A topological interpretation of de Jongh -Parikh theorem and applications

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WQO-BQO What is up?

Lyon, February 21 2023

Abstract

An ordered set P is *well quasi-ordered* (w.q.o.) if it contains no infinite descending chain and if all its antichains are finite. All linear extensions of the order on P are well-ordered (Wolk 1967), hence their order types are ordinals. A famous result, due to de Jongh and Parikh (1977), asserts that among these ordinals, one, the *ordinal length* of P denoted by $o(P)$, is maximum. We give a topological interpretation of the coefficients in $\omega^{\beta_0} \cdot m_0 + \cdots + \omega^{\beta_{k-1}} \cdot m_{k-1}$, the Cantor normal form of $o(P)$, in terms of Cantor-Bendixson ranks of appropriate topological spaces. We illustrate our result with the poset P made of words over a finite alphabet A. We compute the ordinal length of the set $\mathbf{I}_{<\omega}(P)$ of finitely generated initial segments of a w.q.o. P that is embeddable into $[\omega]^{<\omega}$, the poset of finite subsets of ω.
This is a joint work with C.Delhommé (Université de la Réunion) and M.Sobrani (University of Fes, Morocco). The results have been presented at the Meeting of the Canadian Mathematical Society, Session in Honor of Dr. Robert Woodrow on the Occasion of his 70th birthday, Vancouver, December 7 2018.

Introduction

Let $P := (X, \leq)$ be an ordered set (poset). This poset is *well founded* if every non-empty subset of the vertex set X contains some minimal element. It is *well quasi-ordered* (w.q.o for short) or (*partially well-ordered*); if, in addition, it has no infinite antichain i.e., if every infinite set of vertices has comparable elements. Since its introduction by Kurepa, then by Erdös and Rado and Higman in 1952 the notion of w.q.o has attracted considerable interest in various areas of mathematics and computer science. It was observed by Wolk in 1967 that a poset P is w.q.o if and only if its linear extensions are all well orders.

de Jongh and Parikh proved in 1977 that there is a largest ordinal type among these linear extensions, that we denote $o(P)$ and call the *ordinal length* of P.

We give a topological interpretation of the coefficients in $\omega^{\beta_0} \cdot m_0 + \cdots + \omega^{\beta_{k-1}} \cdot m_{k-1}$, the Cantor decomposition of $o(P)$.
For that, we recall that a topological space H is *scattered* if every nonempty subset of H has an isolated point with respect to the induced topology. Equivalently, the Cantor-Bendixson procedure, consisting to take the derivatives of H, (namely $H^{(0)}, \ldots, H^{(\alpha)}, \ldots$ where $H^{(\alpha)}$ is the set of isolated points of $H \setminus \bigcup_{\beta<\alpha} H^{(\beta)}$) terminates on the emptyset.
We recall that if H is a non-empty compact scattered space, there is a largest ordinal β for which the Cantor-Bendixson derivative $H^{(\beta)}$ is non empty; let $\mathrm{rank}^{-}(H)$ be this ordinal, $H^{(\infty-)}$ be this derivative and $d^0(H)$ be the cardinality of $H^{(\infty-)}$.
We also recall that $\mathbf{I}(P)$, the set of initial segments of a poset P, equipped with the topology induced by the power set $\wp(P)$, is a compact space. If P is w.q.o, then $\mathbf{I}(P)$ is topologically scattered (while it is not necessarily w.q.o.). In this case, the set $\mathbf{Id}(P)$ of ideals of P (that is the set of non-empty updirected initial segments of P) is a closed subspace of $\mathbf{I}(P)$, hence compact and scattered.

We prove that if P is w.q.o, $\mathbf{Id}(P)$ and $\mathbf{Id}(o(P))$ have same rank and degree, namely:

Theorem 1

If P is w.q.o then:

$$\mathrm{rank}^{-}(\mathbf{Id}(P)) = \mathrm{rank}^{-}(\mathbf{Id}(o(P))); \tag{1}$$

and

$$d^0(\mathbf{Id}(P)) = d^0(\mathbf{Id}(o(P))). \tag{2}$$

For an example, if P is a well ordered chain of type α then, as a chain, $\mathbf{Id}(P)$ is order isomorphic to $\alpha + 1$ and, as a topological space, isomorphic to the ordinal $\alpha + 1$ equipped with the interval topology, hence its rank, $\mathrm{rank}^{-}(\mathbf{Id}(P))$, and its degree, $d^0(\mathbf{Id}(P))$, are respectively β and m where $\omega^\beta \cdot m$ is the largest term in the Cantor decomposition of the ordinal $\mathbf{Id}(P)$.

