

ω -Lyndon words

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In Memory of the Late Professor Aldo de Luca

Abstract: Let \mathbb{A} be a finite non-empty set and \preceq a total order on $\mathbb{A}^{\mathbb{N}}$ verifying the following lexicographic like condition: For each $n \in \mathbb{N}$ and $u, v \in \mathbb{A}^n$, if $u^\omega \prec v^\omega$ then $ux \prec vy$ for all $x, y \in \mathbb{A}^{\mathbb{N}}$. A word $x \in \mathbb{A}^{\mathbb{N}}$ is called ω -Lyndon if $x \prec y$ for each proper suffix y of x . A finite word $w \in \mathbb{A}^+$ is called ω -Lyndon if $w^\omega \prec v^\omega$ for each proper suffix v of w . In this note we prove that every infinite word may be written uniquely as a non-increasing product of ω -Lyndon words.

Keywords: Lyndon words, lexicographic orders.

1 Introduction

Given a finite non-empty set \mathbb{A} , let $\mathbb{A}^+ = \bigcup_{n \in \mathbb{N}} \mathbb{A}^n$ denote the free semigroup generated by \mathbb{A} consisting of all finite words over \mathbb{A} and let $\mathbb{A}^{\mathbb{N}} = \{a_1 a_2 a_3 \cdots \mid a_i \in \mathbb{A}\}$ be the set of all right infinite words over \mathbb{A} . We also let $\mathbb{A}^* = \mathbb{A}^+ \cup \{\varepsilon\}$ where ε denotes the empty word. Let \preceq be a total order on $\mathbb{A}^{\mathbb{N}}$ verifying the following lexicographic like condition denoted (*): For each $n \in \mathbb{N}$ and $u, v \in \mathbb{A}^n$ if $u^\omega \prec v^\omega$ then $ux \prec vy$ for all $x, y \in \mathbb{A}^{\mathbb{N}}$. A special case of this setting was previously considered in [2]. The authors consider a sequence $(\prec_n)_{n \in \mathbb{N}}$ of total orders on \mathbb{A} . This induces a total lexicographic like order \preceq on $\mathbb{A}^{\mathbb{N}}$ defined by $x \preceq y$ if and only if either $x = y$ or if $x = uax'$ and $y = uby'$ for some $u \in \mathbb{A}^*$, $a, b \in \mathbb{A}$ and $x', y' \in \mathbb{A}^{\mathbb{N}}$ and $a \prec_{|u|+1} b$. It is easily checked that the induced total order on $\mathbb{A}^{\mathbb{N}}$ satisfies condition (*). However the two settings are not equivalent. In fact, in the context of the generalised lexicographic order in [2], if for example $(bb)^\omega \prec (ba)^\omega$ then it would mean that $b \prec_2 a$ and hence $(ab)^\omega \prec (aa)^\omega$. This is no longer true in the setting

considered herein as one may have $(bb)^\omega \prec (ba)^\omega$ and $(aa)^\omega \prec (ab)^\omega$. In [2] the authors introduce the notion of *generalised Lyndon words* with respect to the induced total order on $\mathbb{A}^{\mathbb{N}}$: a finite word $w \in \mathbb{A}^+$ is a generalised Lyndon word if $w^\omega \prec v^\omega$ for every proper suffix v of w . They then prove that every finite word $w \in \mathbb{A}^+$ may be written uniquely in the form $w = l_1 \cdots l_k$ where each l_i is a generalised Lyndon word and $l_1^\omega \succeq l_2^\omega \succeq \cdots \succeq l_k^\omega$ (see Theorem 16 in [2]). Analogously, given a total order on $\mathbb{A}^{\mathbb{N}}$ verifying (*) we adopt the above definition to define the notion of ω -Lyndon words: a finite word w is ω -Lyndon if $w^\omega \prec v^\omega$ for every proper suffix v of w . We may also borrow from the usual definition of infinite Lyndon words to define a class of infinite ω -Lyndon words: An infinite word $x \in \mathbb{A}^{\mathbb{N}}$ is called ω -Lyndon if $x \prec y$ for every proper suffix y of x . Alternatively, to define an infinite ω -Lyndon word, we could have adapted the characterisation given in [1]: An infinite word x is ω -Lyndon if it is non-periodic and contains an infinite number of ω -Lyndon prefixes. It turns out that the two notions are equivalent (see Lemma 2.5).

The main purpose of this note is to extend the results in [4] on factoring infinite words as a non-increasing product of Lyndon words. We prove that every infinite word may be written uniquely as a non-increasing product of ω -Lyndon words. This provides a positive answer to a question raised in [2].

