5} &= \mathbf{1}_E \left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[I_k(s) J * \left(e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} \nu^k_{T_{k-1}}\right)\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star \\
A_{k,6} &= \mathbf{1}_E \left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} \nu^k_{T_{k-1}} J * Z^k_s\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star \\
A_{k,7} &= \mathbf{1}_E \left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[Z^k_s J * \left(e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} \nu^k_{T_{k-1}}\right)\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star \\
A_{k,8} &= \mathbf{1}_E \left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[I_k(s) J * Z^k_s\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star \\
A_{k,9} &= \mathbf{1}_E \left(\int_\star e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[Z^k_s J * I_k(s)\right] \mathrm{d}s, q'_{\psi_{k-1}}\right)_\star ,
\end{aligned}
\tag{4.4.9}
$$

where we have used the shortcuts $E = \{\tau^k_{\sigma_N} \geqslant T_{\psi_{k-1}}\}$, $\int_\star$ stands for $\int_{T_{k-1}}^{T \wedge \tau^k_{\sigma_N}}$ and and $(\cdot,\cdot)_\star$ is $(\cdot,\cdot)_{-1,1/q_{\psi_{k-1}}}$.

The following bound (a direct consequence of Lemma 4.A.2 and 4.A.3) is now going to be of help:

$$
\begin{aligned}
&\left\| \int_{T_{k-1}}^{T_k} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta [h_1(s) J * h_2(s)] \,\mathrm{d}s \right\|_{-1} \\
&\qquad\qquad \leqslant C \int_{T_{k-1}}^{T_k} \left(1 + \frac{1}{\sqrt{T_k - s}}\right) \|h_1(s)\|_{-1} \|h_2(s)\|_{-1} \,\mathrm{d}s \,.
\end{aligned}
\tag{4.4.10}
$$

In fact it is not difficult to see that by using Proposition 4.3.3 and (4.4.10) we can efficiently bound all the $A_{k,j}$'s, except $A_{k,3}$:

$$
\begin{aligned}
&|A_{k,1}| \leqslant \frac{1}{N} N^{5\zeta}, \quad |A_{k,2}| \leqslant \frac{T^5}{N^2} N^{9\zeta}, \quad |A_{k,4}| \leqslant \frac{T^2}{N^{3/2}} N^{7\zeta}, \quad |A_{k,5}| \leqslant \frac{T^2}{N^{3/2}} N^{7\zeta}, \\
&|A_{k,6}| \leqslant \frac{T^{1/2}}{N^{3/2}} N^{4\zeta}, \quad |A_{k,7}| \leqslant \frac{T^{1/2}}{N^{3/2}} N^{4\zeta}, \quad |A_{k,8}| \leqslant \frac{T^{7/2}}{N^{3/2}} N^{6\zeta} \quad \textit{and} \quad |A_{k,9}| \leqslant \frac{T^{7/2}}{N^{3/2}} N^{6\zeta} \,.
\end{aligned}
\tag{4.4.11}
$$

Since $T^4 N^{9\zeta-1} \to 0$ and $N^{5\zeta}/T \to 0$ (see Remark 4.2.6), we get (recall that $n = n_N = \frac{N}{T}$)

$$
\sum_{k=1}^n (\psi_k - \psi_{k-1}) = \sum_{k=1}^n \frac{(Z^k_{T_k}, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q', q')} + \sum_{k=1}^n A_{k,3} + o(1)\,.
\tag{4.4.12}
$$

For the $A_{k,3}$ terms we need to use something more sophisticate. To deal with these terms in fact we rely on an averaging phenomena. This method has been used in [7] for the same kind of problem. We write the Doob decomposition

$$\sum_{k=1}^{m} A_{k,3} \;=\; M_m + \sum_{k=1}^{m} \gamma_k\,, \tag{4.4.13}$$

where

$$\gamma_k \;=\; \mathbb{E}\left[A_{k,3}|\mathcal{F}_{T_{k-1}}\right]\,, \tag{4.4.14}$$

and M_m is a $\mathcal{F}_{T_m}$-martingale with brackets

$$\langle M\rangle_m \;=\; \sum_{k=1}^{m}\left(\mathbb{E}\left[A_{k,3}^2|\mathcal{F}_{T_{k-1}}\right] - \gamma_k^2\right)\,. \tag{4.4.15}$$

We have

$$\begin{aligned}\gamma_k \;=\; \mathbb{E}\Bigg[&\frac{1}{N^2}\sum_{i,j=1}^{N}\int_{T_{k-1}}^{T_k\wedge\tau_{\sigma_N}^k}\mathrm{d}W_s^i\int_{T_{k-1}}^{T_k\wedge\tau_{\sigma_N}^k}\mathrm{d}W_{s'}^j\int_{\mathbb{S}}\mathrm{d}\theta\left(1-\frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)}\right)\\ &\int_{\mathbb{S}}\mathrm{d}\theta''\partial\mathcal{G}_{T_k-s}^{\psi_{k-1}}(\theta,\varphi_s^{i,N})J(\theta-\theta'')\partial\mathcal{G}_{T_k-s'}^{\psi_{k-1}}(\theta'',\varphi_{s'}^{j,N})\Bigg|\mathcal{F}_{T_{k-1}}\Bigg]\mathbf{1}_{\tau_{\sigma_N}^k\geqslant T_{k-1}}\,,\end{aligned} \tag{4.4.16}$$

where $\partial\mathcal{G}_t^{\psi}(\theta,\theta') := \partial_{\theta'}\mathcal{G}_t^{\psi}(\theta,\theta')$, and from this we obtain

$$\begin{aligned}\gamma_k \;=\; \mathbb{E}\Bigg[&\frac{1}{N}\int_0^{T\wedge\widetilde{\tau}_{\sigma_N}}\mathrm{d}s\int_{\mathbb{S}}\widetilde{\mu}_s(\mathrm{d}\theta')\int_{\mathbb{S}}\mathrm{d}\theta\left(1-\frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)}\right)\\ &\int_{\mathbb{S}}\mathrm{d}\theta''\partial_{\theta'}\mathcal{G}_{T-s}^{\psi_{k-1}}(\theta,\theta')J(\theta-\theta'')\partial_{\theta'}\mathcal{G}_{T-s}^{\psi_{k-1}}(\theta'',\theta')\Bigg]\mathbf{1}_{\tau_{\sigma_N}^k\geqslant T_{k-1}}\,,\end{aligned} \tag{4.4.17}$$

with

$$\widetilde{\mu}_s \;:=\; \frac{1}{N}\sum_{j=1}^{N}\delta_{\widetilde{\varphi}_s^{j,N}}\,, \tag{4.4.18}$$

where $\{\widetilde{\varphi}_s^{j,N}\}_{s\geq 0}$ is a solution of (4.1.1) depending on $\mathcal{F}_{T_{k-1}}$ only through the initial condition

$$\widetilde{\varphi}_0^{j,N} \;=\; \varphi_{T_{k-1}}^{j,N}\,. \tag{4.4.19}$$

The stopping time $\widetilde{\tau}_{\sigma_N}$ is defined as follows:

$$\widetilde{\tau}_{\sigma_N} \;:=\; \inf\{s>0,\;\|\widetilde{\mu}_s - q_{\psi_{T_{k-1}}}\|_{-1} > \sigma_N\}\,. \tag{4.4.20}$$

We now write $\widetilde{\mu}_s(\mathrm{d}\theta') = q_{\psi_{k-1}}(\theta')\,\mathrm{d}\theta' + \widetilde{\nu}_s^k(\mathrm{d}\theta')$ and split the for right hand-side of (4.4.17) into the corresponding two terms.

The term coming $q_{\psi_{k-1}}(\theta')\,\mathrm{d}\theta'$ is zero as one can see by using the symmetry:

$$\mathcal{G}_s^{\psi_{k-1}}(\psi_{k-1}+\theta,\psi_{k-1}+\theta') \;=\; \mathcal{G}_s^{\psi_{k-1}}(\psi_{k-1}-\theta,\psi_{k-1}-\theta') \tag{4.4.21}$$

which follows from the same statement with $\psi_{k-1}=0$, which in turn is a consequence of the representation (4.A) and of the fact that if e_j is even (respectively, odd) then f_j is even (respectively, odd) too (see Section 4.2.1 and Section 4.A).

For the term containing $\widetilde{\nu}^k_s(\mathrm{d}\theta')$ instead we get the bound

$$\left|\mathbb{E}\left[\frac{1}{N}\int_{T_{k-1}}^{T_k\wedge\widetilde{\tau}_{\sigma_N}}\mathrm{d}s\int_{\mathbb{S}}\mathrm{d}\widetilde{\nu}^k_s(\theta')\int_{\mathbb{S}}\mathrm{d}\theta\int_{\mathbb{S}}\mathrm{d}\theta''\left(1-\frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)}\right)\right.\right.$$
$$\left.\left.\partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{T_k-s}(\theta,\theta')J(\theta-\theta'')\partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{T_k-s}(\theta'',\theta')\right]\right|$$
$$\leqslant \mathbb{E}\left[\frac{1}{N}\int_{T_{k-1}}^{T_k\wedge\tau^k_{\sigma_N}}\mathrm{d}s\|\widetilde{\nu}^k_s\|_{-1}\|H^k_s\|_{H_1}\right] \quad (4.4.22)$$

where

$$H^k_s(\theta') = \int_{\mathbb{S}}\mathrm{d}\theta\int_{\mathbb{S}}\mathrm{d}\theta''\left(1-\frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)}\right)\partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{T_k-s}(\theta,\theta')J(\theta-\theta'')\partial_{\theta'}\mathcal{G}^{\psi_{k-1}}_{T_k-s}(\theta'',\theta')\,. \quad (4.4.23)$$

We now plug in the explicit representation for the kernels:

$$H^k_s(\theta') = \sum_{l_1,l_2=0}^{\infty} e^{-(\lambda_{l_1}+\lambda_{l_2})(T_k-s)}\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\theta''\left(1-\frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)}\right)$$
$$e_{\psi_{k-1},l_1}(\theta)J(\theta-\theta'')e_{\psi_{k-1},l_2}(\theta'')f'_{\psi_{k-1},l_1}(\theta')f'_{\psi_{k-1},l_2}(\theta')\,. \quad (4.4.24)$$

We obtain

$$\|H^k_s\|_1 \leqslant \sum_{l,m=0}^{\infty} e^{-(\lambda_{l_1}+\lambda_{l_2})(T_k-s)}\left|\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\theta''\left(1-\frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)}\right)\right.$$
$$\left.e_{\psi_{k-1},l_1}(\theta)J(\theta-\theta'')e_{\psi_{k-1},l_2}(\theta'')\right|\left(\|f''_{\psi_{k-1},l_1}\|_2\|f'_{\psi_{k-1},l_2}\|_\infty+\|f'_{\psi_{k-1},l_1}\|_\infty\|f''_{\psi_{k-1},l_2}\|_2\right)\,. \quad (4.4.25)$$

We aim at proving the convergence of this sum. For the integral term, thanks to the rotation symetry, we can limit the study to $\psi_{k-1}=0$. Since $J(\theta-\theta'')=-K\sin(\theta-\theta'')=-K\sin(\theta)\cos(\theta'')+K\cos(\theta)\sin(\theta'')$, we can split these double integrals into products of two simple ones. Corollary 4.B.5 implies that there exists l_0 in $\mathbb{N}$ such that e_{0,l_0+2p} and e_{0,l_0+2p+1} can be writen as

$$e_{0,l_0+2p} = pq_0^{1/2}(c_{1,l_0+2p}v_{1,l_0+p}+c_{2,l_0+2p}v_{2,l_0+p})+O\left(\frac{1}{p}\right)\,, \quad (4.4.26)$$

$$e_{0,l_0+2p+1} = pq_0^{1/2}(c_{1,l_0+2p+1}v_{1,l_0+p}+c_{2,l_0+2p+1}v_{2,l_0+p})+O\left(\frac{1}{p}\right)\,, \quad (4.4.27)$$

where

$$\sup_{l\geqslant l_0}\{|c_{1,l}|,|c_{2,l}|\} < \infty \quad (4.4.28)$$

and the functions $v_{i,l}$ are defined in Proposition 4.B.4. The $v_{i,l}$ are sums and products of sines and cosines, and there exists $h\in\mathbb{N}$ such that the only non-zero Fourier coefficients of v_{i,l_0+p} are of index included between $h+p-2$ and $h+p+2$ and are bounded with respect to p. We deduce that the simple integral terms containing e_{0,l_0+2p}, which are of the form

$$C\int_0^{2\pi}q_0^{\pm 1/2}(\theta)e_{0,l_0+2p}(\theta)g(\theta)\,\mathrm{d}\theta\,, \quad (4.4.29)$$

where g is sine or cosine and C is a constant independent of p, are up to a correction of order $1/p$ a bounded linear combination of the Fourier coefficients of $q_0^{1/2}$ or $q_0^{-1/2}$ of index taken between $h+p-3$ and $h+p+3$. The same argument applies for e_{0,l_0+2p+1}. Since these Fourier terms decrease faster than exponentially (this can be seen by observing that $\int_{\mathbb{S}} \exp(a\cos\theta)\,\mathrm{d}\theta = 2\pi I_n(a)$ and that $\exp(a\cos(\cdot))$ is an entire function), these simple integral terms are of order $1/p$. Using Remark 4.B.3 and Corollary 4.B.6 we deduce the following bound for $\|H_s^k\|_1$:

$$\|H_s^k\|_1 \leqslant C + C\sum_{l_1,l_2=1}^{\infty} \frac{l_1+l_2}{l_1 l_2} e^{(T_k-s)\frac{l_1^2+l_2^2}{C}} + C\sum_{l=1}^{\infty} e^{(T_k-s)\frac{l^2}{C}}\,, \tag{4.4.30}$$

where the first term of the right hand side corresponds to the case $l_1=0, l_2=0$ in (4.4.25), the second term corresponds to $l_1>0, l_2>0$ and the third term to $l_1=0, l_2>0$ or $l_2=0, l_1>0$. Applying (4.4.22) and Proposition 4.3.3, we get:

$$\begin{aligned}|\gamma_k| \leqslant C\frac{T^{1/2}}{N^{3/2}}N^{2\zeta}\int_{T_{k-1}}^{T_k} \mathrm{d}s\|H_s^k\|_1 \leqslant C\frac{T^{1/2}}{N^{3/2}}N^{2\zeta}\left(T + \sum_{h,l=1}^{\infty}\frac{h+l}{hl(h^2+l^2)} + \sum_{l=1}^{\infty}\frac{1}{l^2}\right)\\ \leqslant C\frac{T^{3/2}}{N^{3/2}}N^{2\zeta}\,,\end{aligned} \tag{4.4.31}$$

and thus for N large enough

$$\sum_{k=1}^{n}|\gamma_k| \leqslant \sqrt{\frac{T}{N}}N^{3\zeta}\,. \tag{4.4.32}$$

On the other hand, applying Doob Inequality, (4.4.9) and Proposition 4.3.3, it comes

$$\mathbb{P}\left[\sup_{1\leqslant m\leqslant n}|M_m| \geqslant \sqrt{\frac{T}{N}}N^{3\zeta}\right] \leqslant \frac{N}{TN^{6\zeta}}\mathbb{E}\left[\langle M\rangle_n\right] \leqslant \frac{N^{1-6\zeta}}{T}\sum_{m=1}^{n}\mathbb{E}\left[A_{k,3}^2\right] \leqslant \frac{T^3}{N^{1+2\zeta}}\,. \tag{4.4.33}$$

Since $TN^{-1-2\zeta}\to 0$ (see Remark 4.2.6), the combination of (4.4.32) and (4.4.33) leads to

$$\mathbb{P}\left[\left|\sum_{i=1}^{n}A_{k,3}\right| \geqslant \sqrt{\frac{T}{N}}N^{3\zeta}\right] \to 0\,, \tag{4.4.34}$$

and the proof is complete.

□

4.5 Approach to M

The long time behavior of the solutions to (4.1.4) is rather well understood, so, in particular, we know that if p_0 is not on the set attracted to the unstable solution $\frac{1}{2\pi}$, then it converges to one probability density in M (cf. Proposition 4.5.2) and, in particular, it reaches a given (N independent) neighborhood of M in finite time: this is directly extracted from [9, 43]). This takes care of the first stage of the evolution, because the deterministic result directly extends to the empirical measure by standard arguments since the time horizon is finite. But we do need to get to distances of about $N^{-1/2}$ and this requires a more attentive control of the dynamics. In fact, we exploit the *approximate* contracting properties of the dynamics when the empirical measure is close to q_ψ. We talk about *approximate* contracting properties because the noise plays against getting to M and limits the contraction effect of the linearized operator. Nevertheless, the proof mimics the deterministic proof of nonlinear stability, to which the control of the noise is added. In principle the argument is straightforward: one exploits the spectral gap of the linearized evolution. In practice, one has to set up an iterative procedure similar to the one developed in Section 4.3, because the center of synchronization may change somewhat over long times. This procedure is however substantially easier than the one presented in Section 4.3, mostly because here the control required on the noise is for substantially shorter times ($\log N$ versus $N!$), so we will not go through the arguments in full detail again.

Proposition 4.5.1. *Choose $p_0 \in \mathcal{M}_1 \setminus U$ such that (4.1.9) is satisfied. Then there exists ψ_0 (non random!), that depends on K and $p_0(\cdot)$, C, that depends only on K, and a random variable Ψ_N such that*

$$\lim_{N\to\infty} \mathbb{P}\left(\|\mu_{N,\widetilde{\varepsilon}_N N} - q_{\Psi_N}\|_{-1} \le \frac{N^{2\zeta}}{\sqrt{N}} \right) = 1\,, \tag{4.5.1}$$

where $\widetilde{\varepsilon}_N := \lfloor C \log N \rfloor / N$, and $\lim_N \Psi_N = \psi_0$ in probability. Moreover for ε and ε_N as in Theorem 4.1.1 we have

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{t \in [\varepsilon_N N, \widetilde{\varepsilon}_N N]} \|\mu_{N,t} - q_{\psi_0}\|_{-1} \le \varepsilon \right) = 1\,, \tag{4.5.2}$$

Proof. The proof is divided in two parts. First we prove, using the convergence of $\mu_t = \mu_{N,t}$ to the deterministic solution p_t, that for a given $h > 0$ (arbitrarily small), there exists t_0 such that for ε small enough, $\mathbb{P}(\mathrm{dist}(\mu_{N,t_0}, M) \leqslant h) \to 1$ when $N \to \infty$. Then we show that after a time of order $\log N$, the empirical measure μ_t moves to a distance $N^{\zeta - 1/2}$ from M, without a macroscopic change of the phase.

The first part of the proof relies on the following result:

Proposition 4.5.2. *If $p_0 \in \mathcal{M}_1 \setminus U$ then there exists $\psi \in \mathbb{S}$ such that $\lim_{t\to\infty} p_t = q_\psi$ in $C^k(\mathbb{S};\mathbb{R})$ (for every k).*

Proposition 4.5.2 is essentially taken from [43], in the sense that it follows by piecing together some results taken from [43]. We give below a proof that of course relies on [43]. We point out that the very same result can be proven also by adapting entropy production arguments, like in [2].

Proposition 4.5.2 guarantees that the deterministic solution p_t converges to a element q_{ψ_0} of M. Therefore for $t \ge t_0$, we have that p_t is no farther than $h/2$ from q_{ψ_0} (this is a statement that can be made for example in C^k, but here we just need it in H_{-1}). Actually, it is not difficult to see that one can choose $t_0 = -\frac{2}{\lambda_1} \log h$, for h sufficiently small (λ_1 is the spectral gat of $L_{q_{\psi_0}}$), but this is of little relevance here. Applying the Itô formula

$$\mu_t - p_t = e^{t\frac{\Delta}{2}}(\mu_0 - p_0) - \int_0^t e^{(t-s)\frac{\Delta}{2}} [\mu_s J * \mu_s - p_s J * p_s] \,\mathrm{d}s + z_t\,, \tag{4.5.3}$$

where

$$z_t = \frac{1}{N} \sum_{j=1}^N \partial_{\theta'} \mathcal{H}(\theta, \phi_s^{j,N}) \,\mathrm{d}W_s^j\,, \tag{4.5.4}$$

$e^{t\frac{\Delta}{2}}$ is the semi-group of the Laplacian and $\mathcal{H}$ is the kernel of $e^{s\frac{\Delta}{2}}$ in $\mathbb{L}^2$. Define $W_N = \{w, \|\mu_0 - p_0\|_{-1} \leqslant \varepsilon\}$. Using the classical estimate $\|e^{t\Delta/2}u\|_{-1} \leqslant \frac{C}{\sqrt{t}}\|u\|_{-2}$ and similar argument as in Section 4.3, we deduce that there exist events $\widetilde{W}_N \subset W_N$ such that $\mathbb{P}(\widetilde{W}_N) \to 1$ and that for all outcomes in $\widetilde{W}_N$ we have

$$\sup_{0 \leqslant t \leqslant t_0} \|z_t\|_{-1} \leqslant \sqrt{\frac{t_0}{N}} N^{\zeta} . \tag{4.5.5}$$

From now, we restrict ourselves to $\widetilde{W}_N$. From (4.5.3) we get for all $t \in [0, t_0]$

$$\|\mu_t - p_t\|_{-1} \leqslant \varepsilon + C \int_0^t \frac{1}{\sqrt{t-s}} \|\mu_s - p_s\|_{-1} \,\mathrm{d}s + \sqrt{\frac{t_0}{N}} N^{\zeta} . \tag{4.5.6}$$

The Gronwall-Henry inequality (see [99]) implies that there exists $\gamma > 0$ (independent of ε and N) such that

$$\sup_{t \leq t_0} \|\mu_{t_0} - p_{t_0}\|_{-1} \leqslant \left(\varepsilon + \sqrt{\frac{t_0}{N}} N^{\zeta} \right) e^{\gamma t_0} . \tag{4.5.7}$$

So for $\varepsilon = h/4$ and N large enough, $\|\mu_{t_0} - q_{\psi_0}\|_{-1} \leqslant h$ on the event $\widetilde{W}_N$.

To show that we enter a neighborhood of size *slightly larger* than $N^{-1/2}$, it will be $N^{2\zeta-1/2}$, we set up an iterative scheme. It is very similar to the one given in Section 4.2.4, but with times t_i bounded with respect to N. This times are chosen such that after each iteration, the distance between the empirical measure and M is at least divided by 2. We define $h_0 := h$ and for $m \geqslant 1$

$$t_m := t_{m-1} + \frac{1}{\lambda_1} |\log \alpha| , \tag{4.5.8}$$

$$h_m := \frac{1}{2} h_{m-1} , \tag{4.5.9}$$

until the index m_f defined by

$$m_f := \inf \left\{ m \geqslant 1,\, h_m \leqslant N^{2\zeta - 1/2} \right\} . \tag{4.5.10}$$

The constant α above does not depens on N and will be chosen below. It is now easy to check that m_f is of order $\log N$. Then we define $\widetilde{\tau}_0 := t_0$, and for $1 \leqslant m \leqslant m_f + 1$

$$\widetilde{\psi}_{m-1} := \mathtt{p}(\mu_{t_{m-1}}) , \tag{4.5.11}$$

$$\widetilde{\nu}^m_{t_{m-1}} := \mu_{t_{m-1}} - q_{\widetilde{\psi}_{m-1}} \tag{4.5.12}$$

if $\mathrm{dist}(\mu_{t_{m-1}}, M) \leqslant \sigma$ (see Lemma 4.2.4). We consider for $1 \leqslant m \leqslant m_f$ the stopping times

$$\widetilde{\tau}_m := \widetilde{\tau}_{m-1} \mathbf{1}_{\{\widetilde{\tau}_{m-1} < t_{m-1}\}} + \inf\{s \in [t_{m-1}, t_m],\, \|\mu_s - q_{\widetilde{\psi}_{m-1}}\|_{-1} \geqslant \sigma\} \mathbf{1}_{\{\widetilde{\tau}_{m-1} \geqslant t_{m-1}\}} . \tag{4.5.13}$$

and the process solution of

$$\widetilde{\nu}^m_t = \mathbf{1}_{\{\widetilde{\tau}_m < t_{m-1}\}} \widetilde{\nu}^m_{\widetilde{\tau}_m} + \mathbf{1}_{\{\widetilde{\tau}_m \geqslant t_{m-1}\}} \times \left(e^{-(t\wedge\widetilde{\tau}_m - t_{m-1}) L_{\widetilde{\psi}_{m-1}}} \widetilde{\nu}^m_{t_{m-1}} - \int_{t_{m-1}}^{t\wedge\widetilde{\tau}_m} e^{-(t\wedge\widetilde{\tau}_m - s) L_{\widetilde{\psi}_{m-1}}} \partial_\theta [\widetilde{\nu}^m_s J * \widetilde{\nu}^m_s] \,\mathrm{d}s + \widetilde{Z}^m_{t\wedge\widetilde{\tau}_m} \right) , \tag{4.5.14}$$

where

$$\widetilde{Z}_t^m = \frac{1}{N}\sum_{j=1}^{N}\int_{t_{m-1}}^{t} \partial_{\theta'}\mathcal{G}_{t-s}^{\widetilde{\psi}_{m-1}}\left(\theta, \varphi_s^{j,N}\right) \mathrm{d}W_s^j . \tag{4.5.15}$$

With the same arguments as given in Lemma 4.3.1, we can prove (recall that m_f is of order $\log N$) that the probability of the event

$$\Omega_N := \left\{ \sup_{1\leqslant m\leqslant m_f}\ \sup_{t_{m-1}\leqslant t\leqslant t_m} \left\| \widetilde{Z}_t^m \right\|_{-1} \leqslant \sqrt{\frac{t_m - t_{m-1}}{N}} N^{\zeta} \right\} \tag{4.5.16}$$

tends to 1 when $N \to \infty$. From now, we assume that Ω_N is verified. We insist on the fact that the generic constants C appearing in the following do not depend on N, and if not mentioned do not depend on α. From Lemma 4.A.2 and (4.5.14) we get thet for all $1 \leqslant m \leqslant m_f$,

$$\|\widetilde{\nu}_t^m\|_{-1} \leqslant Ch_{m-1} + C(t+\sqrt{t}) \sup_{s\in[t_{m-1},t]} \|\widetilde{\nu}_s^m\|_{-1} + \sqrt{\frac{t_m - t_{m-1}}{N}} N^{\zeta} . \tag{4.5.17}$$

We now prove that for $1 \leqslant m \leqslant m_f - 1$, $\|\nu_{t_{m-1}}^m\|_{-1} \leqslant h_{m-1}$ implies $\|\nu_{t_m}^{m+1}\|_{-1} \leqslant h_m$, and that $\|\widetilde{\nu}_{t_{m_f-1}}^{m_f}\|_{-1} \leqslant h_{m_f-1}$ implies $\|\widetilde{\nu}_{t_{m_f}}^{m_f+1}\|_{-1} \leqslant N^{2\zeta-1/2}$. Define

$$s_m^* := \sup\{s \in [t_{m-1}, t_m],\ \|\widetilde{\nu}_s^m\|_{-1} \leqslant h_{m-1}^{3/4}\} . \tag{4.5.18}$$

Then for $s < s_m^*$, if $\|\nu_{t_{m-1}}^m\|_{-1} \leqslant h_{m-1}$,we get using (4.5.17)

$$\|\widetilde{\nu}_s^m\|_{-1} \leqslant Ch_{m-1} + C(s+\sqrt{s})h_{m-1}^{3/2} + \sqrt{\frac{t_m - t_{m-1}}{N}} N^{\zeta} . \tag{4.5.19}$$

