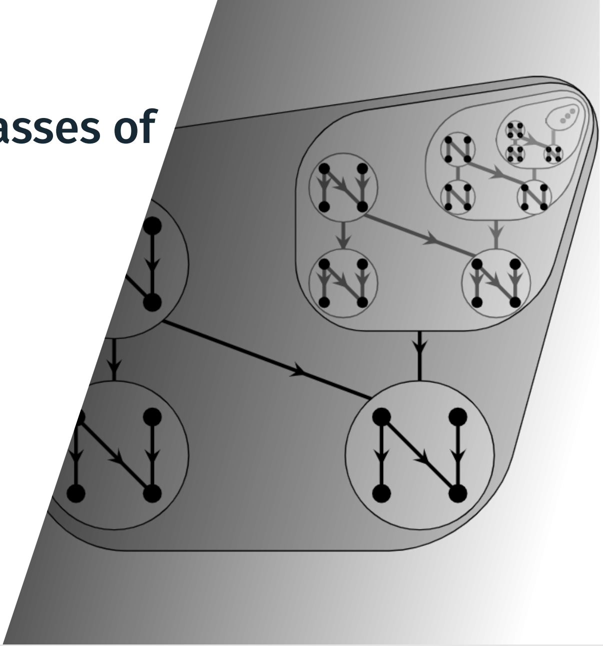
Better-Quasi-Ordering Classes of Partial Orders



Gregory McKay

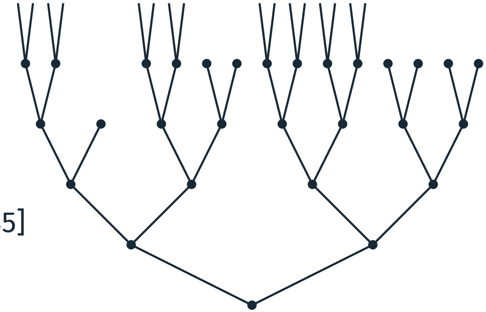
Lyon
February 21, 2023

Find the largest class of partial orders BQO under embeddability.

- These are results I proved in 2014 during my PhD which generalise many theorems that classes partial orders are BQO under embeddability. [McK15]
- More recently written up these results and my paper has been accepted. [McK22]
- Won't have time to give rigorous proofs. For full details see the paper.
- Intuition by pictures.

Previous Results

- *Nash-Williams trees* [NW65]
- Laver σ-scattered linear orders [Lav71]
- Laver σ-scattered trees [Lav78]
- Corominas countable pseudo-trees [Cor85]
- Thomassé countable *N*-free [Tho99]



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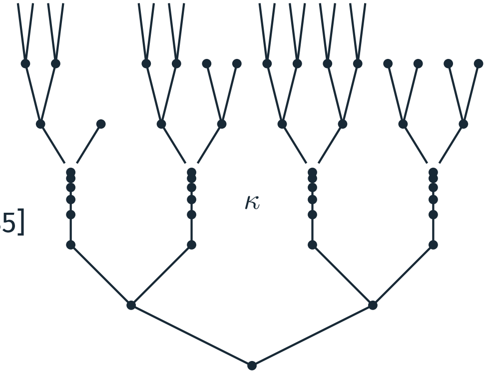

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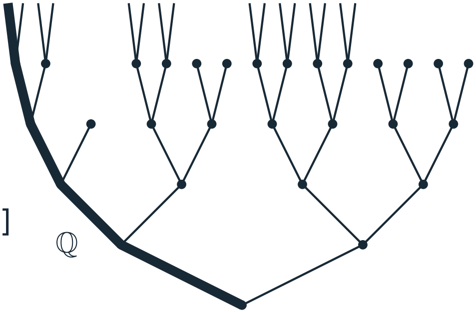

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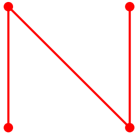
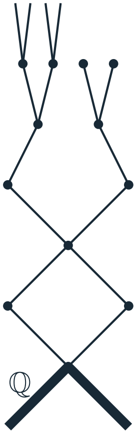


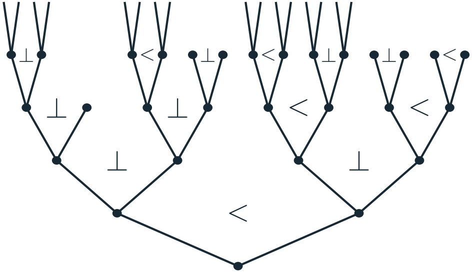


Can we unify these results?

- Can we find a natural class of partial orders that is BQO under embeddability that contains all of the:
  - σ-scattered linear orders,
  - σ-scattered trees,
  - countable N-free partial orders?
- Is there a "σ-scattered" result for N-free partial orders?
- What is special about N?

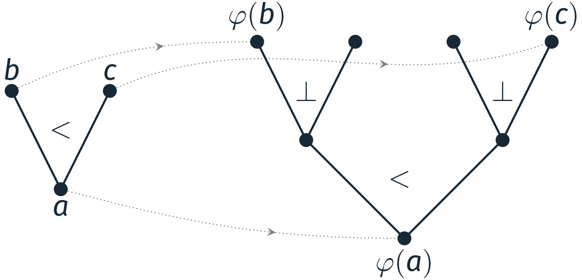
Structured Trees

To build our large class of partial orders, we'll use *structured trees*.
These are trees labelled with partial orderings of to all of the subtrees above each point.