2 Main results

Let us fix once and for all a total order \preceq on $\mathbb{A}^{\mathbb{N}}$ verifying the lexicographic condition (*). Note that if $x, y \in \mathbb{A}^{\mathbb{N}}$ and $x \preceq y$ then for each prefix u of x and v of y with $|u| = |v|$ one has $u^\omega \preceq v^\omega$ with equality if and only if $u = v$. We also observe that if $u^\omega \preceq v^\omega$, and neither u nor v is a prefix of the other, then $u^\omega \prec v^\omega$. In particular, if $u^\omega \preceq v^\omega$ and u and v are primitive, then either $u = v$ or $u^\omega \prec v^\omega$. We begin with the following lemma which is analogous to Lemma 13 in [2]. We omit the proof as it is identical to that of Lemma 13 in [2].

Lemma 2.1. *For each $u, v \in \mathbb{A}^+$ and $\star \in \{=, \prec, \succ\}$ the following are equivalent :*

1. $u^\omega \star v^\omega$;
2. $(uv)^\omega \star v^\omega$;
3. $u^\omega \star (vu)^\omega$;
4. $(uv)^\omega \star (vu)^\omega$;

We remark that a slightly modified version of the above lemma also applies in case one of u and v is infinite and the other finite: For example if $u \in \mathbb{A}^+$ and $v \in \mathbb{A}^{\mathbb{N}}$ then $u^\omega \star v$ if and only if $uv \star v$.

Given a total order on $\mathbb{A}^{\mathbb{N}}$ it is natural to consider the notion of a Lyndon word, namely an element $x \in \mathbb{A}^{\mathbb{N}}$ which is smaller (relative to the prescribed total order) than each of its proper suffixes. It will be useful to extend this notion also to finite words, however given that the order is defined only on infinite words, we shall be required to pass to infinite words by associating to each finite word w its (periodic) infinite counterpart w^ω . Following [2]:

Definition 2.2. An infinite word $x \in \mathbb{A}^{\mathbb{N}}$ is called ω -Lyndon if $x \prec y$ for each proper suffix y of x . A word $w \in \mathbb{A}^+$ is called ω -Lyndon if $w^\omega \prec v^\omega$ for each proper suffix v of w . We let \mathcal{L}_ω denote the set of all ω -Lyndon words (finite and infinite) relative to the total order \preceq .

Remark 1. We note that $\mathbb{A} \subseteq \mathcal{L}_\omega$. If $w \in \mathbb{A}^+$ is ω -Lyndon, then w is primitive and similarly if $x \in \mathbb{A}^{\mathbb{N}}$ is ω -Lyndon, then x is not periodic. It follows from Lemma 2.1 that $w \in \mathbb{A}^+$ is ω -Lyndon if and only if for all factorisations $w = uv$ with $u, v \in \mathbb{A}^+$ we have $u^\omega \prec v^\omega$ (see Theorem 14 in [2]). This in turn implies that if $w \in \mathcal{L}_\omega$, then for each prefix u of w and each factor v of w with $|u| = |v|$, either $u = v$ or $u^\omega \prec v^\omega$. In fact, suppose $u \neq v$ and let z be a suffix of w beginning in v . Then if $w \in \mathbb{A}^+$ we have that $w^\omega \prec z^\omega$ and hence $u^\omega \prec v^\omega$. If $w \in \mathbb{A}^{\mathbb{N}}$, then $w \prec z$ and hence $u^\omega \prec v^\omega$.

Remark 2. As is the case for usual Lyndon words, each finite ω -Lyndon word $w = w_1w_2 \cdots w_n$ of length $n \geq 2$ is a prefix of some infinite ω -Lyndon word w' . In fact, if w is unbordered we may take $w' = ww_n^\omega$. For let s be a suffix of w . Then as $w_1 \cdots w_{|s|} \neq s$ we have $(w_1 \cdots w_{|s|})^\omega \prec s^\omega$ and hence $w' \prec sw_n^\omega$. On the other hand, if w is bordered, we may take $w' = wu$ where u is the longest border of w . In fact, let s be a suffix of w . Then if s is not a border of w then as before we deduce that $w' \prec su^\omega$. While if s is a prefix of w , and hence also a prefix of u , let k be the first position in which w^ω and s^ω first differ. If $k \leq |s|$, then $(w_1 \cdots w_k)^\omega \prec (s_1 \cdots s_k)^\omega$ and hence $w' \prec su^\omega$. If $|s| < k \leq |u|$ then $(w_1 \cdots w_k)^\omega \prec (su_1 \cdots u_{k-|s|})^\omega$ and so again we have $w' \prec su^\omega$. Finally if $k > |u|$ then writing $s = z^a$ and $u = z^b$ for some $z \in \mathbb{A}^+$ and $a, b \in \mathbb{N}$ we again get $w' \prec su^\omega$ as required. A similar proof shows that each finite ω -Lyndon word $w \notin \mathbb{A}$ is a prefix of a longer finite ω -Lyndon word.

