Counting Siblings

Robert Woodrow

Lyon February 21, 2023

Siblings of relational structures

I want to highlight some of the work in two directions. With permission of our student Davoud Abdi Kalow I am going to use slides he made up for a talk given at the Canadian Mathematical Society meeting in December on part of the work, focused on the development of a counterexample of Atsushi Tateno. I will add more results establishing the conjecture in two settings. Both these settings draw heavily on the theory of wqo-bqo.

Because I am a very rusty and bad with technology, I am afraid there will be some jumping around from Davoud's slides with nice pictures, and mine, without. My apologies.

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Counterexample to Conjectures of Bonato-Tardif, Thomassé and Tyomkyn

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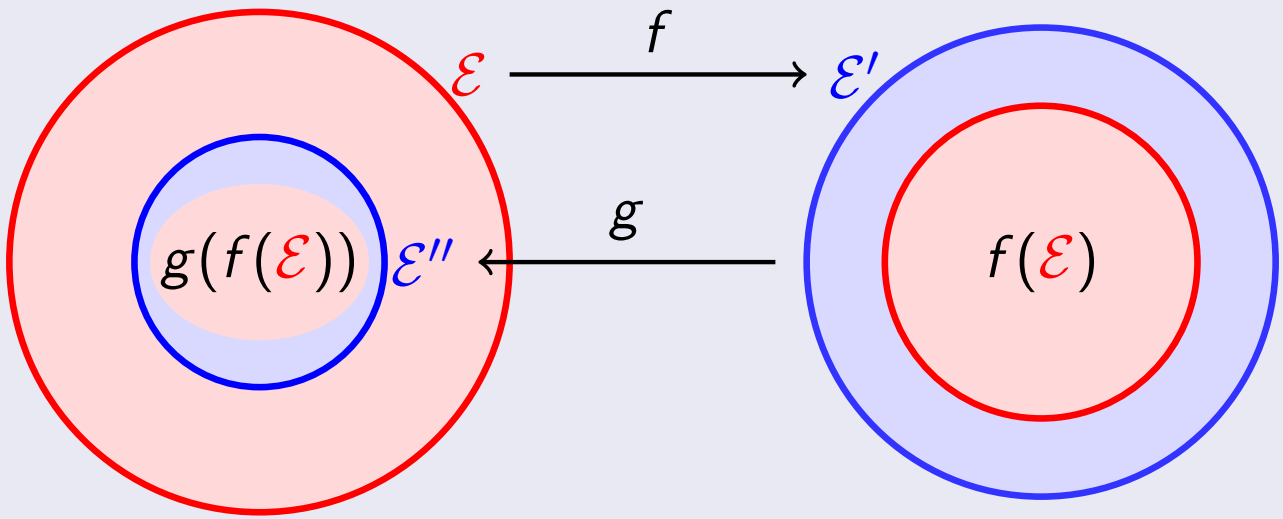
Equimorphism

Embedding

An injective map preserving the structure.

Sibling

Two structures $\mathcal{E}$ and $\mathcal{E}'$ are called *siblings* (or equimorphic), denoted by $\mathcal{E} \approx \mathcal{E}'$, when there are mutual embeddings between them.
$\mathcal{E} \approx \mathcal{E}' \cong \mathcal{E}''$, $g(\mathcal{E}') = \mathcal{E}'' \supseteq (g \circ f)(\mathcal{E}) \cong \mathcal{E}$.



Siblings in Some Categories

Cantor-Schröder-Bernstein Theorem (Sets)

If there exist injective maps $f : A \to B$ and $g : B \to A$ between two sets A and B, then there exists a bijection (isomorphism) $h : A \to B$.

Vector Spaces

If there are mutual injective linear transformations between two vector spaces over a fixed field, then they are isomorphic.