Since $N^{2\zeta-1/2} \leqslant h_{m-1}$, we deduce that $s_m^* = t_m$ if h_0 is small enough. Then using (4.5.14) we get

$$\|\widetilde{\nu}_{t_m}^m\|_{-1} \leqslant C\alpha h_{m-1} + Ch_{m-1}^{3/2} + \sqrt{\frac{t_m - t_{m-1}}{N}} N^{\zeta} . \tag{4.5.20}$$

Since $h_{m-1}^{3/2} \leqslant \alpha h_{m-1}$ for h_0 small enough, it leads us to (recall that $h_{m-1} = 2h_m$)

$$\|\widetilde{\nu}_{t_m}^m\|_{-1} \leqslant 4C\alpha h_m + \sqrt{\frac{t_m - t_{m-1}}{N}} N^{\zeta} . \tag{4.5.21}$$

If $m < m_f$, $\sqrt{\frac{t_m - t_{m-1}}{N}} N^{\zeta} \leqslant C\alpha h_m$ and thus $\|\widetilde{\nu}_{t_m}^m\|_{-1} \leqslant 5C\alpha h_m$. If $m = m_f$, $h_m \leqslant N^{2\zeta-1/2}$ and thus $\|\widetilde{\nu}_{t_m}^m\|_{-1} \leqslant 5C\alpha N^{2\zeta-1/2}$. We now have a good control on $\mu_{t_m} = q_{\widetilde{\psi}_{m-1}} + \widetilde{\nu}_{t_m}^m$, and project it with respect to $\widetilde{\psi}_m$ (writing $\mu_{t_m} = q_{\widetilde{\psi}_m} + \widetilde{\nu}_{t_m}^{m+1}$) to get a bound for $\|\widetilde{\nu}_{t_m}^{m+1}\|_{-1}$. We use the same decomposition as the proof of Proposition 4.3.3:

$$\begin{aligned} \widetilde{\nu}_{t_m}^{m+1} &= q_{\widetilde{\psi}_{m-1}} + \widetilde{\nu}_{t_m}^m - q_{\widetilde{\psi}_m} = P_{\widetilde{\psi}_m}^{\perp}[q_{\widetilde{\psi}_{m-1}} + \widetilde{\nu}_{t_m}^m - q_{\widetilde{\psi}_m}] \\ &= \left(P_{\widetilde{\psi}_m}^{\perp} - P_{\widetilde{\psi}_{m-1}}^{\perp}\right)[q_{\widetilde{\psi}_{m-1}} + \widetilde{\nu}_{t_m}^m - q_{\widetilde{\psi}_m}] + P_{\widetilde{\psi}_{m-1}}^{\perp}[q_{\widetilde{\psi}_{m-1}} - q_{\widetilde{\psi}_m}] + P_{\widetilde{\psi}_{m-1}}^{\perp}\widetilde{\nu}_{t_m}^m . \end{aligned} \tag{4.5.22}$$

Since the projection $\mathtt{p}$ is smouth, we get the bound

$$|\widetilde{\psi}_m - \widetilde{\psi}_{m-1}| = |\mathtt{p}(\mu_{t_m}) - \mathtt{p}(\mu_{t_{m_1}})| \leqslant C\|\mu_{t_m} - \mu_{t_{m_1}}\|_{-1} \leqslant C\|\widetilde{\nu}_{t_m}^m - \widetilde{\nu}_{t_{m-1}}^m\|_{-1} . \tag{4.5.23}$$

But (4.5.21) implies in particular that

$$\|\widetilde{\nu}_{t_m}^{m}\|_{-1} \leqslant C(1+4\alpha)h_{m-1}\,, \tag{4.5.24}$$

which implies, using also (4.5.23),

$$|\widetilde{\psi}_m - \widetilde{\psi}_{m-1}| \leqslant 2C(1+4\alpha)h_{m-1}\,. \tag{4.5.25}$$

Using similar arguments as in the proof of Proposition 4.3.3 (using in particular the smoothness of the projection $P_\psi^\perp$), we see that the two first terms of the right hand side in (4.5.22) are of order h_{m-1}^2. More precisely, there exists a constant $C'[\alpha]$ depending in α (increasing in α) such that

$$\|\widetilde{\nu}_{t_m}^{m+1}\|_{-1} \leqslant C'[\alpha]h_{m-1}^2 + C\|\widetilde{\nu}_{t_m}^{m}\|_{-1}\,. \tag{4.5.26}$$

So, since $\|\widetilde{\nu}_{t_m}^{m}\|_{-1} \leqslant 5C\alpha h_m$ for $m < m_f$ and $\|\widetilde{\nu}_{t_{m_f}}^{m_f}\|_{-1} \leqslant 5C\alpha N^{2\zeta-1/2}$, if h_0 and α are small enough we get $\|\widetilde{\nu}_{t_m}^{m+1}\|_{-1} \leqslant h_m$ for $m < m_f$ and $\|\widetilde{\nu}_{t_{m_f}}^{m_f+1}\|_{-1} \leqslant N^{2\zeta-1/2}$.

We have therefore shown that after a time of order $\log N$, the empirical measure comes at distance $N^{2\zeta-1/2}$ from from $q_{\widetilde{\psi}_{m_f}}$. This angle $\widetilde{\psi}_{m_f}$ corresponds to the angle Ψ_N in the Proposition 4.5.1. So it remains to prove that $\widetilde{\psi}_{m_f}$ converges to ψ_0 in probability as N goes to infinity. We decompose

$$|\widetilde{\psi}_{m_f} - \psi_0| \leqslant |\widetilde{\psi}_0 - \psi_0| + \sum_{m=1}^{m_f} |\widetilde{\psi}_m - \widetilde{\psi}_{m-1}|\,. \tag{4.5.27}$$

We restrict our study on the event $\Omega_N \bigcap \widetilde{W}_N$, whose probability tends to 1. Since $\|\mu_{t_0} - q_{\psi_0}\|_{-1} \leqslant h$ and the projection $\mathtt{p}$ is smouth, we get

$$|\widetilde{\psi}_0 - \psi_0| \leqslant Ch \tag{4.5.28}$$

and (4.5.25) implies (recall $h_0 = h$ and $h_{m-1} = 2h_m$)

$$|\widetilde{\psi}_m - \widetilde{\psi}_{m-1}| \leqslant C2^{1-m}h\,. \tag{4.5.29}$$

Consequently for C large enough $\mathbb{P}[|\widetilde{\psi}_{m_f} - \psi_0| > Ch] \to_{N\to\infty} 0$, which completes the proof of (4.5.1). The bound (4.5.2) is much rougher and it follows directly from the argument we have used for establishing (4.5.1). This completes the proof of Proposition 4.5.1 □

Proof of Proposition 4.5.2 The crucial issues are the gradient flow structure of (4.1.4) and its dissipativity properties. The gradient structure of (4.1.4) [9] implies that the functional

$$\mathcal{F}(p) := \frac{1}{2}\int_{\mathbb{S}} p(\theta)\log p(\theta)\,\mathrm{d}\theta - \frac{K}{2}\int_{\mathbb{S}}\int_{\mathbb{S}} p(\theta)\cos(\theta-\theta')p(\theta')\,\mathrm{d}\theta\,\mathrm{d}\theta'\,, \tag{4.5.30}$$

is non increasing along the time evolution. The dissipativity properties proven in [43, Theorem 2.1] show that for every $k \in \mathbb{N}$ and $a > 0$ we can find $\widetilde{t}$ such that $\|p_t\|_{C^k} < a$ for every $t \geq \widetilde{t}$. Therefore for any k there exists $\{t_n\}_{n=1,2,\dots}$ such that $t_{n+1} - t_n > 1$ and $\lim_n p_{t_n}$ exists in C^k and we call it p_∞. An immediate consequence is that $\lim_n \mathcal{F}(p_{t_n}) = \mathcal{F}(p_\infty)$. But we can go beyond by introducing the semigroup S_t associated to (4.1.4), by setting $S_{t'}p_t = p_{t+t'}$. [43, Theorem 2.2] implies the continuity of this semigroup in C^k, so that, since for $t \in [0,1]$ we have $t_n \leq t_n + t < t_{n+1}$, we obtain $\mathcal{F}(S_t p_\infty) = \mathcal{F}(p_\infty)$. Therefore

$\partial_t \mathcal{F}(S_t p_\infty) = 0$, but the condition $\partial_t \mathcal{F}(p_t) = 0$, for a solution of (4.1.4), directly implies that $\partial^2_\theta p_t = 2\partial_\theta(p_t J * p_t)$, which is the stationarity condition for (4.1.4). Therefore p_t is either q_ψ, for some ψ, or it coincides with $\frac{1}{2\pi}$ (see (4.1.5)-(4.1.6)).

Let us point out that if p_{t_n} converges to $\frac{1}{2\pi}$ then $\{p_t\}_{t>0}$ itself converges to $\frac{1}{2\pi}$. This is just because $\mathcal{F}\left(\frac{1}{2\pi}\right) > \mathcal{F}(q_\psi)$, so that if $\lim_n p_{t'_n} = q_\psi$ and $\lim_n p_{t_n} = \frac{1}{2\pi}$ then it suffices to choose n such that $\mathcal{F}(p_{t'_n}) < \mathcal{F}\left(\frac{1}{2\pi}\right)$ and m such that $t_m > t'_n$ to get $\mathcal{F}(p_{t'_n}) \ge \mathcal{F}(p_{t_m}) \ge \mathcal{F}\left(\frac{1}{2\pi}\right)$, which is impossible.

So we have seen that either $\lim_{t\to\infty} p_t = \frac{1}{2\pi}$ or all limit points are in M. The stronger result we need is the convergence also when the limit point is not $\frac{1}{2\pi}$. This result is provided by the nonlinear stability result [43, Therem 4.6] which says that if p_0 is in a neighborhood of M (the result is proven for a $\mathbb{L}^2$ neighborhood, which is much more than what we need here), then there exists ψ such that $\lim_{t\to\infty} p_t = q_\psi$ in C^k.

To complete the proof we need to characterize the portion of $\mathcal{M}_1$ which is attracted by $\frac{1}{2\pi}$, that is we need to identify the stable manifold of the unstable point with the set U in (4.1.7). But this is the content of [43, Proposition 4.4]. □

4.6 Proof of Theorem 4.1.1

The proof of Theorem 4.1.1 relies on the results of the previous sections and on a convergence argument of the process in the tangent space that we give here.

Proof of Theorem 4.1.1. First of all Proposition 4.5.1 takes care of the evolution up to time $N\widetilde{\varepsilon}_N = C\log N$ and provides an estimate on the closeness of the empirical measure to the manifold M that allows to apply directly Proposition 4.3.3 and then Proposition 4.4.1. Note that the iterative scheme that we have set up in Section 4.2.4 has been presented without asking ψ_0 not to be random or not to depend on N. In fact we start the iterative scheme at time $N\widetilde{\varepsilon}_N$ and from the random phase Ψ_N of Proposition 4.5.1 that converges in probability to the (non random) value ψ_0. Of course there is here an abuse of notation in the use of ψ_0, but notice actually that, by the rotation invariance of the system, we can actually consider without loss of generality that the empirical measure $\mu_{N,C\log N}$ has precisely the phase ψ_0. Moreover we make a time shift of $N\widetilde{\varepsilon}_N$, so that the phase is ψ_0 at time $T_0 = 0$. The result in Theorem 4.1.1 is given for times starting from $N\varepsilon_N$ and not $N\widetilde{\varepsilon}_N$, but as stated in Proposition 4.5.1, the empirical measure stays close to q_{ψ_0} in the time interval $[N\varepsilon_N, N\widetilde{\varepsilon}_N]$. Therefore we have the finite sequence of times $T_0, T_1, \ldots, T_n$, with the corresponding phases $\psi_0, \psi_1, \cdot, \ldots, \psi_n$ and we define ψ_t for every $t \in [0, T_n]$ by linear interpolation. We assume $T_n > \tau_f N$.

We then note that, in view of (4.3.43), the control on the phases, see Proposition 4.4.1, on the times $T_1, T_2, \ldots$ of our iteration scheme suffices not only to control the distance between the empirical measure $\mu_{N,t}$ and q_{ψ_t}, in the H_{-1} norm, for $t = T_k$, but for every $t \in [0, T_n]$. We are now ready to identify the process $W_{N,\cdot}$ of Theorem 4.1.1:

$$W_{N,\tau} := \frac{\psi_{\tau N} - \psi_0}{D_K}\,, \tag{4.6.1}$$

where we recall that $\tau \in [0, T_n/N]$. We are therefore left with showing that $W_{N,\cdot}$ converges to standard Brownian motion.

In proving the convergence to Brownian motion we apply Proposition 4.4.1 and replace the process $\psi_\cdot$ with the cadlag process $\psi_0 + M_{N,\cdot} \in D([0, T_n/N]; \mathbb{R})$ defined by

$$M_{N,\tau} := \sum_{k\in\mathbb{N}:\, T_k \le N\tau} \Delta M_{N,k}\,, \tag{4.6.2}$$

and

$$\Delta M_{N,k} := \frac{(Z^k_{T_k}, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q', q')_{-1,1/q}}. \tag{4.6.3}$$

It is straightforward to see that $M_{N,\cdot}$ is a martingale with respect to the filtration $\widetilde{\mathcal{F}}_\tau := \mathcal{F}_{\lfloor \tau T \rfloor / T}$, where $\mathcal{F}_\cdot$ is the natural filtration of $\{W^j_{N\cdot}\}_{j=1,\ldots,N}$: the martingale is actually in L^p, for every p, as the moment estimates is Section 4.3 show. We can now apply the Martingale Invariance Principle in the form given by [56, Corollary 3.24, Ch. VIII] to $M_{N,\cdot}$ for continuous time martingales: the hypotheses to verify in the case of piecewise constant cadlag martingales boil down to the variance convergence condition that for every $\tau \in [0, \tau_f]$

$$\lim_{N\to\infty} \sum_{k\in\mathbb{N}:\, T_k \le \tau N} \mathbb{E}\left[(\Delta M_{N,T_k})^2 \,\middle|\, \mathcal{F}_{T_{k-1}} \right] = \tau D_K^2 \,, \tag{4.6.4}$$

in probability, and the Lindeberg condition that for every $\varepsilon > 0$ in probability we have

$$\lim_{N\to\infty} \sum_{k\in\mathbb{N}:\, T_k \le \tau N} \mathbb{E}\left[(\Delta M_{N,T_k})^2 \,;\, \Delta M^2_{N,T_k} > \varepsilon \,\middle|\, \mathcal{F}_{T_{k-1}} \right] = 0 \,. \tag{4.6.5}$$

For what concerns (4.6.4) we have

$$\mathbb{E}\left[(\Delta M_{N,T_k})^2 \,\middle|\, \mathcal{F}_{T_{k-1}} \right] = \frac{1}{N \|q'\|^2_{-1,1/q}} \int_{T_{k-1}}^{T_k} \int_{\mathbb{S}} \left(f'_{\psi_{k-1},0}(\theta) \right)^2 \mu_{N,s}(\mathrm{d}\theta)\, \mathrm{d}s \,. \tag{4.6.6}$$

Now take the sum over k and use the uniform estimate (4.3.43) of Proposition 4.3.3 to replace the empirical measure with $q_{\psi_{T_{k-1}}}(\theta)\,\mathrm{d}\theta$. Since a direct computation shows that $\int_{\mathbb{S}} (f'(\theta)_{\psi,0})^2 q_\psi(\theta)\,\mathrm{d}\theta = 1$, (4.6.4) follows.

For what concerns (4.6.5) we remark that, by the Markov inequality, it suffices to show that

$$\lim_{N\to\infty} \sum_{k\in\mathbb{N}:\, T_k \le \tau N} \mathbb{E}\left[(\Delta M_{N,T_k})^4 \,\middle|\, \mathcal{F}_{T_{k-1}} \right] = 0 \,. \tag{4.6.7}$$

Actually one can show that there exists a non random constant C such that almost surely

$$\mathbb{E}\left[(\Delta M_{N,T_k})^4 \,\middle|\, \mathcal{F}_{T_{k-1}} \right] \le C \left(\frac{T}{N} \right)^2 . \tag{4.6.8}$$

This is an immediate consequence of (4.3.32), but of course, since we are projecting on q' and since we are just considering the fourth moment, a similar estimate can be easily obtained explicitly by proceeding like for (4.6.4) and by using the fact that $\|f'_{\psi,0}\|_\infty = \|f'_0\|_\infty < \infty$. Of course (4.6.7) follows from (4.6.8).

Therefore $M_{N,\cdot} \in D([0,\tau_f];\mathbb{R})$ converges in law to $W_\cdot/\|q'\|_{-1,1/q}$, where $W_\cdot$ is a standard Brownian motion. This is almost the result we want (recall that $D_K = 1/\|q'\|_{-1,1/q}$), since $M_{N,\cdot}/D_K$ differs from $W_{N,\cdot}$ just for the fact that they interpolate in a different way between the times T_k (where the coincide) and that in the case of $W_{N,\cdot}$ the convergence is in $C^0([0,\tau_f];\mathbb{R})$. But (4.6.8) guarantees that the sum of the fourth power of the jumps of $M_{N,\cdot}$ adds up to $O(T^2/N) = o(1)$ in probability, so the supremum of the jumps is $o(1)$, and therefore the convergence for $M_{N,\cdot} \in D([0,\tau_f];\mathbb{R})$ implies the convergence of $W_{N,\cdot} \in C^0([0,\tau_f];\mathbb{R})$. The proof of Theorem 4.1.1 is therefore complete. □

4.A The evolution in H_{-1}

In what follows we fix q in the invariant manifold M (see (4.1.6)). Unlike the rest of the paper here we do not identify q with q_ψ and then with ψ, so in particular we write L_q (and not L_ψ), $\mathcal{G}_t^q(\cdot)$ (and not $\mathcal{G}_t^\psi(\cdot)$ like in (4.2.12)), and so on. We work with the signed measure

$$\nu_{N,t}(\mathrm{d}\theta) := \mu_{N,t}(\mathrm{d}\theta) - q(\theta)\,\mathrm{d}\theta\,, \tag{4.A.1}$$

which can be seen as an element of H_{-1}. This is simply because it is the difference of two probability measures. In fact, if $\mu \in \mathcal{M}_1$, $\theta \mapsto \mu([0,\theta])$ is a primitive of μ and, by Remark 4.2.1, $\|\mu-\nu\|_{-1}^2 \le \int_{\mathbb{S}}(\mu([0,\theta])-\nu([0,\theta]))^2\,\mathrm{d}\theta \le 2\pi$. Therefore $\|\mu-\nu\|_{-1} \le \sqrt{2\pi}$: of course this quick argument needs to be cleaned up by first *smoothing* the measures. That is, we introduce an approximate identity $\phi_n \in C^\infty$ ($\phi_n \ge 0$, $\phi_n(\theta) = 0$ for $\theta \in [1/n, 2\pi-1/n]$, $\int_{\mathbb{S}}\phi_n = 1$ and $\lim_n \int_{\mathbb{S}} F\phi_n = F(0)$ for every $F \in C^0$). We then introduce the probability density $\theta \mapsto \mu_n(\theta) := \int_{\mathbb{S}}\phi_n(\theta-\theta')\mu(\mathrm{d}\theta')$ and verify that

$$\|\mu_n - \mu_m\|_{-1}^2 \le \frac{4}{\min(n,m)}\,, \tag{4.A.2}$$

so that $\lim_n \mu_n$ exists in H_{-1} (of course the limit exists also weakly and it is μ).

We aim at proving:

Proposition 4.A.1. *If $\{\varphi_t^{j,N}\}_{t\ge0,\, j=1,\ldots,N}$ solves* (4.1.1) *then $\nu_{N,\cdot} \in C^0([0,\infty);H_{-1})$ and we have*

$$\nu_{N,t} = \exp(tL_q)\nu_{N,0} - \int_0^t \exp((t-s)L_q)\partial\left((J*\nu_{N,s})\nu_{N,s}\right)\mathrm{d}s + Z_{N,t}\,, \tag{4.A.3}$$

where $Z_{N,t}$ is the limit in H_{-1} as $\tau \nearrow t$ of $Z_{N,t,\tau}$, where

$$Z_{N,t,\tau}(\theta) := \frac{1}{N}\sum_{j=1}^N \int_0^\tau \partial_{\theta'}\mathcal{G}_{t-s}^q(\theta,\varphi_s^{j,N})\,\mathrm{d}W_s^j \tag{4.A.4}$$

Moreover all the terms appearing in the right-hand side of (4.A.3)*, as functions of time, are in $C^0([0,\infty);H_{-1})$.*

Proof. For $(t,\theta) \mapsto F_t(\theta)$ in $C^{1,2}(\mathbb{R}^+\times\mathbb{S};\mathbb{R})$, from (4.1.1) we directly obtain

$$\begin{aligned}\int_{\mathbb{S}} F_t(\theta)\nu_{N,t}(\mathrm{d}\theta) &= \int_{\mathbb{S}} F_0(\theta)\nu_{N,0}(\mathrm{d}\theta) + \int_0^t\int_{\mathbb{S}}\left(L_q^*F_s\right)(\theta)\nu_{N,s}(\mathrm{d}\theta)\,\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{S}}\partial_sF_s(\theta)\nu_{N,s}(\mathrm{d}\theta)\,\mathrm{d}s + \int_0^t\int_{\mathbb{S}}\partial_\theta F_s(\theta)(J*\nu_{N,s})(\theta)\nu_{N,s}(\mathrm{d}\theta)\,\mathrm{d}s + Z_{N,t}^F\,, \end{aligned} \tag{4.A.5}$$

where

$$Z_{N,t}^F = \frac{1}{N}\int_0^t\sum_{j=1}^N \partial_\theta F_s(\theta)\Big|_{\theta=\varphi_s^{j,N}}\mathrm{d}W_s^j\,, \tag{4.A.6}$$

and L_q^* is the adjoint in $\mathbb{L}_0^2$ of L_q, that is

$$L_q^*v = \frac{1}{2}v'' + (J*q)v' - J*(qv') - \int_{\mathbb{S}}(J*q)v'\,, \tag{4.A.7}$$

for $v \in C^2(\mathbb{S};\mathbb{R})$ such that $\int_{\mathbb{S}} v = 0$

We sum up here some useful properties of L_q^*:

1. In [9] it is shown that the $\mathbb{L}^2_0$-norm is equivalent to the Dirichlet form norm of L_q: the squared Dirichlet form norm of u is $\|u\|^2_{-1,1/q} + (u,(-L_q)u)_{-1,1/q}$. On the other hand it is straightforward to see that the properties of L_q in $H_{-1,1/q}$, notably the fact that it is self-adjoint and that it has compact resolvent, still hold true in the space of the Dirichlet form. So L_q has compact resolvent in $\mathbb{L}^2_0$, which directly implies that L_q^* has compact resolvent and the very same spectrum (see e.g. [92, VI.5]).
2. Recall that we denote by $\{e_j\}_{j=0,1,\ldots}$ a complete set of eigenvectors of L_q which is orthonormal in $H_{-1,1/q}$ and observe that there is a unique solution f_j to
$$\mathtt{A}_q f_j(\theta) := -\partial_\theta\left(q(\theta)\partial_\theta f_j(\theta)\right) = e_j(\theta)\,, \tag{4.A.8}$$
such that $\int_{\mathbb{S}} f_j = 0$. More generally, $\mathtt{A}_q$ is a bijection from $\{u \in C^\infty : \int_{\mathbb{S}} u = 0\}$ to itself: in fact, $v = \mathtt{A}_q u$ is equivalent to $u' = -\mathcal{V}/q$ in our standard notations, which determines u since $\int_{\mathbb{S}} u = 0$. In particular $f_j' = -\mathcal{E}_j/q$ and $f_j \in C^\infty$, since e_j is C^∞, and one obtains
$$(f_i, e_j)_2 = \int_{\mathbb{S}} f_i e_j = -\int_{\mathbb{S}} f_i' \mathcal{E}_j = \int_{\mathbb{S}} \frac{\mathcal{E}_i\mathcal{E}_j}{q} = \delta_{i,j}\,. \tag{4.A.9}$$
By using the fact that $q(\cdot)$ is even, one verifies directly also that if e_j is even (respectively, odd) – recall from Section 4.2.1 that e_j is either even or odd – the f_j is even (respectively, odd) too.
3. By observing also that $L_q\mathtt{A}_q = \mathtt{A}_q L_q^*$ one verifies that $\{f_j\}_{j=0,1,\ldots}$ is a complete set of eigenfunctions for L_q^* and, of course, $L_q^* f_j = -\lambda_j f_j$.

Therefore for every $t > 0$ and $s \le t$ we can define $F_s(\theta) = (\exp((t-s)L_q^*)F)(\theta)$ for $F \in \mathbb{L}^2_0$ and standard parabolic regularity [35] results imply that $F_s(\cdot)$ is C^∞ for $s < t$ (in our case this can be proven directly by using the Fourier transform, like in [43], but for what follows we choose $F \in C^2$ and the regularity result is even more straightforward). By plugging this choice into (4.A.5) we obtain
$$\begin{aligned}\int_{\mathbb{S}} F(\theta)\nu_{N,t}(\,\mathrm{d}\theta) = &\int_{\mathbb{S}} (\exp(tL_q^*)F)(\theta)\nu_{N,0}(\,\mathrm{d}\theta)\\ &+ \int_0^t \int_{\mathbb{S}} \partial_\theta(\exp((t-s)L_q^*)F)(\theta)(J*\nu_{N,s})(\theta)\nu_{N,s}(\,\mathrm{d}\theta)\,\mathrm{d}s + Z^F_{N,t}\,.\end{aligned} \tag{4.A.10}$$

At this point we step to looking at $\nu_{N,t}$ as an element of H_{-1} and we reconsider (4.A.10) with this novel viewpoint.

First of all $\int_{\mathbb{S}} F(\theta)\nu_{N,t}(\,\mathrm{d}\theta) = \langle F, \nu_{N,t}\rangle_{1,-1}$, where $\langle\,\cdot\,,\,\cdot\,\rangle_{1,-1}$ is the duality between H_1 and H_{-1} (cf. Sec. 4.2.1). For the first term in the right-hand side we observe that, for $v \in H_{-1}$ we have $\langle \exp(tL_q^*)F, v\rangle_{1,-1} = \langle F, \exp(tL_q)v\rangle_{1,-1}$: this is because this relation holds when $v \in \mathbb{L}^2_0$ (in this case the duality can be replaced by the $\mathbb{L}^2$ scalar product) and because one can choose a sequence $\{v_n\}_{n=1,2,\ldots}$, $v_n \in \mathbb{L}^2_0$ such that $v_n \to v$ in H_{-1} (one can choose $v_n = \phi_n * v$) so that
$$\langle \exp(tL_q^*)F, v\rangle_{1,-1} = \lim_n (F, \exp(tL_q)v_n)_2 = \langle F, \exp(tL_q)v\rangle_{1,-1}\,, \tag{4.A.11}$$
where we have used the continuity properties of the duality and of the semigroup operator.

