The Parametric Complexity of Lossy Counter Machines

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Outline

lossy counter machines (LCM) reachability

- canonical Ackermann-complete problem
- complexity gap in fixed dimension d: $\mathbf{F}_d$-hard, in $\mathbf{F}_{d+1}$

complexity using well-quasi-orders (wqo)

- controlled bad sequences
- length function theorem on the length of controlled bad sequences
- $\mathbf{F}_{d+1}$ upper bounds for LCM reachability

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complexity using well-quasi-orders (wqo)

- strongly controlled bad sequences
- antichain factorisation
- width function theorem on the length of controlled antichains
- $\mathbf{F}_d$ upper bounds for LCM reachability

Main Result

$$F_0(x) = x + 1$$

$$F_1(x) = \overbrace{F_0 \circ \cdots \circ F_0}^{x+1 \text{ times}}(x) = 2x + 1$$

$$F_2(x) = \overbrace{F_1 \circ \cdots \circ F_1}^{x+1 \text{ times}}(x) \approx 2^x$$

$$F_3(x) = \overbrace{F_2 \circ \cdots \circ F_2}^{x+1 \text{ times}}(x) \approx \mathrm{tower}(x)$$

$$\vdots$$

$$F_\omega(x) = F_{x+1}(x) \approx \mathrm{ackermann}(x)$$

Upper Bound Theorem

LCM Reachability is F_d-complete in fixed dimension $d \geqslant 3$.

Main Result

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\begin{aligned}
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&\vdots \\
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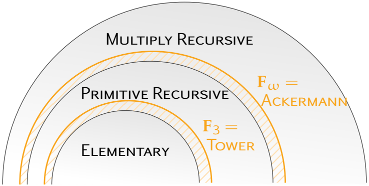
$$



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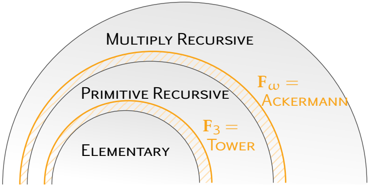

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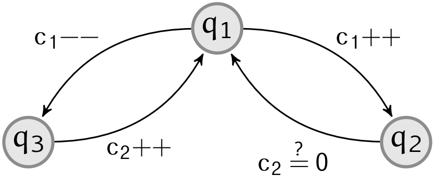
Lossy Counter Machines

Example



Lossy Counter Machines

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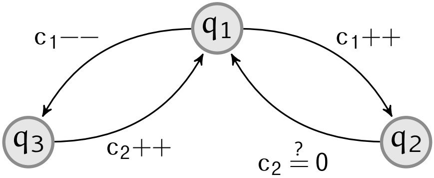
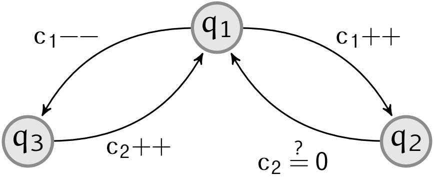


Lossy Semantics

$$q_1(0,2) \xrightarrow{c_1++}_{\ell} q_2(1,1) \xrightarrow{c_2 \stackrel{?}{=} 0}_{\ell} q_1(0,0)$$

Lossy Counter Machines

Example



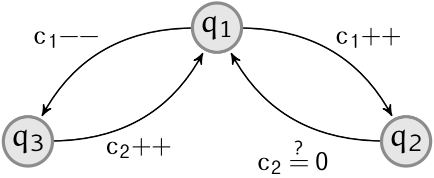
Lossy Semantics

$$q_1(0,2) \xrightarrow{c_1{+}{+}}_{\ell} q_2(1,1) \xrightarrow{c_2 \overset{?}{=} 0}_{\ell} q_1(0,0)$$

$$\begin{array}{ccc} q_1(0,2) & & q_2(1,1) \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ q_1(0,2) & \xrightarrow{c_1{+}{+}} & q_2(1,2) \end{array}$$

Lossy Counter Machines

Example



Lossy Semantics

$$q_1(0,2) \xrightarrow{c_1{+}{+}}_{\ell} q_2(1,1) \xrightarrow{c_2 \stackrel{?}{=} 0}_{\ell} q_1(0,0)$$

$$q_2(1,1) \geqslant q_2(1,0) \qquad q_1(0,0) \leqslant q_1(1,0)$$

$$q_2(1,0) \xrightarrow{c_2 \stackrel{?}{=} 0} q_1(1,0)$$

Reachability and Coverability

Reachability Problem

input an LCM M, initial configuration $q_0(\mathbf{v}_0)$, target configuration $q_f(\mathbf{v}_f)$

question $q_0(\mathbf{v}_0) \rightarrow_{\ell}^{*} q_f(\mathbf{v}_f)$?

Reachability and Coverability

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Remark

equivalent to coverability:

question $\exists \mathbf{v} \geqslant \mathbf{v}_f \,.\, q_0(\mathbf{v}_0) \to_\ell^* q_f(\mathbf{v})$?

Reachability and Coverability

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Lower Bound Theorem (Urquhart'99; Schnoebelen'02,'10)
LCM Reachability is Ackermann-hard.

Upper Bound Theorem (McAloon'84,Clote'86)
LCM Reachability is in Ackermann.

Reachability and Coverability

Reachability Problem

input an LCM M, initial configuration $q_0(\mathbf{v}_0)$, target configuration $q_f(\mathbf{v}_f)$

question $q_0(\mathbf{v}_0) \to_\ell^* q_f(\mathbf{v}_f)$?

Lower Bound Theorem (S.'17)

LCM Reachability is $\mathbf{F}_d$-hard in fixed dimension $d \geqslant 3$.

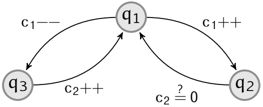
Upper Bound Theorem (Figueira & al.'11, S. & Schnoebelen '12)

LCM Reachability is in $\mathbf{F}_{d+1}$ in fixed dimension $d \geqslant 3$.

Backward Coverability

(Arnold & Latteux'78)

Example: coverability of $q_2(1,1)$ in



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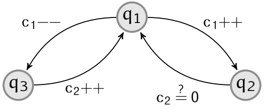

$$U_k \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1) \,.\, q(\mathbf{v}) \to_\ell^{\leqslant k} q_2(\mathbf{v}')\}$$

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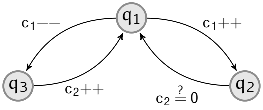
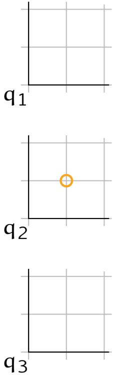


$$U_0 \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1) \,.\, q(\mathbf{v}) \to_{\ell}^{\leqslant 0} q_2(\mathbf{v}')\}$$



$q_2(1,1)$

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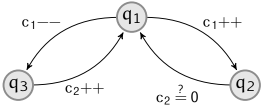

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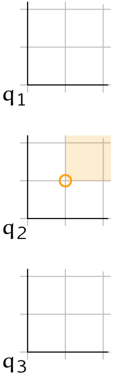