Remark 3. In contrast to the previous remark, many fundamental properties of usual Lyndon words no longer hold for ω -Lyndon words. First of all, every primitive finite word and every non-periodic infinite word is ω -Lyndon relative to some total order \preceq on $\mathbb{A}^{\mathbb{N}}$ verifying (*). As a consequence, a finite ω -Lyndon word need not be unbordered. Or an infinite ω -Lyndon word x may be a product of prefixes of x . Or if $u, v \in \mathbb{A}^+$ are ω -Lyndon and $u^\omega \prec v^\omega$ it need not be the case that uv is ω -Lyndon. For example, let $\mathbb{A} = \{a, b\}$ be ordered by $a < b$. Consider the total order \preceq on $\mathbb{A}^{\mathbb{N}}$ defined as follows: For distinct $x, y \in \mathbb{A}^{\mathbb{N}}$ consider the smallest n with $x_n \neq y_n$. Then set

$$x \prec y \Leftrightarrow \begin{cases} x_n < y_n & \text{if } n \text{ is odd} \\ y_n < x_n & \text{if } n \text{ is even} \end{cases}$$

Then $u = abba \in \mathcal{L}_\omega$ and $v = b \in \mathcal{L}_\omega$ and $u^\omega \prec v^\omega$, yet $uv \notin \mathcal{L}_\omega$. This example also illustrates that ω -Lyndon words may be bordered.

The following proposition is essentially Theorem 16 in [2]. As with Lemma 2.1, we omit the proof as it is identical to that of Theorem 16 in [2].

Proposition 2.3. *Each $w \in \mathbb{A}^+$ admits a unique factorisation $w = l_1 l_2 \cdots l_k$ with $l_i \in \mathcal{L}_\omega$ and $l_1^\omega \succeq l_2^\omega \succeq \cdots \succeq l_k^\omega$.*

Definition 2.4. For $x \in \mathbb{A}^\mathbb{N}$ we say x admits an infinite ω -Lyndon factorisation if $x = \prod_{i=1}^\infty l_i$ with each $l_i \in \mathcal{L}_\omega \cap \mathbb{A}^+$ and $l_1^\omega \succeq l_2^\omega \succeq l_3^\omega \succeq \cdots$. We say x admits a finite ω -Lyndon factorisation if $x = l_1 l_2 \cdots l_k$ with $l_i \in \mathcal{L}_\omega$ and $l_1^\omega \succeq l_2^\omega \succeq \cdots \succeq l_{k-1}^\omega \succ l_k$.

Remark 4. Because of the fact that if u and v are Lyndon in the usual sense and $u < v$ then uv is Lyndon, it follows that the factorisation of a finite word w as a non-increasing product of Lyndon words is also the shortest factorisation of w as a product of Lyndon words. This is no longer true in general for ω -Lyndon words. For example, relative to the total order \preceq defined in Remark 3, the word $w = ababab$ is the product of $ababa$ and b , both of which are ω -Lyndon, yet the ω -Lyndon factorisation of w has length three and is given by $w = (ab)(ab)(ab)$.

The following lemma constitutes a generalisation of a characterisation of infinite Lyndon words given in [1]:

Lemma 2.5. *Let $x \in \mathbb{A}^\mathbb{N}$. Then $x \notin \mathcal{L}_\omega$ if and only if either $x = l^\omega$ for some $l \in \mathcal{L}_\omega$ or only a finite number of prefixes of x are members of \mathcal{L}_ω .*

Proof. Assume $x \notin \mathcal{L}_\omega$ and pick a proper suffix y of x with $y \preceq x$. If $y = x$, then $x = u^\omega$ for some primitive word $u \in \mathbb{A}^+$. As u is primitive, for each nontrivial factorisation $u = u_1 u_2$ one has $u_2 u_1 \neq u$ and hence $(u_2 u_1)^\omega \neq u^\omega$. If $u \in \mathcal{L}_\omega$ we are done. If $u \notin \mathcal{L}_\omega$, pick a factorisation $u = u_1 u_2$ with $(u_2 u_1)^\omega \prec u^\omega$. Then by Remark 1, if v is a prefix of x with $|v| \geq 2|u|$ then $v \notin \mathcal{L}_\omega$. If $y \prec x$, pick a prefix v of y and a prefix u of x with $|u| = |v|$ and $v^\omega \prec u^\omega$. Then again by Remark 1 any prefix of x containing v as a factor cannot belong to \mathcal{L}_ω .