For the second term in the right-hand side of (4.A.10) we write
$$\begin{aligned}\int_0^t \int_{\mathbb{S}} \partial_\theta(\exp((t-s)L_q^*)F)(\theta)(J*\nu_{N,s})(\theta)\nu_{N,s}(\,\mathrm{d}\theta)\,\mathrm{d}s =&\\ \int_0^t \langle (J*\nu_{N,s})\partial\exp((t-s)L_q^*)F, \nu_{N,s}\rangle_{1,-1}\,\mathrm{d}s\,,&\end{aligned} \tag{4.A.12}$$

We now introduce $v_{n,s} := \phi_n * \nu_{N,s}$ so that for every $s \in [0,t)$

$$\begin{aligned}\langle (J*\nu_{N,s})\partial \exp((t-s)L_q^*)F, \nu_{N,s}\rangle_{1,-1} &= -\lim_n \left(F, \exp((t-s)L_q)\partial((J*\nu_{N,s})v_{n,s})\right)_2 \\ &= -\langle F, \exp((t-s)L_q)\partial((J*\nu_{N,s})\nu_{N,s})\rangle_{1,-1},\end{aligned} \tag{4.A.13}$$

where in the last step we have used the fact that $\exp((t-s)L_q)$ is a continuous operator from H_{-2} to H_{-1} (Lemma 4.A.2). Notice moreover that we have

$$|\left(F, \exp((t-s)L_q)\partial((J*\nu_{N,s})v_{n,s})\right)_2| \le \|\phi_n\|_1 \|\partial((J*\nu_{N,s})\partial \exp((t-s)L_q^*)F)\|_2 \|\nu_{N,s}\|_{-1}\,, \tag{4.A.14}$$

and, since $J(\cdot) = -K\sin(\cdot)$, one sees that this expression is bounded by a constant times $\|F''\|_2$, uniformly in n and $s \le t$. Such a bound tells us that one can exchange limit and integration in

$$\int_0^t \lim_n \Big(F, \exp((t-s)L_q)\partial((J*\nu_{N,s})v_{n,s})\Big)_2 \,\mathrm{d}s\,, \tag{4.A.15}$$

and then, for fixed n one can of course exchange integral in $\mathrm{d}s$ and integral in $\mathrm{d}\theta$. At this point we appeal again to Lemma 4.A.2 that guarantees that $\int_0^t \exp((t-s)L_q)\partial((J*\nu_{N,s})v_{n,s})\,\mathrm{d}s$ converges, in H_{-1}, to $\int_0^t \exp((t-s)L_q)\partial((J*\nu_{N,s})\nu_{N,s})\,\mathrm{d}s$: note in fact that $\|\partial((J*\nu_{N,s})v)\|_{-2} \le c_J\|v\|_{-1}$ so that (by Lemma 4.A.2)

$$\begin{aligned}\left\| \int_0^t \exp((t-s)L_q)\partial((J*\nu_{N,s})(v_{n,s}-v_{n',s})\,\mathrm{d}s \right\|_{-1} &\le \\ c_J C\int_0^t \left(1+\frac{1}{\sqrt{t-s}}\right)\|v_{n,s}-v_{n',s}\|_{-1}\,\mathrm{d}s\,,\end{aligned} \tag{4.A.16}$$

and the right-hand side vanishes for $\min(n,n') \to \infty$. Therefore we obtain

$$\begin{aligned}\int_0^t \int_{\mathbb{S}} \partial_\theta(\exp((t-s)L_q^*)F)(\theta)(J*\nu_{N,s})(\theta)\nu_{N,s}(\mathrm{d}\theta)\,\mathrm{d}s &= \\ \langle F, \int_0^t \exp((t-s)L_q)\partial((J*\nu_{N,s})\nu_{N,s})\,\mathrm{d}s\rangle_{1,-1}\,.\end{aligned} \tag{4.A.17}$$

We are left with the last term in (4.A.10). It is now useful to use the kernel of the L_q-semigroup in $\mathbb{L}_0^2$

$$\mathcal{G}_s^q(\theta,\theta') := \sum_{l=0}^\infty \exp(-s\lambda_l)e_l(\theta)f_l(\theta')\,, \tag{4.A.18}$$

so that

$$(u, \exp(sL_q)v)_2 = \big(\exp(sL_q^*)u, v\big)_2 = \int_{\mathbb{S}}\int_{\mathbb{S}} u(\theta)\mathcal{G}_s^q(\theta,\theta')v(\theta')\,\mathrm{d}\theta\,\mathrm{d}\theta'\,. \tag{4.A.19}$$

Note also that, for $s>0$, $\mathcal{G}_s^q$ is C^∞ in both variables, by the standard parabolic regularity results we have mentioned above. So, for every $\tau < t$, $\theta \mapsto Z_{N,t,\tau}(\theta)$ (recall (4.A.4)) is well defined and smooth in θ. But Lemma 4.A.4 tells us that $\lim_{\tau\nearrow t} Z_{N,t,\tau})$ exists in H_{-1}. If we call the limit $Z_{N,t})$ we directly see that (recall (4.A.6))

$$Z_{N,t}^F = \langle F, Z_{N,t}\rangle_{1,-1}\,. \tag{4.A.20}$$

Therefore we have shown that (4.A.10) implies the validity of (4.A.3) if we take the duality with respect to an arbitrary $F \in C^2$. But we have also shown that every term in (4.A.3) is in H_{-1}, therefore the equation extends to $F \in H_1$ and (4.A.3) is proven.

The continuity claimed in the statement follows by the continuity of the three terms in the right-hand side of (4.A.3). The continuity of the first term is immediate from the properties of the semigroup. The continuity of the second term follows from a direct estimate by applying both bounds in Lemma 4.A.2. Finally the continuity of the third term is claimed in Lemma 4.A.4. The proof of Proposition 4.A.1 is therefore complete. □

Lemma 4.A.2. *For $\tau > 0$ the operator $\exp(\tau L_q)$ extends to a bounded operator from H_{-2} to H_{-1} and there exists $C > 0$ such that for every $\tau > 0$*

$$\|\exp(\tau L_q)u\|_{-1} \leq C\left(1 + \frac{1}{\sqrt{\tau}}\right)\|u\|_{-2}, \tag{4.A.21}$$

and such that for every $\varepsilon \in (0, 1/2)$ we have

$$\|\exp((\tau+\delta)L_q)u - \exp(\tau L_q)u\|_{-1} \leq C\delta^{\varepsilon}\left(1 + \frac{1}{\tau^{\varepsilon+1/2}}\right)\|u\|_{-2}, \tag{4.A.22}$$

for every $\tau > 0$ and $\delta \geq 0$.

Proof. We introduce the interpolation spaces associated to L_q that is the (Hilbert) spaces

$$V^m := \left\{u = \sum_{k=0}^{\infty} u_k e_k, \quad \sum_{k=0}^{\infty}(1+\lambda_k)^m u_k^2 < \infty\right\}, \tag{4.A.23}$$

associated with the norms

$$\|u\|_{V^m}^2 := \|(1-L_q)^{m/2}u\|_{-1,1/q}^2 = \sum_{k=0}^{\infty}(1+\lambda_k)^m u_k^2 . \tag{4.A.24}$$

It is proven in [44, Remark A.1] that the norms $\|.\|_{V^n}$ and $\|.\|_{n-1}$ are equivalent. This equivalence can also be deduced from Remark 4.B.3. In particular $\|.\|_{V^{-1}}$ and $\|.\|_{-2}$ are equivalent, so we will prove (4.A.2) with $\|.\|_{-1,1/q}$ and $\|.\|_{V^{-1}}$. For all $u = \sum_{k=0}^{\infty} u_k e_k$, we extend $e^{\tau L_q}u$ as

$$e^{\tau L_q}u = \sum_{k=0}^{\infty} e^{-\lambda_k \tau} u_k e_k , \tag{4.A.25}$$

and we deduce

$$\|e^{\tau L_q}u\|_{-1,1/q}^2 = \sum_{k=0}^{\infty}(1+\lambda_k)e^{-2\lambda_k\tau}\frac{u_k^2}{1+\lambda_k} . \tag{4.A.26}$$

But if we define $f(y) := (1+y)e^{-2y\tau}$, it is easy to see that for all $y \geqslant 0$, there exist C such that $f(y) \leqslant C^2\left(1+\frac{1}{\sqrt{\tau}}\right)^2$, which with (4.A.24). and (4.A.26) gives the first inequality.

For the second inequality we make a similar spectral decomposition and we obtain

$$\|e^{(\tau+\delta)L_q}u - e^{\tau L_q}u\|_{-1,1/q}^2 = \sum_{k=0}^{\infty}(1+\lambda_k)e^{-2\lambda_k\tau}\left(1-\exp(-\delta\lambda_k)\right)^2\frac{u_k^2}{1+\lambda_k} . \tag{4.A.27}$$

We then use $(1-\exp(-x)) \leq x^{\varepsilon}$ for $x \geq 0$ and $(1+x)x^{2\varepsilon}\exp(-x\tau) \leq C^2(1+\tau^{-\varepsilon-1/2})^2$, for a suitable C which can be chosen independent of $\varepsilon \in (0/1/2)$. □

Lemma 4.A.3. *For all $u, v \in H_{-1}$, there exists $C > 0$ such that*

$$\|\partial_\theta(uJ * v)\|_{H_{-2}} \leqslant C\|u\|_{H_{-1}}\|v\|_{H_{-1}}. \tag{4.A.28}$$

Proof. For this proof it is practical to write the H_{-1}-norms by using the Fourier coefficients. In fact if $u \in H_{-s}$, here $s = 1$ or $s = 2$, we can define $u_n = \langle u, b_n\rangle$, where $b_n(\theta) = \exp(in\theta)/2\pi$ (note that $u_0 = 0$) and $\theta \mapsto \sum_{n\in\mathbb{Z}:|n|\leq N} u_n \exp(in\theta)$ converges as $N \to \infty$ in H_s to u. Moreover we have

$$\|u\|_{-s} := \left(\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\frac{u_m^2}{m^{2s}}\right)^{1/2}. \tag{4.A.29}$$

Since $J(\theta) = -K\sin(\theta)$, a direct calculation gives

$$\partial_\theta(uJ * v) = K\pi\left[(m-1)v_{-1}\sum_{m\in\mathbb{Z}} e^{i(m-1)\theta}u_m - (m+1)v_1\sum_{m\in\mathbb{Z}} e^{i(m+1)\theta}u_m\right], \tag{4.A.30}$$

from which we extract

$$\begin{aligned}\|\partial_\theta(uJ * v)\|_{-2}^2 &= \frac{K^2\pi}{2}\sum_{m\in\mathbb{Z},m\neq 0} m^{-4}\left|m(v_{-1}u_{m+1} - v_1u_{m-1})\right|^2 \\ &\leqslant K^2\pi\max(|v_{-1}|^2, |v_1|^2)\sum_{m\in\mathbb{Z},m\neq 0} m^{-2}(u_{m-1}^2 + u_{m+1}^2) \\ &\leqslant 4K^2\pi\|v\|_{-1}^2\|u\|_{-1}^2.\end{aligned} \tag{4.A.31}$$

□

Lemma 4.A.4. *The almost sure limit of $Z_{N,t,\tau}$ as $\tau \nearrow t$ exists in H_{-1} and, if we call the limit point $Z_{N,t}$, we can choose a continuous version of $Z_{N,\cdot}$, that is $Z_{N,\cdot} \in C^0([0,\infty); H_{-1})$.*

Proof. The claim follows from the same estimates as the one that we have obtained for the proof of Lemma 4.3.1, which are however substantially more precise than what we need here: recall that now N is fixed, while in Section 4.3 one of the crucial points is to follow the N dependence of the results. Therefore we will not go through the arguments in detail, but we just point out that that one goes from $Z_{N,t,\tau}$ to $Z_{t,t'}^k$, see (4.3.2) by making obvious changes. So, in particular, proceeding like for (4.3.25) we easily gets

$$\mathbb{E}\left[\|Z_{N,t,\tau} - Z_{N,t,\tau'}\|_{-1}^{2m}\right] \leq Ch_1^m(|\tau - \tau'|), \tag{4.A.32}$$

where C depends on N and m and $0 \leq \tau, \tau' < t$. An estimate like (4.A.32) implies almost sure Hölder continuity of $Z_{N,t,\cdot}$, by a direct application of Kolmogorov continuity Lemma [106] or by using the Garsia-Rodemich-Rumsey Lemma (Lemma 4.3.2). That is, there exists a (positive) random variable X and a positive constant $c > 0$ such that

$$\|Z_{N,t,\tau} - Z_{N,t,\tau'}\|_{-1} \leq X\,|\tau - \tau'|^c, \tag{4.A.33}$$

for every $0 \leq \tau, \tau' < t$. Therefore the almost sure limit of $Z_{N,t,\tau}$, as $\tau \nearrow t$, exists.

The continuity of the limit follows in the same way, this time using also (4.3.31). Actually, in the proof of Lemma 4.3.1 we use Lemma 4.A.4 only to define Z_t^k as almost sure limit in H_{-1}: the proof of continuity is strictly contained in the argument that starts from (4.3.32) and goes till the end of that proof (but, once again, that proof is substantially more informative and involved, since it follows the N-dependence). □

4.A.1 Second order estimates of the projection

As anticipated in § 4.2.3 our approach requires a control up to and including the second order for the projection map $p(\cdot)$ (recall § 4.2.2 for the definition). The expansion is with respect to the H_{-1} distance from the manifold M.

Lemma 4.A.5. *For all $q = q_\psi \in M$ and $h \in H_{-1}$ with $\|h\|_{-1} < \sigma$, we have*

$$p(q+h) \;=\; \psi - \frac{(h,q')_{-1,1/q}}{(q',q')_{-1,1/q}}\left(1 - \frac{1}{2\pi I_0^2(2Kr)}\frac{(h,(\log q)'')_{-1,1/q}}{(q',q')_{-1,1/q}}\right) + O(\|h\|_{-1}^3)\,. \quad (4.A.34)$$

Proof. For h as in the statement we have that

$$(q_\psi + h - q_{\psi+\varepsilon}, q'_{\psi+\varepsilon})_{-1,1/q_{\psi+\varepsilon}} \;=\; 0\,, \quad (4.A.35)$$

for $\varepsilon := p(q_\psi + h) - \psi$. Since $p(\cdot)$ is smooth, we have $\varepsilon = O(\|h\|_{-1})$. By expanding $q_{\psi+\varepsilon}$ with respect to ε we see that (4.A.35) implies

$$\left(h + \varepsilon q'_\psi - \frac{\varepsilon^2}{2} q''_\psi, q'_{\psi+\varepsilon}\right)_{-1,1/q_{\psi+\varepsilon}} \;=\; O(\varepsilon^3)\,. \quad (4.A.36)$$

Let us rewrite (4.A.36) more explicitly (recall Remark 4.2.3) as

$$\int_{\mathbb{S}} \left(\mathcal{H}(\theta) + \varepsilon q_\psi(\theta) - \frac{\varepsilon^2}{2} q'_\psi(\theta)\right)\left(1 - \frac{1}{2\pi I_0^2(2Kr)}\frac{1}{q_{\psi+\varepsilon}(\theta)}\right) \mathrm{d}\theta \;=\; O(\varepsilon^3)\,, \quad (4.A.37)$$

where $\mathcal{H}$ is the primitive of h such that $\int_{\mathbb{S}} \frac{\mathcal{H}}{q_\psi} = 0$. At this point we expand also $q_{\psi+\varepsilon}$ with respect to ε and, using $\varepsilon = O(\|h\|_{-1})$, the parity of $q_\psi(\cdot + \psi)$ and Remark 4.2.3, we get to

$$(h, q'_\psi)_{-1,1/q_\psi} + \varepsilon (q'_\psi, q'_\psi)_{-1,1/q_\psi} + \varepsilon \frac{1}{2\pi I_0^2(2Kr)} (h, (\log q)'')_{-1,1/q_\psi} \;=\; O\left(\|h\|_{-1}^3\right)\,. \quad (4.A.38)$$

Now it suffices to solve this equation for ε and perform one last Taylor expansion. □

4.B Spectral estimates

The aim of this section is to find approximations of the eigenvalues and eigenfunctions of the operators L_ψ for large eigenvalues. In such a regime we expect the Laplacian to dominate and the spectrum of L_ψ should get close to the one of the Laplacian (as long as we deal with large eigenvalues). These are standard estimates, developed for example in [72] that we follow, but we could not find in the literature the result for the non-local operators we consider. Without loss of generality, we can focus on L_0. We have

$$L_0 u \;=\; \frac{1}{2}u'' - (uJ*q_0 + q_0 J*u)' \;=\; \frac{1}{2}u'' - (J*q_0)u' - (J*q_0')u - q_0' J*u - q_0 J'*u\,. \quad (4.B.1)$$

We make a change of variable to get rid of the coefficient of order 1: if we define

$$u \;==\; \sqrt{q_0}\, y\,, \quad (4.B.2)$$

and we observe that $\sqrt{q} = e^{\widetilde{J}*q_0}$, with $\widetilde{J}(\theta) := K\cos(\theta)$, then we get

$$u' \;=\; \sqrt{q_0}\, y' + (J*q_0)\sqrt{q_0}\, y\,, \quad (4.B.3)$$

$$u'' = \sqrt{q_0}y'' + 2(J*q_0)\sqrt{q_0}y' + (J*q_0')\sqrt{q_0}y + (J*q_0)^2\sqrt{q_0}y\,, \tag{4.B.4}$$

and these two last equations together with (4.B.1) give

$$\begin{aligned} L_{q_0}\sqrt{q_0}y, = \ & \frac{1}{2}[\sqrt{q_0}y'' + 2(J*q_0)\sqrt{q_0}y' + (J*q_0')\sqrt{q_0}y + (J*q_0)^2\sqrt{q_0}y] \\ & - (J*q_0)[\sqrt{q_0}y' + (J*q_0)\sqrt{q_0}y] - (J*q_0')\sqrt{q_0}y - q_0'J*(\sqrt{q_0}y) - q_0J'*(\sqrt{q_0}y) \end{aligned} \tag{4.B.5}$$

which leads, after simplification, to the new operator

$$\widetilde{L}y := \frac{1}{2}y'' - m(y)\,, \tag{4.B.6}$$

where we have set

$$m(y) := \frac{1}{2}((J*q_0)^2 + J*q_0')y + \frac{q_0'}{\sqrt{q_0}}J*(\sqrt{q_0}y) + \sqrt{q_0}J'*(\sqrt{q_0}y)\,. \tag{4.B.7}$$

Of course $m(y)$ is a function and when we want to make explicit the θ-dependence we use $m_\theta(y)$. Since the operator L_0 is negative, we are interested in couples (ρ, y) solution of

$$\widetilde{L}y = -\rho^2 y\,, \tag{4.B.8}$$

where ρ is a positive real number. The method of variation of the parameters shows that such solutions exist (for all $\rho > 0$ if we do not restrict the study to the 2π-periodic eigenfunctions of $\widetilde{l}$) and are of the form

$$y(\theta) = c_1 e^{\sqrt{2}\rho i\theta} + c_2 e^{-\sqrt{2}\rho i\theta} - \frac{1}{\sqrt{2}\rho}\int_0^\theta G(\theta, \theta', \rho) m_{\theta'}(y)\,\mathrm{d}\theta' \tag{4.B.9}$$

where

$$G(\theta, \theta', \rho) = ie^{\sqrt{2}\rho i(\theta-\theta')} - ie^{-\sqrt{2}\rho i(\theta-\theta')}\,. \tag{4.B.10}$$

We define y_1 the solution such that $c_1 = 1$, $c_2 = 0$, and y_2 the solution such that $c_1 = 0$, $c_2 = 1$. In what follows we start by getting a first estimate of the eigenfunctions y_1 and y_2 with respect to $\rho \longrightarrow \infty$. This estimate implies a first estimate of the eigenvalue $-\lambda = -\rho^2$, and this leads to a new approximation of the eigenfunctions, and thus a new approximation of $-\lambda$. This procedure can be repeated recursively, but for us two steps will suffice.

Lemma 4.B.1. *For each $\lambda > 0$, there exist y_1 and y_2 independent (non necessarily periodic) eigenfunctions of $\widetilde{L}$ associated to $-\lambda$ such that (recall that ρ=$\sqrt{\lambda}$):*

$$y_1(\theta, \rho) = e^{\sqrt{2}\rho i\theta} + O\left(\frac{1}{\rho}\right)\,, \tag{4.B.11}$$

$$y_2(\theta, \rho) = e^{-\sqrt{2}\rho i\theta} + O\left(\frac{1}{\rho}\right)\,, \tag{4.B.12}$$

$$y_1'(\theta, \rho) = \sqrt{2}\rho i e^{\sqrt{2}\rho i\theta} + O(1)\,, \tag{4.B.13}$$

$$y_2'(\theta, \rho) = -\sqrt{2}\rho i e^{-\sqrt{2}\rho i\theta} + O(1)\,, \tag{4.B.14}$$

where $\theta \in [0, 2\pi]$ and $O(\cdot)$ is as ρ tends to infinity (and we stress that here and below the $O(\cdot)$ term does not depend on θ or, equivalently, it is uniform in $\theta \in [0, 2\pi]$).

Proof. We prove the result for y_1. The proof for y_2 is similar. We define

$$A_0(\theta, \theta', v) \;=\; -\frac{1}{\sqrt{2}\rho} G(\theta, \theta', \rho) m_{\theta'}(v) \mathbf{1}_{\theta' < \theta} \tag{4.B.15}$$

so that for $\theta \in [0, 2\pi]$,

$$y_1(\theta) \;=\; e^{\sqrt{2}\rho i \theta} + \int_0^{2\pi} A_0(\theta, \theta', y_1)\, \mathrm{d}\theta' \,. \tag{4.B.16}$$

The expression for y_1; cf. (4.B.9), can be iterated arbitrarily many times and it leads to a series expression for y_1, at least for ρ sufficiently large. To see this set $f_0(\theta) := e^{\sqrt{2}\rho i \theta}$ and observe that

$$\begin{aligned} y_1(\theta_0) \;=\; & f_0(\theta_0) + \sum_{j=1}^{m} \int_0^{2\pi} \cdots \int_0^{2\pi} A_0(\theta_0, \theta_1, A_0(\theta_1, \theta_2, \cdots A_0(\theta_{i-1}, \theta_i, f_0) \cdots))\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m \\ & + \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} A_0(\theta_0, \theta_1, A_0(\theta_1, \theta_2, \cdots A_0(\theta_m, \theta_{m+1}, y_1) \cdots))\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{m+1} \,. \end{aligned} \tag{4.B.17}$$

One directly verifies that there exists $C = C(K)$ such that for θ, $\theta' \in [0, 2\pi]$,

$$|A_0(\theta, \theta', v)| \;\leqslant\; \frac{C}{\rho} \|v\| \,, \tag{4.B.18}$$

where $\|v\| := \sup_{\theta \in [0,2\pi]} |v(\theta)|$. From (4.B.17) and using $\|f_0(\cdot)| \equiv 1$ we see that

$$\|y_1\| \le 1 + \sum_{j=1}^{m} \left(\frac{2\pi C}{\rho} \right)^m + \left(\frac{2\pi C}{\rho} \right)^{m+1} \|y_1\| \,, \tag{4.B.19}$$

so for $\rho > 2\pi C$ we see that $\|y_1\| < \infty$ and we have a series expression for y_1, from which we directly obtain (4.B.11).

To deal with y_1' we take the derivative of both sides of (4.B.9) with $c_1 = 1$ and $c_2 = 0$, so that

$$y_1'(\theta) \;=\; \sqrt{2}\rho i e^{\sqrt{2}\rho i \theta} - \frac{1}{\sqrt{2}\rho} \int_0^{\theta} \partial_\theta G(\theta, \theta', \rho) m_{\theta'}(y_1)\, \mathrm{d}\theta' \,. \tag{4.B.20}$$

We define the new kernel

$$A_1(\theta, \theta', v) \;:=\; -\frac{1}{\sqrt{2}\rho^2} \partial_\theta G(\theta, \theta', \rho) m_{\theta'}(v) \mathbf{1}_{\theta' < \theta} \,, \tag{4.B.21}$$

so we can write

$$\frac{1}{\rho} y_1'(\theta) \;=\; \sqrt{2} i e^{\sqrt{2}\rho i \theta} + \int_0^{2\pi} A_1(\theta, \theta', y_1)\, \mathrm{d}\theta' \,. \tag{4.B.22}$$

Also A_1 verifies

$$|A_1(\theta, \theta', v)| \;\leqslant\; \frac{C}{\rho} \sup_{\theta \in [0,2\pi]} |v(\theta)| \,, \tag{4.B.23}$$

for a suitable $C = C(K)$ and the same argument as above gives

$$\frac{1}{\rho} y_1'(\theta) \;=\; \sqrt{2} i e^{\sqrt{2}\rho i \theta} + O\left(\frac{1}{\rho} \right) \,, \tag{4.B.24}$$

which is equivalent to (4.B.12). □

Lemma 4.B.2. *There exists $l_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ the eigenvalues of L_0 satisfy*

$$\lambda_{l_0+2p} = \frac{p^2}{2} + O(\sqrt{p}), \tag{4.B.25}$$

$$\lambda_{l_0+2p+1} = \frac{p^2}{2} + O(\sqrt{p}). \tag{4.B.26}$$

Remark 4.B.3. *An immediate consequence of Lemma 4.B.2 and of the basic properties of L_0 is that there exist $C > 1$ such that for $j = 0, 1, \ldots$.*

$$\frac{j^2}{C} \leq \lambda_j \leq Cj^2. \tag{4.B.27}$$

Proof. Let y_1 and y_2 the eigenfunctions of $\widetilde{L}$ given by Lemma 4.B.1 associated to the eigenvalue $-\lambda = -\rho^2$. As a linear combination of y_1 and y_2 is 2π-periodic, the following determinant is equal to zero:

$$\begin{vmatrix} y_1(2\pi) - y_1(0) & y_2(2\pi) - y_2(0) \\ y_1'(2\pi) - y_1'(0) & y_2'(2\pi) - y_2'(0) \end{vmatrix} = 0. \tag{4.B.28}$$

Lemma 4.B.1 implies

$$\begin{vmatrix} e^{2\sqrt{2}\pi\rho i} - 1 + O\left(\frac{1}{\rho}\right) & e^{-2\sqrt{2}\pi\rho i} - 1 + O\left(\frac{1}{\rho}\right) \\ \sqrt{2}\rho i(e^{2\sqrt{2}\pi\rho i} - 1) + O(1) & -\sqrt{2}\rho i(e^{-2\sqrt{2}\pi\rho i} - 1) + O(1) \end{vmatrix} = 0, \tag{4.B.29}$$

and thus we get

$$|e^{2\sqrt{2}\pi\rho i} - 1|^2 = O\left(\frac{1}{\rho}\right). \tag{4.B.30}$$

We deduce that there exits $k \in \mathbb{N}$ such that

$$\rho = \frac{k}{\sqrt{2}} + O\left(\frac{1}{\sqrt{k}}\right). \tag{4.B.31}$$

Reciprocally, all ρ satisfying (4.B.31) satisfies (4.B.29), so the Lemma follows. □

Proposition 4.B.4. *There exists $l_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ the eigenvalues of L_0 satisfy*

$$\lambda_{l_0+2p} = \frac{p^2}{2} - \frac{K^2r^2}{8} + O\left(\frac{1}{p}\right), \tag{4.B.32}$$

$$\lambda_{l_0+2p+1} = \frac{p^2}{2} - \frac{K^2r^2}{8} + O\left(\frac{1}{p}\right), \tag{4.B.33}$$

and any eigenfunction of L_0 associated to λ_{l_0+2p} or λ_{l_0+2p+1} is, up to a correction of order $1/p^2$, a linear combination of the two functions $q_0^{1/2}v_{1,l_0+p}$ and $q_0^{1/2}v_{2,l_0+p}$, where

$$\begin{aligned} v_{1,l_0+p}(\theta) &= \cos(p\theta) - \frac{\sin(p\theta)}{p}\left[\frac{Kr}{2}\sin(\theta) + \frac{K^2r^2}{8}\sin(2\theta)\right], \\ v_{2,l_0+p}(\theta) &= \sin(p\theta) + \frac{\cos(p\theta)}{p}\left[\frac{Kr}{2}\sin(\theta) + \frac{K^2r^2}{8}\sin(2\theta)\right]. \end{aligned} \tag{4.B.34}$$

From Proposition 4.B.4 one can directly extract some important conclusions: let us give them before the proof of the proposition.