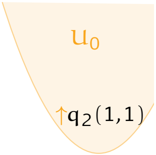

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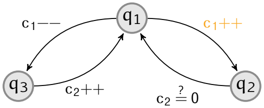
Example: coverability of $q_2(1,1)$ in



$$U_1 \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1)\,.\, q(\mathbf{v}) \to_\ell^{\leqslant 1} q_2(\mathbf{v}')\}$$

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(Arnold & Latteux'78)

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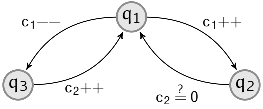

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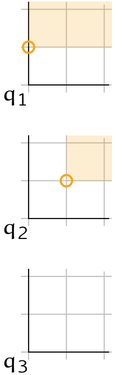

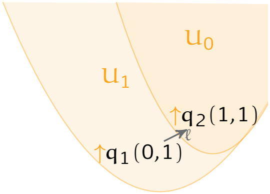

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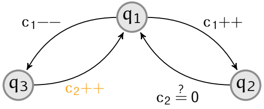
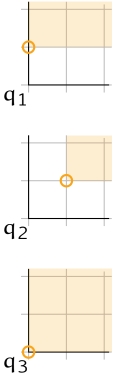
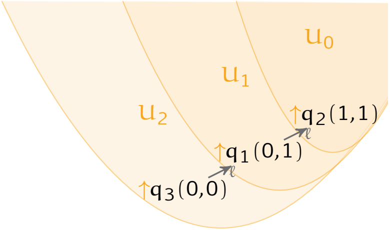


$$U_2 \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1) \,.\, q(\mathbf{v}) \to_\ell^{\leqslant 2} q_2(\mathbf{v}')\}$$





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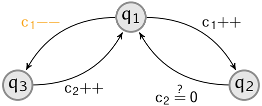
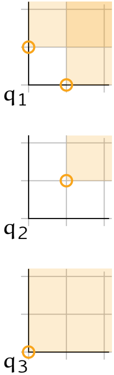
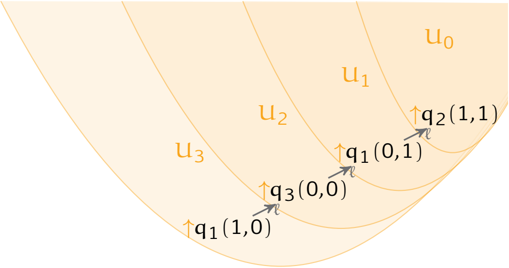


Example: coverability of $q_2(1,1)$ in

$$U_3 \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1) \,.\, q(\mathbf{v}) \rightarrow_{\ell}^{\leqslant 3} q_2(\mathbf{v}')\}$$



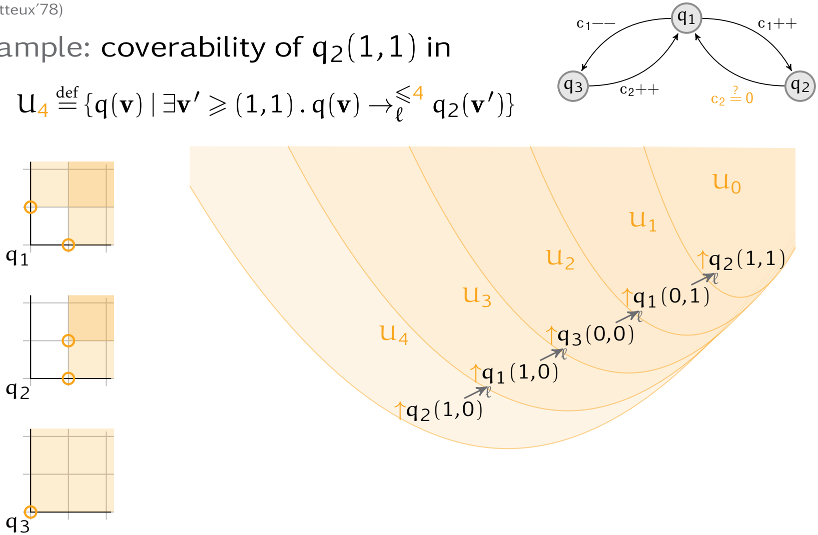


Backward Coverability

(Arnold & Latteux'78)

Example: coverability of $q_2(1,1)$ in

$$U_4 \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1) \,.\, q(\mathbf{v}) \to_\ell^{\leqslant 4} q_2(\mathbf{v}')\}$$



Backward Coverability

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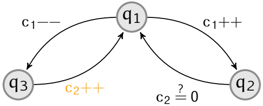

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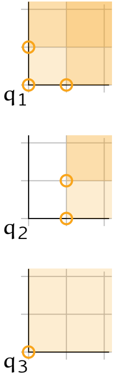

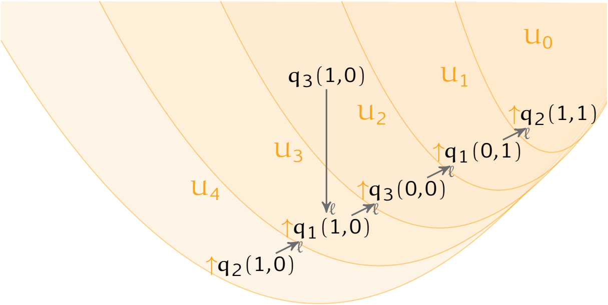

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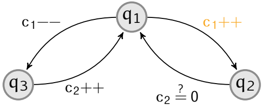
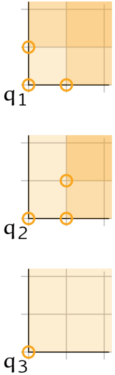
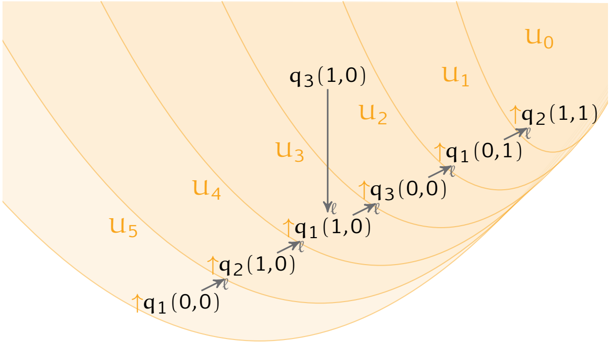


Example: coverability of $q_2(1,1)$ in

$$U_5 \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1)\,.\, q(\mathbf{v}) \to_{\ell}^{\leqslant 5} q_2(\mathbf{v}')\}$$





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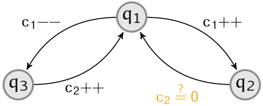
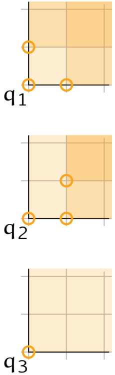
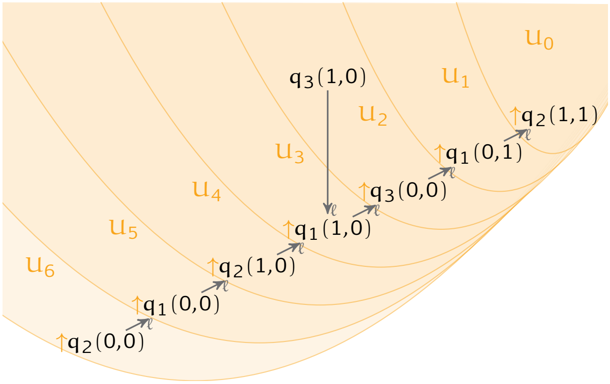


$$U_6 \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1) \,.\, q(\mathbf{v}) \to_{\ell}^{\leqslant 6} q_2(\mathbf{v}')\}$$