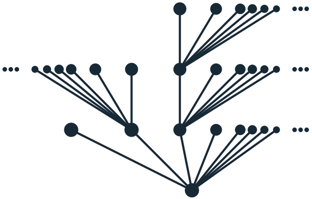
Structured Trees

Structured tree embeddings induce embedding of the orderings of the subtrees.



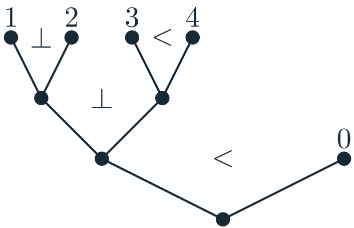
Sums of Partial Orders

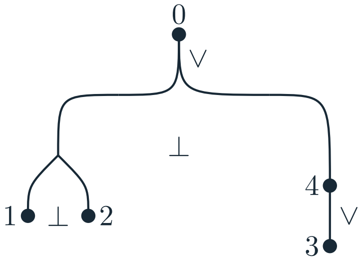
Hausdorff's Theorem on scattered linear orders.



Sums of Partial Orders

Similarly we will use structured trees to index partial order sums.





Well-Behaved

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- We need structured trees need to be more than BQO.
- We will need Q-colourings.
- Q is BQO implies $\mathcal{P}(Q)$ is BQO is also not quite enough.

Well-Behaved

For a class of partial orders $\mathcal{P}$ and quasi-order Q:

- $\mathcal{P}(Q) = \{a : P \to Q \mid P \in \mathcal{P}\}$ is the class of *Q-coloured* members of $\mathcal{P}$.
- $\mathcal{P}$ is *well-behaved* iff for any quasi-order Q and any bad $\mathcal{P}(Q)$-array $f : [\omega]^\omega \to \mathcal{P}(Q)$ there is $M \in [\omega]^\omega$ and a bad Q-array $g : [M]^\omega \to Q$ such that for all $X \in [M]^\omega$ there exists v in the domain of $f(X)$ with

$$g(X) = f(X)(v).$$

- We call g a *witnessing Q-array* for f.

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Similar definition for structured trees.

Well-Behaved

If $\mathcal{P}$ is well-behaved then $\mathcal{P}$ is BQO under embeddability.

Proof.

Suppose there is a bad $\mathcal{P}$-array f. Let 1 be the ordinal $1 = \{0\}$, and let $g : [\omega]^\omega \to \mathcal{P}(1)$ be given by $g(X)$ equals the map $f(X) \to 1$. Then g is a $\mathcal{P}(1)$-array. To see that it is bad, it is required that

$$f(X) \to 1 \;\not\leq\; f(X \setminus \{\min X\}) \to 1$$

for each $X \in [\omega]^\omega$. If not, then $f(X) \to 1$ is embeddable in $f(X \setminus \{\min X\}) \to 1$ which entails that $f(X) \leq f(X \setminus \{\min X\})$, contrary to f bad. Since $\mathcal{P}$ is well-behaved, there is a witnessing bad 1-array for g, which is impossible. □

Well-Behaved

A finite set $\mathcal{P}$ of finite partial orders is well-behaved.

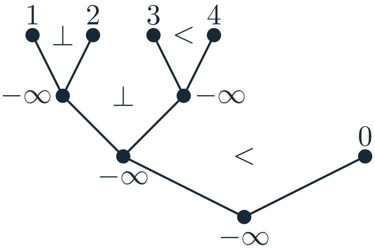
Proof.

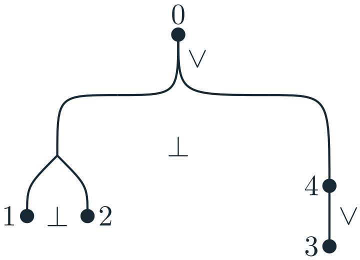
Let Q be an arbitrary quasi-order and let f be a bad $\mathcal{P}(Q)$-array. Since $\mathcal{P}$ is finite, we can repeatedly apply the Galvin and Prikry Theorem to find $A \in [\omega]^\omega$ and some $P \in \mathcal{P}$ such that for all $X \in [A]^\omega$, the domain of $f(X)$ is P.
Since P is finite, for some $n \in \omega$, $P = \{x_i \mid i \leq n\}$. For $i \leq n$ let $f_i : [A]^\omega \to Q$ be given by $f_i(X) = f(X)(x_i)$ for all $X \in [A]^\omega$.
For all $B \in [A]^\omega$, not all $f_i \upharpoonright [B]^\omega$ can be perfect otherwise $f \upharpoonright [B]^\omega$ would be perfect.
By repeatedly restricting such that f_i is either bad or perfect, after n times some f_i must be bad, and this is clearly a witnessing array for f. □

Interval Trees

Our partial order sum construction naturally gives a mapping $\Theta : \mathcal{P}(Q) \to \mathcal{T}(Q \cup \{-\infty\})$. This takes a Q-coloured partial order P into an *interval tree* $\Theta(P)$. For partial orders A and B:

$$\Theta(A) \leq \Theta(B) \quad \longrightarrow \quad A \leq B$$





Well-Behaved

If $\mathcal{T}$ is well-behaved then $\mathcal{P}$ is well-behaved.

Proof.