The main result

More generally, we define inductively a finite sequence $(P_i)_{i<k}$ of subsets of P, that we call the *canonical decomposition* of P. The set P_0 is the union of all members of $\mathbf{Id}(P)^{(\infty-)}$ and $P_i := \bigcup \mathbf{Id}(P \setminus \bigcup_{j<i} P_j))^{(\infty-)}$ while $P \setminus \bigcup_{j<i} P_j \neq \varnothing$. The sequence terminates on the first integer k such that $P = \bigcup_{i<k} P_i$. Let $\beta_i := \mathrm{rank}^{-}(\mathbf{Id}(P_i))$, $m_i := d^0(\mathbf{Id}(P_i))$ for $i < k$, and $\mathrm{s}(\mathbf{Id}(P)) := \omega^{\beta_0} \cdot m_0 + \cdots + \omega^{\beta_{k-1}} \cdot m_{k-1}$.
For an example, if P is a well ordered chain of type α, then $\mathrm{s}(\mathbf{Id}(P)) = \alpha$. We show:

Theorem 2

If P is w.q.o then $\mathrm{s}(\mathbf{Id}(P)) = o(P)$.

Our proof of Theorem 2 above is based on a characterization of $o(P)$ given in terms of the height of P in $\mathbf{I}(P)$.

The height of a well-founded poset

We recall that if Q is a well-founded poset, the *height* of an element $x \in Q$ is the ordinal $\mathrm{h}(x, Q)$ given by the inductive formula:

$$\mathrm{h}(x, Q) = 0 \text{ if } x \text{ is minimal in } Q; \tag{3}$$

$$\mathrm{h}(x, Q) = \mathrm{Sup}\{\mathrm{h}(y, Q) + 1 : y < x\} \tag{4}$$

otherwise.

If P is w.q.o, then $\mathbf{I}(P)$ is well-founded (and conversely) Higman 1952.

Theorem 3

The ordinal length of a w.q.o P is the height of P in the set $\mathbf{I}(P)$ of initial segments of P.

Pouzet and N. Zaguia, 1985, see also Kříž, Thomas,1990, Blass, Gurevich, 2006. With this result at hands, it suffices to prove:

Theorem 4

If P is w.q.o then $h(P, \mathbf{I}(P)) = \mathrm{s}(\mathbf{Id}(P))$.

Proof of Theorem 4

We proceed by induction on $\alpha = \mathsf{s}(\mathbf{Id}(P))$, that is we suppose that the property holds for all posets Q such that $\mathsf{s}(\mathbf{Id}(Q)) < \alpha$.

Case 1 $\mathsf{s}(\mathbf{Id}(P)) < \omega$.
In this case, P is finite and $\mathsf{s}(\mathbf{Id}(P)) = |P|$. Since, in this case, $\mathsf{h}(P, \mathbf{I}(P)) = |P|$, the equality $\mathsf{s}(\mathbf{Id}(P)) = \mathsf{h}(P, \mathbf{I}(P))$ holds.

Case 2 $\mathsf{s}(\mathbf{Id}(P)) \geq \omega$.
In this case, $\mathsf{rank}^-(\mathbf{Id}(P)) \geq 1$. Among ideals $Q \in \mathbf{Id}(P)$ such that $\mathsf{rank}(Q, \mathbf{Id}(P)) = \mathsf{rank}^-(\mathbf{Id}(P))$, take Q minimal with respect to inclusion.