Rational Numbers

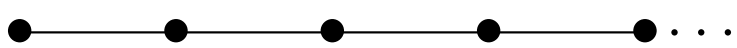
$\mathbb{Q}$ as a chain: there are mutual injective order preserving maps between $\mathbb{Q}$ and $\mathbb{Q} + \infty$, nonetheless, $\mathbb{Q} \ncong \mathbb{Q} + \infty$.

Thomassé's Conjecture

Sibling Number

The number of isomorphism classes of siblings of a relation $\mathcal{E}$, denoted by $Sib(\mathcal{E})$.

If $\mathcal{E}$ is a ray, $Sib(\mathcal{E}) = 1$ in the category of trees,



and $Sib(\mathcal{E}) = \aleph_0$ as a binary relation.

Thomassé's Conjecture (2000)

For a countable relation $\mathcal{E}$, $Sib(\mathcal{E}) = 1$, $\aleph_0$ or $2^{\aleph_0}$.

The Alternate Thomassé Conjecture

For a relation $\mathcal{E}$ of any cardinality, $Sib(\mathcal{E}) = 1$ or ∞.

The Bonato-Tardif Conjecture

Conjectures about Trees

- The Bonato-Tardif Conjecture (2006): If T is a tree, then $Sib(T) = 1$ or ∞ in the category of trees.
- Tyomkyn's Conjecture (2009): If a locally finite tree T has a non-surjective embedding, then $Sib(T) = \infty$, unless T is a ray.

Note [Pouzet]

If for a tree T we have $T \oplus 1 \not\hookrightarrow T$, then the conjectures of Bonato-Tardif and Thomassé are equivalent.

Results towards the conjectures of Bonato-Tardif and Tyomkyn

The Bonato-Tardif conjecture holds for

- rayless trees [Bonato, Tardif] (2006)
- rooted trees [Tyomkyn] (2009)
- scattered trees and stable trees [Laflamme, Pouzet, Sauer] (2017)

Tyomkyn's conjecture holds for

- locally finite scattered trees [Laflamme, Pouzet, Sauer] (2017)

Thomassé and the Alternate Thomassé Conjecture

Positive Results

Thomassé's conjecture holds for

- countable chains [Laflamme, Pouzet, Woodrow] (2017) and countable direct sums of chains [Abdi] (arXiv, 2022^+)

The Alternate Thomassé conjecture holds for

- rayless graphs [Bonato, Bruhn, Diestel, Sprüssel] (2011)
- chains [Laflamme, Pouzet, Woodrow] (2017)
- countable $\aleph_0$-categorical relational structures [Laflamme, Pouzet, Sauer, Woodrow] (2021)
- countable universal theories [Braunfeld, Laskowski] (arXiv, 2022^+)
- countable cographs [Hahn, Pouzet, Woodrow] (arXiv, 2022^+)
- direct sums of chains [Abdi] (arXiv, 2022^+)
- countable *NE*-free posets [Abdi] (arXiv, 2022^+)

Countable cographs (Hahn, Pouzet, Woodrow)

An undirected graph G is a cograph just when it does not embed a path on four vertices. Since the graph complement of a path on four vertices is also a path on four vertices, a graph is a cograph just in case its complement is.

Finite cographs can be obtained from the graph on a single vertex by the operations of direct and complete sum. The direct sum of two graphs G and H is obtained by taking disjoint copies and adding no new edges between them. The complete sum adds all possible edges between the two copies.

To obtain countably infinite cographs another construction tool is needed. The tool arises naturally from consideration of the modular decomposition of the graph.

A subset M of a graph is said to be a module (or interval) just when a not in M and b,c in M we have a is adjacent to b iff a is adjacent to c. The empty set, singletons and the entire set G are always modules, the trivial ones.

The modular decomposition tree arises regarding strong modules. A non-empty module M is said to be strong when it is comparable (with respect to inclusion) to every module N that it meets in a non-empty set. Since the intersection of a family of strong modules containing a common element is always strong, one is led to consider the robust s of the graph---strong modules generated by one or two elements. Ordered by reverse inclusion, the robust modules form a tree.