Suppose f is a bad $\mathcal{P}(Q)$-array. For all $X \in [\omega]^\omega$

$$f(X) \not\leq f(X \setminus \{\min X\}) \quad \text{so} \quad \Theta(f(X)) \not\leq \Theta(f(X \setminus \{\min X\}).$$

Thus $\Theta \circ f$ is a bad $\mathcal{T}(Q \cup \{-\infty\})$-array. Since $\mathcal{T}$ is well-behaved then there is a witnessing bad $Q \cup \{-\infty\}$-array g for $\Theta \circ f$. Then g can be restricted to a bad Q-array that is witnessing for f. □

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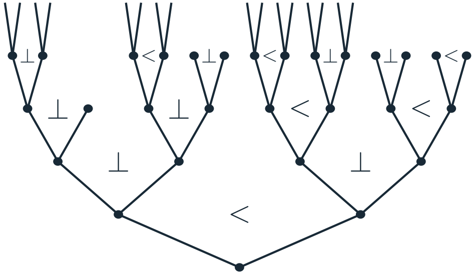
We will appeal to similar correspondences as Θ and similar arguments for well-behavedness can be made.

Choice of Structured Trees

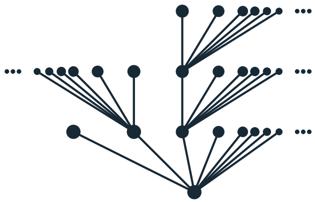
We rely on a result from Kříž [Kří89]:

- Let $\mathcal{R}_\mathcal{P}$ be the class of $\mathcal{P}$-structured trees of height at most ω.
- $\mathcal{R}_\mathcal{P}$ is well-behaved whenever $\mathcal{P}$ is well-behaved.



Linear Order Labels

Kříž used this to show that the class of σ-scattered linear orders $\mathcal{M}$ is well-behaved, strengthening Laver's result. [Kří89]



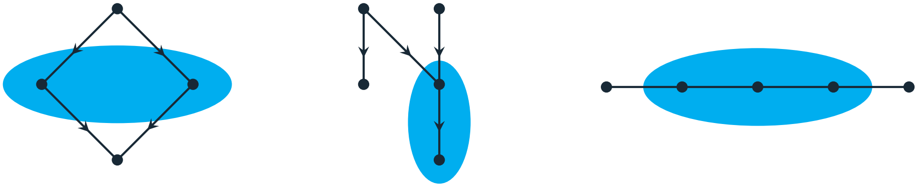
Which labels to choose?

Looking for a class:

- Containing partial orders
- Well-behaved
- Members can’t be written as a sums of other members

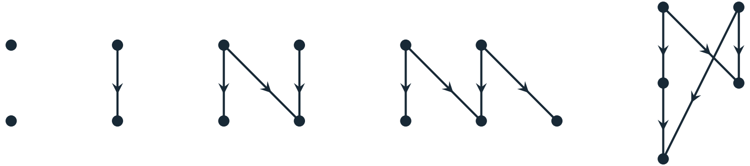
Intervals

If P is a partial order and I is a non-empty subset of P, then call I an *interval* of P if for all x, y in I and for all a in $P \setminus I$, a shares the same relationship to x and y.



Indecomposable Partial Orders

Let P be a non-empty partial order. Then P is *indecomposable* if every interval of P is either P itself or a singleton.



We say that X *decomposes into* $\mathcal{P}$ iff every indecomposable subset of X is isomorphic to a member of $\mathcal{P}$.

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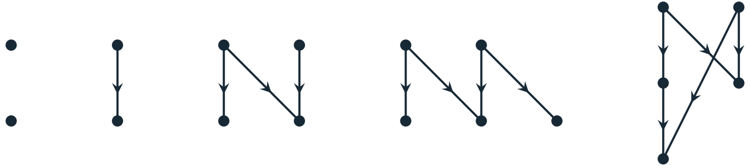

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If an order decomposes into $\{1, 2, 2_\perp\}$ then it is N-free because any indecomposable partial order with at least three vertices embeds N. [Kel85, Tho99]

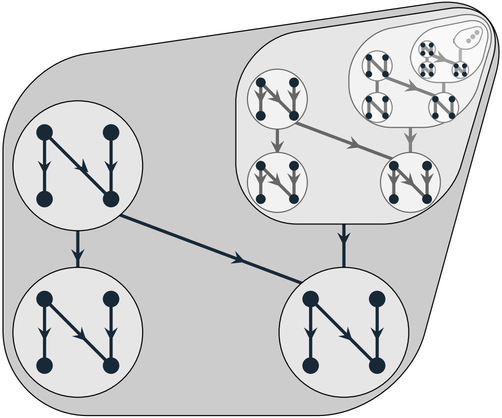
Which labels to choose?

Looking for a class:

- Containing partial orders
- Well-behaved
- Members are indecomposable
- Contains N

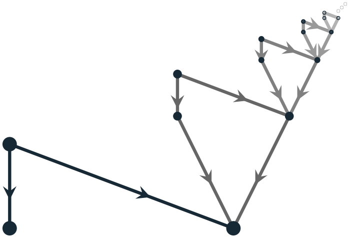
Need to do better than $\mathcal{R}_{\mathcal{P}}$

But there's a problem if we want to capture all countable partial orders.
Branches of $\mathcal{R}$ are at most order type ω. No ω^*. No $\mathbb{Q}$.



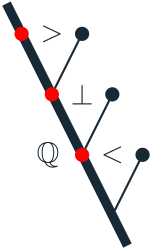
A Shortcut

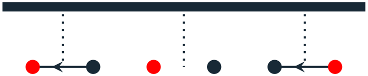
- In [McK22] I take a shortcut by using nested chains of intervals as structure tree labels on the structured trees of $\mathcal{R}$.
- Idea originally came from structured pseudo-trees.
- In [McK15] I proved that σ-scattered $\mathcal{P}$-structured pseudo-trees with branches in $\mathcal{L}$ are well-behaved if $\mathcal{P}$ and $\mathcal{L}$ are both well-behaved.