Subcase 2.1 $Q \neq P$.
By construction, $\mathsf{s}(\mathbf{Id}(Q)) = \omega^{\beta_0}$ where $\beta_0 = \mathsf{rank}^-(\mathbf{Id}(P))$ and $\mathsf{rank}^-(\mathbf{Id}(P \setminus Q)) \leq \beta_0$.
We apply the following formula (that we prove separately):

$$\mathsf{s}(\mathbf{Id}(P)) = \mathsf{s}(\mathbf{Id}(Q)) \oplus \mathsf{s}(\mathbf{Id}(P \setminus Q)). \tag{5}$$

Due to induction, $\mathsf{h}(Q, \mathbf{I}(Q)) = \mathsf{s}(\mathbf{Id}(Q))$ and $\mathsf{h}(P \setminus Q, \mathbf{I}(P \setminus Q)) = \mathsf{s}(\mathbf{Id}(P \setminus Q)$. Hence, $\mathsf{h}(Q, \mathbf{I}(Q)) = \omega^{\beta_0}$ and $\mathsf{rank}^{-}(\mathsf{h}(P \setminus Q, \mathbf{Id}(P \setminus Q))) \leq \beta_0$. On an other hand, we have the following inequalities:

$$\mathsf{h}(Q, \mathbf{I}(Q)) + \mathsf{h}(P \setminus Q, \mathbf{I}(P \setminus Q)) \leq \mathsf{h}(P, \mathbf{I}(P)) \leq \mathsf{h}(Q, \mathbf{I}(Q)) \oplus \mathsf{h}(P \setminus Q, \mathbf{I}(P \setminus Q)).$$

Hence $\mathsf{h}(Q, \mathbf{I}(Q)) \oplus \mathsf{h}(P \setminus Q, \mathbf{I}(P \setminus Q)) = \mathsf{h}(P, \mathbf{I}(P))$. The equality $\mathsf{h}(P, \mathbf{I}(P)) = \mathsf{s}(\mathbf{Id}(P))$ follows. □

Subcase 2.2 $Q = P$. In this case, $\mathsf{s}(\mathbf{Id}(P)) = \omega^{\beta_0}$, $P_0 = P$ and P is an ideal. Let $x \in P$. Set $P_x = P \setminus \uparrow x$. By induction, we have $\mathsf{h}(P_x, \mathbf{I}(P_x)) = \mathsf{s}(\mathbf{Id}(P_x))$. Hence, $\mathsf{h}(P, \mathbf{I}(P)) = \mathsf{Sup}\{\mathsf{h}(P_x, \mathbf{I}(P_x) + 1\} = \mathsf{Sup}\{\mathsf{s}(\mathbf{Id}(P_x)) + 1\} = \mathsf{s}(\mathbf{Id}(P))$. That is $\mathsf{h}(P, \mathbf{I}(P)) = \mathsf{s}(\mathbf{Id}(P))$. □

A special case of Theorem 2 yields:

Corollary 5

Let P be a w.q.o Then:

- (a) $o(P) = \omega^{\beta} \cdot m$ *if and only if* $\mathrm{rank}^{-}(\mathbf{Id}(P)) = \beta$, $d^{0}(\mathbf{Id}(P)) = m$ *and* $P = \bigcup \mathbf{Id}(P)^{(\infty-)}$. *As a consequence,*
- (b) $o(P)$ *is of the form* $\omega^{\beta} \cdot m$ *if and only if* $P = \bigcup \mathbf{Id}(P)^{(\infty-)}$;
- (c) $o(P)$ *is of the form* ω^{β} *if and only if* $\mathbf{Id}(P)^{(\infty-)} = \{P\}$.

A special case of de Jongh-Parikh's theorem, essentially due to Carruth, 1942, is this:

Lemma 6

Given a finite family $(\alpha_i)_{i<k}$ of ordinal numbers, the Hessenberg sum $\bigoplus_{i<k} \alpha_i$ is the largest ordinal admiting a partition in k well-ordered subsets A_i, $i < k$, each of order type α_i.

With this result and Corollary 5, one can derive the following strengthening of Theorem 2.

Corollary 7

Let P be a w.q.o and $P_0, \ldots, P_{k-1}$ be the canonical decomposition of P. Then $o(P) = \bigoplus_{i<k} o(P_i)$. Furthermore, if L is a linear extension of P and for each $i < k$, $L_i := L_{\restriction P_i}$ then:

- (a) *The order type of L is equal to $o(P)$ provided that for $i < k$, the order type of L_i is equal to $o(P_i)$ and $L = \sum_{i<k} L_i$.*
- (b) *The order type of each L_i is equal to $o(P_i)$ provided that the order type of L is equal to $o(P)$.*

Proof.