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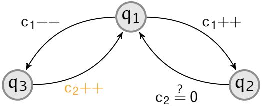
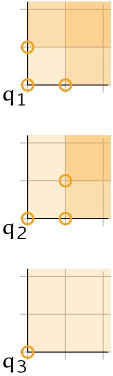
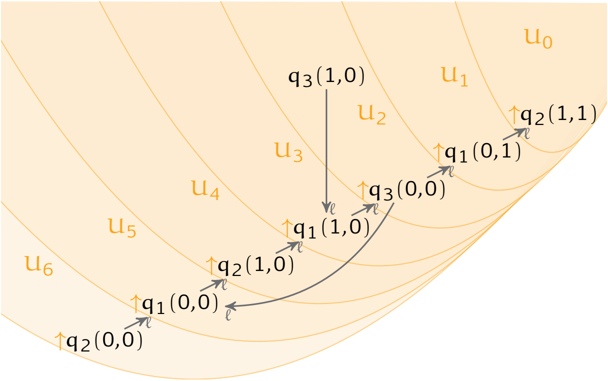


Example: coverability of $q_2(1,1)$ in

$$U_6 \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant (1,1) \,.\, q(\mathbf{v}) \to_\ell^* q_2(\mathbf{v}')\}$$





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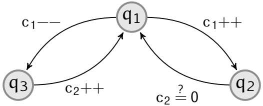

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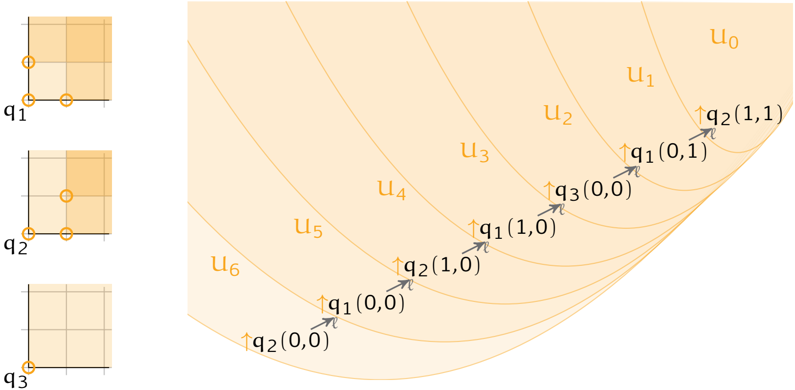

The sequence $q_2(1,1)$, $q_1(0,1)$, $q_3(0,0)$, $q_1(1,0)$, $q_2(1,0)$, $q_1(0,0)$, $q_2(0,0)$ is bad

Bad Sequences

Over a qo $(X, \leqslant)$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $(X, \leqslant)$ wqo if all bad sequences are finite

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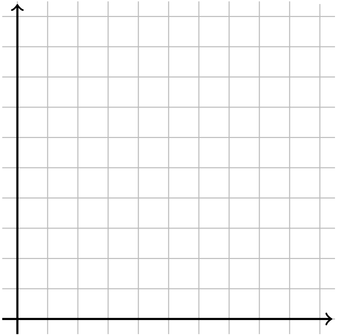
- $(X, \leqslant)$ wqo if all bad sequences are finite...

... but can be of arbitrary length



Example (in $\mathbb{N}^2$)

Bad Sequences

Over a qo $(X,\leqslant)$

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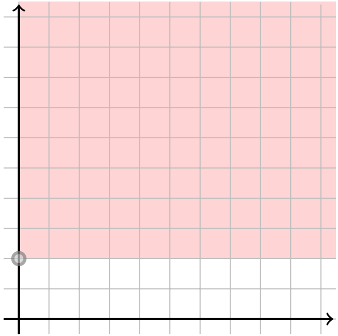

Example (in $\mathbb{N}^2$)

$(0,2)$

Bad Sequences

Over a qo $(X, \leqslant)$

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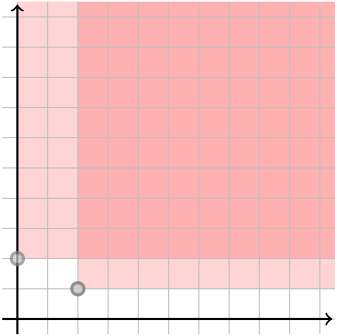

Example (in $\mathbb{N}^2$)

$(0,2), (2,1)$

Bad Sequences

Over a qo $(X, \leqslant)$

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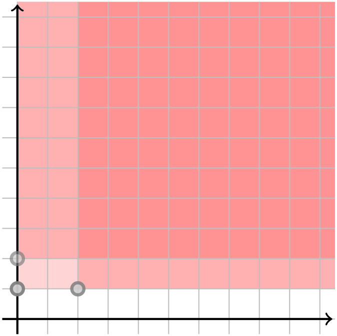

Example (in $\mathbb{N}^2$)

$(0,2), (2,1), (0,1)$

Bad Sequences

Over a qo $(X,\leqslant)$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $(X,\leqslant)$ wqo if all bad sequences are finite...

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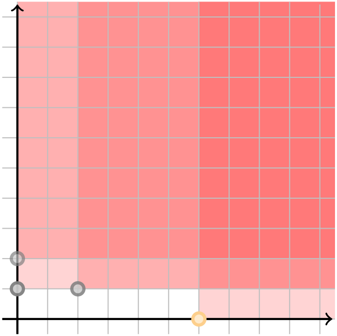

Example (in $\mathbb{N}^2$)

$(0,2), (2,1), (0,1), (6,0)$

Bad Sequences

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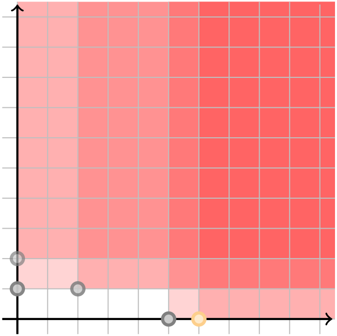

Example (in $\mathbb{N}^2$)

$$(0,2), (2,1), (0,1), (6,0), (5,0)$$

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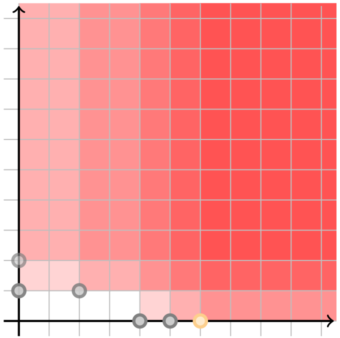

Example (in $\mathbb{N}^2$)