$\mathcal{P}$-Structured $\mathcal{L}$-Branches

For a class of linear orders $\mathcal{L}$ and a class of partial orders $\mathcal{P}$, a single structured branch corresponds to a member of $\mathcal{L}(\mathcal{P}(2_\perp))$.
This gives a mapping with similar properties to Θ. It follows that a single branch is well-behaved whenever $\mathcal{L}$ and $\mathcal{P}$ are.





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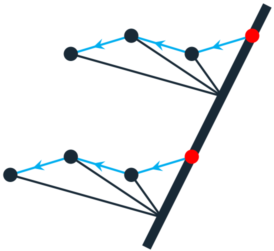
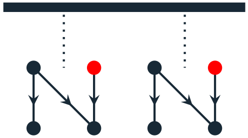
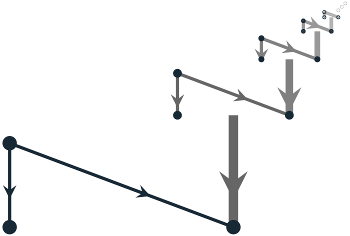






σ-Scattered $\mathcal{P}$-Structured $\mathcal{L}$-Pseudo-Trees

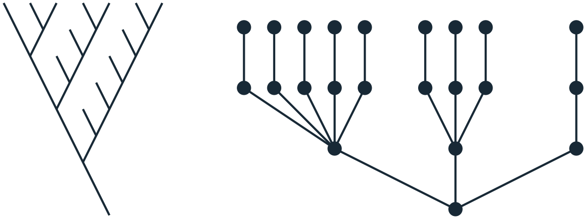
We can now “sum” our individual branches. At no point do the trees embed $2^{<\omega}$.

$\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$ is Well-Behaved

- Taking countable unions gives the class $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$ of σ-scattered $\mathcal{P}$-structured pseudo-trees with branches of order types in $\mathcal{L}$.

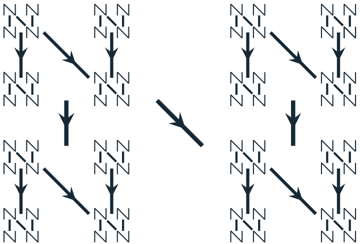
$\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$ is Well-Behaved

- Taking countable unions gives the class $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$ of σ-scattered $\mathcal{P}$-structured pseudo-trees with branches of order types in $\mathcal{L}$.
- Using trees of $\mathcal{R}$ structured using labels in $\mathcal{L}(\mathcal{P}(2_{\perp}))$ to index the sums, we can show similarly that $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$ *is well-behaved* if $\mathcal{L}$ and $\mathcal{P}$ are (by Kříž's Theorem).



New structured trees yield new partial order sums

- $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$ is well-behaved whenever $\mathcal{L}$ and $\mathcal{P}$ are.
- We can now use our new structured pseudo-trees to sum partial orders.
- The resulting class of partial orders will be well-behaved.
- We will use indecomposable partial orders as indexes of these sums.



Hasdorff-esque Theorem

We would like:

- A natural "external" definition for these partial orders - like σ-scattered.
- A theorem that does the same job as Hausdorff's Theorem for our partial orders.

Towards Scattered Partial Orders

Under some basic assumptions on $\mathcal{L}$ and $\mathcal{P}$, we know one of our partial orders X will satisfy:

- X decomposes into $\mathcal{P}$.
- Chains of intervals of X have order type in $\mathcal{L}$.

Given a partial order satisfying these properties, we can find a $\mathcal{P}$-structured $\mathcal{L}$-pseudo-tree T corresponding to its sum construction.

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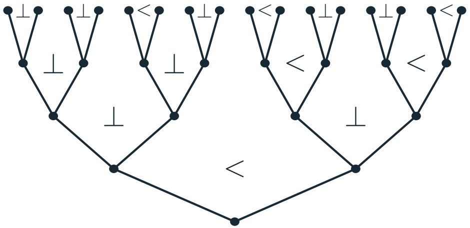
But T may not be a member of $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$.

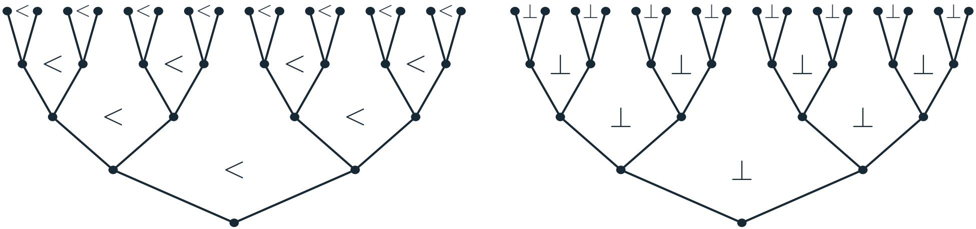
Pathological Trees

Suppose $2^{<\omega} \leq T$. Consider the labels of the image of $2^{<\omega}$ in T.