For the converse, we note that if x is periodic then $x \notin \mathcal{L}_\omega$. So assume x is not periodic and only a finite number of prefixes of x belong to \mathcal{L}_ω . For $n \in \mathbb{N}$, let $x[n]$ denote the prefix of x of length n , and let $l(n)$ denote the length of a longest ω -Lyndon word occurring in the ω -Lyndon factorisation of $x[n]$ (see Proposition 2.3). If $(l(n))_{n \geq 1}$ is unbounded, then pick n such that i) $l(n)$ is greater than the length of the longest ω -Lyndon prefix of x and ii) $l(n)$ is the length of the last ω -Lyndon word in the ω -Lyndon factorisation of $x[n]$. Then $x[n] = l_1 l_2 \cdots l_k$ with $l_i \in \mathcal{L}_\omega$ and $l_1^\omega \succeq l_2^\omega \succeq \cdots \succeq l_k^\omega$, and l_k is not a prefix of x . By iteration of Lemma 2.1, $(l_1 l_2 \cdots l_k)^\omega \succeq l_k^\omega$ and hence $(l_1 l_2 \cdots l_k)^\omega \succ l_k^\omega$. Writing $x = l_1 l_2 \cdots l_{k-1} y$ with $y \in \mathbb{A}^\mathbb{N}$, since l_k is a prefix of y but not of $l_1 l_2 \cdots l_{k-1}$ we have $x \succ y$

and hence $x \notin \mathcal{L}_\omega$. If $(l(n))_{n \geq 1}$ is bounded then pick a finite set $F \subseteq \mathcal{L}_\omega$ such that all ω -Lyndon words occurring in the ω -Lyndon factorisation of $x[n]$ for $n \in \mathbb{N}$ belong to F . Because of the non increasing order condition in the ω -Lyndon factorisation, there exist l_1, l_2, \dots, l_k in F with $l_1^\omega \succ l_2^\omega \succ \dots \succ l_k^\omega$ such that $l_1 l_2 \dots l_{k-1} l_k^m$ is a prefix of x for every $m \in \mathbb{N}$. Pick m such that l_k^m is not a prefix of x and write $x = l_1 l_2 \dots l_{k-1} y$ with $y \in \mathbb{A}^\mathbb{N}$. Then as l_k^m is a prefix of y but not of $l_1 l_2 \dots l_k^m$ and $(l_1 l_2 \dots l_k^m)^\omega \succ l_k^\omega$ we deduce that $x \succ y$ and hence $x \notin \mathcal{L}_\omega$. \square

Proposition 2.6. *Each $x \in \mathbb{A}^\mathbb{N}$ admits either an infinite or a finite ω -Lyndon factorisation.*

Proof. Let $x \in \mathbb{A}^\mathbb{N}$ and assume x does not admit an infinite ω -Lyndon factorisation. We will show that x admits a finite ω -Lyndon factorisation. The result is immediate in case $x \in \mathcal{L}_\omega$ so we may assume that $x \notin \mathcal{L}_\omega$. For $n \in \mathbb{N}$, let $l_i^{(n)}$ be the i 'th ω -Lyndon word occurring in the ω -Lyndon factorisation of $x[n]$, where $x[n]$ is the prefix of length n of x . In other words $x[n] = l_1^{(n)} l_2^{(n)} \dots l_k^{(n)}$. We may take $l_i^{(n)}$ to be the empty word if the ω -Lyndon factorisation of $x[n]$ has fewer than i terms. By Lemma 2.5, the set $L_1 = \{l_1^{(n)} : n \in \mathbb{N}\}$ is finite and hence there exist $l_1 \in \mathcal{L}_\omega$ and an infinite set $A_1 \subseteq \mathbb{N}$ such that for each $n \in A_1$ the ω -Lyndon factorisation of $x[n]$ begins in l_1 . Put $L_2 = \{l_2^{(n)} : n \in A_1\}$. If L_2 is finite, then we may pick $l_2 \in \mathcal{L}_\omega$ and an infinite subset $A_2 \subseteq A_1$ such that for each $n \in A_2$ the ω -Lyndon factorisation of $x[n]$ begins in $l_1 l_2$ and put $L_3 = \{l_3^{(n)} : n \in A_2\}$. Continuing as above, if each L_k is finite then x would admit an infinite ω -Lyndon factorisation contrary to our assumption. And hence, there exists $k \geq 2$ and $l_1, l_2, \dots, l_{k-1} \in \mathcal{L}_\omega$ and an infinite set $A_{k-1} \subseteq \mathbb{N}$ such that for each $n \in A_{k-1}$ the ω -Lyndon factorisation of $x[n]$ begins in $l_1 l_2 \dots l_{k-1}$ and $L_k = \{l_k^{(n)} : n \in A_{k-1}\}$ infinite. Define $l_k \in \mathbb{A}^\mathbb{N}$ by $x = l_1 l_2 \dots l_{k-1} l_k$. We claim $l_k \in \mathcal{L}_\omega$ and $l_{k-1}^\omega \succ l_k$. Observe that $l_k \neq l_{k-1}^\omega$ for otherwise $x = l_1 \dots l_{k-2} l_{k-1}^\omega$ is an infinite ω -Lyndon factorisation of x . Pick $m \in \mathbb{N}$ such that l_{k-1}^m is not a prefix of l_k and $n \in A_{k-1}$ such that $|l_{k-1}^m| < |l_k^{(n)}|$. Since $l_{k-1}^\omega \succ (l_k^{(n)})^\omega$ and l_{k-1}^m is not a prefix of $l_k^{(n)}$, it follows that $l_{k-1}^\omega \succ l_k$. It remains to show that $l_k \in \mathcal{L}_\omega$. By Lemma 2.5, if $l_k \notin \mathcal{L}_\omega$ then $l_k = u^\omega$ for some $u \in \mathcal{L}_\omega$ and hence $x = l_1 l_2 \dots l_{k-1} u^\omega$ is an infinite ω -Lyndon factorisation, a contradiction. \square