Let α be the order type of P, L be a linear extension of P and α_i be the order type of L_i. According to Lemma 6, $\alpha \leq \bigoplus_{i<k} \alpha_i$. Since $o(P_i)$ exists, $\alpha_i \leq o(P_i)$; since the Heissenberg sum in increasing, $\bigoplus_{i<k} \alpha_i \leq \bigoplus_{i<k} o(P_i)$ and thus:

$$o(P) \leq \bigoplus_{i<k} o(P_i). \tag{6}$$

Conversely, since each P_i is convex in P, every lexicographical sum $\sum_{i<k} S_i$, in which each S_i is a linear extensions of P_i, is a linear extension of P. Hence, $\sum_{i<k} o(P_i) \leq o(P)$. By Corollary 5, the $o(P_i)$ are ordinals of the form $\omega^{\beta_i} \cdot m_i$; the β_i's are decreasing, hence $\sum_{i<k} o(P_i) = \bigoplus_{i<k} o(P_i)$ and thus $\bigoplus_{i<k} o(P_i) \leq o(P)$. The equality $\bigoplus_{i<k} o(P_i) = o(P)$ follows.
(a). If each L_i has a order type $o(P_i)$ then, by Corollary 5, its type $o(P_i)$ is $\omega^{\beta_i} \cdot m_i$ and if $L = \sum_{i<k} L_i$ then the order type of L is $\omega^{\beta_0} \cdot m_0 + \cdots + \omega^{\beta_{k-1}} \cdot m_{k-1}$, that is $o(P)$ by the assertion above.

□

(b). Suppose that the order type α of L is $o(P)$. By (6) we have $\alpha \leq \bigoplus_{i<k} \alpha_i \leq \bigoplus_{i<k} o(P_i)$. Hence, $o(P) = \alpha \leq \bigoplus_{i<k} \alpha_i \leq \bigoplus_{i<k} \alpha_i = o(P)$. Since the Heissenberg sum is strictly increasing, the order type of each L_i is $o(P_i)$.

Remark 1

The fact that the order type of L is $o(P) = \sum_{i<k} o(P_i)$ does not ensure that $L = \sum_{i<k} L_i$. For an example, let P be the direct sum of two chains A and B with order types ω^2 and ω respectively. Then $P_0 = A$, $P_1 = B$, $o(P) = \omega^2 + \omega$. Let A_0 be an initial segment of A, A_1 be its complement (in A), let B_0 be an initial segment of B, B_1 its complement (in B), then $L := A_0 + B_0 + A_1 + B_1$ is a linear extension of P. If B_0 is non-empty and distinct of B and if A_1 is non-empty, then the order type of L is $\omega^2 + \omega$, while L_0 is not an interval of L, hence, L is not the sum $L_0 + L_1$.

We give an alternative definition of the canonical partition of a w.q.o poset in terms of ordinal length.
Let P be a poset and β be an ordinal. Set
$F(\beta, P) := \{x \in P : o(\uparrow x) < \omega^{\beta}\}$ and $R(\beta, P) := F(\beta + 1, P) \setminus F(\beta, P)$.

Lemma 8

If P is a non-empty w.q.o, then the set $C(P) := \{\beta : R(\beta, P) \neq \varnothing\}$ is finite and the sets $R(\beta, P)$, with $\beta \in C(P)$, form the canonical decomposition of P.

Theorem 9

Let P be a w.q.o poset and $\beta_0 > \cdots > \beta_{k-1}$ be a finite decreasing sequence of ordinals. Then $o(P) = \omega^{\beta_0} \cdot m_0 + \cdots + \omega^{\beta_{k-1}} \cdot m_{k-1}$, with $m_0, \ldots, m_{k-1} \in \omega \setminus \{0\}$ if and only if $P = R(\beta_0, P) \cup \cdots \cup R(\beta_{k-1}, P)$ with each $R(\beta_i, P)$ non-empty.

A special case of Theorem 9 is essentially Theorem 3.2 page 203 of de Jongh Parikh paper that we recall below.

Corollary 10

If P is wqo, there is a largest final segment $\check{P}$ of P which is finite, furthermore $o(P \setminus \check{P})$ is a limit ordinal and $o(P) = o(P \setminus \check{P}) + |\check{P}|$.