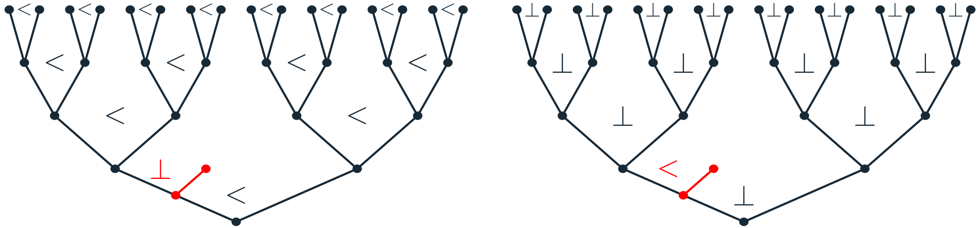
Pathological Trees

Using a Ramsey argument, can restrict so that either all labels are chains or all are antichains.



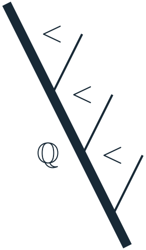
Pathological Trees

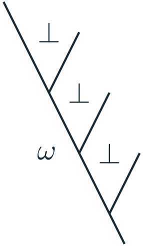
Suppose the labels of T do not alternate from chain to antichain between the points of the embedding of $2^{<\omega}$.



Pathological Trees

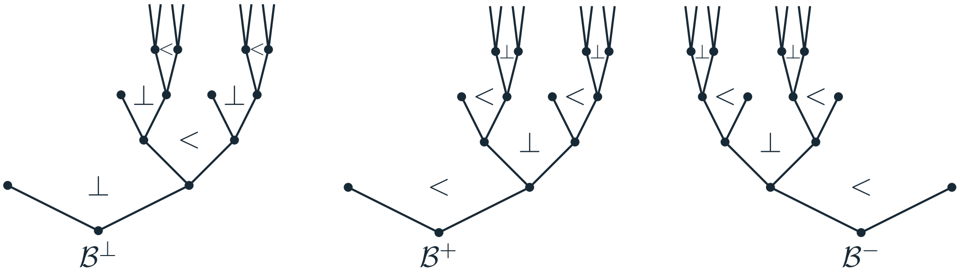
Then we can collapse these binary trees into equivalent single branches and find a scattered pseudo-tree in $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$.





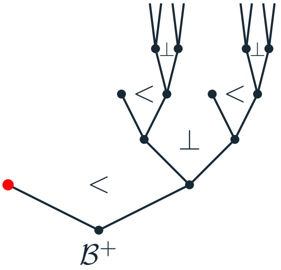
Pathological Trees

So if X has no corresponding scattered tree in $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$, then all corresponding trees embed a binary tree with alternating labels.



Pathological Partial Orders

Which partial orders do these trees correspond to?



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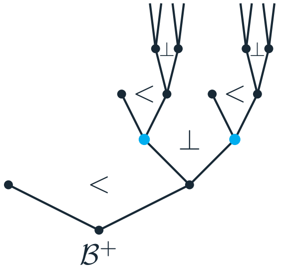

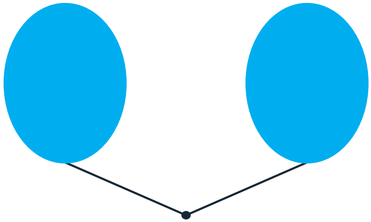

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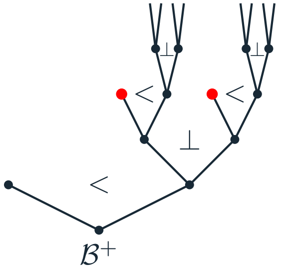

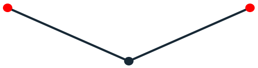

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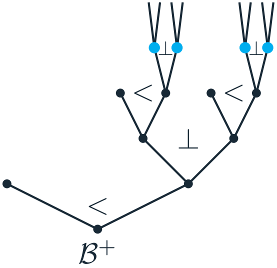
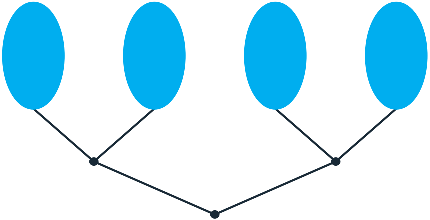
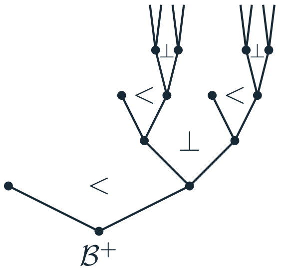
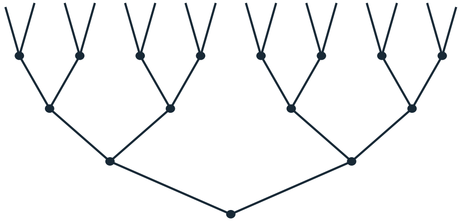




Pathological Partial Orders

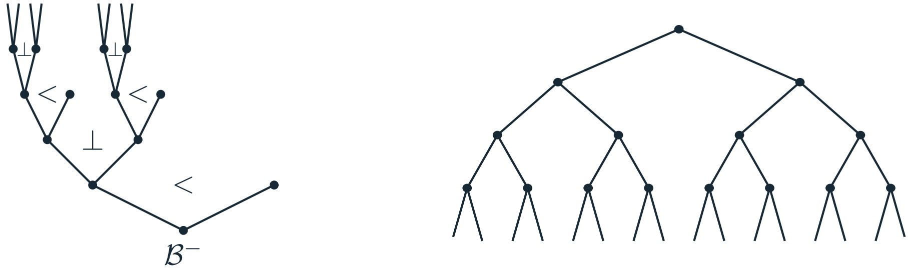
T embeds $\mathcal{B}^+$ means X embeds $2^{<\omega}$.