Gallai Decomposition

The Gallai decomposition theorem applies. Given a robust modules—its maximal proper strong modules form either a complete graph or an independent set, giving a natural label to the robust module 0 or 1, from which the graph structure can be recovered.

Decomposition Tree

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Infinite chain

Given an element x of the graph G one can consider all the robust modules which contain x. In the order of reverse inclusion this may be a chain without a first element.

This leads to the third construction tool, an infinite chain without first element with each element labeled by a countable cograph and the label 0 or 1, (0 for empty and 1 for complete).

HPW result

- A countable cograph has either 1 or infinitely many siblings. We would like to extend this to say 1, aleph zero or the continuum, however some lemmas, such as “if G is connected and G +1 is a sibling of G, then G has infinitely many non isomorphic siblings” would need more.

A key result, that uses wqo-bqo is the following:

If a countably infinite cograph and its complement are both connected, then it has a continuum of siblings.

This situation means that the decomposition tree has no first element. This puts one in the situation of countable trees labelled by countable cographs and 0 or 1, Σ C, the sum of a chain labeled by 0, 1 and cographs. Here we may assume that if i<j in C there is k with I \leq k < j such that the label at k is different than the one at j, a form of density.

Here one applies the result of Laver that countable chains are BQO, and of Thomassé that countable cographs are BQO to obtain that that the labeled order is WQO, This allow us to work with left indecomposability to drive out the results, with some effort.

- The key idea is to take a left indecomposable initial segment J of the order C. We identify a coinitial sequence a_n with label one and add infinitely many new b_n, c_n with b_n covering a_n, and with label 0, and c_n covering b_n with label 1. Next ensure that for i in J we have that in the modified labeled chain, the cograph associated with i is an odd clique, if it was originally associated to an an even clique (and similarly with antichain). We then associate even chains/antichains at b_n and c_n according to a mapping f.

- Fix a mapping f from N to {0,1}. Associate to b_n an independent set of size 2f(n) +2 and to c_n a clique of size 2f(n)+2.

- Now the idea is to argue that we obtain siblings of the original Sum, and that for a suitable collection of f they are pairwise non isomorphic.

A key lemma is the following:

- Lemma 2.12. Let C := (I, ℓ) be a countably infinite labelled chain. If the labels
- belong to a b.q.o. and C is indecomposable then for every positive integer n,
- the ordinal product n.C embeds in C.

This lemma allows us to argue that we do indeed obtain siblings of the initial structures.

- Lemma 2.11. Let C := (I, ℓ) be a countable chain labelled over a b.q.o. Q.
- Then
- (1) C is a finite sum of indecomposable labelled chains;
- (2) If I has no least element, then there is some initial interval J such that
- C↾J is left-indecomposable;
- (3) If C is left-indecomposable then C is an ω∗ sum Σ∗
- n<ωCn where each
- Cn is indecomposable and the set of m such that Cn embeds into Cm is
- infinite;
- (4) If C is indecomposable and the quotient I/ ≡C is dense then C
- is equimorphic to a sum Σq∈QCq such that for every p < q and r
- in Q there are s0, . . . , snm−1 with p < s0 < ··· < snm−1 < q and
- Cr ≤ Cs0 +···+ Csnm−1
- < C.

Obtaining non-isomorphic siblings

- Lemma 2.8. Let G, G′ be two isomorphic cographs such that their tree decomposition
- has no least element. If G = ΣC and G′ = ΣC′ where C := (I, ≤, ℓ)
- and C′ := (I′, ≤′, ℓ′) are two reduced labelled chains then there are two infinite
- initial segments W of I and W′ of I′ and an isomorphism h of the induced
- Labelled chains C↾W and C′ ↾W′ .