Let A be a finite set and let A^* be the set of finite sequence of elements of A viewed as words. Subsets of A^* are called *languages*. A language L over A is said *piecewise testable* if it is a finite Boolean combination of languages of the form:

$$A^* a_1 A^* \dots a_o A^* .$$

A characterization of piecewise testable languages was given by Simon (1972). He proved that a language L is piecewise testable if and only if its syntactic monoid M_L is $\mathcal{J}$-trivial (i.e. $M_L \cdot u \cdot M_L = M_L \cdot u' \cdot M_L$ implies $u = u'$ for every $u, u' \in M_L$). Considering the pseudovariety $\mathcal{J}$ of finite $\mathcal{J}$-trivial semigroups, Almeida (1990) showed that the topological semigroup $\underline{\Omega}_n(\mathcal{J})$, projective limit of all n-generated members of $\mathcal{J}$, is countable. It turns out that, as a topological space, this semigroup is the Stone space associated with the Boolean algebra of piecewise testable languages.

We prove:

Theorem 11

If the alphabet A is formed of n letters then the Stone space of the Boolean algebra of piecewise testable languages is homeomorphic to the ordinal $\omega^{\omega^{n-1}} + 1$ *equipped with the interval topology.*

Proof

Let $P := A^*$. Order P with the subword ordering, that is, for two words$x := x_0, \ldots, x_{n-1}$ and $y := y_0, \ldots, y_{m-1}$ of length n and m, set $x \leq y$ if there is a one-to-one order preserving map $h : \{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$ such that $x_i \leq y_{h(i)}$ for all $i < n$. The Boolean algebra of piecewise testable languages is by definition the Boolean algebra $Tailalg(P)$ generated by the final segments of the form $\uparrow u$ for $u \in P$. According to Theorem 2.1 of M. Bekkali, M. Pouzet, D. Zhani, Incidence structures and Stone-Priestley duality, Ann. Math. Artif. Intell., **49** no.1-4 (2007) 27–38, for every poset P the Stone space, dual of $Tailalg(P)$, is the closure in $\wp(P)$ of the set of initial segments $\downarrow u$ for $u \in P$.

In our case, this closure is the set $\mathbf{Id}(P)$ of ideals of P. Indeed, since A is finite, our poset P is w.q.o. (Higman, 1952), hence, P, as a final segment, is generated by the set of minimal element of P and for every $u \in P$, $v \in P$, the intersection $\uparrow u \cap \uparrow v$ is finitely generated thus belongs to *Tailalg*(P). According to Corollary 2.7 of Bekkali-Pouzet-Zhani's paper, the closure of the set of initial segments $\downarrow u$, for $u \in P$, is $\mathbf{Id}(P)$. In their paper, de Jongh and Parikh proved that $o(P) = \omega^{\omega^{n-1}}$. According to our Corollary above, $rank(\mathbf{Id}(P)) = \omega^{n-1}$ and $d^0(\mathbf{Id}(P)) = 1$. According to a result of Mazurkiewicz and Sierpinski (1920), since $\mathbf{Id}(P)$ is countable, $\mathbf{Id}(P)$ is homeomorphic to an ordinal. This ordinal must be $\omega^{\omega^{n-1}} + 1$ equipped with the interval topology. $\square$

Let Q be a poset and $\mathbf{I}_{<\omega}(Q)$ be the poset made of finitely generated initial segments of Q, ordered by inclusion. Schnoebelen and Schmitz (2019) obtained a beautiful formula:

Theorem 12

Let Q be a w.q.o. Then $o(\mathbf{I}_{<\omega}(Q)) \leq 2^{o(Q)}$.

A short proof is given by Altman and Weiermann. This majoration and a slight improvement can be obtained from our topological approach.

We prove the following result.

Theorem 13

Let Q be a w.q.o and $\check{Q} := \{x \in Q : \uparrow x \text{ is finite}\}$. Let ξ be such that $o(Q) = \omega \cdot \xi + p$ with $p < \omega$ and let $q := |\mathbf{I}(\check{Q})|$. Then:

1. *$p = |\check{Q}|$ and $o(\mathbf{I}_{<\omega}(Q)) \leq \omega^{\xi} \cdot q$;*
2. *Furthermore, if Q is embeddable into $[\omega]^{<\omega}$, the set of finite subsets of ω, ordered by inclusion, then the inequality above is an equality.*