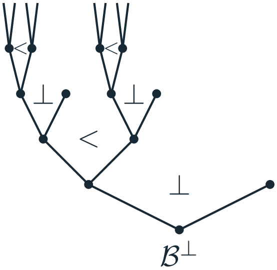
Pathological Partial Orders

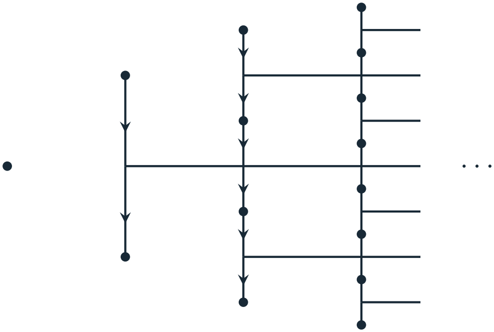
T embeds $\mathcal{B}^-$ means X embeds $-2^{<\omega}$.



Pathological Partial Orders

T embeds $\mathcal{B}^\perp$ means X embeds $2_\perp^{<\omega}$.





Scattered Partial Orders

Define $\mathcal{S}_{\mathcal{P}}^{\mathcal{L}}$ to be the class of non-empty partial orders *X* with the following properties.

1. $2^{<\omega}$, $-2^{<\omega}$ and $2_{\perp}^{<\omega}$ do not embed into *X*
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This class is equivalent to the orders that can be constructed with sums with interval trees in $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$ that do not embed $2^{<\omega}$. [McK15]

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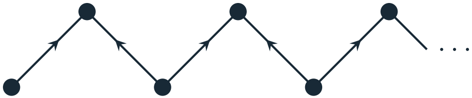
2. *X* decomposes into $\mathcal{P}$
3. ~~For every $x \in X$, there is a maximal chain of intervals of *X* with order type in $\mathcal{L}$ that contains $\{x\}$.~~
3. Every linear subset of *X* is isomorphic to a member of $\mathcal{L}$

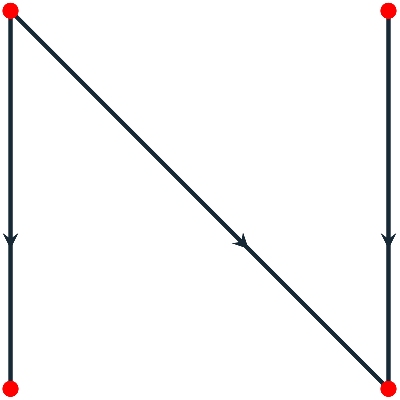
We have a mapping $\Theta : \mathcal{S}_{\mathcal{P}}^{\mathcal{L}}(Q) \to \mathcal{T}_{\mathcal{P}}^{\mathcal{L}}(Q \cup \{-\infty\})$ and by the same argument as before $\mathcal{S}_{\mathcal{P}}^{\mathcal{L}}$ *is well-behaved* whenever $\mathcal{L}$ and $\mathcal{P}$ are both well-behaved.

Towards σ-Scattered

1. Only the scattered members of $\mathcal{T}_{\mathcal{P}}^{\mathcal{L}}$ are used.
2. We wont get all countable partial orders that decompose into $\mathcal{P}$ and have linear subsets in $\mathcal{L}$.
3. We can't take arbitrary countable unions.



Limiting Sequences



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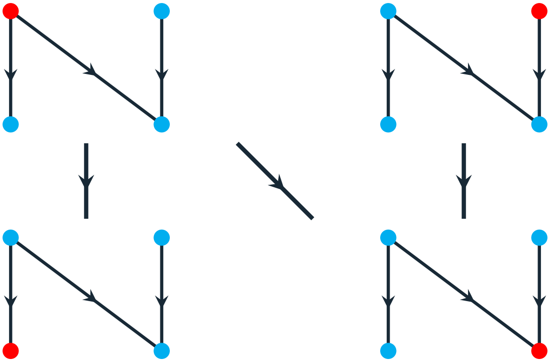

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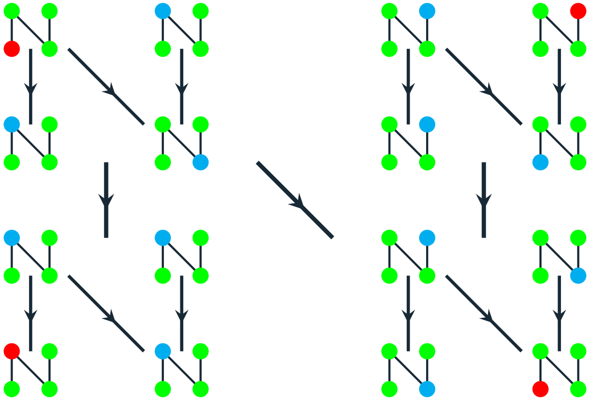

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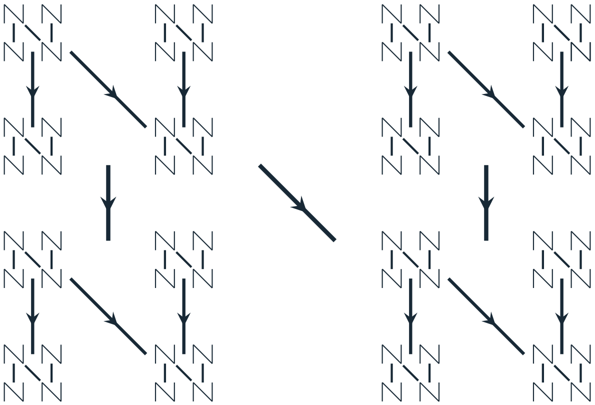

σ-Scattered

Let $\mathcal{M}^{\mathcal{L}}_{\mathcal{P}}$ be the class of unions of limiting sequences of members of $\mathcal{S}^{\mathcal{L}}_{\mathcal{P}}$.