- Having disposed of finite even chains and antichains except those introduced via f for the b_n and c_n, we obtain that the isomorphism h for large enough n must take b_n to some b_m, and we obtain two final segments D and D’ of N such that f’(h(n))=f(n), when f and f’ yield isomorphic cographs.
- This means we only need a continuum of functions f such that for f and f’ distinct and h an isomorphism of a final segment D onto another D’ there is n such that f(n) \neq f’(h(n)). This is possible via an almost disjoint family of subsets of a set X which exhibits a strictly increasing sequence of gaps, so that if A is in the family then A+n is almost disjoint from A.

Davoud Abdi's thesis

Davoud's initial thesis goal was to prove the tree conjecture in the case of locally finite trees. On the way he obtained some extensions of work in the area. As it happens, in personal communications with Tyomkin he learned of a counterexample to Thomassé's conjecture in the Oxford thesis of Tateno. The example was never published, but Tateno did supply a copy. Claude, Davoud, Tateno and I eventually produced the counterexamples reported in last part of the talk.

Of course Davoud's thesis had to take a different tack. He took on generalizing the work on countable cographs to countable NE free posets, these are posets which do not embed an N.

- This was not a straightforward application of the cograph result. While cographs are the comparability graphs of NE free posets, a moments thought highlights the difficulty—all infinite chains have the same comparability graph—a clique.

- Another way to see this is as follows. Suppose P is a summand with incomparable elements p, q and r (not in P) is deemed to be comparable with p and q. (This will arise with either the linear sum or with sums corresponding to robust modules.) We must have r <p and r<q or p<r and q<r. This forces an analysis which splits the two simple situations 0 and 1 from cographs into three 0 1- and 1+, which must be handled. It also makes the discussion about density and applications of the arguments much more subtle.

- Let (I \leq) be a chain with r :I\rightarrow{-1,0,1}. Define Q^I_r = (I \leq') by for i<j
- i is orthogonal to j if r(i) = 0
- i<‘ j if r(i) = -1 and
- j ,’I if r(i) =+1.

- Then we define the poset labelled sum of I w.r.t. r as the poset substitution of the P_i according to Q^I_r

- Davoud did apply the tools of WQOBQO and modified the arguments to drive out the result establishing the “weak” results for N partial orders P—A countable N free partial order has 1 or infinitely many siblings.

- The modular decomposition by robust modules must allow for this split and with a resulting complication in labelling. While the disjoint sum of posets is relatively straight forward, there are complciations that arise for the linear sum.
- Fortunately Laver's result on BQO for countable chains, and Thomassé's proof for NE free posets provide the essential tools needed to carefully extend the approach and carry the day.

Counterexample to the Conjectures of Bonato-Tardif, Tyomkyn, Thomassé, and more

[Abdi, Laflamme, Tateno, Woodrow] (arXiv, 2022^+)

There are locally finite trees having an arbitrary finite number of siblings disproving all conjectures of Bonato-Tardif, Tyomkyn and Thomassé.

Properties of the Tree Examples

For each $n > 0$ there is a locally finite tree T with exactly n siblings such that

- $T \oplus 1 \not\hookrightarrow T$;
- for each self-embedding ϕ of T, $T \setminus \phi(T)$ is finite.

Partial Order Counterexamples

The tree examples can be adapted to construct partial orders with an arbitrary finite number of siblings.

What is going to be constructed?

We inductively build two locally finite trees $\mathcal{T}_i = \bigcup_k \mathcal{T}_i(k)$, $i = 0, 1$ with the following properties:

- at each step k, $\mathcal{T}_i(k) \setminus \phi(\mathcal{T}_i(k))$, $i = 0, 1$, is finite for any self-embedding ϕ of $\mathcal{T}_i(k)$;
- $Sib(\mathcal{T}_i(k)) = \aleph_0$ for all k;
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Then we will have:

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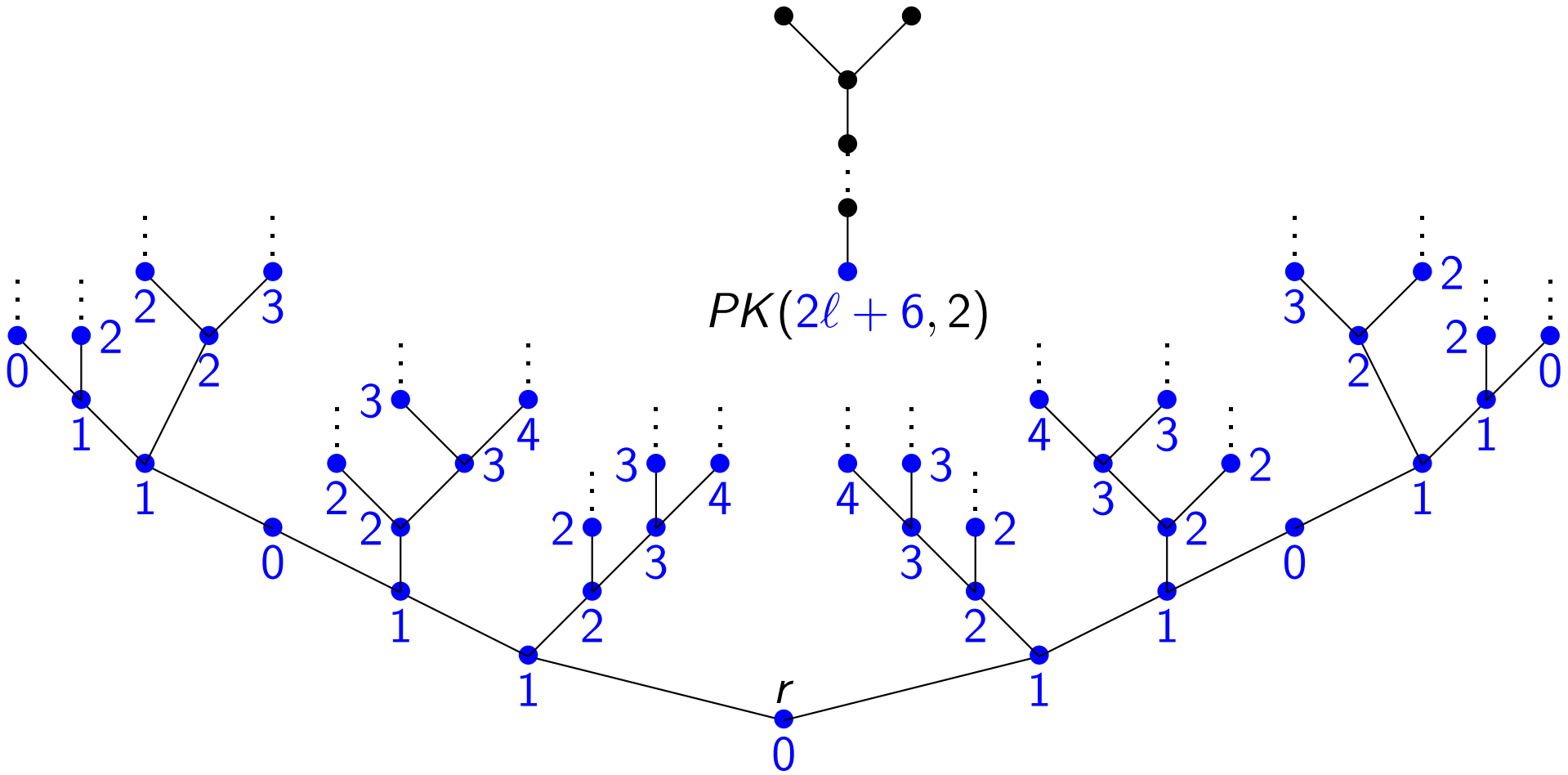
- $\mathcal{T}_i \oplus 1$ does not embed into $\mathcal{T}_i$.

A Labelled Tree

We label the vertices of $\mathcal{R} = (R, r)$ as follows. The 0-labelled vertices of $\mathcal{R}$ are called *tree vertices*, denoted by R_0.