Corollary 4.B.5. *There exists* $l_0 \in \mathbb{N}$ *such that for all* $p \in \mathbb{N}$ *and* $\psi \in \mathbb{S}$, *the unitary (in* $H_{-1,1/q_\psi}$*) eigenfunctions* e_{ψ,l_0+2p} *and* e_{ψ,l_0+2p+1} *of* L_ψ *are up to a correction of order* $1/p$ *a bounded (with respect to* p*) linear combination of* $\theta \mapsto pq_\psi^{1/2}(\theta)v_{1,l_0+p}(\theta-\psi)$ *and* $\theta \mapsto pq_\psi^{1/2}(\theta)v_{2,l_0+p}(\theta-\psi)$ *(see Proposition 4.B.4 for the definition of* $v_{1,l}$ *and* $v_{2,l}$*).*

Proof. We set $\psi = 0$ without loss of generality. Proposition 4.B.4 tells us that the normalized eigenfunctions of L_0 can be written either as

$$c_p\left(\cos(p\theta) - \frac{\sin(p\theta)}{p}\left[\frac{Kr}{2}\sin(\theta) + \frac{K^2r^2}{8}\sin(2\theta)\right] + r_p(\theta)\right) \tag{4.B.35}$$

where $r_p(\theta) = O(1/p^2)$ and c_p is the normalizing constant, or with the analogous expression coming from the second line in (4.B.34) (but we will deal only with (4.B.35) because the other case is treated analogously). To estimate c_p let us observe that the first two addends in (4.B.35) are in H^{-1} (since they are smooth, it suffices to remark that their integral from 0 to 2π is zero), so $r_p \in H_{-1}$, since the eigenfunction is: of course r_p is smooth, since the eigenfunction is. Now we claim that the $H_{-1,1/q}$ norm of $\cos(p\cdot)$, that is the first addendum, is proportional to $1/p$, apart for a correction that is beyond all orders in $1/p$, while the norm of the two other terms is $O(1/p^2)$. In fact if we set $u(\theta) := \cos(p\theta)$, then $\mathcal{U}(\theta) = \sin(p\theta)/p$ so

$$\|u\|_{-1,1/q} = \frac{1}{p}\sqrt{\int_{\mathbb{S}} \frac{1-\cos(2p\theta)}{2q(\theta)}\,\mathrm{d}\theta}\,. \tag{4.B.36}$$

If we use the standard estimate

$$I_k(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(k\theta)e^{x\cos(\theta)}\,\mathrm{d}\theta = \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+k+1)}\left(\frac{x}{2}\right)^{2m+k} \leqslant \frac{C_x}{(k!)^{1/2}}\,. \tag{4.B.37}$$

we readily see that

$$\int_{\mathbb{S}}\frac{\cos(2p\theta)}{q(\theta)}\,\mathrm{d}\theta = O\left(\frac{1}{\sqrt{(2p)!}}\right)\,. \tag{4.B.38}$$

On the other hand

$$\int_{\mathbb{S}}\frac{1}{q(\theta)}\,\mathrm{d}\theta = \left(2\pi I_0(2Kr)\right)^2\,, \tag{4.B.39}$$

so $\|u\|_{-1,1/q}$ is equal to $c(K)/p$, $c(K) := \sqrt{2}\pi I_0(2Kr)$, up to a correction that decays faster than any power of $1/p$.

For the second addendum it suffices to observe that it can be rewritten as a linear combination of terms of $\cos(p'\theta)$, with $|p-p'| = 1$ and 2. But then the computation is very similar to the one that we have done for the first addendum (or, easier, one can explicitly compute the H_-1 norm, without weight). Therefore this term is $O(1/p^2)$.

For the third addendum we recall that $|r_p(\theta)| \le C/p^2$, so that if we set $\mathcal{R}(\theta) := \int_0^\theta r_p(\theta')\,\mathrm{d}\theta'$, we have $\|\mathcal{R}(\theta)| \le C\theta/p^2$. Of course $\mathcal{R}$ is not necessarily centered, but, by using Remark 4.2.1, we see that $\|r_p\|_{-1} \le 2C^2\pi^2/p^2$.

By collecting the estimates of the three addends we see that

$$c_p = c(K)p\left(1+O(1/p)\right)\,, \tag{4.B.40}$$

and this completes the proof of Corollary 4.B.5. □

By putting Corollary 4.B.5 and (4.A.8) together we obtain

Corollary 4.B.6. *With* $\{f_j\}_{j=0,1,\dots}$ *defined as in Appendix 4.A, we have* $\sup_j \|f_j'\|_\infty < \infty$ *and* $\sup_j \|f_j''\|_\infty / j < \infty$.

Proof. From (4.A.8), see also the discussion right after that, we see that $f_j' = -\mathcal{E}_j/q$ and $f_j'' = -e_j/q + \mathcal{E}_j q'/q^2$. Where e_j is the j^{th} (normalized) eigenvector. Taking into account the normalization, see proof of Corollary 4.B.5, the claim is readily proven. □

Proof of Proposition 4.B.4. Injecting (4.B.11) in the integral term of (4.B.16) leads to

$$y_1(\theta) \;=\; e^{\sqrt{2}\rho i\theta} - \frac{1}{\sqrt{2}\rho}\left[ie^{\sqrt{2}\rho i\theta}\int_0^\theta e^{-\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot})\,\mathrm{d}\theta' - ie^{-\sqrt{2}\rho i\theta}\int_0^\theta e^{\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot})\,\mathrm{d}\theta'\right] + O\left(\frac{1}{\rho^2}\right). \tag{4.B.41}$$

Similarly, we obtain

$$y_1'(\theta) \;=\; \sqrt{2}\rho i e^{\sqrt{2}\rho i\theta} + \left[e^{\sqrt{2}\rho i\theta}\int_0^\theta e^{-\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot})\,\mathrm{d}\theta' + e^{-\sqrt{2}\rho i\theta}\int_0^\theta e^{\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot})\,\mathrm{d}\theta'\right] + O\left(\frac{1}{\rho}\right), \tag{4.B.42}$$

and similar expressions for y_2 and y_2', which actually are just the complex conjugate of y_1 and y_1'. We define

$$H_1 \;=\; \int_0^{2\pi} e^{-\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot})\,\mathrm{d}\theta', \quad H_2 \;=\; \int_0^{2\pi} e^{\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot})\,\mathrm{d}\theta' \quad \text{and} \quad \Omega \;=\; e^{2\sqrt{2}\pi\rho i}. \tag{4.B.43}$$

With the higher estimates (4.B.41) and (4.B.42), we see that (4.B.28) becomes

$$\begin{vmatrix} \Omega - 1 - \frac{1}{\sqrt{2}\rho}\left[i\Omega H_1 - i\bar{\Omega}H_2\right] + O\left(\frac{1}{\rho^2}\right) & \bar{\Omega} - 1 - \frac{1}{\sqrt{2}\rho}\left[-i\bar{\Omega}\bar{H}_1 + i\Omega\bar{H}_2\right] + O\left(\frac{1}{\rho^2}\right) \\ \Omega - 1 - \frac{1}{\sqrt{2}\rho}\left[i\Omega H_1 + i\bar{\Omega}H_2\right] + O\left(\frac{1}{\rho^2}\right) & -\bar{\Omega} + 1 - \frac{1}{\sqrt{2}\rho}\left[i\bar{\Omega}\bar{H}_1 + i\Omega\bar{H}_2\right] + O\left(\frac{1}{\rho^2}\right) \end{vmatrix} \;=\; 0, \tag{4.B.44}$$

which implies

$$|\Omega - 1|^2 - \frac{\sqrt{2}}{\rho}\Im((\Omega - 1)H_1) \;=\; O\left(\frac{1}{\rho^2}\right). \tag{4.B.45}$$

We now use the expansion of ρ given by (4.B.31). In particular, the $O(1/\rho^2)$ above becomes a $O(1/k^2)$. The second term of the left hand side above is of order $1/k^2$. In fact, we get the first order of H_1 :

$$H_1 \;=\; \int_0^{2\pi} e^{-ki\theta'} m_{\theta'}(e^{ki\cdot})\,\mathrm{d}\theta' + O\left(\frac{1}{\sqrt{k}}\right), \tag{4.B.46}$$

where the non local terms in the integral are negligible, since we have

$$J * (\sqrt{q_0}e^{ki\cdot})(\theta) \;=\; \frac{iK}{2(2\pi I_0(2Kr))^{1/2}}(e^{i\theta}I_{k-1}(Kr) - e^{-i\theta}I_{k+1}(Kr)) \tag{4.B.47}$$

and we can apply (4.B.37). A similar bound apply for $J' * (\sqrt{q_0} e^{ki\cdot})$. So it remains the (real !!) first order (remark that $J * q_0(\cdot) = -Kr\sin(\cdot)$):

$$H_1 = \int_0^{2\pi} \frac{1}{2}((J * q_0)^2 + J * q_0')(\theta')\,\mathrm{d}\theta' + O\left(\frac{1}{\sqrt{k}}\right) = \frac{\pi K^2 r^2}{2} + O\left(\frac{1}{\sqrt{k}}\right). \quad (4.B.48)$$

But since (using (4.B.31))

$$\Omega - 1 = 2\pi i(\sqrt{2}\rho - k) + O\left(\frac{1}{k}\right), \quad (4.B.49)$$

where the first term of the right hand side is of order $1/\sqrt{k}$, we have improved the result of Lemma 4.B.2, since using (4.B.45), (4.B.48), (4.B.49) and (4.B.31) we obtain

$$|e^{2\sqrt{2}\pi\rho i} - 1|^2 - \frac{2\pi^2 K^2 r^2}{k}(\sqrt{2}\rho - k) = O\left(\frac{1}{k^2}\right) \quad (4.B.50)$$

which implies

$$\sqrt{2}\rho = k + O\left(\frac{1}{k}\right). \quad (4.B.51)$$

Taking (4.B.51) into account, (4.B.44) yields

$$|\Omega - 1|^2 - \frac{2}{k}\Im((\Omega - 1)H_1) + \frac{1}{k^2}(|H_1|^2 - |H_2|^2) = O\left(\frac{1}{k^3}\right). \quad (4.B.52)$$

The non local terms in H_2 are negligible as for H_1 (see above) and a direct calculation shows that the local terms are of order $1/k$, so from (4.B.52), (4.B.48) and (4.B.49) we get

$$(\sqrt{2}\rho - k)^2 - \frac{K^2 r^2}{2k}(\sqrt{2}\rho - k) + \frac{K^4 r^4}{16k^2} = \left(\sqrt{2}\rho - k - \frac{K^2 r^2}{4k}\right)^2 = O\left(\frac{1}{k^3}\right), \quad (4.B.53)$$

which implies

$$\sqrt{2}\rho = k + \frac{K^2 r^2}{4}\frac{1}{k} + O\left(\frac{1}{k^{3/2}}\right). \quad (4.B.54)$$

We now go further in the expansion to prove that the $O(1/k^{3/2})$ in (4.B.54) is in fact a $O(1/k^2)$. Using (4.B.17), we get the the second order expansion of y_1 (recall (4.B.15) and $f_0 = e^{\sqrt{2}\rho i\cdot}$)

$$y_1(2\pi) = \Omega + \int_0^{2\pi} A_0(2\pi, \theta_1, f_0)\,\mathrm{d}\theta_1 + \int_0^{2\pi}\int_0^{2\pi} A_0(2\pi, \theta_1, A_0(\theta_1, \theta_2, f_0))\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 + O\left(\frac{1}{\rho^3}\right). \quad (4.B.55)$$

From (4.B.51), we deduce

$$\begin{aligned}\int_0^{2\pi} A_0(\theta_1, \theta_2, f_0)\,\mathrm{d}\theta_2 &= -\frac{1}{\sqrt{2}\rho}\Bigg[ie^{\sqrt{2}\rho i\theta_1}\int_0^{\theta_1} e^{-\sqrt{2}\rho i\theta_2} m_{\theta_2}(e^{\sqrt{2}\rho i\cdot})\,\mathrm{d}\theta_2 \\ &\qquad - ie^{-\sqrt{2}\rho i\theta_1}\int_0^{\theta_1} e^{\sqrt{2}\rho i\theta_2} m_{\theta_2}(e^{\sqrt{2}\rho i\cdot})\,\mathrm{d}\theta_2\Bigg] \\ &= -\frac{i}{k}\left[e^{ki\theta_1}\int_0^{\theta_1} e^{-ki\theta_2} m_{\theta_2}(e^{ki\cdot})\,\mathrm{d}\theta_2 - e^{-ki\theta_1}\int_0^{\theta_1} e^{ki\theta_2} m_{\theta_2}(e^{ki\cdot})\,\mathrm{d}\theta_2\right] + O\left(\frac{1}{k^2}\right), \quad (4.B.56)\end{aligned}$$

and since the non local terms are negligible (see (4.B.47)), we get

$$\int_0^{2\pi} A_0(\theta_1, \theta_2, f_0)\, \mathrm{d}\theta_2 \;=\; \frac{iKre^{ki\theta_1}}{2k}\left(\sin\theta_1 + \frac{Kr}{4}\sin(2\theta_1) - \frac{Kr}{2}\theta_1\right) + O\left(\frac{1}{k^2}\right). \quad (4.B.57)$$

We deduce the following expansion for the third term of the right hand side of (4.B.55):

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} & A_0(2\pi, \theta_1, A_0(\theta_1, \theta_2, f_0))\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \;= \\ & \frac{Kr}{2k^2}\left(\Omega \int_0^{2\pi} e^{-ki\theta_1} m_{\theta_1}\left[e^{ki\cdot}\left(\sin\cdot + \frac{Kr}{4}\sin(2\cdot) - \frac{Kr}{2}\cdot\right)\right]\mathrm{d}\theta_1\right. \\ & \left. - \bar{\Omega}\int_0^{2\pi} e^{ki\theta_1} m_{\theta_1}\left[e^{ki\cdot}\left(\sin\cdot + \frac{Kr}{4}\sin(2\cdot) - \frac{Kr}{2}\cdot\right)\right]\mathrm{d}\theta_1\right) + O\left(\frac{1}{k^3}\right). \quad (4.B.58)\end{aligned}$$

Using similar arguments as before, we get to

$$\int_0^{2\pi} e^{-ki\theta_1} m_{\theta_1}\left(e^{ki\cdot}\left(\sin(\cdot) + \frac{Kr}{4}\sin(2\cdot)\right)\right)\mathrm{d}\theta_1 \;=\; O\left(\frac{1}{k^3}\right), \quad (4.B.59)$$

$$\int_0^{2\pi} e^{ki\theta_1} m_{\theta_1}\left(e^{ki\cdot}\left(\sin(\cdot) + \frac{Kr}{4}\sin(2\cdot)\right)\right)\mathrm{d}\theta_1 \;=\; O\left(\frac{1}{k^3}\right). \quad (4.B.60)$$

Moreover, the non local terms of $m_{\theta_1}(e^{ki\cdot}\cdot)$ are of order $1/k$. In fact, these non local terms are finite sums of the form

$$\int_0^{2\pi} e^{mi\theta}\sqrt{q_0}(\theta)\theta\, \mathrm{d}\theta\,, \quad (4.B.61)$$

where $|m|$ is included in $[k-1, k+1]$, and it is easy to see that since the Fourier coefficients of $\sqrt{q_0}$ decay very quickly (see (4.B.37)), (4.B.61) is of order $1/k$. So (4.B.58) becomes

$$\int_0^{2\pi}\int_0^{2\pi} A_0(2\pi, \theta_1, A_0(\theta_1, \theta_2, f_0))\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \;=\; -\frac{K^4r^4\pi^2}{8k^2}\Omega + \left(\frac{1}{k^3}\right), \quad (4.B.62)$$

and we deduce from (4.B.55)

$$y_1(2\pi) - y_1(0) \;=\; \Omega - 1 - \frac{i}{k}(\Omega H_1 - \bar{\Omega}H_2) - \frac{K^4r^4\pi^2}{8k^2}\Omega + O\left(\frac{1}{k^3}\right). \quad (4.B.63)$$

Similarly, we obtain

$$\frac{y_1'(2\pi) - y_1'(0)}{\sqrt{2}\rho i} \;=\; \Omega - 1 - \frac{i}{k}(\Omega H_1 + i\bar{\Omega}H_2) - \frac{K^2r^2\pi^2}{8k^2}\Omega + O\left(\frac{1}{k^3}\right). \quad (4.B.64)$$

Using these new estimates, (4.B.44) becomes

$$\begin{vmatrix} \Omega - 1 - \frac{1}{\sqrt{2}\rho}\left[i\Omega H_1 - i\bar{\Omega}H_2\right] & \bar{\Omega} - 1 - \frac{1}{\sqrt{2}\rho}\left[-i\bar{\Omega}\bar{H}_1 + i\Omega\bar{H}_2\right] \\ -\frac{K^4r^4\pi^2}{8k^2}\Omega + O\left(\frac{1}{k^3}\right) & -\frac{K^4r^4\pi^2}{8k^2}\bar{\Omega} + O\left(\frac{1}{k^3}\right) \\ \Omega - 1 - \frac{1}{\sqrt{2}\rho}\left[i\Omega H_1 + i\bar{\Omega}H_2\right] & -\bar{\Omega} + 1 - \frac{1}{\sqrt{2}\rho}\left[i\bar{\Omega}\bar{H}_1 + i\Omega\bar{H}_2\right] \\ -\frac{K^4r^4\pi^2}{8k^2}\Omega + O\left(\frac{1}{k^3}\right) & +\frac{K^4r^4\pi^2}{8k^2}\bar{\Omega} + O\left(\frac{1}{k^3}\right) \end{vmatrix} \;=\; 0\,, \quad (4.B.65)$$

which leads to

$$\begin{aligned}|\Omega - 1|^2 - \frac{\sqrt{2}}{\rho}\Im((\Omega - 1)H_1) + \frac{1}{k^2}(|H_1|^2 - |H_2|^2) + \frac{K^4r^4\pi^2}{8k^2}\left(4 - 2\Omega - 2\bar{\Omega}\right) \\ + \frac{K^4r^4\pi^2}{2k^3}\Im(H_1) \;=\; O\left(\frac{1}{k^4}\right). \quad (4.B.66)\end{aligned}$$

The last term of (4.B.66) is of order $1/k^4$ since using (4.B.51) we get

$$\Omega = 1 + i2\pi(\sqrt{2}\rho - k) + O\left(\frac{1}{k^2}\right). \tag{4.B.67}$$

Moreover using (4.B.51) we have

$$H_1 = \int_0^{2\pi} e^{-ki\theta} m_\theta(e^{ki\cdot})\,\mathrm{d}\theta + i(\sqrt{2}\rho - k)\Bigg(- \int_0^{2\pi} e^{ki\theta}\theta m_\theta(e^{ki\cdot})\,\mathrm{d}\theta \\ + \int_0^{2\pi} e^{ki\theta} m_\theta(e^{ki\cdot}\cdot)\,\mathrm{d}\theta\Bigg) + O\left(\frac{1}{k^2}\right). \tag{4.B.68}$$

As before, the non local terms of $m_\theta(e^{ki}\cdot)$ are of order $1/k$, so the last two integrals in (4.B.68) are equal up to a correction of order $1/k$, and thus (recall (4.B.48) for the first order term), using (4.B.51),

$$H_1 = \frac{\pi K^2 r^2}{2} + O\left(\frac{1}{k^2}\right). \tag{4.B.69}$$

We deduce that the first term of the second row of (4.B.66) is of order $1/k^4$, and that, using (4.B.67),

$$\frac{\sqrt{2}}{\rho}\Im((\Omega - 1)H_1) = \frac{2\pi^2 K^2 r^2}{k}(\sqrt{2}\rho - k) + O\left(\frac{1}{k^4}\right), \tag{4.B.70}$$

and

$$\frac{1}{k^2}|H_1|^2 = \frac{\pi^2 K^4 r^4}{4k^2} + O\left(\frac{1}{k^4}\right). \tag{4.B.71}$$

Since $|H_2|$ is of order $1/k$ and that (4.B.67) implies

$$|\Omega - 1|^2 = 4\pi^2(\sqrt{2} - \rho)^2 + O\left(\frac{1}{k^4}\right), \tag{4.B.72}$$

(4.B.66) becomes

$$(\sqrt{2}\rho - k)^2 - \frac{K^2 r^2}{2k}(\sqrt{2}\rho - k) + \frac{K^4 r^4}{16k^2} = O\left(\frac{1}{k^4}\right), \tag{4.B.73}$$

and we deduce

$$\sqrt{2}\rho = k + \frac{K^2 r^2}{4}\frac{1}{k} + O\left(\frac{1}{k^2}\right). \tag{4.B.74}$$

Now we are able to get a second expansion of the eigenvectors: using (4.B.41), (4.B.57) and (4.B.74), we get the following expansion for y_1

$$y_1(\theta) = e^{ki\theta}\left(1 + \frac{Kri}{2k}\sin(\theta) + \frac{K^2 r^2 i}{8k}\sin(2\theta)\right) + O\left(\frac{1}{k^2}\right) \tag{4.B.75}$$

and y_2 is the complex conjugate. So if we define w_1 and w_2 the real and imaginary parts, we get

$$w_1(\theta) = \cos(k\theta) - \frac{\sin(k\theta)}{k}\left(\frac{Kr}{2}\sin\theta + \frac{K^2 r^2}{8}\sin(2\theta)\right) + O\left(\frac{1}{k^2}\right), \tag{4.B.76}$$

$$w_2(\theta) = \sin(k\theta) + \frac{\cos(k\theta)}{k}\left(\frac{Kr}{2}\sin\theta + \frac{K^2 r^2}{8}\sin(2\theta)\right) + O\left(\frac{1}{k^2}\right). \tag{4.B.77}$$

Therefore the proof of Proposition 4.B.4 is complete. □

Chapter 5

Noise induced escape problem and phase reduction

Contents

5.1 Introduction

5.1.1 Phase reduction and escape problem

For dynamical systems with an attracting limit cycle, the phase reduction method consists in simplifying the system by projecting the dynamics on the limit cycle, and neglecting the distance between the trajectory and the limit cycle [62]. Such an approximation allows to reduce the dynamics to a one dimensional self-contained equation satisfied by the phase parameterizing the limit cycle. Such a reduction is widely used in the context of noisy oscillators (see [45, 108, 113] and references therein).

The aim of this paper is to show that the phase reduction can be made in a rigorous way for the escape problem for a class of systems close to reversibility. For a smooth dynamical system

$$\mathrm{d}X_t = F[X_t]\,\mathrm{d}t\,, \tag{5.1.1}$$

where $X_t \in \mathbb{R}^n$ (we use the notation $f[\cdot]$ for functions with domain $\mathbb{R}^n$), including a stable fixed point A with basin of attraction D, the escape problem is the study of the metastable behavior of A under a small noisy perturbation

$$\mathrm{d}X_t = F[X_t]\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t\,, \tag{5.1.2}$$

where B_t is a Brownian motion in $\mathbb{R}^n$. The natural questions arising are where, when and how do trajectories of (5.1.2) escape from D. This problem has been much studied in the literature. The fundamental reference is of course [34], where it is shown that these questions are related to the large deviation behavior of (5.1.2), and more precisely to the corresponding "quasipotential". For a connected domain K of $\mathbb{R}^n$ and two points P_1 and P_2 of K, the quasipotential $W_K(P_1, P_2)$ is defined by

$$W_K(P_1, P_2) = \inf\left\{I_T^{P_1}(Y) : Y \in C([T,0],K),\, T<0,\, Y_T = P_1,\, Y_0 = P_2\right\}\,, \tag{5.1.3}$$

where I is the large deviation rate function of (5.1.2), that is

$$I_T^x(Y) = \begin{cases} \frac{1}{2}\int_T^0 \left\|\dot{Y}_t - F[Y_t]\right\|^2 \mathrm{d}t & \text{if } Y \text{ is absolutely continuous} \\ & \text{and } Y_T = x\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{5.1.4}$$

where $\|\cdot\|$ is the norm associated to the canonical scalar product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^n$. If K is a compact neighborhood of A with smooth boundaries included in D (and thus attracted to A), and if we suppose moreover that $\langle F[X], n[X]\rangle < 0$ for all $X \in \partial K$, where n is the exterior normal vector of K at the point X, then it is shown in [34] that the escape from K will take place, with probability tending to 1 as $\varepsilon \to 0$, very close to the points B of the boundary of K satisfying $W_K(A,B) = \inf_{E\in\partial K} W_K(A,E)$. By a compactness argument and since $W_K(A,\cdot)$ is continuous [34], there exists at least one point B satisfying this property. Moreover for each starting point $x \in K$, the exit time τ^ε satisfies

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_x \tau^\varepsilon = W_K(A,B)\,. \tag{5.1.5}$$

Further work has been made to weaken the hypothesis on K, to study the escape at saddle points and the switching between the basins of attraction of several stable fixed points (e.g. [24, 39, 75]). The escape time, after renormalization, is in fact asymptotically exponential (e.g. [23, 70, 75]).

When (5.1.2) is reversible ($F = -\nabla V$ with V smooth), the quasipotential is proportional to the potential driving the dynamics: if B is in the basin of attraction of A, then

$$W_K(A,B) = 2(V(B) - V(A))\,. \tag{5.1.6}$$

In this case analytic approaches (in particular potential theory, see [12, 6]) show that the factor preceding $e^{W_K(A,B)/\varepsilon}$ in (5.1.5) satisfies the Eyring-Kramer's law [31, 61]. We point out that in the one-dimensional case, since the escape problem only depends on the value of F on a bounded domain, we can always consider that the dynamics is driven by a gradient flow.

The purpose of this paper is to show that for a class of systems close to reversibility and containing an attracting curve M, the escape from a stable fixed point A located on M occurs close to M, and that the quasipotential at the escape point can be approximated by the one corresponding to the dynamics constrained to M. We point out that the closeness to reversibility is a central point in our work. For what may happen far from reversibility, see for example [68]: the trajectories may make a non negligible profit by performing excursions away from M, and in this case the quasipotential can not be reduced anymore.

In principle the results we prove here should also be true in infinite dimension, and this generalization would be particularly relevant (see [99, 44] for systems for which the infinite dimensional result would be of great interest).