We now turn to the question of unicity of ω -Lyndon factorisations for infinite words. We begin by establishing unicity for words admitting a finite ω -Lyndon factorisation.

Lemma 2.7. *Let $x \in \mathbb{A}^\mathbb{N}$ and $u_1 u_2 \dots u_k$ be a prefix of x such that $u_1^\omega \succ u_2^\omega \succ \dots \succ u_k^\omega$. If $x \in \mathcal{L}_\omega$ then each u_i is a prefix of x .*

Proof. By iteration of Lemma 2.1, for $1 \leq i \leq k$ we have that $(u_1 \dots u_i)^\omega \succ u_i^\omega$. Let v_i denote the prefix of x of length $|u_i|$. If $v_i \neq u_i$ then $v_i^\omega \succ u_i^\omega$ contradicting that $x \in \mathcal{L}_\omega$. \square

Lemma 2.8. *Let $x \in \mathbb{A}^{\mathbb{N}}$ and $k \geq 3$. Assume $u_1 u_2 \cdots u_k$ is a prefix of x such that $u_1^\omega \succ u_2^\omega \succ \cdots \succ u_k^\omega$. If $x \in \mathcal{L}_\omega$, then either $|u_1 \cdots u_{k-2}| \leq |u_k|$ or $u_1 \cdots u_{k-2} u_k$ is a prefix of x .*

Proof. Assume $|u_1 \cdots u_{k-2}| > |u_k|$. By Lemma 2.7, we have that u_k is a prefix of x and hence a prefix of $u_1 \cdots u_{k-2}$. By Lemma 2.1 we have that $(u_1 \cdots u_{k-2})^\omega \succeq u_{k-2}^\omega \succeq u_{k-1}^\omega$ and hence $(u_1 \cdots u_{k-2} u_{k-1})^\omega \succeq (u_{k-1} u_1 \cdots u_{k-2})^\omega$. Since $x \in \mathcal{L}_\omega$ it follows that $u_{k-1} u_k$ is a prefix of x and hence u_k is a prefix of $u_{k-1} u_k$. Thus $u_1 \cdots u_{k-2} u_k$ is a prefix of x . \square

Lemma 2.9. *Let $(u_i)_{i \in \mathbb{N}}$ be a sequence in \mathbb{A}^+ with $u_1^\omega \succ u_2^\omega \succ \cdots$. Then $\prod_{i=1}^{\infty} u_i \notin \mathcal{L}_\omega$.*

Proof. Put $x = u_1 u_2 \cdots$ and suppose to the contrary that $x \in \mathcal{L}_\omega$. We will show that $|u_k| < |u_1|$ for each $k \geq 3$ and hence the sequence $(u_i)_{i \in \mathbb{N}}$ is ultimately constant, a contradiction. To see that $|u_k| < |u_1|$ for each $k \geq 3$, suppose to the contrary that $|u_k| \geq |u_1|$ for some $k \geq 3$. By iteration of Lemma 2.8, there exists $2 \leq j \leq k-1$ such that $u_1 \cdots u_j u_k$ is a prefix of x and $|u_1 \cdots u_{j-1}| \leq |u_k|$. By Lemma 2.7 we have that u_k is a prefix of x and hence $u_1 \cdots u_{j-1}$ is a prefix of u_k . As $(u_1 \cdots u_{j-1})^\omega \succ u_j^\omega$ we have that $(u_1 \cdots u_{j-1} u_j)^\omega \succ (u_j u_1 \cdots u_{j-1})^\omega$ and hence $u_1 \cdots u_{j-1} u_j \neq u_j u_1 \cdots u_{j-1}$. Since $u_1 \cdots u_{j-1}$ is a prefix of u_k it follows that the suffix of x beginning in $u_j u_k$ is smaller than x contradicting that $x \in \mathcal{L}_\omega$. \square

