The Chain Covering Number of a poset with No Infinite Antichains

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21-23 February, Lyon Colloquium
In honour of Robert Bonnet

Report on joint work with Maurice Pouzet.

Some Definitions.

(1) Points $x, y \in P$ are said to be comparable if $x \leq y$ or $y \leq x$. The points are incomparable if they are not comparable. We write $x \perp y$ to say that x and y are incomparable.

(2) A subset $A \subseteq P$ is a chain if any two points of A are comparable. We say that A is an antichain if any two distinct points of A are incomparable.

(3) A collection $\mathscr{A}$ of chains of P covers P when $P = \bigcup \mathscr{A}$.
I.e. every $x \in P$ is a member of some chain $A \in \mathscr{A}$.

The *Covering Number*, $\mathrm{Cov}(P)$, of a poset P is the least cardinality of a collection $\mathscr{A}$ of chains that covers P.

For any poset $P = (V, \leq_P)$ we notate with $P^* = (V, \leq^*)$ the dual of P i.e. the inverse ordering.
Surely $\mathrm{Cov}(P^*) = \mathrm{Cov}(P)$. $\oplus$

The aim of our work is to give some information on $\mathrm{Cov}(P)$ when P satisfies the Finite Antichain Condition (FAC) meaning that every antichain is finite. All posets in this lecture satisfy the FAC.
The classical theorem of Dilworth is the paradigm of our work.

Theorem (R. P. Dilworth, 1950)

For any poset P and integer n, $\mathrm{Cov}(P) \geq n$ iff P contains an antichain of cardinality n.

So all posets that cannot be covered by n chains have the same 'obstruction', an antichain of size n.

We search similar theorems when $\mathrm{Cov}(P) > n$ is replaced by $\mathrm{Cov}(P) \geq \kappa$ where κ is an infinite cardinal.

M. Perles and E. S. Wolk have independently shown that a naive extension of Dilworth's theorem is not true. It is possible that all antichains of a poset are finite and yet $\mathrm{Cov}(P) \geq \omega$.
Let ν be any infinite cardinal, denote with $[\nu]^2$ the collection of all unordered pairs (a, b) of members of ν such that $a \neq b$. For clarity, we equate $[\nu]^2$ with the poset of all ordered pairs (a, b) such that $a < b$ and we order this collection by

$$(a_1, b_1) \leq (a_2, b_2) \text{ iff } a_1 \leq a_2 \wedge b_1 \leq b_2.$$

We say that $[\nu]^2$ is a ν-Perles ordering.

Theorem (Perles, Wolk)

The ν-Perles ordering $[\nu]^2$ satisfies the Finite Antichain Condition, and yet $\mathrm{Cov}([\nu]^2) = \nu$.

Proof of Perles's theorem on the blackboard for example for the case of $\nu = \aleph_1$. $\oplus$

It is time to state our theorem : Covering of FAC posets.

Theorem

The following holds for every poset P that has only finite antichains.

- *<u>The successor case</u>. For every successor cardinal κ^+, $\mathrm{Cov}(P) \geq \kappa^+$ iff the poset P contains a copy of $[\kappa^+]^2$ or a copy of the dual (inverse) of $[\kappa^+]^2$.* ⊕
- *<u>The limit case</u>. If ν is a limit uncountable cardinal, then $\mathrm{Cov}(P) \geq \nu$ iff P contains a poset Q, or its dual, where Q has the form $\Sigma_{c \in C} Q_c$ where*
  - *C is a linear ordering of cardinality $\mathrm{cf}(\nu)$, and*
  - *$Q_c = [\kappa_c^+]^2$ for every $c \in C$, where the cardinals κ_c are such that $\nu = \sup\{\kappa_c \mid c \in C\}$.*

⊕

Item 1 of this theorem is satisfying. There are only two obstructions : $[\omega_1]^2$ inverse of this poset. Item 2 however is incomplete : we have no knowledge about the linear ordering C (except for its cardinality).