Let α be an ordinal. We recall that there is unique pair of ordinals β, r such that $\alpha = \omega \cdot \beta + r$ and $r < \omega$. The ordinal β, denoted by $\frac{1}{\omega} \cdot \alpha$, is the *quotient* of α by ω, the ordinal $\omega \cdot \beta$, denoted by $l(\alpha)$, is the *limit part* of α, the ordinal r, denoted α *mod* ω, is the *remainder*. The formula above becomes

$$o(\mathbf{I}_{<\omega}(Q)) \leq \omega^{\frac{1}{\omega} \cdot o(Q)} \cdot |\mathbf{I}(\check{Q})|.$$

Hint for a proof of Theorem 13

In order to prove that the inequality holds, it suffices to prove that $o(\mathbf{I}_{<\omega}(Q \setminus \check{Q})) \leq \omega^{\xi}$.
We prove by induction on ξ that for every w.q.o Q: uparrow

$$o(Q) \leq \omega \cdot \xi \Rightarrow o(\mathbf{I}_{<\omega}(Q)) \leq \omega^{\xi}. \tag{7}$$

If $\xi = 0$, Q is the empty set, hence $|\mathbf{I}_{<\omega}(Q)| = 1$ and the inequality holds. If $\xi = 1$, then either Q is finite, in which case $\mathbf{I}_{<\omega}(Q)$ is finite, and the implication holds, or Q is infinite. In this case $Q \setminus \uparrow x$ is finite for every $x \in Q$, from which follows that $\mathbf{I}_{<\omega}(Q)$ has the same property, hence $o(\mathbf{I}_{<\omega}(Q)) \leq \omega$. If $\xi = \xi' + 1$ and $o(Q) \leq \omega \cdot \xi$ then $Q = Q' \cup Q''$, where $o(Q') \leq \xi'$ and $o(Q'') \leq \omega$. The poset $\mathbf{I}_{<\omega}(Q)$ embeds into the product $\mathbf{I}_{<\omega}(Q') \times \mathbf{I}_{<\omega}(Q'')$, hence $o(\mathbf{I}_{<\omega}(Q)) \leq o(\mathbf{I}_{<\omega}(Q')) \otimes o(\mathbf{I}_{<\omega}(Q''))$. Via the induction hypothesis, $o(\mathbf{I}_{<\omega}(Q')) \leq \omega^{\xi'}$ and $o(\mathbf{I}_{<\omega}(Q'')) \leq \omega$, hence $o(\mathbf{I}_{<\omega}(Q)) \leq \omega^{\xi'} \cdot \omega = \omega^{\xi}$. ETC

1)Suppose that $Q = \mathbb{N}^{\times m}$ with $m < \omega$, the direct product of m copies of $\mathbb{N}$. Then $\check{Q} = \varnothing$; $q = 1$; $o(Q) = \omega^m$ (Carruth), hence $\xi = \omega^{m-1}$.
Since Q is embeddable in $[\omega]^{<\omega}$, we have:
$o(\mathbf{I}_{<\omega}(Q)) = \omega^{\omega^{m-1}}$. A formula obtained by Altman and Weierman in 2019.

2 **Monomial ideals in m variables**

The set $\mathbf{I}(P)$ of initial segments of a w.q.o. poset P is not necessarily w.q.o. If it is, the ordinal length can be difficult to compute, even for simple P. The value of the ordinal length of $Mon_K(m)$, the set of monomial ideals in m variables was asked by Aschenbrenner and Pong 2004, and computed by Altman and Weiermann in 2019.
Let m be a positive integer, K be a field and $R := K[X] = K[X_1, \ldots, X_m]$ be the ring of polynomials in indeterminates $X = \{X_1, \ldots, X_m\}$ with coefficients from K. A *monomial* is any expression of the form $X^{n_1} \cdot \ldots X^{n_i} \cdot \ldots X^{n_m}$ with $n_i \in \mathbb{N}$. As usual, $X_i^0 = 1$.