Lemma 2.10. *Let $x \in \mathbb{A}^{\mathbb{N}}$. If $x = v_1 v_2 v_3 \cdots$ with $v_1^\omega \succeq v_2^\omega \succeq \cdots$ then $x \notin \mathcal{L}_\omega$.*

Proof. Assume to the contrary that $x \in \mathcal{L}_\omega$. Without loss of generality we may assume that each v_i is primitive. We claim $(v_i)_{i \geq 1}$ is ultimately periodic. In fact, if the sequence $(v_i)_{i \geq 1}$ is not ultimately periodic, then by concatenating together the consecutive terms of the sequence which are equal, we may write $x = u_1 u_2 \cdots$ with $u_1^\omega \succ u_2^\omega \succ \cdots$ in contradiction with Lemma 2.9. As $x \in \mathcal{L}_\omega$ and hence not periodic, write $x = v_1 \cdots v_k v_{k+1}^\omega$ with $v_1^\omega \succeq \cdots \succeq v_k^\omega \succ v_{k+1}^\omega$ and pick m such that v_{k+1}^m is not a prefix of x . As $(v_1 \cdots v_k v_{k+1}^m)^\omega \succ v_{k+1}^\omega$, it follows that the suffix $v_{k+1}^\omega \prec x$ contradicting that $x \in \mathcal{L}_\omega$. \square

Lemma 2.11. *If x admits an infinite ω -Lyndon factorisation, then no suffix of x belongs to \mathcal{L}_ω . In particular x does not admit a finite ω -Lyndon factorisation.*

Proof. Suppose x admits an infinite ω -Lyndon factorisation $x = l_1 l_2 l_3 \cdots$. Then any suffix y of x may be written as $y = s_i l_{i+1} l_{i+2} \cdots$ with $i \geq 1$ and s_i a suffix of l_i . Since $s_i^\omega \succeq l_i^\omega$ it follows that $s_i^\omega \succeq l_{i+1}^\omega \succeq \cdots$. By Lemma 2.10, it follows that $y \notin \mathcal{L}_\omega$. \square

Corollary 2.12. *If an infinite word $x \in \mathbb{A}^{\mathbb{N}}$ admits a finite ω -Lyndon factorisation $x = l_1 l_2 \cdots l_k$, then it is the unique ω -Lyndon factorisation of x .*

Proof. It follows from Lemma 2.11 that x does not admit an infinite ω -Lyndon factorisation. It remains to show that x admits no other finite ω -Lyndon factorisation. For this purpose, write $x = u l_k$ with $u \in \mathbb{A}^*$ and observe that if $v \in \mathbb{A}^+$ is any suffix of u , then (by

iteration of Lemma 2.1) $l_k \preceq vl_k$. In other words, $vl_k \notin \mathcal{L}_\omega$ and hence l_k is necessarily the first ω -Lyndon suffix of x . Unicity now follows from Proposition 2.3. \square

We next prove unicity of ω -Lyndon factorisations for those infinite words x not admitting a finite ω -Lyndon factorisation. We first consider the case that x is ultimately periodic:

Lemma 2.13. *Assume $x \in \mathbb{A}^{\mathbb{N}}$ is ultimately periodic. Then x admits a unique ω -Lyndon factorisation.*

Proof. By Corollary 2.12 we may suppose that x does not admit a finite ω -Lyndon factorisation. Using Proposition 2.6 let $x = l_1 l_2 \cdots$ be an infinite ω -Lyndon factorisation of x . We claim the sequence $(l_i)_{i \in \mathbb{N}}$ is ultimately constant. It suffices to show that $\liminf_{i \rightarrow \infty} |l_i| < +\infty$. So pick a suffix x' of x and an infinite set $I \subseteq \mathbb{N}$ such that l_i is a prefix of x' for each $i \in I$. Using lemmas 2.5 and 2.11 it follows that either $x' = l^\omega$ for some $l \in \mathcal{L}_\omega$ or x' has only finitely many ω -Lyndon prefixes. In the latter case $\{|l_i| : i \in I\}$ is clearly bounded. In the former case pick $j < k$ in I such that $x' = \prod_{i \geq j} l_i$ and $\min\{|l_j|, |l_k|\} \geq 2|l|$. Then as ω -Lyndon words are primitive it follows that $w = l_j \cdots l_{k-1} = l^r$ for some $r \in \mathbb{N}$ contradicting Proposition 2.3.