Intriguing corollaries were observed by Maurice :

Corollary (Pouzet)

(1) For any poset P and cardinal κ, if $\mathrm{Cov}(P) \geq \kappa$ then there exists a subposet $Q \subseteq P$ of cardinality κ such that $\mathrm{Cov}(Q) \geq \kappa$. (This resembles compactness : If every subposet of cardinality κ^+ is coverable with k chains, then the whole poset is so coverable.) $\oplus$
(2) For any cardinal κ, the poset $P = [\kappa^+]^2$ is indecomposable.

That is, whenever P is decomposed, $P = P_1 \cup P_2$, into two subposets, then one of the two posets contains a copy of P itself.

Proof. (1) holds because the cardinality of $[\kappa]^2$ is κ.
(2) By Perles's theorem $P = [\kappa^+]^2$ is not a union of a collection of κ chains. Hence P_1 or else P_2 is not a union of κ chains. But any FAC poset that cannot be presented as a union of κ chains contains a copy of $[\kappa^+]$. Hence P_1 or else P_2 contain a copy of $[\kappa^+]$.

Some proofs are necessary. Let's return to our theorem in the limit case on page 5 :

where

You may ask, what can be said about the ω case ? By Dilworth's theorem if a FAC poset P has no cover by a finite family of chains, then it must have antichains of arbitrarily large size. But we want to have a single poset Q such that either P or its dual has Q as a subposet. Such a poset is demonstrated by Milner and Pouzet.

Theorem (Milner – Pouzet)

Let P be a poset with no bound on the size of its antichains. Then P or else P^* contains a poset of the form $\Sigma_{i \in \omega} A_i$ where A_i is the the antichain with i elements, and for every $i < j$, $A_i < A_j$.

I give a new proof which illustrate the basic method of proving fact about FAC posets, i.e. by induction on the rank of antichains of the poset.

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Proof. We use the following notation. $p \perp q$ say that p and q are incomparable P. For $p \in P$ $p^\perp = \{q \in P \mid p \perp q\}$. For antichains A and B of a poset P we write $A < B$ if $\forall a \in A\ \forall b \in B\ (a <_P b)$.
In case there is $p \in P$ such that the antichains of $p^\perp$ have no bound on their cardinality, we get the theorem immediately by the induction assumption.
So we may assume that for all $p \in P$ there is a bound on the antichain size. in $p^\perp$.
Hence, for every finite $X \subset P$, there is a finite bound on the size of antichains in $X^\perp = \bigcup\{p^+ \mid p \in X\}$. Hence there is no bound on the antichains of $P \setminus X^\perp$, namely, for every finite $X \subset P$ there is no bound on the antichains of $P \setminus X^\perp = comp(X)$ where $comp(X)$ is the set of $q \in P$ that are strictly comparable with every member of X.

Property of P : For every finite $X \subset P$, there is no bound on the antichains in $comp(X)$ = the set of $q \in P$ that are strictly comparable with every member of X.

Define by induction on $i \geq 1$ antichains A_i of P such that $|A_i| = i$ and for every $i \neq j$, $A_i < A_j$ or $A_j < A_i$.
Suppose that $A_1, \dots, A_n$ are defined. Apply the property to $X = A_1 \cup \dots \cup A_n$. Since there is no bound on the antichain size of $comp(X)$, we can pick and antichain A in $comp(X)$ of size $|A| > (n+1)^2$.
Every point $x \in A$ has a "cut" with respect to the antichains $A_1, \dots, A_n$, that is, the two sets of indexes $\{1 \leq i \leq n \mid A_i < \{x\}\}$ and $\{1 \leq i \leq n \mid A_i > \{x\}\}$. There are $n+1$ possible cuts, and so there is a set $A_{n+1} \subset A$ all of whose cuts are the same.
Finally, the set of antichains obtained $\{A_i \mid i \in \omega\}$ are linearly ordered and being an infinite set, an increasing or else a decreasing sequence can be foun. Renaming this sequence, we get a subposet Q of P of the form $\Sigma_{i \in \omega} B_i$ where every B_i is an antichain of size i and the sets B_i are either increasing in the $<$ order, or else they form a decreasing sequence.