Slow/fast dynamics and periodic behaviors for mean-field excitable systems

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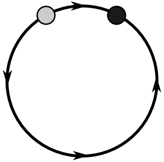
In collaboration with E. Luçon (Université Paris Descartes)

An excitable system :

- possesses a **stable rest position**.
- **threshold phenomenon** : after a sufficiently large perturbation, follows a complex trajectory before coming back to the rest state.



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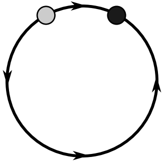

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A large population of noisy excitable systems in mean field interaction may possess a **synchronized periodic behavior**.

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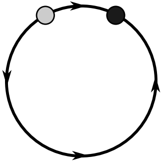

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Aim

Rigorous proof of periodic behavior for noisy neurons in mean field interaction ?

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$$d\varphi_{i,t} = -\delta V'(\varphi_{i,t})dt - \frac{K}{N}\sum_{j=1}^{N}\sin(\varphi_{i,t} - \varphi_{j,t})dt + dB_{i,t}.$$

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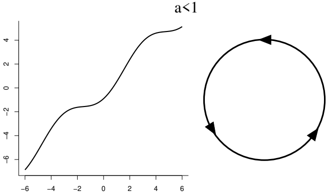

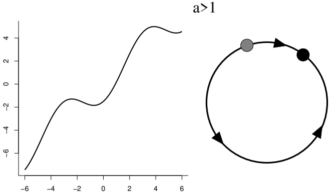

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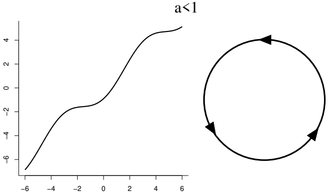

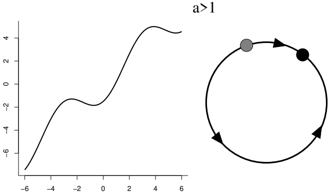

On any time interval $[0,T]$, the empirical measure $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^{N}\delta_{\varphi_{i,t}}$ converges weakly to the solution of

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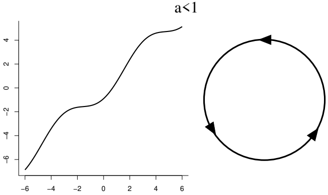

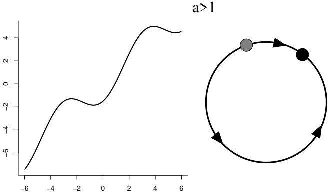

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For accurate choices of parameters (a may be larger than one) and δ small enough, this non-linear Fokker Planck PDE **admits a limit cycle**. [Giacomin, Pakdaman, Pellegrin and P., 2012]

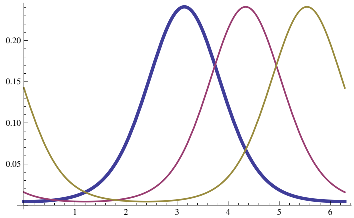
Simulation : $N = 4000$, $K = 2$, $a = 1.1$, $\delta = 0.5$

For $\delta = 0$ we have

$$\partial_t \mu_t(\theta) = \frac{1}{2}\partial_\theta^2 p_t(\theta) + K\partial_\theta \left(\mu_t \int_{\mathcal{S}} \sin(\theta - \psi) d\mu_t(\psi) \right),$$

and if $K > 1$, the model admits moreover a stable curve of synchronized stationary solutions

$$M_0 = \{q_\psi(\cdot) : \psi \in \mathcal{S}\}, \quad \text{where} \quad q_\psi(\cdot) = q_0(\cdot - \psi).$$



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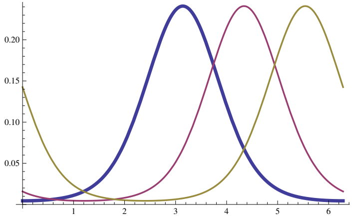

For δ small the model admits an invariant curve $M_\delta = \{q_\psi^\delta : \psi \in \mathcal{S}\}$, perturbation of M_0, with phase dynamics

$$\dot{\psi}_t^\delta \approx \delta \left(1 + \frac{a}{a_K} \sin(\psi_t^\delta) \right).$$

