WQO-BQO What is up ?   Lyon 21–23

Vietoris and scattered hyperspaces

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Some references

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Definition 1.

For a topological space X, we say that a family $\mathscr{U} := \{U_x : x \in X\}$ is a clopen selector if each U_x is a closed and open (clopen) subset of X and if $\mathscr{U}$ satisfies :

(1) If $x \neq y$ then $x \notin U_y$ or $y \notin U_x$ (Hausdorff Separation Axiom),

(2) $x \in U_x$ for every $x \in X$,

(3) if $y \in U_x$ then $U_y \subseteq U_x$ (Transitivity / Absorption).

A Skula space X is a compact 0-dimensional space having a clopen selector (so X is a Priestley space). ■

Note that, given $\mathscr{U}$, $y \leq^{\mathscr{U}} x$ whenever $y \in U_x$ is a partial order on X, and thus $U_x = \{y \in U_x : y \leq x\}$.

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Basic examples :

1. Successor ordinals, and 2. Countable compact and scattered spaces.

Main Example : Final subsets of a WQO

A poset P is a Well-Quasi-Ordering (WQO) if P is well-founded with no infinite antichain (FAC).

Let P be a WQO. Denote by $\mathrm{FS}(P)$ the set of all finite final subsets of P. So $\emptyset, P \in \mathrm{FS}(P)$. Moreover

– $\mathrm{FS}(P)$ ordered by $\supseteq$ is well-founded.

– $\mathrm{FS}(P)$ is endowed with the pointwise topology : a basic clopen set is $\{\, z \in \mathrm{FS}(P) \colon \sigma \subseteq z \text{ and } \tau \cap z = \emptyset\}$ where $\sigma, \tau \subseteq P$ are finite.

Theorem 1.

Let P be a WQO. Then $\langle \mathrm{FS}(P), \supseteq \rangle$ is Skula, i.e. it has a clopen selector.

Proof. For $F \in \mathrm{FS}(P)$ let σ_F be the finite subset of P such that F is the final subset of P generated by σ_F, i.e. $F = \bigcup_{t \in \sigma_F} \uparrow t$. So F is clopen in and

$$U_F := \{\, G \in \mathrm{FS}(P) \colon G \supseteq F\} \;=\; \{\, G \in \mathrm{FS}(P) \colon \sigma_F \subseteq G\}$$

is clopen in $\mathrm{FS}(P)$. So $\{U_F \colon F \in \mathrm{FS}(P)\}$ is a clopen selector.

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Let $\mathscr{U}$ be a clopen selector of a Skula space X then :

- $\mathscr{U}$ defines the topology on the compact space X.
- $\langle \mathscr{U}, \subseteq \rangle$ is well-founded and
  If U_x is minimal in $\mathscr{U}$ then $U_x = \{x\}$..

Proof Let $(U_{x_n})_{n\in\omega}$ be s strictly decreasing sequence of members of $\mathscr{U}$. Let $F = \bigcap_n U_{x_n}$. For $y \in F$, $U_y \subseteq F$ and thus $F = \bigcup_{y\in F} U_y$ is open. Then $U_{x_n} \setminus F \neq \emptyset$, but $\bigcap_n U_{x_n} \setminus F = \emptyset$: a contradiction. ∎

- X is a (topologically) scattered space : every nonempty closed set F has an isolated point (for the induced topology)

- Every closed initial subset K of X (in particular $U_x \cap U_y$) is a finite union of U_z and thus, K is clopen.

  So in some sense $\mathscr{U}$ is an "almost-meet-semilattice".

Denote by $D(Y)$ the set of non-isolated points of Y. Moreover set

$$D^0(X) = X \text{ and } D^\alpha(X) = D\big(\bigcap_{\beta<\alpha} D^\beta(X)\big).$$

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Two invariants for a Skula space.

Let $\mathscr{U}$ be a clopen selector of a Skula space X. Then

(1) $\mathscr{U}$ is well-founded. Therefore $\langle X, \subseteq \rangle$ has a (well-founded) rank :

$$\mathrm{rk}_{\mathsf{WF}X}(x) = \sup\{\mathrm{rk}_{\mathsf{WF}X}(y) \colon y < x\}.$$

So $\mathrm{rk}_{\mathsf{WF}X}(x) = 0$ if and only if x is minimal, i.e. $U_x = \{x\}$. Moreover $\mathrm{rk}_{\mathsf{WF}}(X)_{:=} \sup_{x \in X} \mathrm{rk}_{\mathsf{WF}X}(x)$.

By compactness $\mathrm{rk}_{\mathsf{WF}}(X) = \sup_x \mathrm{rk}_{\mathsf{WF}X}(x)$ is the last (ordered) derivative is nonempty and finite.

(2) X is compact and scattered. Therefore X has a (Cantor-Bendixson) height :

$$\mathrm{ht}_{\mathsf{CB}X}(x) = \gamma \text{ iff } x \in D^{\gamma}(X) \setminus D^{\gamma+1}(X).$$

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Hyperspace $H(X)$ of a Skula space X.

We define the Vietoris hyperspace $H(X)$ over a Skula space X as follows :

- $H(X)$ is the set of all nonempty closed initial subsets of $\langle X, \leq \rangle$. Therefore $\mathscr{U} \subseteq H(X)$.
- For $F, G \in H(X)$, we set $F \leq G$ if and only if $F \subseteq G$.
- The topology on $H(X)$ is the topology generated by the sets
$$U^+ := \{K \in H(X)\colon K \subseteq U\}$$
declared to be clopen where U is any clopen initial subset in X. ■

So $V^- := \{K \in H(X) : K \cap V \neq \emptyset\}$ is clopen in $H(X)$ if V is clopen final in X.