$(0,2), (2,1), (0,1), (6,0), (5,0), (4,0)$

Bad Sequences

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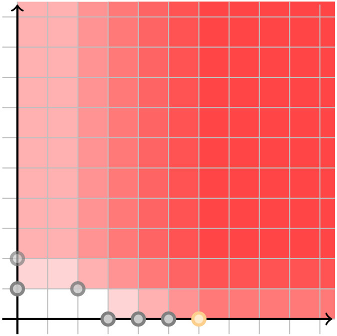

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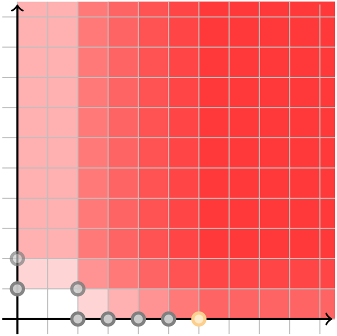

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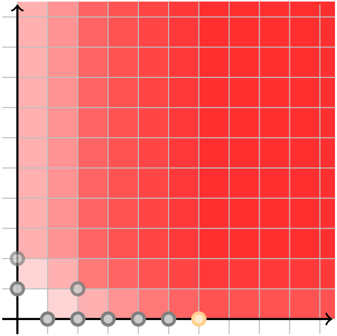

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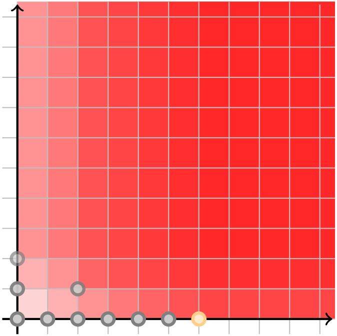

Example (in $\mathbb{N}^2$)

$(0,2), (2,1), (0,1), (6,0), (5,0), (4,0), (3,0), (2,0), (1,0), (0,0)$

Controlled Bad Sequences

Over a qo $(X, \leqslant)$ with norm $\|\cdot\|$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $x_0, x_1, \ldots$ is amortised controlled by $g\colon \mathbb{N} \to \mathbb{N}$ and $n_0 \in \mathbb{N}$ if
$$\forall i \,.\, \|x_i\| \leqslant g^i(n_0)$$
[Cichoń & Tahhan Bittar'98]

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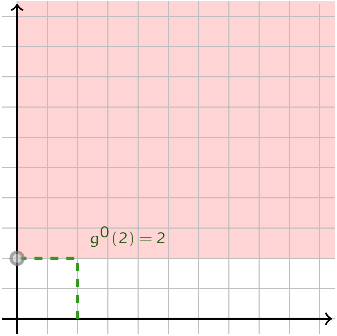

Example (in $\mathbb{N}^2$ with $n_0 = 2$ and $g(n) = n + 1$)

$(0, 2)$

Controlled Bad Sequences

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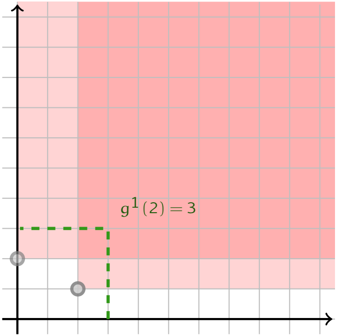
[Cichoń & Tahhan Bittar'98]



Example (in $\mathbb{N}^2$ with $n_0 = 2$ and $g(n) = n + 1$)

$$(0,2), (2,1)$$

Controlled Bad Sequences

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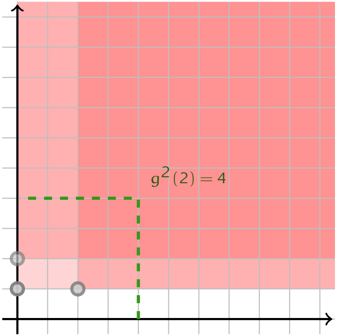

Example (in $\mathbb{N}^2$ with $n_0 = 2$ and $g(n) = n+1$)

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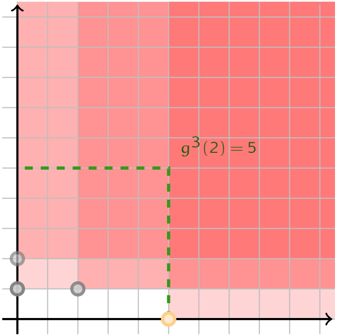

Example (in $\mathbb{N}^2$ with $n_0 = 2$ and $g(n) = n + 1$)

$$(0,2), (2,1), (0,1), (5,0)$$

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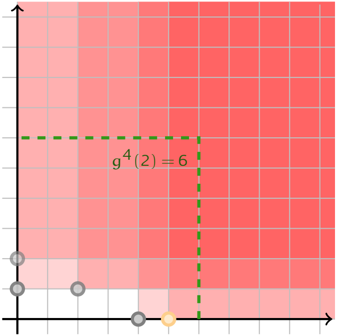

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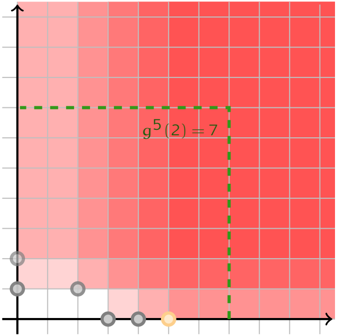

Example (in $\mathbb{N}^2$ with $n_0=2$ and $g(n)=n+1$)

$$(0,2),(2,1),(0,1),(5,0),(4,0),(3,0)$$

Controlled Bad Sequences

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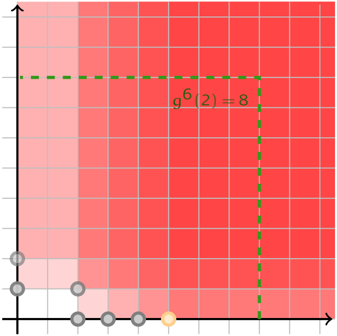

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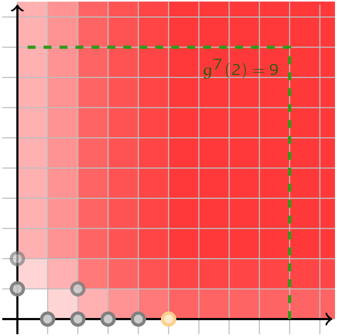

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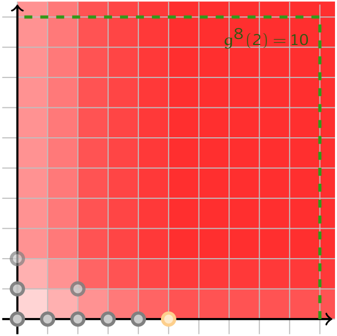

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[Cichoń & Tahhan Bittar'98]

Proposition

In a wqo $(X,\leqslant)$, if $\forall n \{x \in X \mid \|x\| \leqslant n\}$ is finite, then amortised (g,n_0)-controlled bad sequences have a maximal length, denoted $L^a_{g,X}(n_0)$.