Consider a population of N interacting units in $\mathbb{R}^d$ with dynamics

$$dX_{i,t} = \delta F(X_{i,t})dt - K\left(X_{i,t} - \frac{1}{N}\sum_{j=1}^{N} X_{j,t}\right)dt + \sqrt{2}\sigma dB_{i,t},$$

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- $\delta \geqslant 0$, $K = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_d \end{pmatrix} > 0$, $\sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_d \end{pmatrix} > 0$,

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[Baladron, Fasoli, Faugeras, Touboul, 2012], [Luçon, Stannat, 2014], [Bossy, Faugeras, Talay, 2015], [Mehri, Scheutzow, Stannat, Zangeneh, 2018]] : On any time interval $[0, T]$, the empirical measure $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_{i,t}}$ converges weakly to the solution of

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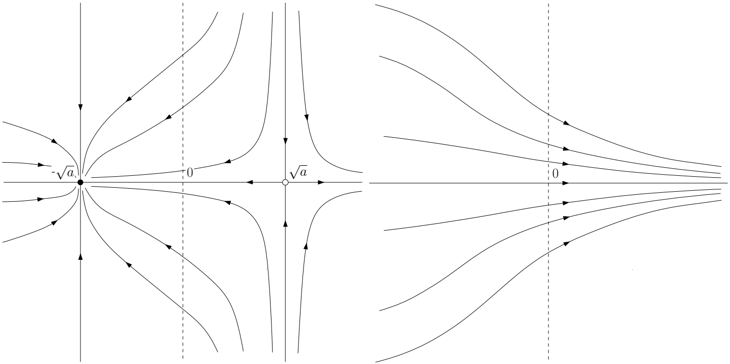
Consider $F(x,y) = \begin{pmatrix} x^2 - a \\ -by \end{pmatrix}$ with $a \in \mathbb{R}$, $b > 0$.



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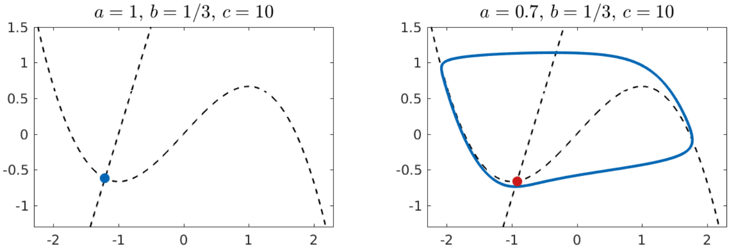

A look at the literature

- [Scheutzow, 1985], [Touboul, Hermann, Faugeras, 2012] **noise-induced phenomena** for non-linear Fokker-Planck equations admitting **Gaussian solutions**.
- [Scheutzow, 1986] **existence of periodic solutions** for the mean-field Brusselator model (for large interaction, when each unit has a periodic behavior).
- [Pakdaman, Perthame, Salort, 2011] **existence of periodic solutions** for time elapsed neuron network model.
- [Giacomin, Pakdaman, Pellegrin and P., 2012] **noise-induced periodicity** for the *Active rotators* model.
- [Mischler, Quiñinao, Touboul, 2016] existence of stationary solutions for the **kinetic mean-field FitzHugh Nagumo model**, uniqueness and stability for small coupling.
- [Quiñinao, Touboul, 2018] for large coupling, the **kinetic mean-field FitzHugh Nagumo model** behaves as a single FitzHugh Nagumo unit.

Recall

$$\partial_t \mu_t = \nabla \cdot (\sigma^2 \nabla \mu_t) + \nabla \cdot \left(\mu_t K(x - \int_{\mathbb{R}^d} z d\mu_t(z) \right) - \delta \nabla \cdot (\mu_t F).$$

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Denote $m_t = \mathbb{E}[X_t] = \int x d\mu_t(x)$, and p_t the distribution of $X_t - m_t$.

(m_t, p_t) is solution of the system

$$\left\{ \begin{array}{lcl} \dot{m}_t & = & \delta \int F(x + m_t) dp_t(x) \\ \partial_t p_t & = & \nabla \cdot (\sigma^2 \nabla p_t) + \nabla \cdot (p_t K x) + \nabla \cdot (p_t (\dot{m}_t - \delta F(x + m_t)) \end{array} \right. ,$$

which is a slow/fast system when $\delta \to 0$ with m_t the slow variable, p_t the fast one.