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The correspondence implies $\mathcal{M}^{\mathcal{L}}_{\mathcal{P}}$ *is well-behaved* whenever $\mathcal{L}$ and $\mathcal{P}$ are both well-behaved.

Countable partial orders

- Let $\mathcal{C}$ be the class of countable linear orders.
- Let $\mathcal{C}_{\mathcal{P}}$ be the class of countable partial orders that decompose into $\mathcal{P}$.
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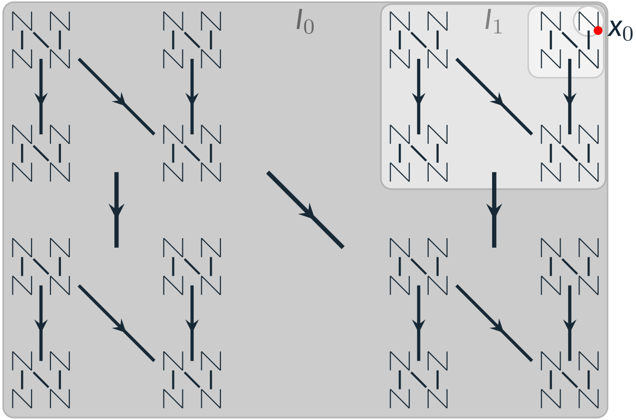
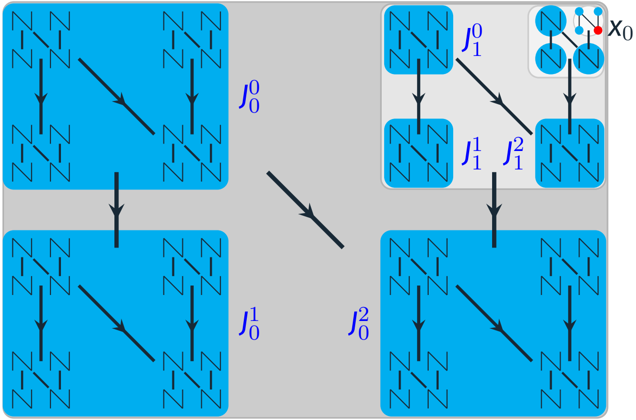
Countable partial orders

Enumerate $P = \{x_i \mid i \in \omega\} \in \mathcal{C}_{\mathcal{P}}$. Pick a maximal chain $\langle I_\alpha \mid \alpha \in \gamma \rangle$ of intervals of P containing $\{x_0\}$.



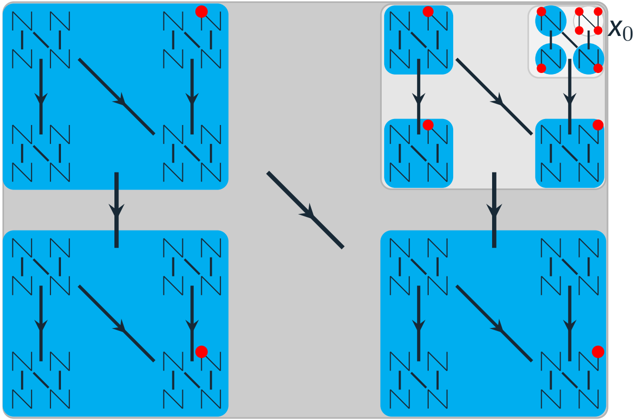
Countable partial orders

For each α, let C^γ_α be a maximal chain of intervals of $I_\alpha \setminus \bigcup_{\beta>\alpha} I_\beta$.
Consider the intervals $J^\gamma_\alpha = \bigcup(C^\gamma_\alpha \setminus \{\max C^\gamma_\alpha\})$.



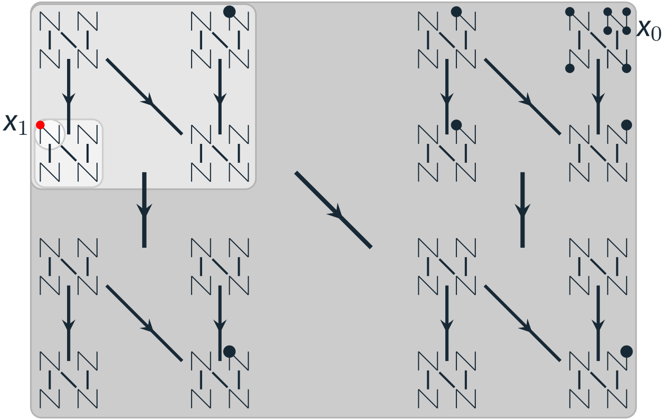
Countable partial orders

Pick a point from each distinct J^{γ}_{α} to make $P_0 \in \mathcal{S}^{\mathcal{L}}_{\mathcal{P}}$.