5.1.2 Mathematical set-up and main result

We will consider dynamical systems of the type

$$\mathrm{d}X_t \;=\; (-\nabla V[X_t] + \delta G[X_t])\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t\,, \tag{5.1.7}$$

where δ is meant to be small, $G \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $V \in C^4(\mathbb{R}^n, \mathbb{R}^n)$. The rate function associated to (5.1.7) is

$$I^x_{\delta,T}(Y) \;=\; \begin{cases} \frac{1}{2}\int_T^0 \|\dot{Y}_t + \nabla V[Y_t] - \delta G[Y_t]\|^2\,\mathrm{d}t & \text{if } Y \text{ is absolutely continuous} \\ & \text{and } Y_T = x\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{5.1.8}$$

We suppose that the unperturbed deterministic dynamical system

$$\mathrm{d}X_t \;=\; -\nabla V[X_t]\,, \tag{5.1.9}$$

contains a stable compact one-dimensional manifold of stationary solutions. More precisely, we suppose that there exists a curve M (C^3 by Local Inverse Theorem, since V is C^4) without crossings such that for all $X \in M$ we have

$$\nabla V[X] \;=\; 0\,. \tag{5.1.10}$$

For convenience we take $V \equiv 0$ on M. Moreover we will suppose the existence of a spectral gap for the linearized evolution in the neighborhood of M: we suppose that if v is a tangent vector for M at the point X and w a vector orthogonal to v, then for $H[X]$ the Hessian matrix of V at the point X

$$H[X]v \;=\; 0 \tag{5.1.11}$$

and there exists a positive constant λ (independent from the vector w) such that

$$\langle H[X]w, w\rangle \;\geqslant\; \lambda \|w\|^2\,. \tag{5.1.12}$$

With these hypothesis, M is a normally hyperbolic manifold, which is a structure stable under small perturbations (see [46, 51, 99]). The perturbed deterministic dynamical system

$$\mathrm{d}X_t \;=\; (-\nabla V[X_t] + \delta G[X_t])\,\mathrm{d}t \tag{5.1.13}$$

thus also contains a stable normally hyperbolic curve M^δ. However this new stable invariant manifold in general won't be a manifold of stationary solutions, but may contain a slow dynamics (of speed of order δ). Moreover M^δ is located at distance of order δ from M (more details will be given in Section 5.2.3).

We consider a parametrization $\{q_\delta(\varphi),\, \varphi \in \mathbb{R}/L_\delta\mathbb{R}\}$ of M^δ satisfying $\|q'_\delta(\varphi)\| = 1$ for all φ. L_δ denotes the length of the curve. The perturbed deterministic dynamical system (5.1.13) reduced on M^δ is given by

$$\mathrm{d}\varphi^\delta_t \;=\; b_\delta\left(\varphi^\delta_t\right)\,\mathrm{d}t\,, \tag{5.1.14}$$

where

$$b_\delta(\varphi) \;:=\; \Big\langle -\nabla V[q_\delta(\varphi)] + \delta G[q_\delta(\varphi)], q'_\delta(\varphi)\Big\rangle\,. \tag{5.1.15}$$

$b_\delta = O(\delta)$ and b_δ/δ is in a certain sense a smooth perturbation of the function

$$b(\theta) \;=\; \langle G[q(\theta)], q'(\theta)\rangle\,, \tag{5.1.16}$$

where q is a parametrization of M defined on $\mathbb{R}/L\mathbb{R}$ (where L is the length of M) and satisfying $\|q'(\theta)\| = 1$, so b characterizes the perturbed dynamics projected on M. The smoothness of the perturbation ensures that the dynamics induced by b_δ on M^δ will be conjugated to the one induced by b on M (and thus have the same properties). We consider the case when there exists a stable fixed point θ^0 for b such that $b'(\theta^0) < 0$, and such that the interval $[\theta^0 - \Delta_1 - \zeta, \theta^0 + \Delta_2 + \zeta]$ is included in the basin of attraction of θ^0 for positive reals Δ_1, Δ_2, ζ. Then for δ small enough (see Lemma 5.2.7 and the associated discussion), there exists a phase φ_{A^δ} (corresponding to a point $A^\delta = q_\delta(\varphi_{A^\delta}) \in M^\delta$) also stable for b_δ and such that $[\varphi_{A^\delta} - \Delta_1 - \zeta, \varphi_{A^\delta} + \Delta_2 + \zeta]$ is included in its basin of attraction. Moreover, since M^δ is stable, A^δ is a stable fixed point also for (5.1.13). Our goal is to justify that a phase reduction for the escape problem for this fixed point A^δ is valid, for δ small.

For each Z close enough to M^δ there exists a unique $q_\delta(\varphi)$ such that $\|Z - q_\delta(\varphi)\| = \mathrm{dist}(Z, M^\delta)$. We denote by $p_\delta(Z) := \varphi$ the phase of this projection. We will study the quasipotential W_δ associated to the rate function $I^x_{\delta,T}$ (see (5.1.8)) and to the tubular domain U^δ defined by

$$U^\delta = \left\{Z \in \mathbb{R}^n,\ \mathrm{dist}(Z, M^\delta) \leqslant C_0 \delta^{1/2},\ p_\delta(Z) \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]\right\}, \qquad (5.1.17)$$

depending on a constant C_0. In particular , since we are interested in the escape problem for the fixed point A^δ, we are looking for the location of the points $B^\delta \in \partial U^\delta$ satisfying

$$W_\delta\left(A^\delta, B^\delta\right) = \inf_{E \in \partial U^\delta} W_\delta\left(A^\delta, E\right). \qquad (5.1.18)$$

Since the "length" of such a tube U^δ may be of order 1, whereas its "slice" is of order $\delta^{1/2}$, a trajectory exiting the tube at a point B^δ satisfying either $p_\delta(B^\delta) = \varphi_{A^\delta} - \Delta_1$ or $p_\delta(B^\delta) = \varphi_{A^\delta} + \Delta_2$ stays very close to M^δ. We aim to show that the most probable paths verify this property.

To justify the validity of the phase reduction for this escape problem, we also have to prove that the value $W_\delta\left(A^\delta, B^\delta\right)$ of the quasipotential at these minimizing points may be well approximated by the quasipotential of the dynamics reduced on M^δ. Since the flow (5.1.13) is tangent to M^δ, for a trajectory Y^δ staying in M^δ (that is of the form $q_\delta(\varphi^\delta_t)$) linking two point A^δ and B^δ of M^δ, the rate function is reduced to

$$I^{A^\delta}_{\delta,T}(Y^\delta) = \int_T^0 \left|\dot\varphi^\delta_t - \Big\langle -\nabla V[q_\delta(\varphi^\delta_t)] + \delta G[q_\delta(\varphi^\delta_t)], q'_\delta(\varphi^\delta_t)\Big\rangle\right|^2 \mathrm{d}t. \qquad (5.1.19)$$

This functional coincides with the large deviation rate function one obtains by considering the one-dimensional diffusion

$$\mathrm{d}\varphi^\delta_t = b_\delta\left(\varphi^\delta_t\right)\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B^1_t, \qquad (5.1.20)$$

where B^1 is a standard one-dimensional Brownian motion. We denote $W^{red}_\delta(\varphi^1, \varphi^2)$ the associated quasipotential, i.e for all $\delta > 0$, $\varphi_1 \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ and $\varphi_2 \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$, W^{red}_δ is defined as follows:

$$W^{red}_\delta(\varphi_1, \varphi_2) = \inf\Bigg\{\int_T^0 |\dot\varphi_t - b_\delta(\varphi_t)|^2\,\mathrm{d}t : \varphi \in C([T,0], [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]) \text{ and absolutely continuous, } T < 0,\ \varphi_T = \varphi_1,\ \varphi_0 = \varphi_2\Bigg\}. \qquad (5.1.21)$$

The noise induced escape associated to (5.1.20) can be viewed as a reversible one, and since $[\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ is included in the domain of attraction of φ_{A^δ} for the reduced deterministic dynamics $\dot{\varphi}_t = b_\delta(\varphi_t)$, for $\varphi \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ the quasipotential W_δ^{red} is simply given by

$$W_\delta^{red}(\varphi_{A^\delta}, \varphi) = \int_{\varphi_{A^\delta}}^{\varphi} b_\delta(\varphi')\,\mathrm{d}\varphi' . \tag{5.1.22}$$

Since W_δ^{red} is the infimum of the rate function taken on the subset made of the trajectories staying in M^δ, we have the immediate bound

$$W_\delta(q_\delta(\varphi_1), q_\delta(\varphi_2)) \leqslant W_\delta^{red}(\varphi_1, \varphi_2) . \tag{5.1.23}$$

We prove the following theorem.

Theorem 5.1.1. *There exist δ_0 and a constant C_0 such that for all $\delta \leqslant \delta_0$, for each $B^\delta \in \partial U^\delta$ realizing the minimum of $W_\delta(A^\delta, \cdot)$ on U^δ, if we denote $\varphi_{B^\delta} := p_\delta(B^\delta)$ we have either $\varphi_{B^\delta} = \varphi_{A^\delta} - \Delta_1$ or $\varphi_{B^\delta} = \varphi_{A^\delta} + \Delta_2$, and*

$$\begin{aligned} W_\delta\left(A^\delta, B^\delta\right) &= W_\delta^{red}\Big(\varphi_{A^\delta}, \varphi_{B^\delta}\Big) + O\left(\delta^3 |\log \delta|^3\right) \\ &= \int_{\varphi_{A^\delta}}^{\varphi_{B^\delta}} b_\delta(\varphi)\,\mathrm{d}\varphi + O\left(\delta^3 |\log \delta|^3\right) . \end{aligned} \tag{5.1.24}$$

This theorem proves that the quasipotential can be well approximated for the points satisfying the minimum of the quasipotential $W_\delta(A^\delta, \cdot)$ in the boundary of tube U^δ. It is quite natural to think that this approximation is also possible for the points lying on the attracting curve M^δ of (5.1.13) (that is the points B^δ of the type $B^\delta = q_\delta(\varphi^\delta)$, but not necessarily belonging to ∂U^δ). This is the purpose of the following Corollary, obtained by carrying out a slight modification of the proof of Theorem 5.1.1.

Corollary 5.1.2. *There exist δ_0 and a constant C_0 such that for all $\delta \leqslant \delta_0$ and for each $\varphi^\delta \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ we have*

$$\begin{aligned} W_\delta\left(A^\delta, q_\delta\left(\varphi^\delta\right)\right) &= W_\delta^{red}\left(\varphi_{A^\delta}, \varphi^\delta\right) + O\left(\delta^3 |\log \delta|^3\right) \\ &= \int_{\varphi_{A^\delta}}^{\varphi^\delta} b_\delta(\varphi')\,\mathrm{d}\varphi' + O\left(\delta^3 |\log \delta|^3\right) . \end{aligned} \tag{5.1.25}$$

These results are obtained by quantitative estimates on the most probable paths. To understand why these paths stay at a distance of order $\delta^{1/2}$ from M^δ (or equivalently at distance $\delta^{1/2}$ from M, since M^δ is located at distance δ from M), remark that for a point Z in the neighborhood of M, (5.1.10) and (5.1.12) imply that $V[Z]$ is equivalent up to a constant factor to $\mathrm{dist}(Z, M)^2$, where "dist" denotes the distance associated to the norm $\|\cdot\|$. Since $A^\delta \in M^\delta$ and thus $\mathrm{dist}(A^\delta, M) = O(\delta)$, the contribution to the quasipotential of the reversible part of the dynamics for such a point Z is (see (5.1.6)) $V[Z] - V[A^\delta] = V[Z] + O(\delta^2)$. On the other hand, the deterministic dynamics restraint on M^δ is slow (b_δ is of order δ), and it implies that leaving U^δ following the curve M^δ has a cost of order δ. This suggests that reaching a point located at a distance larger than $\delta^{1/2}$ is more expensive than following M^δ. This idea is used in particular in the proof of Lemma 5.4.7.

5.2 Preliminary results of geometrical nature

5.2.1 Projection and local coordinates

We first give more details about the orthogonal projection on smooth curves. We are here in a particular case, since the topology we use is induced by a scalar product. For the existence in more general cases, based on the Local Inverse Theorem, we refer for example to [46]. We will denote dist the distance associated to the norm $\|.\|$.

Lemma 5.2.1. *Let $\mathcal{C}$ be a 1-dimensional C^r manifold of $\mathbb{R}^n$ ($r \geqslant 2$). Let $s \mapsto g(s)$ be a C^r parametrization of $\mathcal{C}$ satisfying $\|g'(s)\| = 1$. Then there exists a neighborhood N of $\mathcal{C}$ such that for all Y in N there exists a unique $s = p_g(Y)$ such that*

$$\|Y - g(s)\| \;=\; dist(Y, \mathcal{C})\,. \tag{5.2.1}$$

Moreover, for $s := p_g(Y)$,

$$\langle Y - g(s), g'(s)\rangle \;=\; 0\,, \tag{5.2.2}$$

the mapping $Y \mapsto p_g(Y)$ is C^{r-1}, and for all $\beta \in \mathbb{R}^n$

$$Dp_g[Y]\beta \;=\; \frac{1}{1 - \langle Y - g(s), g''(s)\rangle}\langle g'(s), \beta\rangle\,. \tag{5.2.3}$$

Proof. The uniqueness of the projection for a sufficiently small neighborhood is ensured by the smoothness of $\mathcal{C}$. (5.2.2) is obtained by simply taking the derivative of $\|Y - g(u)\|^2$ with respect to u and the Implicit Function Theorem and (5.2.2) imply that p_g is C^{r-1}. Let $h \in \mathbb{R}^n$ such that $\langle h, g'(\theta)\rangle = 0$. Then it is clear that if h is small enough such that the projection is well defined, $p_g(Y) = \theta$ for $Y = g(\theta) + h$. For a small perturbation $g(\theta) + h + \beta$, we are looking for the real α satisfying

$$\langle g(\theta) + h + \beta - g(\theta + \alpha), g'(\theta + \alpha)\rangle \;=\; 0\,. \tag{5.2.4}$$

Since p_g is C^{r-1}, we already know that $\alpha = O(\|\beta\|)$. Now a first order expansion of (5.2.4) with respect to α gives

$$\langle -\alpha g'(\theta) + h + \beta + O(\beta^2), g'(\theta) + \alpha g''(\theta) + O(\beta^2)\rangle \;=\; 0\,, \tag{5.2.5}$$

which, since $\langle h, g'(\theta)\rangle = 0$ and $\|g'(\theta)\| = 1$ (which implies also $\langle g''(\theta), g'(\theta)\rangle = 0$), leads to

$$\alpha(-1 + \langle h, g''(\theta)\rangle) + \langle g'(\theta), \beta\rangle + O(\beta^2) \;=\; 0\,. \tag{5.2.6}$$

□

In Theorem 5.1.1 and in the rest of the paper, we consider a parametrization of M (respectively of M^δ) $\theta \mapsto q(\theta)$ for $\theta \in \mathbb{R}/L\mathbb{R}$ (respectively $\varphi \mapsto q_\delta(\varphi)$ for $\varphi \in \mathbb{R}/L_\delta\mathbb{R}$) satisfying $\|q'(\theta)\| = 1$ (respectively $\|q'_\delta(\varphi)\| = 1$) and we use the notations

$$p_\delta \;:=\; p_{q_\delta} \tag{5.2.7}$$

$$p \;:=\; p_q\,. \tag{5.2.8}$$

We stress out that the size of the neighborhood of a curve $\mathcal{C}$ where the projection is defined depends continuously on its curvature and the sizes of its bottlenecks (which quantify in particular the non-crossing property of the curve). As we will see in Theorem 5.2.4, for the family of curves M^δ these quantities have continuous variations of order δ. So if the projection p is defined in a ε-neighborhood of M , this ensures the existence of the projections p_δ on a $(\varepsilon + O(\delta))$-neighborhood of M^δ (ε fixed with respect to δ), and in particular at distance $\delta^{1/2}$ from M^δ for δ small enough.

5.2.2 Stable Normally Hyperbolic Manifolds

We now quickly review the notion of Stable Normally Hyperbolic manifolds (SNHM) (see [46] for more details). SNHMs are invariant manifolds, linearly stable, and such that the attraction they apply on their neighborhood is stronger than their inner dynamics. Consider a C^r flow on $\mathbb{R}^n$

$$\dot{X} = F(X) \tag{5.2.9}$$

and suppose that it admits a compact invariant manifold M. Define for each $Q \in M$ its tangent space T_Q, its normal space N_Q and the corresponding orthogonal projections P_Q^T and P_Q^N. To each initial condition Q on M we associate the linearized evolution semi-group $\Phi(Q,t)$ defined by

$$\Phi(Q,0)u = u \tag{5.2.10}$$

for all $u \in \mathbb{R}^n$ and

$$\partial_t \Phi(Q,t) = DF(Q_t)\Phi(Q,t) \tag{5.2.11}$$

where Q_t is the trajectory of (5.2.9) with initial condition Q, and thus a trajectory staying on M.

Definition 5.2.2. *For all $Q \in M$, we define the generalized Lyapunov-type numbers*

$$\nu(Q) := \inf\left\{ a : \left(\frac{\|w\|}{\|P_{Q_t}^N \Phi(Q,t) w\|} \right) \Big/ a^{-t} \to 0 \quad as \quad t \downarrow -\infty \quad \forall w \in N_Q \right\} \tag{5.2.12}$$

and when $\nu(Q) < 1$

$$\sigma(Q) := \inf\left\{ b : \frac{\|w\|^b / \|v\|}{\|P_{Q_t}^N \Phi(Q,t) w\|^b / \|P_{Q_t}^T \Phi(Q,t) v\|} \to 0 \right.$$
$$\left. as \quad t \downarrow -\infty \quad \forall v \in T_Q,\, w \in N_Q \right\}. \tag{5.2.13}$$

The number ν characterizes the linear stability of M, and σ compares the normal and tangential linear evolution in the neighborhood of M. ν and σ are C^r functions (see [46]), so they are bounded on the compact M, and attain their suprema $\bar{\nu}(M)$ and $\bar{\sigma}(M)$ on M.

Definition 5.2.3. *M is called a Stable Normally Hyperbolic Manifold if $\bar{\nu}(M) < 1$ and $\bar{\sigma}(M) < 1$.*

It is clear that in our specific problem, the curve M is a SNHM, since (5.1.11) and (5.1.12) imply $\bar{\nu}(M) \leqslant e^{-\lambda}$ and $\bar{\sigma}(M) = 0$.

5.2.3 Persistence of hyperbolic manifolds

We now formulate the persistence result of our 1-dimensional manifold M under perturbation. We refer to [32, 46] for the general proof of persistence in the finite-dimension case. For more general cases (infinite dimensions), see for example [51, 99]. Recall that $\theta \mapsto q(\theta)$ is a parametrization of M satisfying $\|q'(\theta)\| = 1$.

Theorem 5.2.4. *If G is C^2, then for all δ small enough, there exists a C^2 mapping $\theta \mapsto \phi_\delta(\theta)$ satisfying*

$$\langle \phi_\delta(\theta), q'(\theta)\rangle = 0 \,, \tag{5.2.14}$$

$$\sup_{\theta\in\mathbb{R}/L\mathbb{R}} \{\|\phi_\delta(\theta)\|,\, \|\phi_\delta'(\theta)\|,\, \|\phi_\delta''(\theta)\|\} = O(\delta)\,, \tag{5.2.15}$$

and such that

$$M^\delta = \{q(\theta) + \phi_\delta(\theta),\, \theta \in \mathbb{R}/L\mathbb{R}\} \tag{5.2.16}$$

is a stable normally hyperbolic manifold for (5.1.13).

This result implies in particular that $\theta \mapsto q(\theta) + \phi_\delta(\theta)$ is a parametrization of M^δ (possibly $\|q'(\theta) + \phi_\delta'(\theta)\| \neq 1$). In the following Lemma, we give the first order of the mapping ϕ_δ.

Lemma 5.2.5. *For all δ small enough,*

$$\sup_{\theta\in\mathbb{R}/L\mathbb{R}} \|\phi_\delta(\theta) - \delta h^1(\theta)\| = O(\delta^2)\,, \tag{5.2.17}$$

where for all $\theta \in \mathbb{R}/L\mathbb{R}$ the vector $h^1(\theta)$ is the unique solution of (recall that H denotes the Hessian matrix of V)

$$\langle h^1(\theta), q'(\theta)\rangle = 0 \qquad \textit{and} \qquad H[q(\theta)]h^1(\theta) = G[q(\theta)] - \langle G[q(\theta)], q'(\theta)\rangle q'(\theta)\,. \tag{5.2.18}$$

Proof. Let $Y_0^\delta = q(\theta_0) + \phi_\delta(\theta_0) \in M^\delta$ be the initial condition of a the trajectory Y^δ of (5.1.13). If we denote $\theta_t := p(Y_t)$, then (5.1.13) at time $t = 0$ in this case becomes

$$(q'(\theta_0) + \phi_\delta'(\theta_0))\dot\theta_0^\delta = -\nabla V[q(\theta_0) + \phi_\delta(\theta_0)] + \delta G[q(\theta_0) + \phi_\delta(\theta_0)]\,. \tag{5.2.19}$$

We view here $\dot\theta^\delta$ as a function of θ_0, and we first look for uniform estimations of $\dot\theta_0^\delta$ with respect to θ_0. After a projection on the tangent space of M we get

$$(1 + \langle \phi_\delta'(\theta_0), q'(\theta_0)\rangle)\dot\theta_0^\delta = \langle -\nabla V[q(\theta_0) + \phi_\delta(\theta_0)] + \delta G[q(\theta_0) + \phi_\delta(\theta_0)], q'(\theta_0)\rangle\,. \tag{5.2.20}$$

Recalling Lemma 5.2.1 and Theorem 5.2.4 we deduce that $\dot\theta_0^\delta$ is C^2 with respect to θ_0, and we get the first order expansion (using (5.1.11))

$$\dot\theta_0^\delta = \delta\langle G[q(\theta_0)], q'(\theta_0^\delta)\rangle + O(\|\phi_\delta(\theta_0)\|, \|\phi_\delta'(\theta_0)\|)\,. \tag{5.2.21}$$

So we deduce from Theorem 5.2.4

$$\sup_{\theta_0\in\mathbb{R}/L\mathbb{R}} |\dot\theta_0^\delta| = O(\delta)\,. \tag{5.2.22}$$

Now we can prove Lemma 5.2.5: projecting (5.2.19) on the normal space we get

$$\begin{aligned}\dot\theta_0^\delta\Big(\phi_\delta'(\theta_0) - \langle\phi_\delta'(\theta_0), q'(\theta_0)\rangle q'(\theta_0)\Big) &= -H[q(\theta_0)]\phi_\delta(\theta_0) + \delta G[q(\theta_0)] \\ &\quad - \delta\langle G[q(\theta_0)], q'(\theta_0)\rangle q'(\theta_0) \\ -\Big(\nabla V[q(\theta_0) + \phi_\delta(\theta_0)] - \langle \nabla V[q(\theta_0) + \phi_\delta(\theta_0)]&, q'(\theta_0)\rangle q'(\theta_0) - H[q(\theta_0)]\phi_\delta(\theta_0)\Big) + O(\delta^2)\,.\end{aligned} \tag{5.2.23}$$

The last line in the previous equation is of order δ^2, due to Lemma 5.2.5, and thus for h^1 defined as in the statement of the Lemma we have (recall (5.2.22))

$$H[q(\theta_0)](\phi_\delta(\theta_0) - \delta h^1(\theta_0)) \;=\; O(\delta^2)\,. \tag{5.2.24}$$

Since both vectors $\phi_\delta(\theta_0)$ and $h^1(\theta_0)$ belong to the normal space of M at the point $q(\theta_0)$, the spectral gap (5.1.12) together with (5.2.24) imply

$$\phi_\delta(\theta_0) - \delta h^1(\theta_0) \;=\; O(\delta^2)\,. \tag{5.2.25}$$

By a compactness argument the $O(\delta^2)$ in the previous equation is in fact uniform with respect to θ, so we get (5.2.17). □

5.2.4 Choice of projection

The proof of Theorem 5.1.1 we develop is based on perturbation arguments around the manifold M. We will thus use the orthogonal projection on M rather than the one on M^δ: for a point Y located in a neighborhood of M, we will use the coordinates (θ, h) defined as follows

$$\theta \;:=\; p(Y)\,, \tag{5.2.26}$$

$$h \;:=\; Y - q(\theta)\,. \tag{5.2.27}$$

We will use the notations θ_t^δ and h_t^δ for a path Y_t^δ depending on δ. We stress that these coordinates satisfy

$$\langle h, q'(\theta)\rangle \;=\; 0\,. \tag{5.2.28}$$

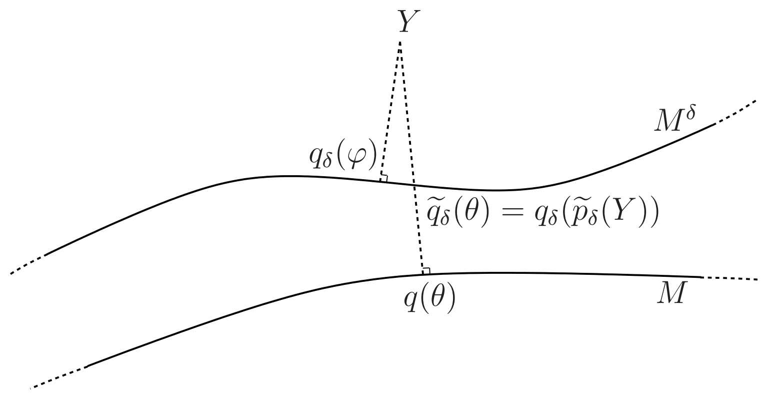


FIGURE 5.1. M^δ parametrized by $\widetilde{q}_\delta$. Here $\varphi = p_\delta(Y)$ and $\theta = p(Y)$.