The set of monomonials ordered by divisibility is isomorphic to the cartesian product of m copies of $\mathbb{N}$ ordered componentwise, that we will denote by $\mathbb{N}^{\times m}$. A *monomial ideal* is any ideal of R generated by monomials. The set $Mon_K(m)$ of monomial ideals, ordered by inclusion, is isomorphic to the set $\mathbf{F}(\mathbb{N}^{\times m})$ of final segments of $\mathbb{N}^{\times m}$. Hence, it is dually isomorphic to the set $\mathbf{I}(\mathbb{N}^{\times m})$ of initial segments of $\mathbb{N}^{\times m}$. The set of monomials, ordered by divisibility, is a w.q.o., this fact is known as Dickson's Lemma, it appears also in Janet, 1920. The set $Mon_K(m)$ of monomial ideals, ordered by reverse of inclusion, is w.q.o too. This fact, stated by Maclagan, 2001, follows from a basic result of the theory of better quasi-ordering (b.q.o) invented by Nash-Williams, namely that b.q.o. are w.q.o., that $\mathbb{N}$ is b.q.o and a finite product of b.q.o is b.q.o.

For an ordinal α denote by $\alpha^{\otimes m}$ the m-th Hessenberg power of α; hence $o(\alpha^{\times m}) = \alpha^{\otimes m}$.

Aschenbrenner and Pong proved that

$$\omega^{\omega^{m-1}} + 1 \leq o(Mon_K(m)) \leq \omega^{(\omega+1)^{\otimes m}} + 1$$

and they asked for the correct value.

Altman and Weiermann proved

Theorem 14

$o(Mon_K(m)) = \omega^{\sum_{k=1,m}\omega^{m-k}\binom{m}{k-1}} + 1.$

The fact that $o(Mon_K(m)) \leq \omega^{\sum_{k=1,m}\omega^{m-k}\binom{m}{k-1}} + 1$ follows from Theorem 13

Indeed, as for every w.q.o, $\mathbf{I}(Q)$ is isomorphic to $\mathbf{I}_{<\omega}(\mathbf{Id}(Q))$. Hence, $Mon_K(m)$ is order isomorphic to $\mathbf{I}_{<\omega}(\mathbf{Id}(\mathbb{N}^{\times m}))$. Since $\mathbf{Id}(\mathbb{N})$ is isomorphic to $\mathbb{N}+1$, $Mon_K(m)$ is order isomorphic to $\mathbf{I}_{<\omega}((\omega+1)^{\times m})$.

According to Theorem 13

$o(\mathbf{I}_{<\omega}((\omega+1)^{\times m}) \leq \omega^{\frac{1}{\omega}\cdot o((\omega+1)^{\times m})} + 1 = \omega^{\frac{1}{\omega}\cdot((\omega+1)^{\otimes m})} + 1.$

$(\omega+1)^{\otimes m} = \omega^m + \omega^{m-1}\cdot(m-1) + \cdots + \omega^{m-u}\cdot\binom{m}{u} + \cdots + \omega\cdot m + \cdots + 1$

$\frac{1}{\omega}\cdot((\omega+1)^{\otimes m}) = \omega^{m-1} + \omega^{m-2}\cdot(m-1) + \cdots + \omega^{m-u-1}\cdot\binom{m}{u} + \cdots + m.$

Setting $k = u+1$, we get $\frac{1}{\omega}\cdot((\omega+1)^{\otimes m}) = \omega^{\sum_{k=1,m}\omega^{m-k}\binom{m}{k-1}} + 1.$

Hence we get the upper bound given by Altman and Weiermann.

Instead of looking for the minoration, one can do as follows. Let p, q be two non-negative integers. Let $A_{p,q} := (\omega + 1)^{\times p} \times \omega^{\times q}$ be the cartesian product of p copies of the chain $\omega + 1$ with q copies of the chain ω. If $q = 0$, $\omega^{\times q}$ is the one-element chain 1; hence $A_{p,0} = (\omega + 1)^{\times p}$; similarly, $A_{0,q} := \omega^{\times q}$. Using induction and our theorem on finitely generatred segments one can prove easily that:

Theorem 15

For every non-negative integers p, q:

$$o(\mathbf{I}_{<\omega}(A_{p,0})) = \omega^{\frac{1}{\omega} \cdot o(A_{p,0})} + 1; \tag{8}$$

and if q is not zero, then:

$$o(\mathbf{I}_{<\omega}(A_{p,q})) = \omega^{\frac{1}{\omega} \cdot o(A_{p,q})}. \tag{9}$$

Since $Mon_K(m)$ is order isomorphic to $\mathbf{I}_{<\omega}((\omega + 1)^{\times m}) = A_{m,0}$, this yields the researched result.

This is not much different from Altman and Weiermann strategy of proof.