For $\delta = 0$ we get

$$\begin{cases} \dot{m}_t = 0 \\ \partial_t p_t = \nabla \cdot (\sigma^2 \nabla p_t) + \nabla \cdot (p_t K x) \end{cases} .$$

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In this case p_t is the distribution of the Ornstein Uhlenbeck process

$$dX_t = -KX_t dt + \sqrt{2}\sigma dB_t,$$

which has stationnary distribution $q \sim \mathcal{N}(0, \Gamma)$ with $\Gamma = \sigma^2 K^{-1}$, and satisfies in particular

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Approximation for δ small :

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This corresponds to the approximation

$$\mu_t \approx \mathcal{N}(m_t, \Gamma), \quad \text{with} \quad \dot{m}_t \approx \delta F_\Gamma(m_t),$$

which reduces the problem to a d-dimensional dynamics.

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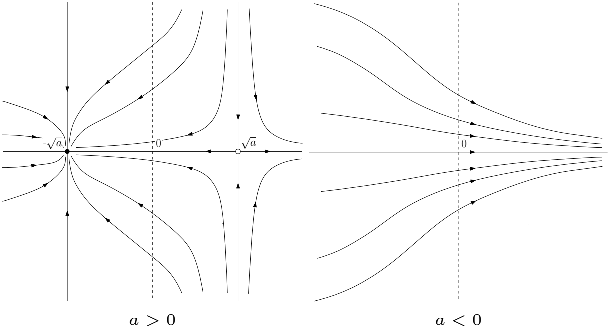

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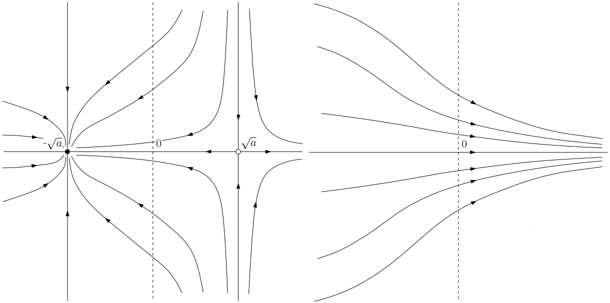

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we get $F_\Gamma(m_x, m_y) = \begin{pmatrix} m_x^2 - \left(a - \frac{\sigma_1^2}{k_1}\right) \\ -b\, m_y \end{pmatrix}$.

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- For $\frac{\sigma_1^2}{K_1}$ small, if (a, b, c) is not a bifurcation point of $\dot{z}_t = F(z_t)$, then $\dot{z}_t = F(z_t)$ and $\dot{m}_t = \delta F_\Gamma(m_t)$ have the same type of dynamics.

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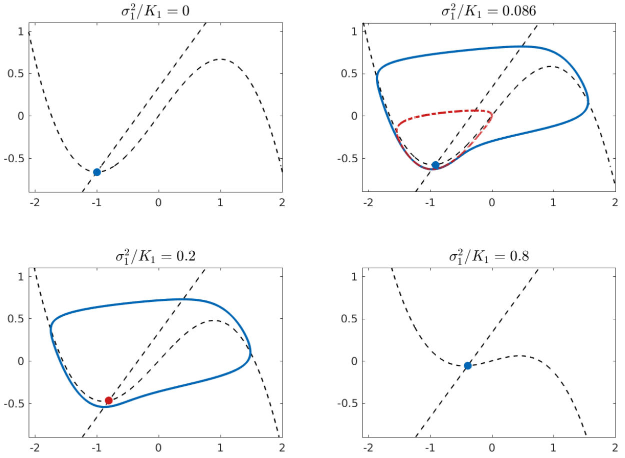
  - For larger values of $\frac{\sigma_1^2}{k_1}$, this two dynamics may differ.