Theorem 2.

Let X be a Skula space. Then $H(X)$ is a Skula space. ■

Main order property ($A, B \in H(X)$) :

- $H(X)$ is a continuous join-semillattice where $A \vee B := A \cup B$.

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Canonical Skula spaces

Recall that for a compact space Z, we denote by $\mathrm{ht}_{\mathrm{CB}}(Z)$ the (unique) finite and nonempty Cantor-Bendixson derivative.
We say that Z is unitary if $\left|D^{\mathrm{ht}_{\mathrm{CB}}(Z)}\right| = 1$.

A space X is canonically Skula if X has a clopen selector $\mathscr{U} := \{U_x : x \in X\}$ such that for $D^{\mathrm{ht}_{\mathrm{CB}}(U_x)}(U_x) = \{x\}$ for $x \in X$ and $\mathscr{U}$ is called a canonical clopen selector.
Recall that if X is Skula then $\mathrm{ht}_{\mathrm{CB}X}(x) \leq \mathrm{rk}_{\mathrm{WF}X}(x)$.

Fact 3.

Let X be a Skula space and $\mathscr{U} := \{U_x : x \in X\}$ be a clopen selector. The following are equivalent :

(i) *X is canonically Skula. That is*
Each U_x is unitary and satisfies $D^{\mathrm{ht}_{CB}}(U_x) = \{x\}$.

(ii) *Each U_x is unitary and $\mathrm{ht}_{CB}(U_x) = \mathrm{rk}_{WF}(U_x)$.*

(iii) *Each x is the maximum of U_x and $\mathrm{ht}_{CB}(U_x) = \mathrm{rk}_{WF}(U_x)$.* ■

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Theorem 4.

If X is canonically Skula then $H(X)$ is canonically Skula.

A computation of height and rank on canonical Skula space.

Fact 5. *Let* $\mathscr{U} := \{U_x : x \in X\}$ *be a canonical selector for* X *and let* $x \in X$ *(so* $\mathscr{U} \subseteq H(X)$*). Then*

$$\mathrm{rk}_{WF}(U_x) = \mathrm{rk}_{WF_X}(x) = \mathrm{ht}_{CB_X}(x) = \mathrm{ht}_{CB}(U_x) \text{ and}$$

(1) *If* $\mathrm{rk}_{WF_X}(x) = 0$ *then* $\mathrm{ht}_{CB_{H(X)}}(U_x) = 0$.

(2) *If* $\mathrm{rk}_{WF_X}(x) = 1$ *then* $\mathrm{ht}_{CB_{H(X)}}(U_x) = 1$.

(3) *If* $\mathrm{rk}_{WF_X}(x) = 1+\alpha \geq 2$ *then* $\mathrm{ht}_{CB_{H(X)}}(U_x) = \omega^{\alpha}$.

Application. Let $U_\sigma := \bigcup_{x\in\sigma} U_x$ where $\sigma = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ is an antichain of X satisfying

$\mathrm{ht}_{CB_X}(x_0) = 0$ $\quad$ $\mathrm{ht}_{CB_X}(x_1) = 1 = \mathrm{ht}_{CB_X}(x_2)$ $\quad$ $\mathrm{ht}_{CB_X}(x_3) = 2$

$\mathrm{ht}_{CB_X}(x_4) = 10 = \mathrm{ht}_{CB_X}(x_5)$ $\quad$ $\mathrm{ht}_{CB_X}(x_6) = \omega + 7$ $\quad$ $\mathrm{ht}_{CB_X}(x_7) = 3.$

By Fact 5 and Telgàsky Theorem, we have :

$$\mathrm{ht}_{CB_{H(X)}}(U_\sigma) = \omega^{\omega+7} + \omega^9 \cdot 2 + \omega^2 + \omega + 2.$$

Examples of canonically Skula spaces.

Claim 6. Let α be an ordinal. Then $\mathrm{FS}(\alpha) = \alpha{+}1$ is canonically Skula.
If D is a scattered chain then $\mathrm{FS}(D)$ is (canonically) Skula.
But if $D' = \omega_1{+}\omega^*$ then
$\mathrm{FS}(D')$ not homeomorphic to $\mathrm{FS}(P)$ for any wqo P. [ABK, § 3.3]
Hint. For $\beta \leq \alpha$ consider $\delta_\beta < \beta$ such that (δ_β, β) is indecomposable and cofinal in $[0, \beta)$. Then $\mathscr{U} := \{(\delta_\beta, \beta] \colon \beta \leq \alpha\}$ is a canonical clopen selector. For scatteredness, use variant in the proof of Hausdorff. ... ∎

Claim 7. Let P be a countable wqo. Then $\mathrm{FS}(P)$ is canonically Skula.
Hint. Each member of $\mathrm{FS}(P)$ is generated by a finite subset of P, and thus $\mathrm{FS}(P)$ is countable. Hence $\mathrm{FS}(P)$ is homeomorphic to a compact subset K of the reals $\mathbb{R}$. Again, since K is countable, K is scattered. Therefore K is homeomorphic to some $\alpha + 1$. So $K := \mathrm{FS}(P)$ is canonically Skula. ∎

Main Question [M. Pouzet].

Let P be a well-quasi-ordering (w.q.o.) or a better-quasi-ordering (b.q.o.). Is $\mathrm{FS}(P)$ canonically Skula ?

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