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Theorem (S. & Schnoebelen'12)

For LCM Reachability, $g(x)\overset{def}{=}x+1$ and $n_0\overset{def}{=}\|\mathbf{v}_f\|$ fit, and

$$L^a_{g,Q\times\mathbb{N}^d}(n_0)\approx F_{d+1}(n).$$

Controlled Bad Sequences

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[Cichoń & Tahhan Bittar'98]
- $x_0,x_1,\ldots$ is strongly controlled by $g\colon\mathbb{N}\to\mathbb{N}$ and $n_0\in\mathbb{N}$ if $\|x_0\|\leqslant n_0$ and $\forall i\,.\,\|x_{i+1}\|\leqslant g(\|x_i\|)$

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Corollary

In a wqo $(X, \leqslant)$, if $\forall n \{x \in X \mid \|x\| \leqslant n\}$ is finite, then strongly (g, n_0)-controlled bad sequences have a maximal length, denoted $L^s_{g,X}(n_0)$.

Controlled Bad Sequences

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Corollary

In a wqo $(X,\leqslant)$, if $\forall n\,\{x\in X\mid\|x\|\leqslant n\}$ is finite, then strongly (g,n_0)-controlled bad sequences have a maximal length, denoted $L^s_{g,X}(n_0)$ and a maximal norm, denoted $N^s_{g,X}(n_0)$.

Controlled Bad Sequences

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Corollary

In a wqo $(X,\leqslant)$, if $\forall n\,\{x\in X \mid \|x\|\leqslant n\}$ is finite, then (g,n_0)-controlled bad sequences have a maximal length, denoted $L^s_{g,X}(n_0)$ and a maximal norm, denoted $N^s_{g,X}(n_0)$.

Theorem

For LCM Reachability, $g(x)\stackrel{def}{=}x+1$ and $n_0\stackrel{def}{=}\|\mathbf{v}_f\|$ fit, and

$$L^s_{g,Q\times\mathbb{N}^d}(n_0)\approx F_d(n).$$

Controlled Antichains

Over a qo $(X,\leqslant)$

- $x_0, x_1, \ldots$ is an antichain if
  $\forall i < j \,.\, x_i \perp x_j$ (i.e. $x_i \not\leq x_j$ and $x_j \not\leq x_i$)

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- $(X,\leqslant)$ wqo iff it is well-founded and all antichains are finite

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- $x_0, x_1, \ldots$ is amortised controlled by $g\colon \mathbb{N} \to \mathbb{N}$ and $n_0 \in \mathbb{N}$ if
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Controlled Antichains

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Corollary

In a wqo $(X,\leqslant)$, if $\forall n \{x \in X \mid \|x\| \leqslant n\}$ is finite, then amortised (g, n_0)-controlled antichains have a maximal length, denoted $W^a_{g,X}(n_0)$.

Antichain Factorisation

Example (strongly $(x \mapsto x+1, 4)$-controlled bad sequence over $\mathbb{N}^2$)

(3,4), (5,2), (4,3), (4,2), (5,1), (2,3), (4,1), (5,0), (1,4), (3,1), (0,4), (3,0), (1,1)

Antichain Factorisation

Example (strongly $(x \mapsto x+1, 4)$-controlled bad sequence over $\mathbb{N}^2$)

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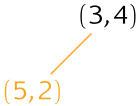
$(3,4)$

Antichain Factorisation

Example (strongly $(x \mapsto x+1, 4)$-controlled bad sequence over $\mathbb{N}^2$)

(3,4), (5,2), (4,3), (4,2), (5,1), (2,3), (4,1), (5,0), (1,4), (3,1), (0,4), (3,0), (1,1)



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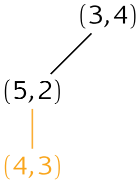

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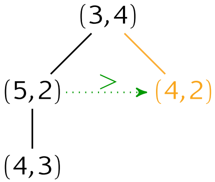

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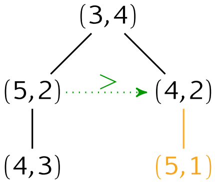

Antichain Factorisation

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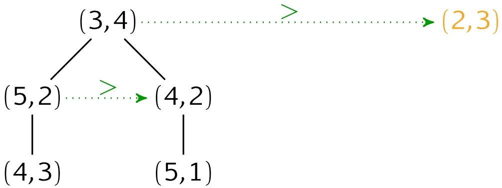

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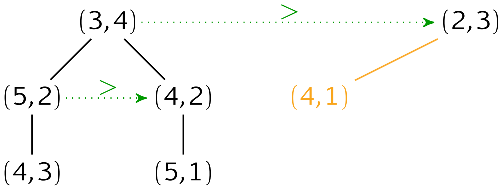

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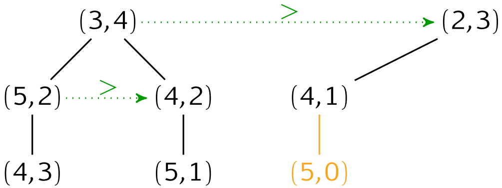

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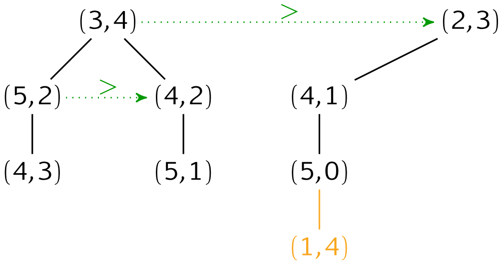

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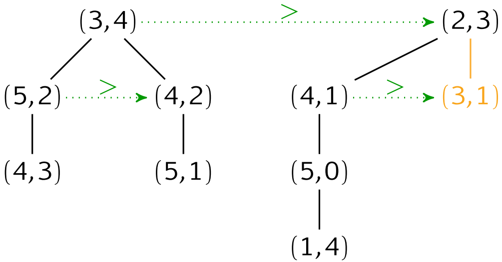

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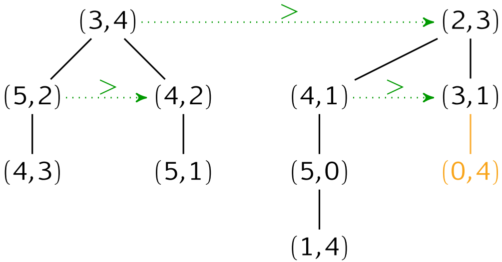

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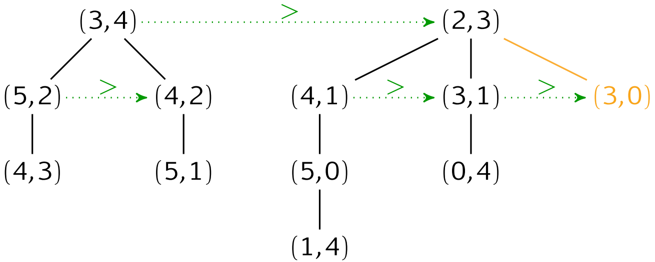

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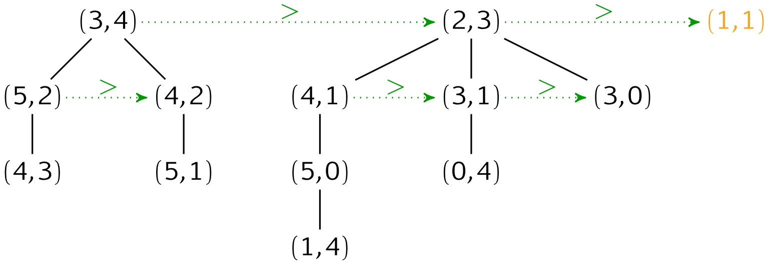

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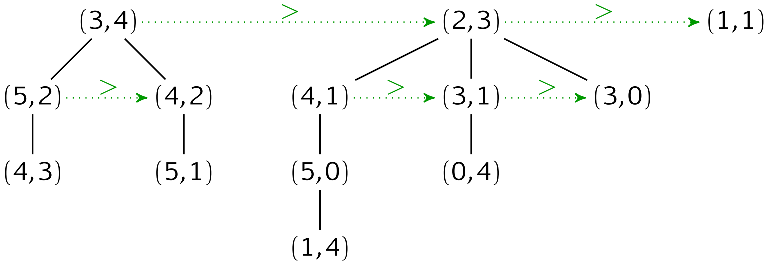

Property

Every branch is a strongly $(x \mapsto x+1, 4)$-controlled antichain over $\mathbb{N}^2$.