Dynamics of $\dot{m}_t = F_\Gamma(m_t)$, FitzHugh Nagumo model

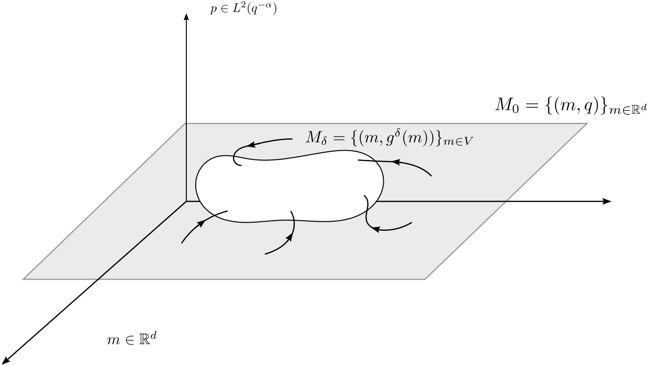
Parameters : $a = \frac{1}{3}$, $b = 1$, $c = 10$.



Parameters : $N = 100000$, $k_1 = 1$, $k_2 = 1$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.03$, $\delta = 0.1$.

[Luçon, P., 2018a] We suppose that there exists a bounded smooth subset V such that

$$n_{\partial V}(m) \cdot F_\Gamma(m) < 0.$$



If $p_0 = g^\delta(m_0) \in M_\delta$, then $p_t = g^\delta(m_t) \in M_\delta$ and $\dot{m}_t \approx \delta F_\Gamma(m_t)$.

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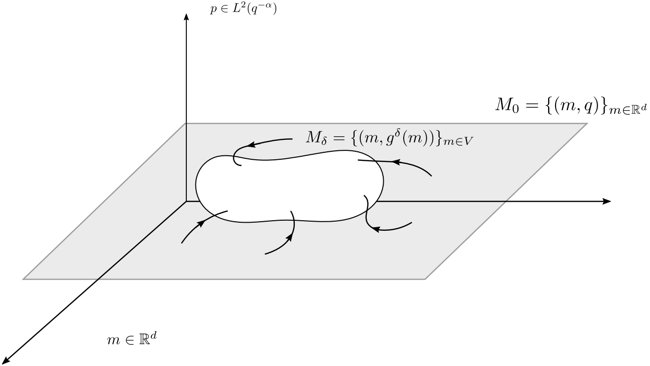

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Persistence of normally hyperbolic manifolds under perturbation : [Fénichel, 1971], [Hirsh, Pugh, Shub, 1977], [Wiggins 1994], [Bates, Lu, Zeng, 1998], [Sell, You, 2002].

Simulation for N particles, kinetic FitzHugh Nagumo model

Parameters : $N = 100000$, $k_1 = 1$, $k_2 = 0$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0$, $\delta = 0.01$.

Consider the nonlinear process

$$\begin{cases} dV_t &= \delta \left(V_t - \frac{V_t^3}{3} - W_t \right) dt - k_1 (V_t - \mathbb{E}[V_t]) dt + \sqrt{2} \sigma_1 dB_t \\ dW_t &= \frac{\delta}{c} (V_t + a - bW_t) dt \end{cases},$$

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Slow/fast approximation :

Here $V_t - \mathbb{E}[V_t]$ is the fast variable, we have

$$V_t - \mathbb{E}[V_t] \approx \mathcal{N}(0, \sigma_1^2/k_1),$$

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and we make the approximation (the dynamics of W_t is linear) :

$$\begin{cases} d\widetilde{V}_t &= \delta \dot{m}_{v,t}dt - K(\widetilde{V}_t - m_{v,t})dt + \sqrt{2}\sigma dB_t \\ d\widetilde{W}_t &= \frac{\delta}{c}(\widetilde{V}_t + a - b\widetilde{W}_t)dt \end{cases},$$

with

$$\begin{pmatrix} \dot{m}_{v,t} \\ \dot{m}_{w,t} \end{pmatrix} = \delta F_{\sigma_1^2/k_1}(m_{v,t}, m_{w,t}), \quad \text{and} \quad (\widetilde{V}_t, \widetilde{W}_t) \sim \mathcal{N}((m_{v,t}, m_{w,t}), \Gamma_\delta).$$

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We rely on Wasserstein type distances for this model [Luçon, P., 2018b].

Open questions :

- long time behavior for finite but large population ?
- consider other models (Morris-Lecar) ?
- consider other interactions ?

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Thank you for your attention