We define

$$\widetilde{q}_\delta(\theta) = q(\theta) + \phi_\delta(\theta)\,. \tag{5.2.29}$$

The parametrization $\{\widetilde{q}_\delta(\theta),\, \theta \in \mathbb{R}/L\mathbb{R}\}$ is close in a certain sense to the one given by q_δ. In fact if we define, for each point Y in the neighborhood of M^δ, $\widetilde{p}_\delta(Y)$ as the phase $\varphi \in \mathbb{R}/L_\delta\mathbb{R}$ satisfying $q_\delta(\varphi) = \widetilde{q}_\delta(\theta)$, where $\theta = p(Y)$, then we have the following Lemma:

Lemma 5.2.6. *For δ small enough and Y in a neighborhood of M^δ (such that p and p_δ are well defined)*

$$|\widetilde{p}_\delta(Y) - p_\delta(Y)| \;=\; O(\delta\, dist(Y, M^\delta))\,. \tag{5.2.30}$$

Proof. We denote $\varphi := p_\delta(Y)$ (recall $\theta = p(Y)$), and $\alpha := \widetilde{p}_\delta(Y) - \varphi$. We have thus $q_\delta(\varphi+\alpha) = \widetilde{q}_\delta(\theta)$, and Theorem 5.2.4 implies $q'_\delta(\varphi+\alpha) = q'(\theta) + O(\delta)$. From the identity $\langle Y - q_\delta(\varphi+\alpha), q'(\theta)\rangle = 0$ we thus get $\langle Y - q_\delta(\varphi+\alpha), q'_\delta(\varphi+\alpha) + O(\delta)\rangle = 0$. Expanding this last equation with respect to α we obtain

$$\langle Y - q_\delta(\varphi) - \alpha q'_\delta(\varphi) + O(\alpha^2), q'_\delta(\varphi) + \alpha q''_\delta(\varphi) + O(\alpha^2) + O(\delta)\rangle \;=\; 0\,. \tag{5.2.31}$$

Expanding the scalar product, recalling the identities $\langle Y - q_\delta(\varphi), q'_\delta(\varphi)\rangle = 0$ and $\|q'_\delta(\varphi)\| = 1$ this reduces to

$$-\alpha(1 - \langle Y - q_\delta(\varphi), q''_\delta(\varphi)\rangle) + O(\alpha^2) + O(\|Y - q_\delta(\varphi)\|\delta) \;=\; 0\,, \tag{5.2.32}$$

which implies the expected bound for α. □

It will be useful to consider the restriction of the dynamics (5.1.13) on M^δ with respect to the parametrization $\theta \mapsto \widetilde{q}_\delta(\theta)$, and thus we introduce the function $\widetilde{b}_\delta$ defined on $\mathbb{R}/L\mathbb{R}$ by

$$\widetilde{b}_\delta(\theta) \;=\; \left\langle -\nabla V[\widetilde{q}_\delta(\theta)] + G[\widetilde{q}_\delta(\theta)], \frac{\widetilde{q}'_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|}\right\rangle. \tag{5.2.33}$$

This drift satisfies the following Lemma:

Lemma 5.2.7.

$$\sup_{\theta\in\mathbb{R}/L\mathbb{R}} \{|\widetilde{b}_\delta(\theta) - \delta b(\theta)|, |\widetilde{b}'_\delta(\theta) - \delta b'(\theta)|\} \;=\; O(\delta^2) \tag{5.2.34}$$

This Lemma allows us to study the escape problem on tubes U^δ defined on intervals $[\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ with constant length with respect to δ: we will suppose in the rest of the paper that b has a stable hyperbolic fixed point θ^0 with domain of attraction I, and with this hypothesis the previous Lemma ensures that $\widetilde{b}_\delta$ has a stable fixed θ_0^δ located in a δ-neighborhood of θ^0 and whose domain of attraction is a δ-perturbation of I. Since (recall Theorem 5.2.4) $\|\widetilde{q}'_\delta(\theta)\| = 1 + O(\delta)$, using the parametrization $\theta \mapsto \widetilde{q}_\delta(\theta)$ instead of $q_\delta(\varphi)$ only induces an error of order δ in the phases. Thus if φ_{A^δ} denotes the phase satisfying $q_\delta(\varphi_{A^\delta}) = \widetilde{q}_\delta(\theta_0^\delta)$, then φ_{A^δ} is an hyperbolic fixed point for b_δ and its domain of attraction is also a δ-perturbation of I. Of course, if we denote $A^\delta = q_\delta(\varphi_{A^\delta})$, then since M^δ is a SNHM, A^δ is a stable fixed point for (5.1.13).

Proof. Using Theorem 5.2.4 and (5.1.10), it is clear that $|\widetilde{b}_\delta(\theta) - \delta b(\theta)|$ is of order δ^2. Taking the derivative with respect to θ, we obtain

$$\begin{aligned}\widetilde{b}'_\delta(\theta) - \delta b'(\theta) \;=\; &\left\langle -H[\widetilde{q}_\delta(\theta)]\widetilde{q}'_\delta(\theta), \frac{\widetilde{q}'_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|}\right\rangle + \left\langle -\nabla V[\widetilde{q}_\delta(\theta)], \frac{\widetilde{q}''_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|} - \frac{\langle \widetilde{q}''_\delta(\theta), \widetilde{q}'_\delta(\theta)\rangle \widetilde{q}'_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|^3}\right\rangle \\ &+ \delta\left(\left\langle DG[\widetilde{q}_\delta(\theta)]\widetilde{q}'_\delta(\theta), \frac{\widetilde{q}'_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|}\right\rangle - \langle DG[q(\theta)]q'(\theta), q'(\theta)\rangle\right) \\ &+ \delta\left(\left\langle G[\widetilde{q}_\delta(\theta)], \frac{\widetilde{q}''_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|} - \frac{\langle \widetilde{q}''_\delta(\theta), \widetilde{q}'_\delta(\theta)\rangle \widetilde{q}'_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|^3}\right\rangle - \langle G[q(\theta)], q''(\theta)\rangle\right).\end{aligned} \tag{5.2.35}$$

Using Theorem 5.2.4 we get the following expansion for the first term of the right hand side (recall that it implies in particular $\|\widetilde{q}'_\delta(\theta)\| = 1 + O(\delta)$):

$$\begin{aligned}\left\langle H[\widetilde{q}_\delta(\theta)]\widetilde{q}'_\delta(\theta), \frac{\widetilde{q}'_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|}\right\rangle \;=\; &\langle H[q(\theta)]q'(\theta), q'(\theta)\rangle + \langle H[q(\theta)](\widetilde{q}'_\delta(\theta) - q'(\theta)), q'(\theta)\rangle \\ &+ \left\langle H[q(\theta)]q'(\theta), \frac{\widetilde{q}'_\delta(\theta)}{\|\widetilde{q}'_\delta(\theta)\|} - q'(\theta)\right\rangle + D^3V[q(\theta)](\phi_\delta(\theta), q'(\theta), q'(\theta)) + O(\delta^2)\,,\end{aligned} \tag{5.2.36}$$

and (5.1.11) implies that the three first term of the right hand side in this expansion are equal to zero. Using similar argument to treat the other terms of (5.2.35) (recalling in particular (5.1.10)), we see that it reduces to

$$\widetilde{b}'_\delta(\theta) - \delta b'(\theta) \;=\; -D^3V[q(\theta)](\phi_\delta(\theta), q'(\theta), q'(\theta)) - \langle H[q(\theta)]\phi_\delta(\theta), q''(\theta)\rangle + O(\delta^2)\,. \quad (5.2.37)$$

Now two derivations with respect to θ of the identity $\nabla V[q(\theta)] = 0$ imply that for all $u \in \mathbb{R}^n$

$$D^3V[q(\theta)](q'(\theta), q'(\theta), u) + \Big\langle H[q(\theta)]q''(\theta), u\Big\rangle \;=\; 0\,, \quad (5.2.38)$$

and for $u = q'(\theta)$ this implies the expected bound for $\widetilde{b}'_\delta(\theta) - \delta b'(\theta)$. □

5.3 Quasipotential and optimal path

As shown in [25], a continuity argument allows us to define $W_\delta(A^\delta, \cdot)$ as the infimum of the rate function over the paths defined on $(-\infty, 0]$ with limit A^δ at $-\infty$. In fact, extending the paths $Y \in C([-T, 0], U^\delta)$ with $Y_{-T} = A^\delta$ by $Y_t = A^\delta$ for $t \leqslant -T$, we get that for all $E \in U^\delta$

$$W_\delta(A^\delta, E) \;\geqslant\; \inf\left\{ I^{A^\delta}_{\delta,-\infty}(Y) : \, Y \in C((-\infty, 0], U^\delta),\, \lim_{t\to-\infty} Y_t = A^\delta,\, Y_0 = E \right\}. \quad (5.3.1)$$

On the other hand, for each path $Y \in C((-\infty, 0], U^\delta)$ with $\lim_{t\to-\infty} Y_t = A^\delta$ and $Y_0 = E$, we have for all $t \leqslant 0$

$$W_\delta(A^\delta, E) \;\leqslant\; W_\delta(A^\delta, Y_t) + I^{Y_t}_{\delta,t}(Y)\,. \quad (5.3.2)$$

But $W_\delta(A, \cdot)$ is Lipschitz continuous (see [34] Lemma 2.3), so $W_\delta(A^\delta, Y_t) \to 0$ when $t \to -\infty$, and thus (5.3.1) is in fact an equality.

For a point $E \in U^\delta$, we call an optimal path a path $Y^\delta \in C((-\infty, 0], U^\delta)$ with $\lim_{t\to-\infty} Y^\delta_t = A^\delta$, $Y_0 = E$ and satisfying

$$I^{A^\delta}_{\delta,-\infty}(Y^\delta) \;=\; W_\delta(A^\delta, E)\,. \quad (5.3.3)$$

In [25] it is explained that if U^δ is included in the domain of attraction of A^δ, then there exists an optimal path for all $E \in U^\delta$. We will follow this idea. In our case we can not ensure that U^δ is included in the domain of attraction of A^δ, but we prove in the following Lemma that a solution of (5.1.13) starting in a neighborhood of size of order 1 of M^δ converges to one of the fixed points of M^δ (and thus it is of course the case for the solutions starting from U^δ for δ small enough). Theorem 5.2.4 already implies that it is the case for the solution starting in some (unknown) neighborhood of M^δ, but we prove in Lemma 5.3.1 that we can take this neighborhood of size 1.

Lemma 5.3.1. *There exists $x_0 > 0$ independent from δ such that if dist$(X^\delta_0, M^\delta) \leqslant x_0$ and if δ is small enough, the solution of* (5.1.13) *starting from X^δ_0 converges to one of the fixed point of M^δ.*

Proof of Lemma 5.3.1. It is sufficient to prove that (where λ is the constant defined in (5.1.12))

$$\langle -\nabla V[q_\delta(\varphi) + u] + \delta G[q_\delta(\varphi) + u], u\rangle \;\leqslant\; -\frac{\lambda}{2}\|u\|^2\,, \quad (5.3.4)$$

for $\varphi \in \mathbb{R}/L_\delta\mathbb{R}$, δ small and $u \neq 0$ small (independently from δ) and satisfying

$$\langle u, q_\delta'(\varphi)\rangle = 0 . \tag{5.3.5}$$

Indeed if (5.3.4) is satisfied, then for a solution $(X_t^\delta)_{t\geqslant 0}$ of (5.1.13) and a time $t \geqslant 0$ such that X_t^δ is sufficiently close to M^δ (in particular such that the projection $p_\delta(X_t^\delta)$ is well defined, but as it has already been discussed in Section 5.2 this is true at a distance of order 1 from M^δ), we have (using (5.3.4) with $u = X_t^\delta - q_\delta(p_\delta(X_t^\delta))$ and the orthogonality $\langle q_\delta'(\varphi), q_\delta(\varphi)\rangle = 0$):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|X_t^\delta - q_\delta(p_\delta(X_t^\delta))\|^2 &= 2\Big\langle \dot{X}_t^\delta - Dp_\delta[X_t]\dot{X}_t^\delta \cdot q_\delta'(p_\delta(X_t^\delta)), X_t^\delta - q_\delta(p_\delta(X_t^\delta))\Big\rangle \\ &= 2\Big\langle \dot{X}_t^\delta, X_t^\delta - q_\delta(p_\delta(X_t^\delta))\Big\rangle \\ &\leqslant -\lambda\|X_t^\delta - q_\delta(p_\delta(X_t^\delta))\|^2 . \end{aligned} \tag{5.3.6}$$

We deduce that there exists a $x_0 > 0$ independent from δ such that if $\mathrm{dist}(X_0^\delta, M^\delta) \leqslant x_0$ we have for all $t \geqslant 0$

$$\mathrm{dist}(X_t^\delta, M^\delta) \leqslant e^{-\lambda t}\mathrm{dist}(X_0^\delta, M^\delta) , \tag{5.3.7}$$

which implies the Lemma.

We now prove (5.3.4). Let $u \in \mathbb{R}^n$ satisfying (5.3.5). Since $-\nabla V[q_\delta(\varphi)] + \delta G[\varphi_\delta]$ is tangent to $q_\delta'(\varphi)$, a second order expansion leads to

$$\langle -\nabla V[q_\delta(\varphi) + u] + \delta G[q_\delta(\varphi) + u], u\rangle = -\langle H[q_\delta(\varphi)]u, u\rangle + O(\delta\|u\|^2) + O(\|u\|^3) . \tag{5.3.8}$$

Define $\theta := p(q_\delta(\varphi))$. We have $q_\delta(\varphi) = q(\theta) + \phi_\delta(\theta)$, and (5.3.5) and Theorem 5.2.4 imply

$$\langle u, q'(\theta)\rangle = \langle u, q'(\theta) + \phi_\delta'(\theta)\rangle - \langle u, \phi_\delta'(\theta)\rangle = O(\delta\|u\|) , \tag{5.3.9}$$

which induces

$$u = u - \langle u, q'(\theta)\rangle q'(\theta) + O(\delta\|u\|) . \tag{5.3.10}$$

We are now able to prove (5.3.4): this last assertion taken together with Theorem 5.2.4 and (5.1.12) implies

$$\begin{aligned}\langle -\nabla V[q_\delta(\varphi)+u]+\delta G[q_\delta(\varphi)+u], u\rangle &= -\Big\langle H[q(\theta)]\Big(u-\langle u, q'(\theta)\rangle q'(\theta)\Big), u-\langle u, q'(\theta)\rangle q'(\theta)\Big\rangle \\ &\quad + O(\delta\|u\|^2) + O(\|u\|^3) \\ &\leqslant -(\lambda + O(\delta) + O(\|u\|))\|u\|^2 . \end{aligned} \tag{5.3.11}$$

□

A compactness argument applied to a sequence of paths interpolating $W_\delta(A^\delta, E)$, where $E \in U^\delta$, will allow us to get an infinite time path limit. To prove that this limit path is an optimal path for E (and more precisely that it converges to A^δ for $t \to -\infty$), we rely on the following Lemma.

Lemma 5.3.2. *For all δ small enough and for all $\mu > 0$ small enough (depending on δ) there exists $0 < T_{exit}^\delta(\mu) < \infty$ such that for all $E \in U^\delta$ the solution $(X_t^\delta)_{t\geqslant 0}$ of the perturbed deterministic dynamical system* (5.1.13) *starting at E satisfies one of the following properties:*

- $\|X^\delta_{T^\delta_{exit}(\mu)} - A^\delta\| < \mu$,
- *there exists $t \in [0, T_{exit(\mu)}]$ such that $dist(X_t^\delta, U^\delta) > \mu$.*

Proof. Since $[\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ is included in the domain of attraction of φ_{A^δ}, it also the case of $[\varphi_{A^\delta} - \Delta_1 - \mu, \varphi_{A^\delta} + \Delta_2 + \mu]$ for μ small enough. Choose $\zeta > 0$ such that a solution of (5.1.13) staring in the ζ neighborhood of A^δ stays in the μ neighborhood of A^δ. Define the domain

$$D^\delta = \mathcal{C}(\mathcal{N}^\mu(U^\delta)) \cup B(A^\delta, \zeta)\,, \tag{5.3.12}$$

where $\mathcal{C}(\mathcal{A})$ denotes the complementary of $\mathcal{A}$, $\mathcal{N}^\mu(U^\delta)$ the closed μ-neighborhood of U^δ and $B(A^\delta, \zeta)$ the open ball with center A^δ and radius ζ. Lemma 5.3.1 implies that the reaching time of D^δ for the solutions of (5.1.13) with initial condition in U^δ is finite. Since this reaching time is upper semicontinuous with respect to the initial condition it is uniformly bounded by a $T^\delta_{exit}(\mu) < \infty$.

□

We now state the existence of an optimal path for each point of U^δ.

Lemma 5.3.3. *For δ small enough there exists an optimal path Y^δ for each E of U^δ. Moreover if Y^δ is an optimal path associated to a point $B^\delta \in \partial U^\delta$ realizing the minimum of $W_\delta(A^\delta, \cdot)$ on ∂U^δ, then Y^δ_t is in the interior of U^δ for all $t < 0$.*

Before proving this Lemma, we recall in the following Lemma the classical Gronwall's Inequality.

Lemma 5.3.4. Gronwall's Inequality. *Let u be a non-negative continuous function defined on a compact interval I and suppose that there exist positive constants a and b such that for all t_0, $t \in I$ with $t_0 \leqslant t$*

$$u_t \leqslant a + b \int_{t_0}^{t} u_s \,\mathrm{d}s\,. \tag{5.3.13}$$

Then for all t_0, $t \in I$ with $t_0 \leqslant t$ we have

$$u_t \leqslant a e^{b(t-t_0)}\,. \tag{5.3.14}$$

Proof of Lemma 5.3.3. Let $Y^{\delta,k}$ be a sequence of paths with endpoint $E \in U^\delta$ and such that $I^{A^\delta}_{\delta,-\infty}(Y^{\delta,k})$ converges to $W_\delta(A^\delta, E)$. The family $(Y^{\delta,k}_t)_{t \leqslant 0}$ is equicontinuous (the convergence of the rate function and the compactness ensures a uniform control of $\int_{t-\varepsilon}^{t} \|\dot{Y}^{\delta,k}_s\|^2 \,\mathrm{d}s$). Thus using the Arzelà-Ascoli Theorem on each compact interval $[-m, 0]$ and a diagonal procedure we can show that there exists a subsequence Y^{δ,ψ_k}, that we will also denote $Y^{\delta,k}$, and that converges in $C((-\infty, 0), \mathbb{R}^n)$ to a path Y^δ, with endpoint E. The lower semicontinuity of the large deviation rates (see [34], Lemma 2.1) induces the following inequality for all $m > 0$:

$$I^{Y^\delta_{-m}}_{\delta,-m}(Y^\delta) \leqslant \liminf_k I^{Y^{\delta,k}_{-m}}_{\delta,-m}(Y^{\delta,k})\,, \tag{5.3.15}$$

and it is clear that

$$\liminf_k I^{Y^{\delta,k}_{-m}}_{\delta,-m}(Y^{\delta,k}) \leqslant \liminf_k I^{A^\delta}_{\delta,-\infty}(Y^{\delta,k}) \leqslant W_\delta(A^\delta, E)\,. \tag{5.3.16}$$

So for all $m > 0$

$$I^{Y^\delta_{-m}}_{\delta,-m}(Y^\delta) \leqslant W_\delta(A^\delta, E)\,. \tag{5.3.17}$$

Remark that the left hand side of the last equation is increasing in m, and thus we deduce

$$\frac{1}{2}\int_{-\infty}^{0}\|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2\,\mathrm{d}t \leqslant W_\delta(A^\delta, E)\,. \tag{5.3.18}$$

Y^δ is thus an optimal path for E if it converges to A^δ when $t \to -\infty$. We proceed by contradiction to show this convergence. If it is not the case, there exists $\zeta > 0$ and a strictly decreasing sequence $(T_k)_{k\geqslant 1}$ converging to $-\infty$ such that $\|Y^\delta_{T_k} - A^\delta\| > \zeta$ for all k. It is easy to see that $W_\delta(A^\delta, E_1) > 0$ for $E_1 \neq A^\delta$, and thus by compactness and by the continuity of $W_\delta(A^\delta, \cdot)$ there exists $\alpha > 0$ such that $W_\delta(A^\delta, E_2) > 2\alpha$ for all E_2 satisfying $\|E_2 - A^\delta\| > \zeta$. Let $\mu > 0$ such that $W_\delta(A^\delta, E_1) \leqslant \alpha$ for all E_1 satisfying $\|E_1 - A^\delta\| \leqslant 2\mu$. With this choice of μ we have $W_\delta(E_1, E_2) \geqslant \alpha$ for all E_1 and E_2 such that $\|E_1 - A^\delta\| \leqslant 2\mu$ and $\|E_2 - A^\delta\| > \zeta$.

We can suppose without loss of generality that $T_k - T_{k+1} \geqslant T^\delta_{exit}(\mu)$ (see Lemma 5.3.2 for the definition of $T^\delta_{exit}(\mu)$). Since $\sum_k \int_{T_{k+1}}^{T_k} \left\|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\right\|^2 \mathrm{d}t < \infty$, for k large enough we have

$$\int_{T_{k+1}}^{T_k} \left\|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\right\|^2 \mathrm{d}t \leqslant \frac{\mu^2}{T^\delta_{exit}(\mu) e^{\kappa T^\delta_{exit}(\mu)}}\,, \tag{5.3.19}$$

where

$$\kappa := \max_{\mathrm{dist}(Z, M^\delta) \leqslant C_0\delta^{1/2}+1} |||H[Z] - \delta DG[Z]|||\,, \tag{5.3.20}$$

and $|||\cdot|||$ denotes the norm of linear operators in $\mathbb{R}^n$. Define $(X_t^\delta)_{t\geqslant 0}$ the solution of (5.1.13) with initial condition $Y^\delta_{T_{k+1}}$. For $0 \leqslant t \leqslant T^\delta_{exit}(\mu)$ and such that $\mathrm{dist}(Y^\delta_{T_{k+1}+s}, M^\delta) \leqslant C_0\delta^{1/2} + 1$ for all $0 \leqslant s \leqslant t$, we get, using the Cauchy-Schwarz Inequality,

$$\begin{aligned}
\|Y^\delta_{T_{K+1}+t} - X_t^\delta\| \leqslant\; & \int_0^t \left\|\dot{Y}^\delta_{T_{k+1}+s} + \nabla V[Y^\delta_{T_{k+1}+s}] - \delta G[Y^\delta_{T_{k+1}+s}]\right\| \mathrm{d}s \\
& + \int_0^t \left\|\nabla V[Y^\delta_{T_{k+1}+s}] - \delta G[Y^\delta_{T_{k+1}+s}] - \nabla V[X_s^\delta] + \delta G[X_s^\delta]\right\| \mathrm{d}s \\
\leqslant\; & \left(T^\delta_{exit}(\mu)\right)^{1/2} \left(\int_{T_{k+1}}^{T_k} \left\|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\right\|^2 \mathrm{d}t\right)^{1/2} + \kappa \int_0^t \left\|Y^\delta_{T_{K+1}+s} - X_s^\delta\right\| \mathrm{d}s\,.
\end{aligned} \tag{5.3.21}$$

Applying the Gronwall's inequality we get

$$\begin{aligned}
\|Y^\delta_{T_{k+1}+t} - X_t^\delta\| \leqslant & \left(T^\delta_{exit}(\mu)\right)^{1/2} \left(\int_{T_{k+1}}^{T_k} \left\|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\right\|^2 \mathrm{d}t\right)^{1/2} e^{T^\delta_{exit}(\mu)\kappa} \\
& \leqslant \mu\,.
\end{aligned} \tag{5.3.22}$$

We can choose $\mu \leqslant 1$, and in this case (5.3.22) and (5.3.7) show that $\mathrm{dist}(Y^\delta_{T_{k+1}+s}, M^\delta) \leqslant C_0\delta^{1/2} + 1$ for all $0 \leqslant s \leqslant T^\delta_{exit}(\mu)$, which induces, using again (5.3.22), $\|Y^\delta_{T_{k+1}+t} - X_t^\delta\| \leqslant \mu$ for all $t \leqslant T^\delta_{exit}(\mu)$. Since Y^δ does not leave U^δ, Lemma 5.3.2 implies that $\|Y^\delta_{T_{exit}(\mu)^\delta} - A^\delta\| \leqslant \mu$, and thus we have

$$\int_{T^\delta_{exit}(\mu)}^{T_k} \left\|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\right\|^2 \mathrm{d}t \geqslant W_\delta\left(Y^\delta_{T^\delta_{exit}(\mu)}, Y^\delta_{T_k}\right) \geqslant \alpha\,. \tag{5.3.23}$$

This last assertion contradicts $\sum_k \int_{T_{k+1}}^{T_k} \left\| \dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta] \right\|^2 \mathrm{d}t < \infty$.

Suppose now that Y^δ is an optimal path associated to a point $B^\delta \in \partial U^\delta$ realizing the minimum of $W_\delta(A^\delta, \cdot)$ on ∂U^δ, and suppose that it reaches ∂U^δ at a time $t < 0$ at some point B_2^δ. Then B_2^δ realizes also the minimum of $W_\delta(A^\delta, \cdot)$ on ∂U^δ, and $W_\delta(B_2^\delta, B^\delta) = 0$. But a simple argument involving the Gronwall's Inequality shows that the only point E satisfying $W_\delta(B_2^\delta, E) = 0$ is the fixed point of M^δ to which the solution of (5.1.13) starting from B_2^δ converges (see Lemma 5.3.1). This concludes the proof, by contradiction.

□

We now give the Euler Lagrange type equation satisfied by the optimal paths. It corresponds to Theorem 1 in [25]. We denote by $A^\dagger$ the transpose of a square matrix A.

Lemma 5.3.5. *Let Y^δ be an optimal path associated to a point $E \in U^\delta$. Then for all $T_1 < T_2 < 0$ such that Y_t^δ belongs to the interior of U^δ for all $t \in [T_1, T_2]$, $Y^\delta_{|(T_1,T_2)} \in C^2((T_1, T_2), \mathbb{R}^n)$ and satisfies for all $t \in (T_1, T_2)$*

$$\ddot{Y}_t^\delta = \left(H[Y_t^\delta] - \delta DG^\dagger[Y_t^\delta]\right)\left(\nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\right) + \delta\left(DG[Y_t^\delta] - DG^\dagger[Y_t^\delta]\right)\dot{Y}_t^\delta \,. \quad (5.3.24)$$

In particular, if $E \in U^\delta$ and realizes the minimum of $W_\delta(A^\delta, \cdot)$ on U^δ, then $Y^\delta \in C^2((-\infty, 0), \mathbb{R}^n)$ and satisfies (5.3.24) for all $t < 0$.