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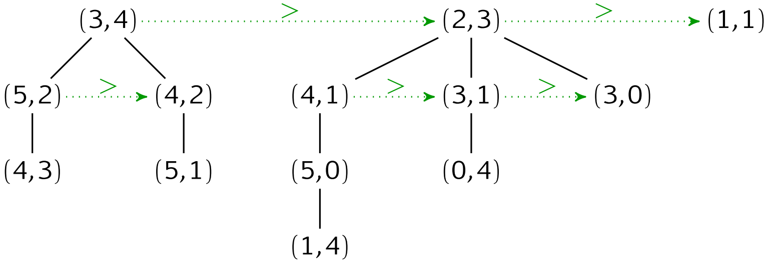

Here, height is 3, hence

- maximal norm at most $g^3(n_0) = n_0 + 3 = 7$,
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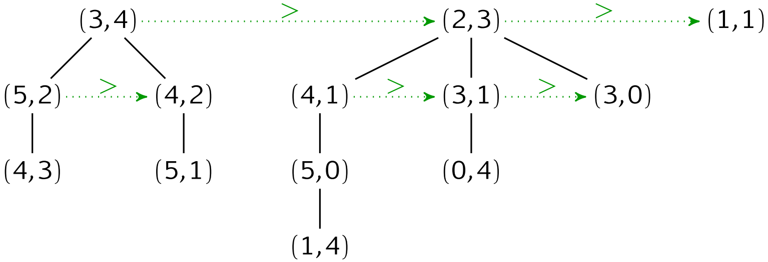

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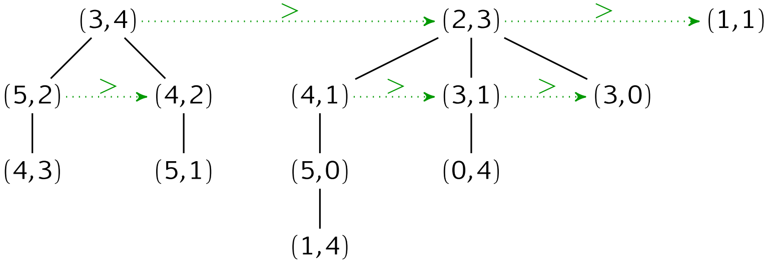

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Factorisation Lemma

Lemma

Let $(X,\leqslant)$ *be a wqo with norm* $\|.\|:X\to\mathbb{N}$ *monotone. Then* $N^{s}_{g,X}(n_0)\leqslant g^{W^{s}_{g,X}(n_0)}(n_0)$.

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Width Function Theorem

Informal statement

$W^{a}_{g,Q\times\mathbb{N}^d}(n_0) \approx F_d(n)$ in the case of LCMs.

Proof ingredients

1. descent equation
2. normed reflections
3. subrecursive hierarchies

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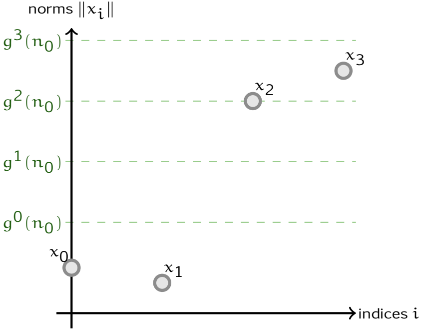
Ingredient 1: Descent Equation

amortised (g, n_0)-controlled antichain $x_0, x_1, x_2, x_3, \ldots$ over a wqo $(X, \leqslant)$:



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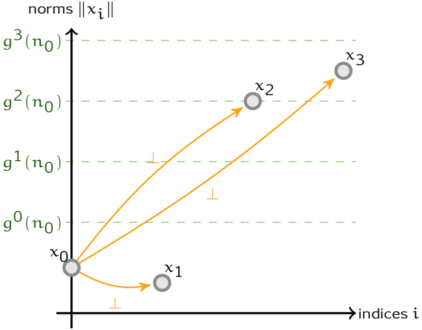

over the suffix $x_1, x_2, x_3, \ldots, \forall i > 0$,

$$x_0 \perp x_i$$

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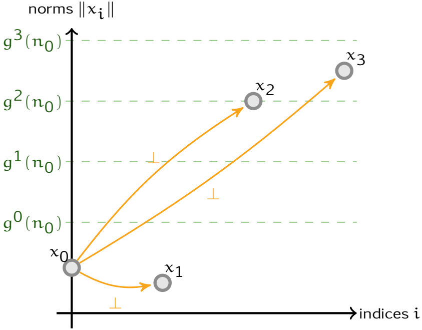

over the suffix $x_1, x_2, x_3, \ldots, \forall i > 0$,

$$x_i \in X_{\perp x_0} \stackrel{\text{def}}{=} \{x \in X \mid x_0 \perp x\}$$

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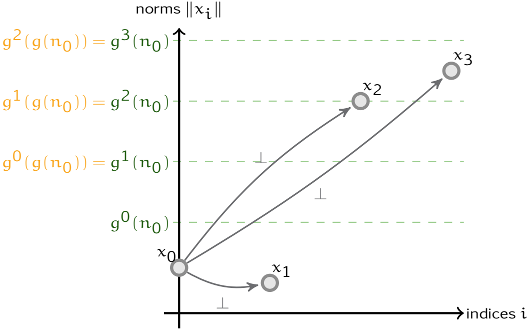

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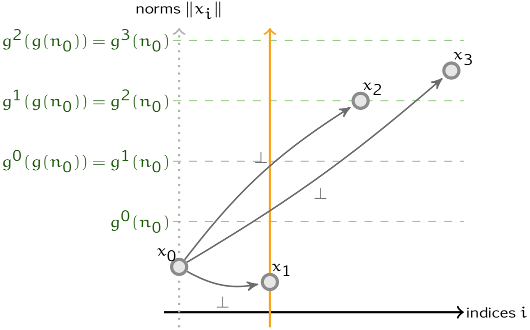