Our motivation for the study of the sets of ideals of a poset comes from the theory of relations

First, recall that a *relational structure* is a pair $R := (E, (\rho_i)_{i \in I})$, where for each $i \in I$, ρ_i is a *n_i-ary relation* on E (that is a subset of E^{n_i}) and n_i is a non-negative integer; the family $\mu := (n_i)_{i \in I}$ is called the *signature* of R. An *induced substructure* of R is any relational structure of the form $R_{\restriction F} := (F, (\rho_i \cap F^{n_i})_{i \in I})$. One may define the notion of *relational isomorphism* and then the notion of *embeddability* between relational structures with the same signature (e.g. R embeds into R' if R is isomorphic to an induced substructure of R'). According to Fraïssé, the *age* of a relational structure R is the collection $\mathcal{A}(R)$ of its finite induced substructures, considered up to isomorphism.

A first order sentence (in the language associated with the signature μ) is *universal* whenever it is equivalent to a sentence of the form $\forall x_1 \cdots \forall x_n \varphi(x_1, \ldots, x_n)$ where $\varphi(x_1, \ldots, x_n)$ is a formula built with the variable $x_1, \ldots, x_n$, the logical connectives $\neg$, $\vee$, $\wedge$ and predicates $=$, $\rho_i, i \in I$.

Let Ω_μ be the set of finite relational structures with signature μ, these structures being considered up to isomorphism and ordered by embeddability. If μ is finite, then Ω_μ is a ranked poset with a least element. The set $\mathbf{Id}(\Omega_\mu)$ is a closed subset of $\wp(\Omega_\mu)$ equipped with the product topology. The consideration of $\mathbf{Id}(\Omega_\mu)$ is justified by the following:

Proposition 16

If μ is finite, $\mathbf{Id}(\Omega_\mu)$ is the set of ages of relational structures with signature μ. As a topological space, its dual is the Boolean algebra made of Boolean combinations of universal sentences, considered up to elementary equivalence.

If μ is constant and equal to 1 ($I := \{0, \ldots, k-1\}$ and $n_i = 1$ for $i \in I$) then Ω_μ can be identified to the direct product of 2^k copies of the chain ω of non-negative integers. From this it follows that $\mathbf{Id}(\Omega_\mu)$ is homeomorphic to the ordinal $\omega^{2^k} + 1$, equipped with the interval topology. If μ contains some integer larger than 1, then Ω_μ embeds as a subposet the set $[\omega]^{<\omega}$ of finite subsets of ω hence $\mathbf{Id}(\Omega_\mu)$ embeds the Cantor space $\wp(\omega)$.
If μ is finite, then, like Ω_μ, ages are ranked. The function $\varphi_{\mathcal{A}}$, which counts for each nonnegative integer *n*, the number of elements of rank *n*, is the *profile* of $\mathcal{A}$, (also called the profile of any *R* with age $\mathcal{A}$). It enjoys quite striking properties: for an example, if the domain is infinite, φ_A is a non decreasing function of the rank (Pouzet 1971, see the Fraïsse's book on logic, t1, Exercice 8, p. 113, see Pouzet 2006 for a survey on the profile). Still, the problem of characterizing ages among posets is largely unsolved.

An apparently easier task is to classify ages, distinguishing, first, those which are complicated from those which are not and, next classifying the simpler ones. The basic idea is that if an age is included into an other one, then it is simpler. From this, the simplest ages must be the finite ones, next the infinite ages such that all proper subages are finite, and so on. The continuation of this process relies on the notion of well-foundedness; the parent notion, namely well-quasi-ordering, emerges then, but also topological notions, like scatteredness.
Let $\mathcal{A} \in \mathbf{Id}(\Omega_\mu)$. If $\mathbf{Id}(\mathcal{A})$ is well-founded, let $H(\mathcal{A})$ be the height of $\mathcal{A}$ in $\mathbf{Id}(\mathcal{A})$. If $\mathcal{A}$ is countable (which is the case if μ is finite), the ordinal $H(\mathcal{A})$ is countable. Every countable ordinal can be attained, but known examples require an infinite signature Pouzet-Sobrani, 2001. If $\mathcal{A}$ is w.q.o it is natural to use $H(\mathcal{A})$ or $o(\mathcal{A})$ as a measure of the complexity of $\mathcal{A}$. This leeds to look at the relationship between these parameters.

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Thank you for your attention.