Proof. Define

$$I_{\delta,T_1,T_2}(Y) = \frac{1}{2}\int_{T_1}^{T_2} \|\dot{Y}_t + \nabla V[Y_t] - \delta G[Y_t]\|^2 \,\mathrm{d}t \,. \quad (5.3.25)$$

Y^δ must be a local minimum for $I_{\delta,T_1,T_2}(Z)$ viewed as a functional on the space of absolutely continuous paths Z satisfying $Z_{T_1} = Y_{T_1}^\delta$ and $Z_{T_2} = Y_{T_2}^\delta$. We denote $H^k((T_1, T_2), \mathbb{R}^n)$ the usual Sobolev spaces on the interval (T_1, T_2). Remark that since $I_{\delta,T_1,T_2}(Y^\delta) < \infty$, $Y^\delta_{|(T_1,T_2)} \in H^1((T_1, T_2), \mathbb{R}^n)$, and in particular the right hand side of (5.3.24) is well defined in the sense of distributions. Let $f \in C^\infty((T_1, T_2), \mathbb{R}^n)$ with compact support. We get the expansion for $\eta \in \mathbb{R}$

$$\begin{aligned} I_{\delta,T_1,T_2}(Y^\delta + \eta f) \;=\; & I_{\delta,T_1,T_2}(Y^\delta) + \eta \int_{T_1}^{T_2} \left\langle \dot{Y}_t^\delta, \dot{f}_t \right\rangle + \left\langle \nabla V[Y_t^\delta] - \delta G[Y_t^\delta], \dot{f}_t \right\rangle \\ & + \left\langle H[Y_t^\delta] f_t - \delta DG[Y_t^\delta] f_t, \dot{Y}_t^\delta \right\rangle + \left\langle H[Y_t^\delta] f_t - \delta DG[Y_t^\delta] f_t, \nabla V[Y_t^\delta] - \delta G[Y_t^\delta] \right\rangle \mathrm{d}t \\ & + O(\eta^2) \,. \end{aligned} \quad (5.3.26)$$

Since $Y^\delta + \eta f \in U^\delta$ for η small enough and since Y^δ is a local minimum, the term of order η in the right hand side of previous equation is equal to 0, and it implies (5.3.24) on the interval (T_1, T_2) in the sense of distributions. But since $Y^\delta_{|(T_1,T_2)} \in H^1((T_1, T_2), \mathbb{R}^n)$, V is C^4 and G is C^2, (5.3.24) implies that $\ddot{Y}^\delta_{|(T_1,T_2)} \in L^2((T_1, T_2), \mathbb{R}^n)$, or in other words $Y^\delta_{|(T_1,T_2)} \in H^2((T_1, T_2), \mathbb{R}^n)$. But again in this case (5.3.24) implies that $\ddot{Y}^\delta_{|(T_1,T_2)} \in H^1((T_1, T_2), \mathbb{R}^n)$, and thus admits a continuous representation. □

5.4 Proof of Theorem 5.1.1 and Corollary 5.1.2

5.4.1 Sketch of the proof

The aim of the proof is to make a expansion of $I^{A^\delta}_{-\infty,\delta}(Y^\delta_t)$ for the optimal paths linking A^δ to the points $B^\delta \in \partial U^\delta$ satisfying (5.1.18), and to compare this expansion to (5.1.19). The main idea we follow is that when a trajectory Y_t is located at a distance of order δ^2 from M^δ on a time interval $[T_1, T_2]$, then it is possible to make an accurate expansion of

$$\int_{T_1}^{T_2} \|\dot{Y}_t + \nabla V[Y_t] - \delta G[Y_t]\|^2 \,\mathrm{d}t\,. \tag{5.4.1}$$

Unfortunately we are not able to prove that a optimal trajectory linking A^δ to a point $B^\delta \in \partial U^\delta$ satisfying (5.1.18) stays at distance δ^2 from M^δ. However we are able to prove that such a optimal path stays at distance of order δ^2 for most of the time. To do that we rely on the fact that the optimal paths satisfy the Euler Lagrange type equation (5.3.24). When $\delta = 0$ (5.3.24) reduces to

$$\ddot{Y}_t \;=\; H[Y_t]\nabla V[Y_t]\,, \tag{5.4.2}$$

and a solution of (5.4.2) starting in a neighborhood of M but not in M moves away from M. Indeed if $q(\theta)$ is the projection on M of a point Y located in a neighborhood of M, then

$$H[Y]\nabla V[Y] \;=\; H[Y]^2(Y - q(\theta)) + O(\|Y - q(\theta)\|^2)\,, \tag{5.4.3}$$

and (5.1.11) and the spectral gap (5.1.12) imply that the vector $H[Y]^2(Y - q(\theta))$ lies in the normal space of M at the point $q(\theta)$ and has a norm bounded from below by $\lambda^2\|Y - q(\theta)\|$. Using perturbation arguments, we show that when $\delta \neq 0$, a solution of the Euler Lagrange type equation (5.3.24) starting from a point located at a distance from M^δ bounded from below by $C\delta^2$ behaves similarly: it moves away from M^δ. So, for an optimal path Y^δ associated to a point $B^\delta \in \partial U^\delta$ realizing the minimum of $W_\delta(A^\delta, \cdot)$ on ∂U^δ, there exists a time $\tau_1^\delta \leqslant 0$ (maybe equal to 0) such that for $t \leqslant \tau_1^\delta$ Y^δ is located at distance of order δ^2 from M^δ, and if $\tau_1^\delta < 0$ then for $t > \tau_1^\delta$ $\mathrm{dist}(Y_t^\delta, M^\delta) \geqslant C\delta^2$. It is the purpose of Lemma 5.4.5, where we also control $|\tau_1^\delta|$ when it is non null. The Lemmas 5.4.1, 5.4.3 and 5.4.4 are intermediate results needed to prove Lemma 5.4.5. In Lemma 5.4.6 we control the derivate in time of Y^δ for $t \leqslant \tau_1^\delta$.

In Lemma 5.4.7 we control the behavior of the optimal paths Y^δ on the time interval $[\tau_1^\delta, 0]$ (if $\tau_1^\delta < 0$). The estimations we obtain are sufficient to allow a good expansion of

$$\int_{\tau_1^\delta}^{0} \|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2 \,\mathrm{d}t\,. \tag{5.4.4}$$

Finally the expansion of (5.4.1) we are able to make depends on the length of the time interval $[T_1, T_2]$, so we can not simply take $T_1 = -\infty$ and $T_2 = \tau_1^\delta$. We have to restrict the expansion on a time interval $[\tau_0^\delta, \tau_1^\delta]$, choosing τ_0^δ in such a way that $\int_{-\infty}^{\tau_0^\delta} \|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2 \,\mathrm{d}t$ is negligible.

5.4.2 Preliminary results

We can easily find a first upper bound for $W_\delta^{red}(\varphi_{A^\delta}, \varphi)$ for all φ: since there exists $C > 0$ such that for δ small enough $|b_\delta| \leqslant C\delta$ (recall the definition of b_δ (5.1.15)) and

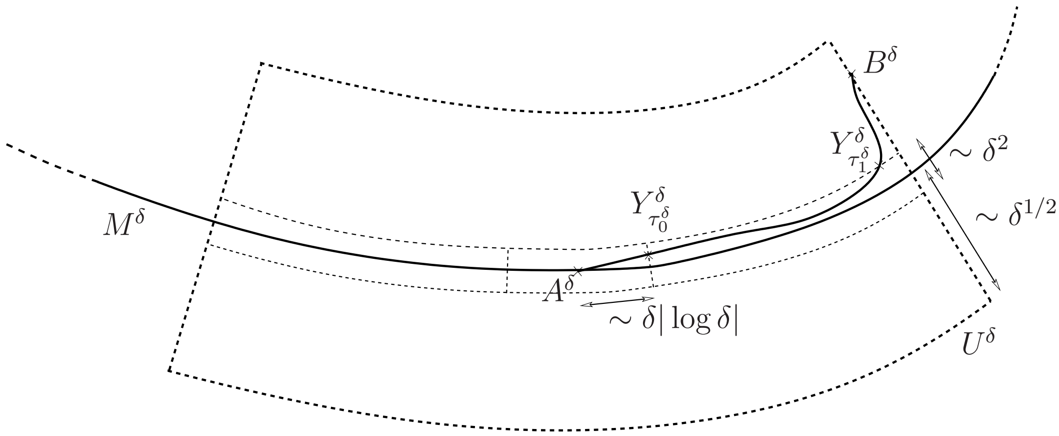


FIGURE 5.2. The times τ_0^δ and τ_1^δ for an optimal path Y^δ linking A^δ to B^δ.

thus, since the lengths of the curves M^δ are also bounded, there exists $C_1 > 0$ such that for all δ small enough

$$\sup_{\varphi \in \mathbb{R}/L_\delta\mathbb{R}} W_\delta^{red}(\varphi_{A^\delta}, \varphi) \leqslant C_1 \delta\,. \tag{5.4.5}$$

Recalling (5.1.23) it implies in particular that for $B^\delta \in \partial U^\delta$ satisfying (5.1.18) we have

$$W_\delta(A^\delta, B^\delta) \leqslant C_1\delta\,. \tag{5.4.6}$$

The spectral gap (5.1.12) implies that for points E sufficiently close to M we have

$$V[E] \geqslant \frac{\lambda}{2}(\mathrm{dist}(E, M))^2\,. \tag{5.4.7}$$

So we can choose the value of C_0 (recall the definition of U^δ in Theorem 5.1.1) such that for δ small enough

$$\inf_{\mathrm{dist}(E,M^\delta)=C_0\delta^{1/2}} V[E] \geqslant 2C_1\delta\,. \tag{5.4.8}$$

This choice of C_0 will be useful in the proof of Lemma 5.4.7 below, to ensure that the projection $p_\delta(B^\delta)$ coincides either with $\varphi_{A^\delta} - \Delta_1$ or with $\varphi_{A^\delta} + \Delta_2$.

In the following B^δ will denote a point of ∂U^δ realizing the minimum of $W_\delta(A^\delta, \cdot)$ on ∂U^δ, and Y^δ an associated optimal path. Since M^δ is located at a distance of order δ from M (see Theorem 5.2.4), this implies (recall the coordinates introduced in Section 5.2.4) that h_t^δ is of order $\delta^{1/2}$ for all t: there exists $C > 0$ such that

$$\sup_{t \leqslant 0} \|h_t^\delta\| \leqslant C\delta^{1/2}\,. \tag{5.4.9}$$

Y^δ converges to A^δ when $t \to -\infty$, so Lemma 5.3.5 ensures that its second derivate is bounded on $(-\infty, 0)$, and thus $\|\dot{Y}^\delta\|$ is also bounded and since Y^δ stays in a compact, $\|\dot{Y}^\delta\|$ reaches its maximum on $(-\infty, 0)$. The following Lemmas give some properties satisfied by Y^δ and its derivative in time. We will drop the dependence in the initial value in the large deviation rate for simplicity:

$$I_{\delta,T}(Z) := I_{\delta,T}^{Z_T}(Z)\,. \tag{5.4.10}$$

The constant C is a generic constant independent from δ, and whose value may change during the proof.

Lemma 5.4.1. *There exists $C_2 > 0$ such that for all δ small enough*

$$\sup_{t \leqslant 0} \|\dot{Y}_t^\delta\| \leqslant C_2 \delta^{1/2} . \tag{5.4.11}$$

Remark 5.4.2. *Due to the smoothness of the projection p (see Lemma 5.2.1), the result of the previous Lemma is also true for the coordinates θ^δ and h^δ associated with Y^δ:*

$$\sup_{t \leqslant 0} \{|\dot{\theta}_t^\delta|, \|\dot{h}_t^\delta\|\} = O(\delta^{1/2}) . \tag{5.4.12}$$

The same argument will be true for Lemma 5.4.4 below.

Proof. From Lemma 5.3.5, (5.1.10) and (5.4.9) we deduce that there exists $C > 0$ such that

$$\|\ddot{Y}_t^\delta\| \leqslant C(\delta^{1/2} + \delta\|\dot{Y}_t^\delta\|) . \tag{5.4.13}$$

Now suppose that $\sup_{t \leqslant 0} \|\dot{Y}_t^\delta\|$ is reached at $\|\dot{Y}_{t_0}^\delta\|$. Then, for $s \in [t_0 - 1, t_0]$, $\|\dot{Y}_s^\delta\|$ satisfies

$$\|\dot{Y}_s^\delta\| \geqslant \|\dot{Y}_{t_0}^\delta\| - C(\delta^{1/2} + \delta\|\dot{Y}_{t_0}^\delta\|) . \tag{5.4.14}$$

So for δ small enough

$$\|\dot{Y}_s^\delta\| \geqslant \frac{1}{2}\|\dot{Y}_{t_0}^\delta\| - C\delta^{1/2} . \tag{5.4.15}$$

Using the elementary bound $(a+b)^2 \geqslant \frac{1}{2}a^2 - b^2$, we can bound $I_{\delta,-\infty}^{A^\delta}(Y^\delta)$ from below (recall (5.1.8)):

$$I_{\delta,-\infty}(\varphi^\delta) \geqslant \frac{1}{4}\int_{t_0-1}^{t_0} \|\dot{Y}_t^\delta\|^2 \, \mathrm{d}t - \frac{1}{2}\int_{t_0-1}^{t_0} \|\nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2 \, \mathrm{d}t . \tag{5.4.16}$$

Taking into account (5.4.15), (5.1.10) and (5.4.9) we get

$$I_{\delta,-\infty}(\varphi^\delta) \geqslant \frac{1}{4}\left(\frac{1}{2}\|\dot{Y}_{t_0}^\delta\| - C\delta^{1/2}\right)^2 - C\delta . \tag{5.4.17}$$

So there exists $C_2 > 0$ such that if $\dot{Y}_{t_0}^\delta \geqslant C_2\delta^{1/2}$, (5.4.17) contradicts (5.4.6). $\square$

Lemma 5.4.3. *There exists $C_3 > 0$ such that for all δ small enough, if we define*

$$\tau_2^\delta := \inf\{t < 0, \|h_t^\delta - \phi_\delta(\theta_t^\delta)\| \geqslant C_3\delta\} , \tag{5.4.18}$$

then $\|h_t^\delta - \phi_\delta(\theta_t^\delta)\| \geqslant C_3\delta$ for all $\tau_2^\delta \leqslant t \leqslant 0$, and if $\tau_2^\delta < 0$ then $\|h_t^\delta - \phi_\delta(\theta_t^\delta)\|$ increases strictly on $(\tau_2^\delta, 0]$.

Proof. Using Lemma 5.2.5, (5.1.10) and (5.4.9) we get

$$\begin{aligned} H[Y_t^\delta](\nabla V[Y_t^\delta] - \delta G[Y_t^\delta]) &= H[q(\theta_t^\delta)](H[q(\theta_t^\delta)]h_t^\delta - \delta G[q(\theta_t^\delta)]) + O(\delta) \\ &= \Big(H[q(\theta_t^\delta)]\Big)^2 (h_t^\delta - \phi_\delta(\theta_t^\delta)) + O(\delta) , \end{aligned} \tag{5.4.19}$$

thus using furthermore Lemma 5.4.1 we get the following first order expansion of (5.3.24):

$$\ddot{Y}_t^\delta = \Big(H[q(\theta_t^\delta)]\Big)^2 (h_t^\delta - \phi_\delta(\theta_t^\delta)) + O(\delta) . \tag{5.4.20}$$

Now define

$$\alpha_t^\delta = \|h_t^\delta - \phi_\delta(\theta_t^\delta)\|^2 . \tag{5.4.21}$$

A straightforward calculation gives

$$\ddot{\alpha}_t^\delta = 2\|\dot{h}_t^\delta - \dot{\theta}_t^\delta \phi'_\delta(\theta_t^\delta)\|^2 + 2\langle h_t^\delta - \phi_\delta(\theta_t^\delta), \ddot{h}_t^\delta - \ddot{\theta}_t^\delta \phi'_\delta(\theta_t^\delta) - (\dot{\theta}_t^\delta)^2 \phi''_\delta(\theta_t^\delta)\rangle . \tag{5.4.22}$$

Using Lemma 5.2.1 we express $\dot{h}_t^\delta$ in with respect to $\dot{Y}_t^\delta$ and h_t^δ:

$$\dot{h}_t^\delta = \dot{Y}_t^\delta - \frac{1}{1-\langle h_t^\delta, q''(\theta_t^\delta)\rangle}\langle \dot{Y}_t^\delta, q'(\theta_t^\delta)\rangle q'(\theta_t^\delta) , \tag{5.4.23}$$

and after a derivation in time it leads to

$$\begin{aligned}\ddot{h}_t^\delta = \ddot{Y}_t^\delta - \dot{\theta}_t^\delta \frac{1}{1-\langle h_t^\delta, q''(\theta_t^\delta)\rangle} q''(\theta_t^\delta) + \Bigg(\frac{\langle \dot{h}_t^\delta, q''(\theta_t^\delta)\rangle + \dot{\theta}_t^\delta \langle h_t^\delta, q'''(\theta_t^\delta)\rangle}{(1-\langle h_t^\delta, q''(\theta_t^\delta)\rangle)^2} \\ - \frac{1}{1-\langle h_t^\delta, q''(\theta_t^\delta)\rangle}(\langle \dot{Y}_t^\delta, q'(\theta_t^\delta)\rangle + \dot{\theta}_t^\delta \langle Y_t^\delta, q''(\theta_t^\delta)\rangle\Bigg) q'(\theta_t^\delta) .\end{aligned} \tag{5.4.24}$$

So taking together (5.4.20), (5.4.22), (5.4.24), Lemma 5.4.1 and Remark (5.4.2), we get (recall $\langle h_t^\delta, q'(\theta_t^\delta)\rangle = 0$)

$$\ddot{\alpha}_t^\delta = 2\|\dot{h}_t^\delta - \dot{\theta}_t^\delta \phi'_\delta(\theta_t^\delta)\|^2 + 2\|H[q(\theta_t^\delta)](h_t^\delta - \phi_\delta(\theta_t^\delta))\|^2 + O(\delta\|h_t^\delta - \phi_\delta(\theta_t^\delta)\|) , \tag{5.4.25}$$

where the second term of the right hand side is bounded below by $\lambda^2\|h_t^\delta - \phi_\delta(\theta_t^\delta)\|^2$, due to (5.1.12). So there exists C_3 such that if $\|h_t^\delta - \phi_\delta(\theta_t^\delta)\| \geqslant C_3\delta$ and $\dot{\alpha}_t^\delta \geqslant 0$ for a $t<0$, then α_s^δ is strictly increasing for $s>t$. □

Lemma 5.4.4. *There exists $C_4 > 0$ such that for all δ small enough*

$$\sup_{t \leqslant \tau_2^\delta} \|\dot{Y}_t^\delta\| \leqslant C_4\delta . \tag{5.4.26}$$

Proof. From Lemma 5.3.5, Lemma 5.4.3 and (5.1.10) we deduce that there exists $C>0$ such that for $t \leqslant \tau_2^\delta$

$$\|\ddot{Y}_t^\delta\| \leqslant C(\delta + \delta\|\dot{Y}_t^\delta\|) . \tag{5.4.27}$$

Suppose that $\sup_{t \leqslant \tau_2^\delta} \|\dot{Y}_t^\delta\|$ is reached at $\|\dot{Y}_{t_0}^\delta\|$. For δ small enough, for $s \in [t_0-1, t_0]$ we have

$$\|\dot{Y}_s^\delta\| \geqslant \frac{1}{2}\|\dot{Y}_{t_0}^\delta\| - C\delta . \tag{5.4.28}$$

Proceeding as in Lemma 5.4.1, we can bound the cost of the path Y^δ on the time interval $[t_0-1, t_0]$ from below by

$$\frac{1}{4}\int_{t_0-1}^{t_0} \|\dot{Y}_t^\delta\|^2\, \mathrm{d}t - \frac{1}{2}\int_{t_0-1}^{t_0} \|\nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2\, \mathrm{d}t \geqslant \frac{1}{4}\left(\frac{1}{2}\|\dot{Y}_{t_0}^\delta\| - C\delta\right)^2 - C\delta^2 . \tag{5.4.29}$$

On the other hand, we consider the path defined on the time interval $[0, \|\dot{Y}_t^\delta\|/\delta]$ by

$$Z_t = Y^\delta_{t_0-1+\delta t/\|\dot{Y}_t^\delta\|} , \tag{5.4.30}$$

which connects the points $Y^\delta_{t_0-1}$ and $Y^\delta_{t_0}$. Using Lemma 5.4.3 we can bound the cost of this path:

$$\int_0^{\|\dot{Y}^\delta_t\|/\delta} \|\dot{Z}_t + \nabla V[Z_t] - \delta G[Z_t]\|^2 \,\mathrm{d}t \;\leqslant\; C\frac{\|\dot{Y}^\delta_t\|}{\delta}\left(\sup_{t\in[0,\|\dot{Y}^\delta_t\|/\delta]} \|\dot{Z}_t\|^2 + \delta^2\right) \;\leqslant\; C\delta\|\dot{Y}^\delta_t\|\,. \tag{5.4.31}$$

So if $\|\dot{Y}^\delta_t\|$ is too large, that is $\|\dot{Y}^\delta_t\| \geqslant C_4\delta$ for C_4 large enough, we get a contradiction between (5.4.29) and (5.4.31): by distending time and replacing $Y^\delta_{|(t_0-1,t_0)}$ by Z we can create better path than Y^δ for B^δ. □

Lemma 5.4.5. *There exists $C_5 > 0$ such that for all δ small enough, if we define*

$$\tau^\delta_1 \;:=\; \inf\{t < 0, \|h^\delta_t - \phi_\delta(\theta^\delta_t)\| \geqslant C_5\delta^2\}\,, \tag{5.4.32}$$

then $\|h^\delta_t - \phi_\delta(\theta^\delta_t)\| \geqslant C_5\delta^2$ for all $\tau^\delta_1 \leqslant t \leqslant 0$ and if $\tau^\delta_1 < 0$, $\|h^\delta_t - \phi_\delta(\theta^\delta_t)\|$ increases strictly on $(\tau^\delta_1, 0]$. Moreover there exists $C_6 > 0$ such that

$$\tau^\delta_1 \;\geqslant\; -C_6|\log\delta|\,. \tag{5.4.33}$$

Proof. We proceed as for Lemma 5.4.3. Using Lemma 5.2.5, Lemma 5.4.3, Lemma 5.4.4 and (5.1.10) we get for $t \leqslant \tau^\delta_2$ the following first order expansion of (5.3.24):

$$\ddot{Y}^\delta_t \;=\; \Big(H[q(\theta^\delta_t)]\Big)^2(h^\delta_t - \phi_\delta(\theta^\delta_t)) + O(\delta^2)\,. \tag{5.4.34}$$

This time we have (recall (5.4.21))

$$\ddot{\alpha}^\delta_t \;=\; 2\|\dot{h}^\delta_t - \dot{\theta}^\delta_t\phi'_\delta(\theta^\delta_t)\|^2 + 2\|H[q(\theta^\delta_t)](h^\delta_t - \phi_\delta(\theta^\delta_t))\|^2 + O(\delta^2\|h^\delta_t - \phi_\delta(\theta^\delta_t)\|)\,. \tag{5.4.35}$$

So (recall (5.1.12)) there exists $C > 0$ such that

$$\ddot{\alpha}^\delta_t \;\geqslant\; 2\lambda^2\alpha^\delta_t - C\delta^2(\alpha^\delta_t)^{1/2}\,. \tag{5.4.36}$$

In particular, for $(\alpha^\delta_t)^{1/2} \geqslant C_5\delta^2$ where $C_5 := C/\lambda^2$, we have

$$\ddot{\alpha}^\delta_t \;\geqslant\; \lambda^2\alpha^\delta_t\,. \tag{5.4.37}$$

We deduce that if $\|h^\delta_t - \phi_\delta(\theta^\delta_t)\| \geqslant C_5\delta^2$ and $\dot{\alpha}^\delta_t \geqslant 0$ for a $t < 0$, then α^δ_s is strictly increasing for $s > t$. Moreover, it takes at most a time of order $|\log\delta|$ for α^δ to reach $C_0\delta^{1/2}$.

□

Lemma 5.4.6. *There exists $C_7 > 0$ such that for all δ small enough*

$$\sup_{t\leqslant\tau^\delta_1} \|\dot{h}^\delta_t\| \;\leqslant\; C_7\delta^2\,. \tag{5.4.38}$$

Proof. Suppose that $\sup_{t \leqslant \tau_1^\delta} \|\dot{h}_t^\delta\|$ is reached at $\|\dot{h}_{t_0}^\delta\|$. Then the mean value Theorem implies

$$\Big\| h_{t_0-1}^\delta - \phi_\delta(\theta_{t_0-1}^\delta) - \Big(h_{t_0}^\delta - \phi_\delta(\theta_t^\delta) + \dot{h}_{t_0}^\delta - \dot{\theta}_{t_0}^\delta \phi_\delta'(\theta_{t_0}^\delta)\Big)\Big\| \leqslant \sup_{t\in[t_0-1,t_0]} \|\ddot{h}_t^\delta - \ddot{\theta}_t^\delta \phi_\delta'(\theta_t^\delta) - (\dot{\theta}_t^\delta)^2 \phi_\delta''(\theta_t^\delta)\|\,. \quad (5.4.39)$$

Theorem 5.2.4, Lemma 5.4.4, Lemma 5.4.5 and (5.4.34) imply that the right hand side is of order δ^2. But the same arguments imply that all the right hand side with $\dot{h}_{t_0}^\delta$ taken away is also of order δ^2. We deduce that $\dot{h}_{t_0}^\delta$ must also be of order δ^2. □

Lemma 5.4.7. *For δ small enough, the projection $p_\delta(B^\delta)$ on M^δ coincides either with $\varphi_{A^\delta} - \Delta_1$ or with $\varphi_{A^\delta} + \Delta_2$. Moreover, there exist C_8, C_9 and C_{10} such that for δ small enough and for all $\tau_1^\delta \leqslant t \leqslant 0$*

$$|\dot{\theta}_t^\delta| \leqslant C_8 \delta |\log\delta|\,, \quad (5.4.40)$$

$$\|h_t^\delta\| \leqslant C_9 \delta |\log\delta|\,, \quad (5.4.41)$$

and

$$\|\dot{Y}_t^\delta\| \leqslant C_{10} \delta |\log\delta|\,. \quad (5.4.42)$$

Proof. Isolating the δG term in $I_{\delta,\tau_1^\delta}(Y^\delta)$, we get

$$I_{\delta,\tau_1^\delta}(Y^\delta) \geqslant \frac{1}{4}\int_{\tau_1^\delta}^0 \|\dot{Y}_t^\delta + \nabla V[Y_t^\delta]\|^2\,\mathrm{d}t - \frac{\delta^2}{2}\int_{\tau_1^\delta}^0 \|G[Y_t^\delta]\|^2\,\mathrm{d}t\,. \quad (5.4.43)$$

Thanks to Lemma 5.4.5, we know that the last term of the right hand side is of order $\delta^2|\log\delta|$, so inserting the reversed time dynamics (using the identity $\|u-v\|^2 = \|u+v\|^2 - 4\langle u, v\rangle$), we obtain for a $C > 0$

$$I_{\delta,\tau_1^\delta}(Y^\delta) \geqslant \frac{1}{4}\int_{\tau_2^\delta}^0 \|\dot{Y}_t^\delta - \nabla V[Y_t^\delta]\|^2\,\mathrm{d}t + V[B^\delta] - V[Y_{\tau_1^\delta}^\delta] - C\delta^2|\log\delta|\,. \quad (5.4.44)$$

Since $Y_{\tau_1^\delta}^\delta$ is located at a distance of order δ from M, $V[Y_{\tau_1^\delta}^\delta]$ is of order δ^2. Thus (5.4.44) implies

$$I_{\delta,\tau_1^\delta}(Y^\delta) \geqslant V[B^\delta] - C\delta^2|\log\delta|\,. \quad (5.4.45)$$

Taking (5.4.6) and (5.4.8) into account, this implies the first assertion of the Lemma for δ small enough.

Lemma 5.2.1 implies for all $t \leqslant 0$

$$\dot{\theta}_t^\delta = \frac{1}{1 - \langle h_t^\delta, q''(\theta_t^\delta)\rangle}\langle \dot{Y}_t^\delta, q'(\theta_t^\delta)\rangle\,. \quad (5.4.46)$$

So a derivation in time gives

$$\ddot{\theta}_t^\delta = \frac{1}{1 - \langle h_t^\delta, q''(\theta_t^\delta)\rangle}\langle \ddot{Y}_t^\delta, q'(\theta_t^\delta)\rangle + \frac{\dot{\theta}_t^\delta}{1 - \langle h_t^\delta, q''(\theta_t^\delta)\rangle}\langle \dot{Y}_t^\delta, q''(\theta_t^\delta)\rangle + \frac{\langle \dot{h}_t^\delta, q''(\theta_t^\delta)\rangle + \dot{\theta}_t^\delta \langle h_t^\delta, q'''(\theta_t^\delta)\rangle}{(1 - \langle h_t^\delta, q''(\theta_t^\delta)\rangle)^2}\langle \dot{Y}_t^\delta, q'(\theta_t^\delta)\rangle\,. \quad (5.4.47)$$

Lemma 5.4.1 implies that the two last terms of the right hand side are of order δ. Moreover (5.4.20) and (5.1.11) imply that the first term of the right hand side is also of order δ. We deduce that theres exists $C > 0$ such that for all $\tau_1^\delta \leqslant t \leqslant 0$

$$|\dot\theta_t^\delta - \dot\theta_{\tau_1^\delta}^\delta| \leqslant C\delta|\log\delta| . \tag{5.4.48}$$

To deduce (5.4.40), we just have to remind Lemma 5.4.4, and more precisely that $\dot\theta_{\tau_1^\delta}^\delta$ is of order δ.