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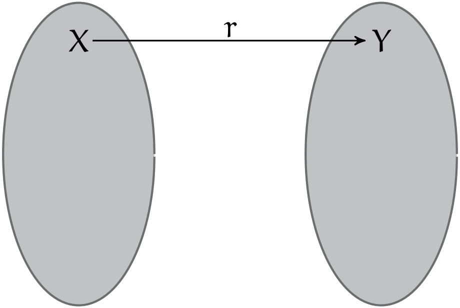
$$W^a_{g,X}(n_0) = \max_{x_0 \in X, \|x_0\| \leqslant n_0} 1 + W^a_{g,X_{\perp x_0}}(g(n_0))$$

Ingredient 2: Normed Reflections



Definition

Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be qos with norms $\|.\|_X$ and $\|.\|_Y$. A normed reflection is a function $r{:}X \to Y$ such that

1. $\forall x, x' \in X \,.\, x \not\leqslant_X x'$ implies $r(x) \not\leqslant_Y r(x')$
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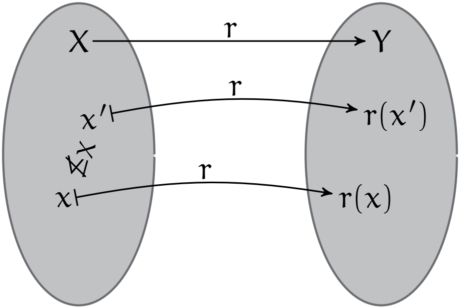

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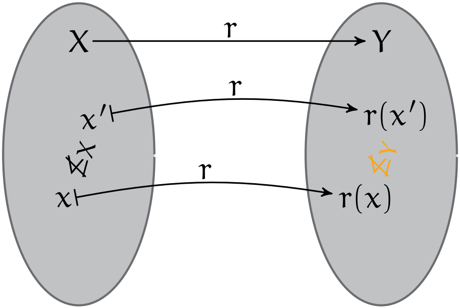

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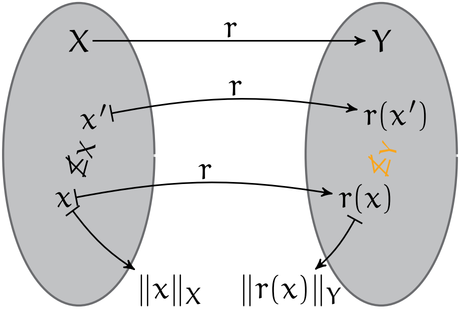

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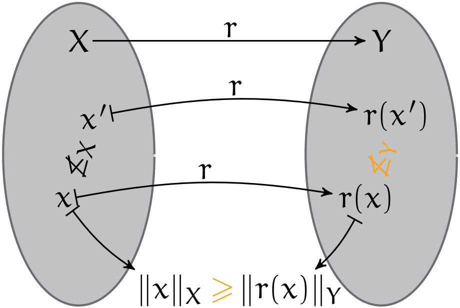

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Fact

If $r: X \to Y$ is a normed reflection, then for all g, n_0

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Outline

lossy counter machines (LCM) reachability

- canonical Ackermann-complete problem
- complexity gap in fixed dimension d:
  $\mathbf{F}_d$-hard, in $\mathbf{F}_{d+1}$

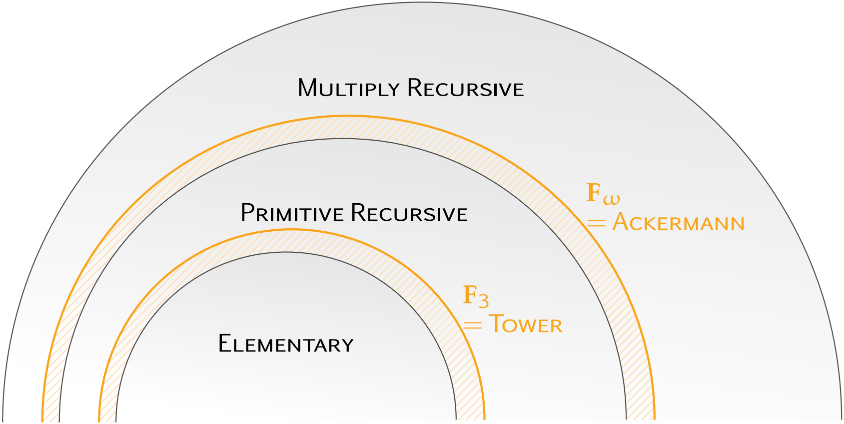
complexity using well-quasi-orders (wqo)

- strongly controlled bad sequences
- antichain factorisation
- width function theorem
  on the length of controlled antichains
- $\mathbf{F}_d$ upper bounds for LCM reachability

Technical Appendix

Complexity Classes Beyond Elementary

[S.'16]



Complexity Classes Beyond Elementary

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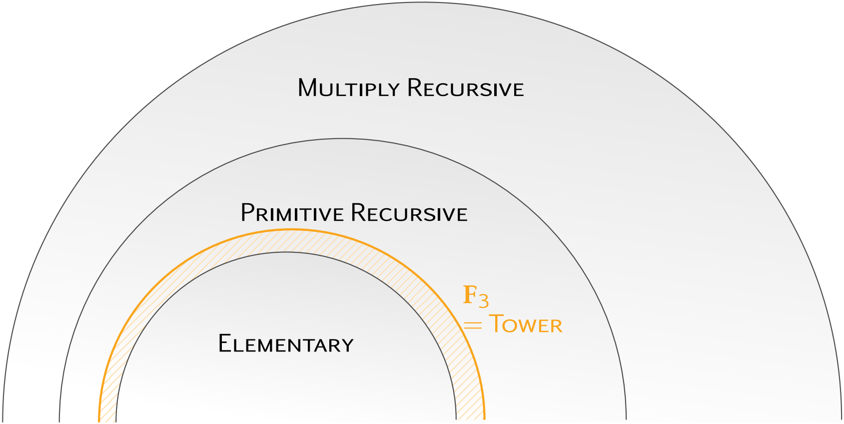

$$\mathbf{F}_3 \stackrel{\text{def}}{=} \bigcup_{e \text{ elementary}} \text{DTime}(\text{tower}(e(n)))$$

Complexity Classes Beyond Elementary

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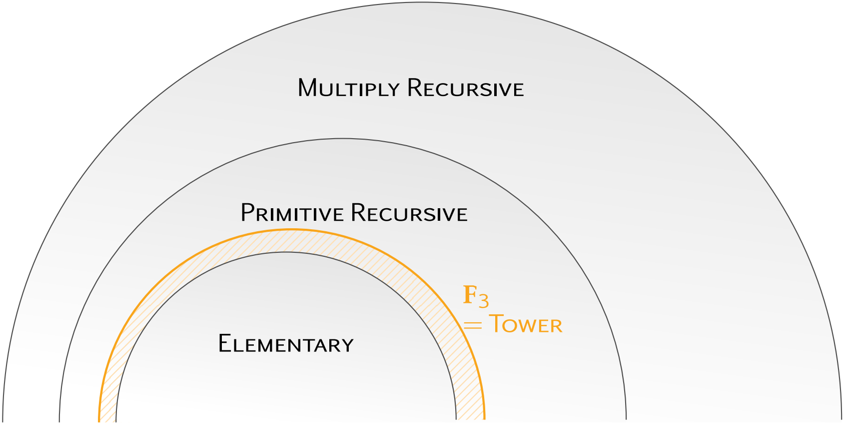