Having proved that any infinite ω -Lyndon factorisation of x is ultimately constant, unicity of the factorisation now follows. In fact, suppose $x = l'_1 l'_2 \cdots$ is another ω -Lyndon factorisation with $l'_i = l'$ for all i greater than some k' and $l' \in \mathcal{L}_\omega$. Then since l and l' are each primitive, it follows that $|l| = |l'|$ whence $l = l'$ and the two factorisations must ultimately synchronise, i.e., $l_i = l'_i$ for all sufficiently large i . The rest now follows from Proposition 2.3. \square

Definition 2.14. A factor $u \in \mathbb{A}^+$ of an infinite word x is said to be minimal in x if $u^\omega \preceq v^\omega$ for all factors v of x with $|v| = |u|$.

We note that if u is a minimal factor of x then so is every prefix of u . The following lemma will be applied to show that any infinite aperiodic word x admits at most one infinite ω -Lyndon factorisation, and how to construct it.

Lemma 2.15. *Assume $x \in \mathbb{A}^{\mathbb{N}}$ and $u \in \mathbb{A}^+$ is a minimal factor of x . Let $w \in \mathbb{A}^*$ be the longest prefix of x preceding the first occurrence of u in x . Assume x admits an infinite ω -Lyndon factorisation $x = l_1 l_2 l_3 \cdots$ with $\limsup_{i \rightarrow \infty} |l_i| = +\infty$. Then either $w = \varepsilon$ or $w = l_1 \cdots l_k$ for some $k \in \mathbb{N}$.*

Proof. Put $n = |u|$ and write $u = u_1 u_2 \cdots u_n$. Also write $x = w u x'$ with $x' \in \mathbb{A}^{\mathbb{N}}$; by assumption wu contains exactly one occurrence of u . Assume $w \neq \varepsilon$ and let k be the least positive integer such that $\sum_{i=1}^k |l_i| \geq |w|$. We must show that $\sum_{i=1}^k |l_i| = |w|$. Suppose to the contrary that $\sum_{i=1}^k |l_i| > |w|$. We first note that u cannot be fully contained inside l_k for otherwise, if v denotes the prefix of l_k with $|v| = |u|$, then as $v \neq u$ and $l_k \in \mathcal{L}_\omega$ we

have $v^\omega \prec u^\omega$ which contradicts that u is minimal. Thus we may write $l_k = zu_1 \cdots u_p$ for some $z \neq \varepsilon$ and $p < n$. Let $r = \min\{|l_i| : i \geq k+1\}$ and pick $j \geq k+1$ with $|l_j| = r$. Also pick $j' > j$ such that $|l_{j'}| \geq p+r$.

Case 1: $n \geq p+r$

By definition of r it follows that $u_{p+1} \cdots u_{p+r}$ is a prefix of l_{k+1} . We first claim that

$$u_{p+1} \cdots u_{p+r} = u_1 \cdots u_r = (u_1 \cdots u_p)^{\frac{r}{p}} \quad (1)$$

where $(u_1 \cdots u_p)^{\frac{r}{p}}$ denotes the prefix of length r of $(u_1 \cdots u_p)^\omega$. In fact, as u is a minimal factor of x we have that $(u_1 \cdots u_r)^\omega \preceq (u_{p+1} \cdots u_{p+r})^\omega$. Furthermore since $l_{k+1}^\omega \preceq l_k^\omega \prec (u_1 \cdots u_p)^\omega$ and $u_{p+1} \cdots u_{p+r}$ is a prefix of l_{k+1} it follows that $(u_{p+1} \cdots u_{p+r})^\omega \preceq ((u_1 \cdots u_p)^{\frac{r}{p}})^\omega$. Combining we get $(u_1 \cdots u_r)^\omega \preceq (u_{p+1} \cdots u_{p+r})^\omega \preceq ((u_1 \cdots u_p)^{\frac{r}{p}})^\omega$ from which (1) follows.

We also claim that

$$l_j = u_1 \cdots u_r. \quad (2)$$

Indeed, since u is a minimal factor of x we have $(u_1 \cdots u_r)^\omega \preceq l_j^\omega$. On the other hand $l_j^\omega \preceq l_{k+1}^\omega$ and so by taking the prefix of length r of both words we obtain $l_j^\omega \preceq (u_{p+1} \cdots u_{p+r})^\omega$. So combining and using (1) we deduce that $(u_1 \cdots u_r)^\omega \preceq l_j^\omega \preceq (u_1 \cdots u_r)^\omega$ from which (2) follows.