To get the two last results of Lemma 5.4.7, we find an upper bound for $I_{\delta,\tau_1^\delta}(Y^\delta)$ by studying the cost of a specifically chosen path. We consider a path Z_t defined on the time interval $[T,0]$, starting from $Y_{\tau_1^\delta}^\delta$, linking in a linear way $Y_{\tau_1^\delta}^\delta$ to $\widetilde{q}_\delta(\theta_{\tau_1^\delta}^\delta)$, and then following the curve M^δ to exit U^δ at the point $q_\delta(\varphi_{B^\delta})$ (we know that it is possible to exit U^δ this way, thanks to the first assertion of the Lemma). Since Y^δ is an optimal path for the exit from U^δ, it is clear that $\int_T^0 \|\dot Z_t + \nabla V[Z_t] - \delta G[Z_t]\|^2\,\mathrm{d}t$ is an upper bound for $I_{\delta,\tau_1^\delta}(Y^\delta)$. More precisely we define for $T \leqslant t \leqslant T+\delta$:

$$Z_t := Y_{\tau_1^\delta}^\delta + \frac{t-T}{\delta}\Big(\widetilde{q}_\delta(\theta_{\tau_1^\delta}^\delta) - Y_{\tau_1^\delta}^\delta\Big) = Y_{\tau_1^\delta}^\delta - \frac{t-T}{\delta}h_{\tau_1^\delta}^\delta . \tag{5.4.49}$$

In this case, using (5.1.10) and the fact that $h_{\tau_1^\delta}^\delta$ is of order δ^2, we get the bound

$$\int_T^{T+\delta} \|\dot Z_t + \nabla V[Z_t] - \delta G[Z_t]\|^2\,\mathrm{d}t \leqslant 2\int_T^{T+\delta}\|\dot Z_t\|^2\,\mathrm{d}t + 2\int_T^{T+\delta}\|\nabla V[Z_t] - \delta G[Z_t]\|^2\,\mathrm{d}t = O(\delta^3) \tag{5.4.50}$$

On the order hand, if we define $\varphi_{\tau_1^\delta}$ the phase satisfying $q_\delta(\varphi_{\tau_1^\delta}) = \widetilde{q}_\delta(\theta_{\tau_1^\delta}^\delta)$, since for $t \geqslant T+\delta$ Z_t follows the curve M^δ, we can make $\int_{T+\delta}^0 \|\dot Z_t + \nabla V[Z_t] - \delta G[Z_t]\|^2\,\mathrm{d}t$ as close to $\int_{\varphi_{\tau_1^\delta}}^{\varphi_{B^\delta}} b_\delta(\varphi)\,\mathrm{d}\varphi$ as we want. But (5.4.44) implies that $\theta_0^\delta - \theta_{\tau_1^\delta}^\delta$ is of order $\delta|\log\delta|^2$, and Lemma 5.2.6 implies that $\varphi_{B^\delta} - \varphi_{\tau_1^\delta}$ is of the same order. We deduce that $\int_{\varphi_{\tau_1^\delta}}^{\varphi_{B^\delta}} b_\delta(\varphi)\,\mathrm{d}\varphi$ is of order $\delta^2|\log(\delta)|^2$, and thus that $I_{\delta,\tau_1^\delta}(Y^\delta)$ is also at most of order $\delta^2|\log(\delta)|^2$.

Recalling (5.4.45), (5.1.10) and (5.1.11), we deduce that $\mathrm{dist}(B^\delta, M^\delta)$ is at most of order $\delta|\log\delta|$, and thus we get (5.4.41), since M^δ is located at a distance of order δ from M. The proof of (5.4.42) is similar to the proof of Lemma 5.4.4, keeping (5.4.41) in mind. □

5.4.3 Proof of Theorem 5.1.1

We will show that $I_{\delta,-\infty}(Y^\delta)$ can be well approximated by $I_{\delta,\tau_0^\delta}(Y^\delta)$ with a good choice of $\tau_0^\delta \leqslant \tau_1^\delta$, and then we will study the cost of the path Y^δ on the time intervals $[\tau_0^\delta, \tau_1^\delta]$ (when $\tau_0^\delta < \tau_1^\delta$) and $[\tau_1^\delta, 0]$.

Recalling (5.1.10), Lemma 5.2.5, Lemma 5.4.5 and Lemma 5.4.4, we can expand

(5.3.24) in the following way for all $t \leqslant \tau_1^\delta$:

$$\begin{aligned}\ddot{Y}_t^\delta \; = \; & \Big(H[q(\theta_t^\delta)] + D^3V[q(\theta_t^\delta)](h_t^\delta, .,.) - \delta DG^\dagger[q(\theta_t^\delta)] + O(\delta^2)\Big) \\ & \times \Big(H[q(\theta_t^\delta)]h_t^\delta - \delta\Big(G[q(\theta_t^\delta)] - \langle G[q(\theta_t^\delta)], q'(\theta_t^\delta)\rangle q'(\theta_t^\delta)\Big) + D^3V[q(\theta_t^\delta)](h_t^\delta, h_t^\delta, .) \\ & \quad - \delta\langle G[q(\theta_t^\delta)], q'(\theta_t^\delta)\rangle q'(\theta_t^\delta) - \delta DG[q(\theta_t^\delta)]h_t^\delta + O(\delta^3)\Big) \\ & + \delta\Big(DG[q(\theta_t^\delta)] - DG^\dagger[q(\theta_t^\delta)]\Big)\dot{Y}_t^\delta + O(\delta^3)\,, \end{aligned} \qquad (5.4.51)$$

and (recall (5.1.11)) we expand the product in the following way:

$$\begin{aligned}\ddot{Y}_t^\delta \; = \; & H[q(\theta_t^\delta)]\Big(H[q(\theta_t^\delta)]h_t^\delta - \delta\Big(G[q(\theta_t^\delta)] - \langle G[q(\theta_t^\delta)], q'(\theta_t^\delta)\rangle q'(\theta_t^\delta)\Big)\Big) \\ & + H[q(\theta_t^\delta)]D^3V[q(\theta_t^\delta)](h_t^\delta, h_t^\delta, .) + \delta H[q(\theta_t^\delta)]DG[q(\theta_t^\delta)]h_t^\delta \\ & + \delta\langle G[q(\theta_t^\delta)], q'(\theta_t^\delta)\rangle D^3V[q(\theta_t^\delta)](h_t^\delta, q'(\theta_t^\delta), .) + \delta^2\langle G[q(\theta_t^\delta)], q'(\theta_t^\delta)\rangle DG^\dagger[q(\theta_t^\delta)]q'(\theta_t^\delta) \\ & + \delta\Big(DG[q(\theta_t^\delta)] - DG^\dagger[q(\theta_t^\delta)]\Big)\dot{Y}_t^\delta + O(\delta^3) \end{aligned} \qquad (5.4.52)$$

Using again (5.1.11) we get the projection

$$\begin{aligned}\langle \ddot{Y}_t^\delta, q'(\theta_t^\delta)\rangle \; = \; & \delta\Big\langle G[q(\theta_t^\delta)], q'(\theta_t^\delta)\Big\rangle D^3V[q(\theta_t^\delta)](h_t^\delta, q'(\theta_t^\delta), q'(\theta_t^\delta)) \\ & + \delta^2\Big\langle G[q(\theta_t^\delta)], q'(\theta_t^\delta)\Big\rangle\Big\langle DG^\dagger[q(\theta_t^\delta)]q'(\theta_t^\delta), q'(\theta_t^\delta)\Big\rangle \\ & + \delta\Big\langle\Big(DG[q(\theta_t^\delta)] - DG^\dagger[q(\theta_t^\delta)]\Big)\dot{Y}_t^\delta, q'(\theta_t^\delta)\Big\rangle + O(\delta^3)\,. \end{aligned} \qquad (5.4.53)$$

But Lemma 5.4.6 implies the following first order expansion of $\dot{Y}_t^\delta$ for $t \leqslant \tau_1^\delta$:

$$\dot{Y}_t^\delta \; = \; \dot{\theta}_t^\delta q'(\theta_t^\delta) + O(\delta^2)\,, \qquad (5.4.54)$$

and thus the first term of the last line in (5.4.53) is in fact

$$\begin{aligned}\Big\langle\Big(DG[q(\theta_t^\delta)] - DG^\dagger[q(\theta_t^\delta)]\Big)\dot{Y}_t^\delta, q'(\theta_t^\delta)\Big\rangle \; = \; & \dot{\theta}_t^\delta\langle DG[q(\theta_t^\delta)]q'(\theta_t^\delta), q'(\theta_t^\delta)\rangle \\ & - \dot{\theta}_t^\delta\langle q'(\theta_t^\delta), DG[q(\theta_t^\delta)]q'(\theta_t^\delta)\rangle + O(\delta^3) \; = \; O(\delta^3). \end{aligned} \qquad (5.4.55)$$

Applying (5.2.38) to (5.4.53), we get

$$\begin{aligned}\langle \ddot{Y}_t^\delta, q'(\theta_t^\delta)\rangle \; = \; & \delta^2\Big(\langle DG[q(\theta_t^\delta)q'(\theta_t^\delta), q'(\theta_t^\delta)\rangle + \langle G[q(\theta_t^\delta)], q''(\theta_t^\delta)\rangle\Big)\langle G[q(\theta_t^\delta)], q'(\theta_t^\delta)\rangle + O(\delta^3) \\ = \; & \delta^2 b'(\theta_t^\delta)b(\theta_t^\delta) + O(\delta^3)\,. \end{aligned} \qquad (5.4.56)$$

Now recalling (5.4.47), Lemma 5.4.5, Lemma 5.4.4 and Lemma 5.4.6 we get the same expansion for $\ddot{\theta}_t^\delta$: for all $t \leqslant \tau_1^\delta$ we have

$$\ddot{\theta}_t^\delta \; = \; \delta^2 b'(\theta_t^\delta)b(\theta_t^\delta) + O(\delta^3)\,. \qquad (5.4.57)$$

Remind that we have supposed that the dynamics $\dot{\theta} = b(\theta)$ (and equivalently the dynamics $\dot{\theta} = \delta b(\theta)$) admits an hyperbolic stable fixed point θ^0, and remind (recall Lemma 5.2.6

and Lemma 5.2.7) that $\theta_{A^\delta} := p(A^\delta)$ is located at a distance of order δ from θ^0. So when θ_t^δ is close to θ_{A^δ}, though at a distance greater than δ, the leading term in (5.4.57) is up to a constant factor equivalent to $\delta^2(\theta_t^\delta - \theta_{A^\delta})$. If we suppose (without loss of generality) that $p_\delta(B^\delta)$ is located on the side of the increasing θ with respect to θ_{A^δ}, and we define τ_0^δ the first time $t \leqslant \tau_1^\delta$ such that $\theta_t^\delta - \theta_{A^\delta}^\delta = \delta|\log\delta|$, then θ_t^δ is exponentially increasing (at rate $\exp(C\delta t)$) for $t \geqslant \tau_0^\delta$, at least until it reaches $1/|\log\delta|$ at a time t_1. So $t_1 - \tau_0^\delta = O(|\log\delta|/\delta)$. We now want to bound the difference $\tau_1^\delta - \tau_0^\delta$. If $t_1 < \tau_1^\delta$, then for $t_1 \leqslant t \leqslant \tau_1^\delta$, by multiplying each side of (5.4.57) by $\dot\theta_t^\delta$ and integrating in time, we get (recall that $\dot\theta_t^\delta = O(\delta)$ for $t \leqslant \tau_1^\delta$)

$$(\dot\theta_t^\delta)^2 - (\dot\theta_{\tau_0^\delta}^\delta)^2 \;=\; \delta^2 b^2(\theta_t^\delta) - \delta^2 b^2(\theta_{\tau_0^\delta}^\delta) + O(\delta^4(t-\tau_0^\delta))\,. \tag{5.4.58}$$

By construction $b(\theta_{\tau_0^\delta}^\delta) = O(\delta|\log\delta|)$ and $b(\theta_{t_1})$ is up to a constant factor equivalent to $1/|\log\delta|$. Moreover, if we denote $\theta_{B^\delta} := p(B^\delta)$, since $[\theta_{t_1}^\delta, \theta_{B^\delta}]$ is strictly included in the domain of attraction of θ^0 for $\dot\theta = b(\theta)$, for δ small enough $|b(\theta)|$ is greater than $1/|\log\delta|$ for $\theta \in [\theta_{t_1}^\delta, \theta_{B^\delta}]$. Thus θ_t^δ keeps increasing for $t \geqslant t_1$ and (5.4.58) implies

$$\dot\theta_t^\delta \;\geqslant\; \delta|b(\theta_t^\delta)| + O(\delta^3|\log\delta|, \delta^3|\log\delta|(t-\tau_0^\delta))\,. \tag{5.4.59}$$

Dividing by $|b(\theta_t^\delta)|$ and integrating in time between t_1 and τ_1^δ, we see that $\tau_1^\delta - t_1 = O(|\log\delta|/\delta)$. We deduce

$$\tau_1^\delta - \tau_0^\delta \;=\; O(|\log\delta|/\delta)\,. \tag{5.4.60}$$

We easily see that the cost of the path Y^δ restricted to the times smaller than τ_0^δ is negligible. Since Y^δ is an optimal path, this cost is exactly $W(A^\delta, Y_{\tau_0^\delta}^\delta)$, and this quasipotential is of order $\delta^3(\log\delta)^2$. Indeed, reaching $\widetilde q_\delta(\theta_{\tau_0^\delta}^\delta)$ following M^δ costs $\int_{\varphi_{A^\delta}}^{\varphi_{\tau_0^\delta}} b_\delta(\varphi)\,\mathrm{d}\varphi$ where $\varphi_{\tau_0^\delta}$ satisfies $q(\varphi_{\tau_0^\delta}) = \widetilde q_\delta(\theta_{\tau_0^\delta}^\delta)$, and (recall Lemma 5.2.6 and the definition of τ_0^δ) $\varphi_{\tau_0^\delta} - \varphi_{A^\delta} \sim \delta|\log\delta|$. Since $b_\delta(\varphi_{A^\delta}) = 0$ and $b_\delta(\phi) = O(\delta)$, we indeed get $\int_{\varphi_{A^\delta}}^{\varphi_{\tau_0^\delta}} b_\delta(\varphi)\,\mathrm{d}\varphi = O(\delta^3|\log\delta|^2)$. Moreover it is possible to link linearly $q_\delta(\theta_{\tau_0^\delta}^\delta)$ to $Y_{\tau_0^\delta}^\delta$ at cost δ^3 (see the proof of Lemma 5.4.7).

We now study the cost of the Y_t^δ on the time interval $[\tau_0^\delta, \tau_1^\delta]$. It is possible to get a lower bound by projecting on the tangent space of M:

$$\frac12 \int_{\tau_0^\delta}^{\tau_1^\delta} \|\dot Y_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2\,\mathrm{d}t \;\geqslant\; \frac12 \int_{\tau_0^\delta}^{\tau_1^\delta} |\langle \dot Y_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta], q'(\theta_t^\delta)\rangle|^2\,\mathrm{d}t\,. \tag{5.4.61}$$

It is now enough to simply expand the integrand. Lemma 5.2.1, Lemma 5.4.5 and Lemma 5.4.4 imply

$$\langle \dot Y_t^\delta, q'(\theta_t^\delta)\rangle \;=\; \dot\theta_t^\delta(1 - \langle h_t^\delta, q''(\theta_t^\delta)\rangle) \;=\; \dot\theta_t^\delta(1 - \langle \phi_\delta(\theta_t^\delta), q''(\theta_t^\delta)\rangle) + O(\delta^3) \tag{5.4.62}$$

and (using moreover (5.1.10) and (5.1.11))

$$\begin{aligned}\langle \nabla V[Y_t^\delta] - \delta G[Y_t^\delta], q'(\theta_t^\delta)\rangle \;&=\; D^3V[q(\theta_t^\delta)](h_t^\delta, h_t^\delta, q'(\theta_t^\delta))\\ &\quad - \delta\langle G[q(\theta_t^\delta)] + DG[q(\theta_t^\delta)]h_t^\delta, q'(\theta_t^\delta)\rangle + O(\delta^2)\\ &=\; D^3V[q(\theta_t^\delta](\phi_\delta(\theta_t^\delta), \phi_\delta(\theta_t^\delta), q'(\theta_t^\delta))\\ &\quad - \delta\langle G[q(\theta_t^\delta)] + DG[q(\theta_t^\delta)]\phi_\delta(\theta_t^\delta), q'(\theta_t^\delta)\rangle + O(\delta^2)\,.\end{aligned} \tag{5.4.63}$$

Remark that Theorem 5.2.4, (5.1.10) and (5.1.11) also imply

$$\begin{aligned}\langle \nabla V[\widetilde{q}_\delta(\theta_t^\delta)] - \delta G[\widetilde{q}_\delta(\theta_t^\delta)], q'(\theta_t^\delta)\rangle &= D^3V[q(\theta_t^\delta)](\phi_\delta(\theta_t^\delta), \phi_\delta(\theta_t^\delta), q'(\theta_t^\delta)) \\ &\quad - \delta\langle G[q(\theta_t^\delta)] + DG[q(\theta_t^\delta)]\phi_\delta(\theta_t^\delta), q'(\theta_t^\delta)\rangle + O(\delta^2)\,, \end{aligned} \tag{5.4.64}$$

so

$$\langle \nabla V[Y_t^\delta] - \delta G[Y_t^\delta], q'(\theta_t^\delta)\rangle = \langle \nabla V[\widetilde{q}_\delta(\theta_t^\delta)] - \delta G[\widetilde{q}_\delta(\theta_t^\delta)], q'(\theta_t^\delta)\rangle + O(\delta^2)\,. \tag{5.4.65}$$

Since M^δ is invariant under (5.1.13), $\nabla V[\widetilde{q}_\delta(\theta_t^\delta)] - \delta G[\widetilde{q}_\delta(\theta_t^\delta)]$ and $\widetilde{q}'_\delta(\theta_t^\delta)$ are collinear vectors, and thus

$$\langle \nabla V[Y_t^\delta] - \delta G[Y_t^\delta], q'(\theta_t^\delta)\rangle = \langle \nabla V[\widetilde{q}_\delta(\theta_t^\delta)] - \delta G[\widetilde{q}_\delta(\theta_t^\delta)], \widetilde{q}'_\delta(\theta_t^\delta)\rangle \frac{\langle \widetilde{q}'_\delta(\theta_t^\delta), q'(\theta_t^\delta)\rangle}{\|\widetilde{q}'_\delta(\theta_t^\delta)\|^2} + O(\delta^2)\,. \tag{5.4.66}$$

Now $\langle \phi_\delta(\theta), q'(\theta)\rangle = 0$ implies $\langle \phi'_\delta(\theta), q'(\theta)\rangle = -\langle \phi_\delta(\theta), q''(\theta)\rangle$ and thus using Theorem 5.2.4 we get

$$\|\widetilde{q}'_\delta(\theta_t^\delta)\| = (1 + 2\langle \phi'_\delta(\theta_t^\delta), q'(\theta_t^\delta)\rangle + O(\delta^2))^{1/2} = 1 - \langle \phi_\delta(\theta_t^\delta), q''(\theta_t^\delta)\rangle + O(\delta^2)\,, \tag{5.4.67}$$

and similarly

$$\langle \widetilde{q}'_\delta(\theta_t^\delta), q'(\theta_t^\delta)\rangle = 1 - \langle \phi_\delta(\theta_t^\delta), q''(\theta_t^\delta)\rangle + O(\delta^2) = \|\widetilde{q}'_\delta(\theta_t^\delta)\| + O(\delta^2)\,. \tag{5.4.68}$$

So in view of (5.4.62), (5.4.66), (5.4.67), (5.4.68) and (5.4.60), (5.4.61) becomes

$$\begin{aligned}&\frac{1}{2}\int_{\tau_0^\delta}^{\tau_1^\delta} \|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2 \,\mathrm{d}t \\ &\quad\geqslant \frac{1}{2}\int_{\tau_0^\delta}^{\tau_1^\delta} \left|\dot{\theta}_t^\delta \|\widetilde{q}'_\delta(\theta_t^\delta)\| - \left\langle \nabla V[\widetilde{q}_\delta(\theta_t^\delta)] - \delta G[\widetilde{q}_\delta(\theta_t^\delta)], \frac{\widetilde{q}'_\delta(\theta_t^\delta)}{\|\widetilde{q}'_\delta(\theta_t^\delta)\|}\right\rangle\right|^2 \mathrm{d}t + O(\delta^3|\log\delta|)\,.\end{aligned} \tag{5.4.69}$$

We proceed similarly on the time interval $[\tau_1^\delta, 0]$: we also use the lower bound

$$\frac{1}{2}\int_{\tau_1^\delta}^{0} \|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2 \,\mathrm{d}t \geqslant \frac{1}{2}\int_{\tau_1^\delta}^{0} |\langle \dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta], q'(\theta_t^\delta)\rangle|^2 \,\mathrm{d}t\,. \tag{5.4.70}$$

This time, using Lemma 5.4.7, (5.4.67) and (5.4.68) we get

$$\dot{Y}_t^\delta = \dot{\theta}_t^\delta \|\widetilde{q}'_\delta(\theta_t^\delta)\| + O(\delta^2(\log\delta)^2) \tag{5.4.71}$$

and

$$\langle \nabla V[Y_t^\delta] - \delta G[Y_t^\delta], q'(\theta_t^\delta)\rangle = \left\langle \nabla V[\widetilde{q}_\delta(\theta_t^\delta)] - \delta G[\widetilde{q}_\delta(\theta_t^\delta)], \frac{\widetilde{q}'_\delta(\theta_t^\delta)}{\|\widetilde{q}'_\delta(\theta_t^\delta)\|}\right\rangle + O(\delta^2(\log\delta)^2)\,. \tag{5.4.72}$$

Since $\tau_1^\delta = O(|\log\delta|)$, we deduce

$$\begin{aligned}&\frac{1}{2}\int_{\tau_1^\delta}^{0} \|\dot{Y}_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2 \,\mathrm{d}t \\ &\quad\geqslant \frac{1}{2}\int_{\tau_1^\delta}^{0} \left|\dot{\theta}_t^\delta \|\widetilde{q}'_\delta(\theta_t^\delta)\| - \left\langle \nabla V[\widetilde{q}_\delta(\theta_t^\delta)] - \delta G[\widetilde{q}_\delta(\theta_t^\delta)], \frac{\widetilde{q}'_\delta(\theta_t^\delta)}{\|\widetilde{q}'_\delta(\theta_t^\delta)\|}\right\rangle\right|^2 \mathrm{d}t + O(\delta^3|\log\delta|^3)\,.\end{aligned} \tag{5.4.73}$$

In conclusion we have proved

$$I^{A^\delta}_{\delta,-\infty}(Y^\delta) \geqslant \frac{1}{2}\int_{\tau_0^\delta}^0 \left|\dot\theta_t^\delta \|\widetilde{q}'_\delta(\theta_t^\delta)\| - \left\langle \nabla V[\widetilde{q}_\delta(\theta_t^\delta)] - \delta G[\widetilde{q}_\delta(\theta_t^\delta)], \frac{\widetilde{q}'_\delta(\theta_t^\delta)}{\|\widetilde{q}'_\delta(\theta_t^\delta)\|}\right\rangle\right|^2 \mathrm{d}t + O(\delta^3|\log\delta|^3)\,. \tag{5.4.74}$$

If we define $\varphi_t^\delta := \widetilde{p}_\delta(\widetilde{q}_\delta(\theta_t^\delta))$ for all $\tau_0^\delta \leqslant t \leqslant 0$ (i.e $q_\delta(\varphi_t^\delta) = \widetilde{q}_\delta(\theta_t^\delta)$), then (5.4.74) becomes

$$I^{A^\delta}_{\delta,-\infty}(Y^\delta) \geqslant \frac{1}{2}\int_{\tau_0^\delta}^0 |\dot\varphi_t^\delta - b_\delta(\varphi_t^\delta)|^2\,\mathrm{d}t + O(\delta^3|\log\delta|^3)\,. \tag{5.4.75}$$

We stress that φ_t^δ is not exactly $p_\delta(Y_t^\delta)$, but Lemma 5.2.6 and Lemma 5.4.7 ensure that the induced error is negligible. We have indeed $\varphi_0^\delta - \varphi_{B^\delta} = O(\delta^2|\log\delta|)$. On the other hand φ^δ does not begin at the point φ_{A^δ}, but remind that $\varphi^\delta_{\tau_0^\delta} - \varphi_{A^\delta} = O(\delta|\log\delta|)$, so $W^{red}(\varphi_{A^\delta}, \varphi^\delta_{\tau_0^\delta}) = O(\delta^3|\log\delta|^2)$. These two observations imply

$$\inf\left\{\frac{1}{2}\int_{\tau_0^\delta}^0 |\dot\varphi_t - b_\delta(\varphi_t)|^2\,\mathrm{d}t : \varphi \text{ is } C^2,\ \varphi_{\tau_0^\delta} = \varphi^\delta_{\tau_0^\delta} \text{ and } \varphi_0 = \varphi_0^\delta\right\}$$
$$\geqslant W_\delta^{red}(\varphi_{A^\delta}, \varphi_{B^\delta}) + O(\delta^3|\log\delta|^2)\,, \tag{5.4.76}$$

and this concludes the proof of the Theorem 5.1.1 (the reversed inequality is evident, as already stated in the Introduction).

5.4.4 Proof of Corollary 5.1.2

We follow almost the same procedure as for the proof of the Theorem. Let Y^δ be an optimal path associated to a $q_\delta(\varphi)$ with $\varphi \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$. Since Lemmas 5.4.1 to 5.4.6 only rely on the fact that $I_{\delta,-\infty}(Y^\delta) \leqslant C_1\delta$ (which corresponds to (5.4.5)) and on the the fact that Y^δ satisfies the Euler Lagrange type equation (5.3.24), the statements of these Lemmas are true for $\left(Y^\delta_{t+\tau^\delta_{\partial U^\delta}}\right)_{t\leqslant 0}$ where $\tau^\delta_{\partial U^\delta}$ is the first hitting time of ∂U^δ (and is equal to 0 if Y^δ never reaches ∂U^δ).

If $\tau^\delta_{\partial U^\delta} = 0$, a similar expansion as the one made in the proof of Theorem 5.1.1 gives the expected result. If $\tau^\delta_{\partial U^\delta} < 0$, Lemma 5.4.6 implies that $\varphi_{\tau^\delta_{\partial U^\delta}} := p_\delta\left(Y^\delta_{\tau^\delta_{\partial U^\delta}}\right)$ is equal either to $\varphi_{A^\delta} - \Delta_1$ or to $\varphi_{A^\delta} + \Delta_2$, and after expansion we get

$$\frac{1}{2}\int_{-\infty}^{\tau^\delta_{\partial U^\delta}} \|\dot Y_t^\delta + \nabla V[Y_t^\delta] - \delta G[Y_t^\delta]\|^2\,\mathrm{d}t \geqslant W_\delta^{red}\Big(\varphi_{A^\delta}, \varphi_{\tau^\delta_{\partial U^\delta}}\Big) + O(\delta^3|\log\delta^3|)\,. \tag{5.4.77}$$

Moreover similar arguments as the ones used between (5.4.57) and (5.4.60) show that once $p_\delta(Y^\delta)$ has left a $O(\delta^2)$-neighborhood of A^δ, it never comes back to this neighborhood and thus always stays on the same side of A^δ. We thus have $W_\delta^{red}\Big(\varphi_{A^\delta}, \varphi_{\tau^\delta_{\partial U^\delta}}\Big) \geqslant W_\delta^{red}(\varphi_{A^\delta}, \varphi)$, which together with (5.4.77) concludes the proof.

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