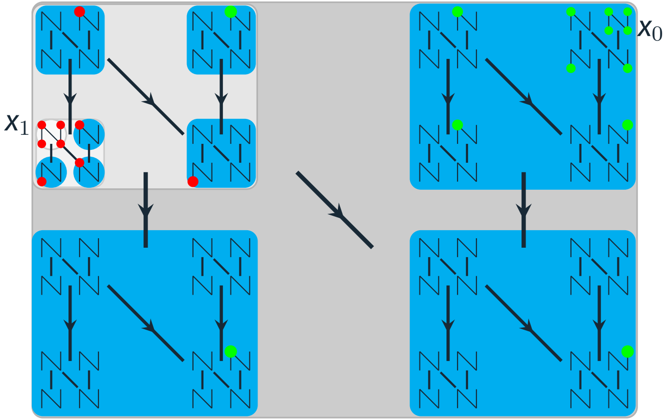
Countable partial orders

Pick a maximal chain of intervals of P that contains $\{x_{n+1}\}$.



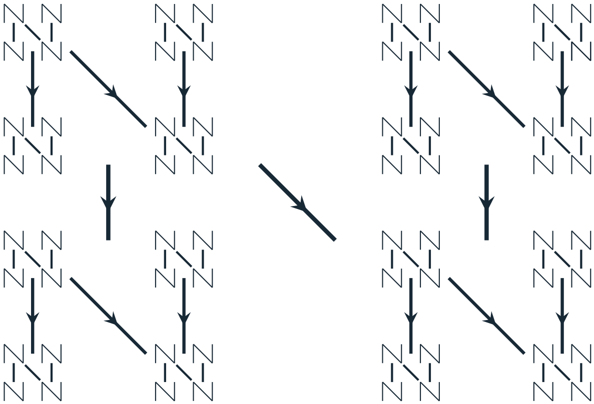
Countable partial orders

Build P_{n+1} similarly. Including all of the points of P_n.



Countable partial orders

$\langle P_n \mid n \in \omega \rangle$ is a limiting sequence with union $P \in \mathcal{M}^{\mathcal{C}}_{\mathcal{P}}$.



Recap

- $\mathcal{L}$ is a well-behaved class of linear orders.
- $\mathcal{P}$ is a well-behaved class of indecomposable partial orders.
- $\mathcal{S}_{\mathcal{P}}^{\mathcal{L}}$ is the class of orders that decompose into $\mathcal{P}$, have linear subsets in $\mathcal{L}$ and don't embed $2^{<\omega}$, $-2^{<\omega}$ or $2_{\perp}^{<\omega}$.
- $\mathcal{M}_{\mathcal{P}}^{\mathcal{L}}$, countable unions limiting sequences of members of $\mathcal{S}_{\mathcal{P}}^{\mathcal{L}}$.
- $\mathcal{M}_{\mathcal{P}}^{\mathcal{L}}$ is well-behaved.
- $\mathcal{M}_{\mathcal{P}}^{\mathcal{L}}$ contains all countable partial orders that decompose into $\mathcal{P}$.

Making it concrete

What are the largest well-behaved classes we can choose for $\mathcal{L}$?

- The class of σ-scattered linear orders $\mathcal{M}$ is well-behaved by Kříž. [Kří89]
- Aronszajn lines under PFA? [Bar20]

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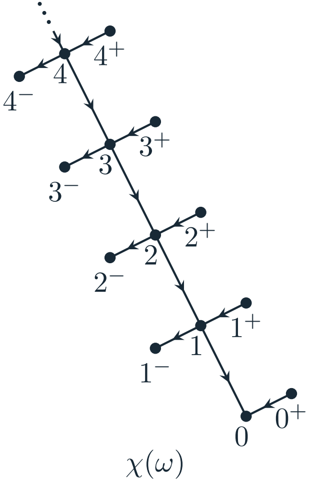
Can we do better?

- Is there an infinite well-behaved class of indecomposable partial orders?
- Is there an infinite indecomposable partial order whose singleton is well-behaved?

We can do better

If P is a partial order, define a new order $\chi(P)$ as:

- $\chi(P) = P \cup P^+ \cup P^-$
- $P^+ = \{x^+ \mid x \in P, (\exists y \in P) y > x\}$
- $P^- = \{x^- \mid x \in P, (\exists y \in P) y < x\}$
- $a^- < a \leq b < b^+$ for any $a, b \in P$ with $a \leq_P b$
- if $a \neq b$ then:
  - $a^- \perp b^-$
  - $a^+ \perp b^+$
  - $b^- \perp a$
  - $a^+ \perp b$
  - $a^+ \perp b^-$



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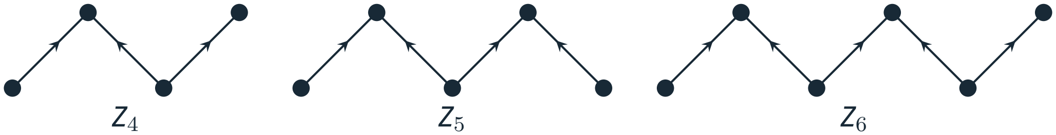
- Therefore $(\mathcal{M}^{\mathcal{M}}_{\mathcal{P}_n})^{\chi}$ contains infinitely many infinite indecomposable partial orders.
- $\mathcal{M}^{\mathcal{M}}_{(\mathcal{M}^{\mathcal{M}}_{\mathcal{P}_n})^{\chi}}$ *is well-behaved* for every $n \in \omega$.

Where to go next?

- Which classes of indecomposable partial orders are well-behaved? Arbitrarily large zigzags seem to be the issue.
- For $n \in \omega$ is the class of countable indecomposable partial orders that don't embed Z_n well-behaved?
- In the definition of scattered orders can we replace "decomposes into $\mathcal{P}$" condition by saying there is some $n \in \omega$ such that Z_n does not embed?



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