Thus we have

$$(u_1 \cdots u_r)^\omega = l_j^\omega \preceq l_k^\omega \prec (u_1 \cdots u_p)^\omega$$

and hence by Lemma 2.1

$$(u_1 \cdots u_r u_1 \cdots u_p)^\omega \prec (u_1 \cdots u_p u_1 \cdots u_r)^\omega. \quad (3)$$

Using the fact $u_1 \cdots u_p u_{p+1} \cdots u_{p+r}$ is a minimal factor of x together with (1) and (2) gives

$$(u_1 \cdots u_p u_1 \cdots u_r)^\omega = (u_1 \cdots u_{p+r})^\omega \preceq (l_{j'}[p+r])^\omega \preceq (l_j^\omega[p+r])^\omega = ((u_1 \cdots u_r)^{\frac{p+r}{r}})^\omega.$$

Together with (3) gives

$$(u_1 \cdots u_r u_1 \cdots u_p)^\omega \prec (u_1 \cdots u_p u_1 \cdots u_r)^\omega \preceq ((u_1 \cdots u_r)^{\frac{p+r}{r}})^\omega. \quad (4)$$

It follows from (4) that $u_1 \cdots u_p = (u_1 \cdots u_r)^{\frac{p}{r}}$ and hence

$$u_1 \cdots u_r u_1 \cdots u_p = (u_1 \cdots u_r)^{\frac{p+r}{r}}$$

which by (4) gives $((u_1 \cdots u_r)^{\frac{p+r}{r}})^\omega \prec ((u_1 \cdots u_r)^{\frac{p+r}{r}})^\omega$, a contradiction.

Case 2: $n < p+r$.

In this case $u_{p+1} \cdots u_n$ is a prefix of l_{k+1} and the same arguments used to prove (1) shows that

$$u_1 \cdots u_n = (u_1 \cdots u_p)^{\frac{n}{p}}. \quad (5)$$

As $l_k^\omega = (zu_1 \cdots u_p)^\omega \prec (u_1 \cdots u_p)^\omega$, it follows that $|l_k| = |z| + p < n$ for otherwise $u = u_1 \cdots u_n$ would be a prefix of l_k which would imply an earlier occurrence of u in x . Thus

$$zu_1 \cdots u_p = (u_1 \cdots u_p)^{\frac{|z|+p}{p}} = (u_1 \cdots u_p)^a u_1 \cdots u_q \quad (6)$$

for some choice of integers a, q and as l_k is primitive we have that $1 \leq q < p$.

Finally, we have $z = (u_1 \cdots u_p)^{a-1} u_1 \cdots u_q$ and hence

$$u_1 \cdots u_p u_1 \cdots u_q = u_1 \cdots u_q u_1 \cdots u_p \quad (7)$$

from which it follows that l_k is not primitive, a contradiction. \square

Proposition 2.16. *Let $x \in \mathbb{A}^{\mathbb{N}}$ be an aperiodic infinite word and $x = l_1 l_2 l_3 \cdots = l'_1 l'_2 l'_3 \cdots$ two infinite ω -Lyndon factorisations of x . Then $l_i = l'_i$ for each $i \in \mathbb{N}$.*

Proof. Suppose to the contrary that $l_i \neq l'_i$ for some $i \in \mathbb{N}$. Short of replacing x by some suffix of x , we may assume that $l_1 \neq l'_1$. By Lemma 2.11 it follows that $x \notin \mathcal{L}_\omega$ and hence x contains a minimal factor u which is not a prefix of x . Let $w \in \mathbb{A}^+$ denote the prefix of x which precedes the first occurrence of u in x . As x is aperiodic it follows that $\limsup_{i \rightarrow \infty} |l_i| = \limsup_{i \rightarrow \infty} |l'_i| = +\infty$. By Lemma 2.15 it follows that there exist $k, k' \in \mathbb{N}$ such that $w = l_1 \cdots l_k = l'_1 \cdots l'_{k'}$ contradicting Proposition 2.3. \square

Corollary 2.17. *Each infinite word $x \in \mathbb{A}^{\mathbb{N}}$ admits precisely one ω -Lyndon factorisation.*

Proof. Existence follows from Proposition 2.6. For unicity, if x admits a finite ω -Lyndon factorisation, then unicity follows from Corollary 2.12. So we may suppose that x admits only infinite ω -Lyndon factorisations. If x is ultimately periodic unicity follows from Lemma 2.13 while if x is aperiodic unicity follows from Proposition 2.16. \square

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