Examples of Tower-Complete Problems:

- satisfiability of first-order logic on words [Meyer'75]
- β-equivalence of simply typed λ terms [Statman'79]
- model-checking higher-order recursion schemes [Ong'06]

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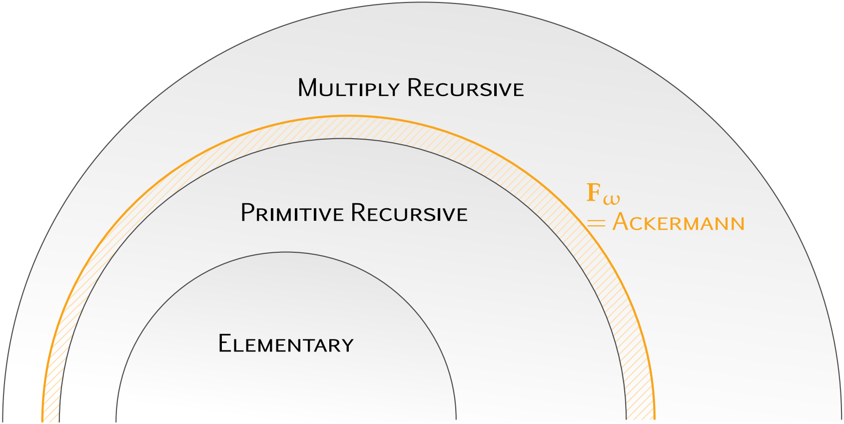

$$\mathbf{F}_\omega \stackrel{\text{def}}{=} \bigcup_{p \text{ primitive recursive}} \text{DTime}\left(\mathrm{ack}(p(n))\right)$$

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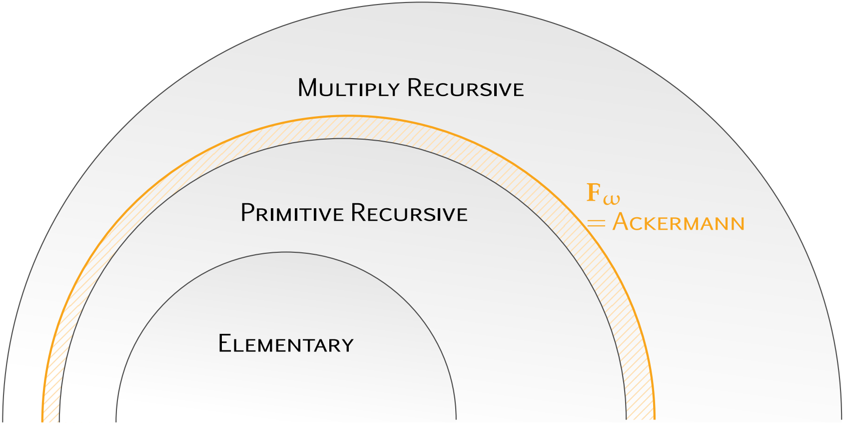

Examples of Ackermann-Complete Problems:

- reachability in lossy Minsky machines [Urquhart'98, Schnoebelen'02]
- satisfiability of safety Metric Temporal Logic [Lazić et al.'16]
- satisfiability of Vertical XPath [Figueira and Segoufin'17]

Backward Coverability

(Arnold & Latteux'78)

Goal: Check whether

$$q_0(\mathbf{v}_0) \in \mathrm{Pre}^*_\exists(\uparrow q_f(\mathbf{v}_f)) \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists \mathbf{v}' \geqslant \mathbf{v}_f \,.\, q(\mathbf{v}) \to^*_\ell q_f(\mathbf{v}')\}$$

Fixed-point computation

$$U_0 \stackrel{\text{def}}{=} \uparrow q_f(\mathbf{v}_f) \qquad U_{n+1} \stackrel{\text{def}}{=} U_n \cup \mathrm{Pre}_\exists(U_n)$$

where $\mathrm{Pre}_\exists(S) \stackrel{\text{def}}{=} \{q(\mathbf{v}) \mid \exists q'(\mathbf{v}') \in S \,.\, q(\mathbf{v}) \to_\ell q'(\mathbf{v}')\}$

Ascending chain

$$U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_L = U_{L+1} = \mathrm{Pre}^*_\exists(\uparrow q_f(\mathbf{v}_f))$$

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$$U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_L = U_{L+1} = \mathrm{Pre}^*_\exists(\uparrow q_f(\mathbf{v}_f))$$

Backward Coverability

(Arnold & Latteux'78)

Goal: Check whether

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Fixed-point computation

$$U_0 \stackrel{\text{def}}{=} \uparrow q_f(\mathbf{v}_f) \qquad U_{n+1} \stackrel{\text{def}}{=} U_n \cup \mathrm{Pre}_\exists(U_n)$$

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Fixed-point computation

$$B_0 \stackrel{\text{def}}{=} \{q_f(\mathbf{v}_f)\} \qquad B_{n+1} \stackrel{\text{def}}{=} \min(B_n \cup \mathrm{Pre}_\exists(\uparrow B_n))$$

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$$c_0 \stackrel{\text{def}}{=} q_f(\mathbf{v}_f) \qquad c_{n+1} \in B_{n+1} \setminus \uparrow B_n$$

Then $\forall i < j \,.\, c_i \not\leqslant c_j$

$(Q \times \mathbb{N}^d, \leqslant)$ is a well-quasi-order (wqo), thus

finite basis property: each $B_n = \min U_n$ is finite

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Subrecursive Functions

Definition (Cichoń Hierarchy)

For $g\colon \mathbb{N} \to \mathbb{N}$, define $(g_\alpha\colon \mathbb{N} \to \mathbb{N})_\alpha$ by

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Example

$$P_3(\omega^2) = \omega \cdot 3 + 3$$

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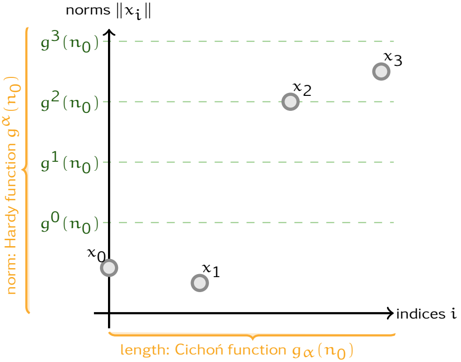
If $g(x) = x + 1$, then $g^{\omega^\alpha}(x) = F_\alpha(x)$

Relating Norm and Length

[Cichoń & Tahhan Bittar'98]



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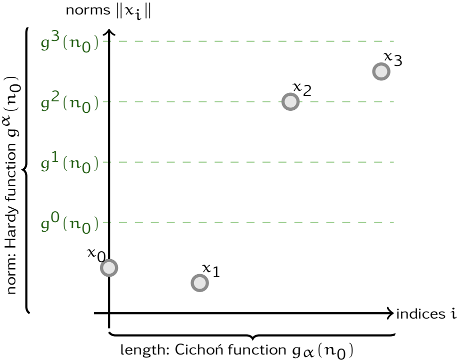

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