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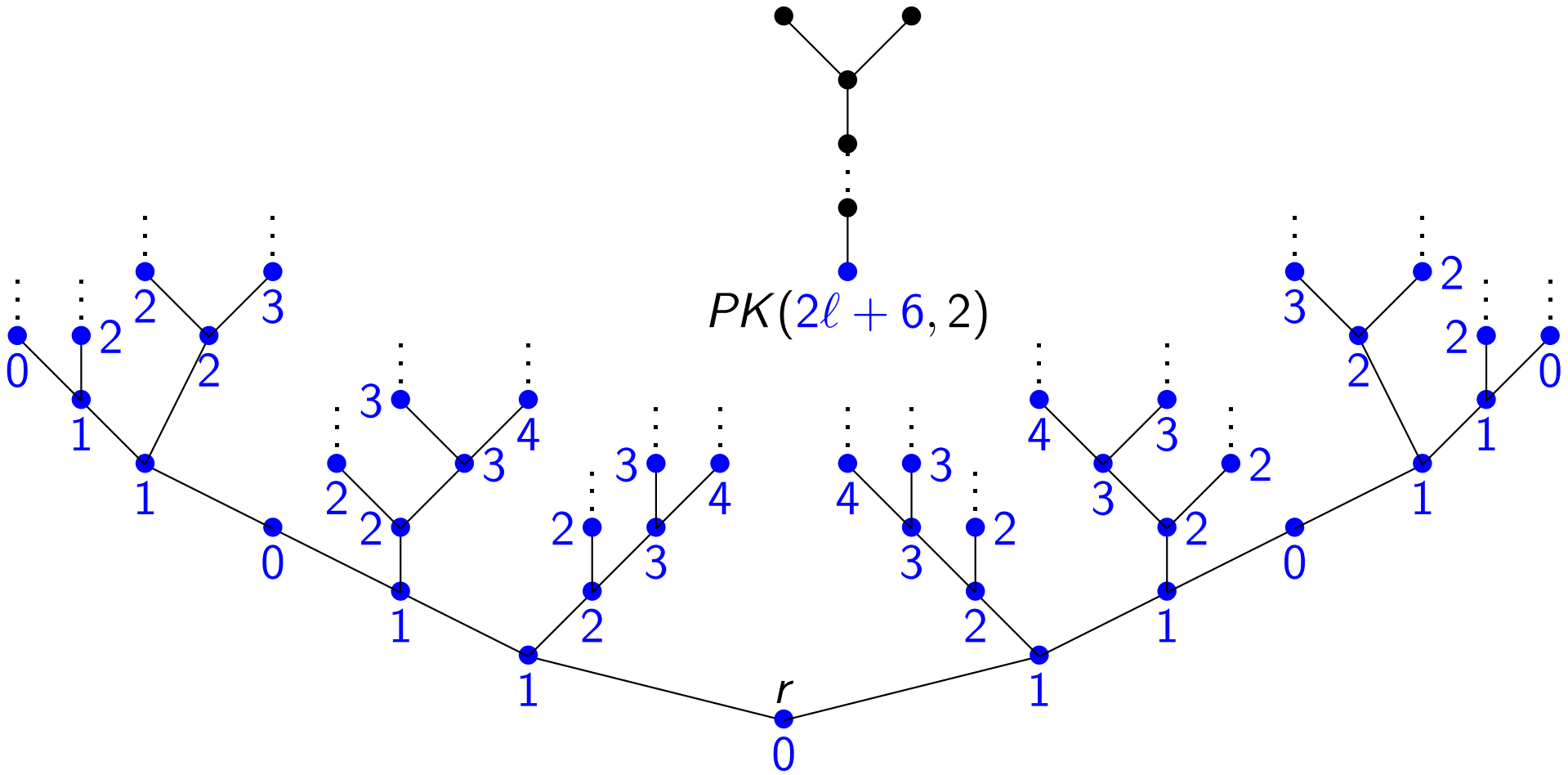

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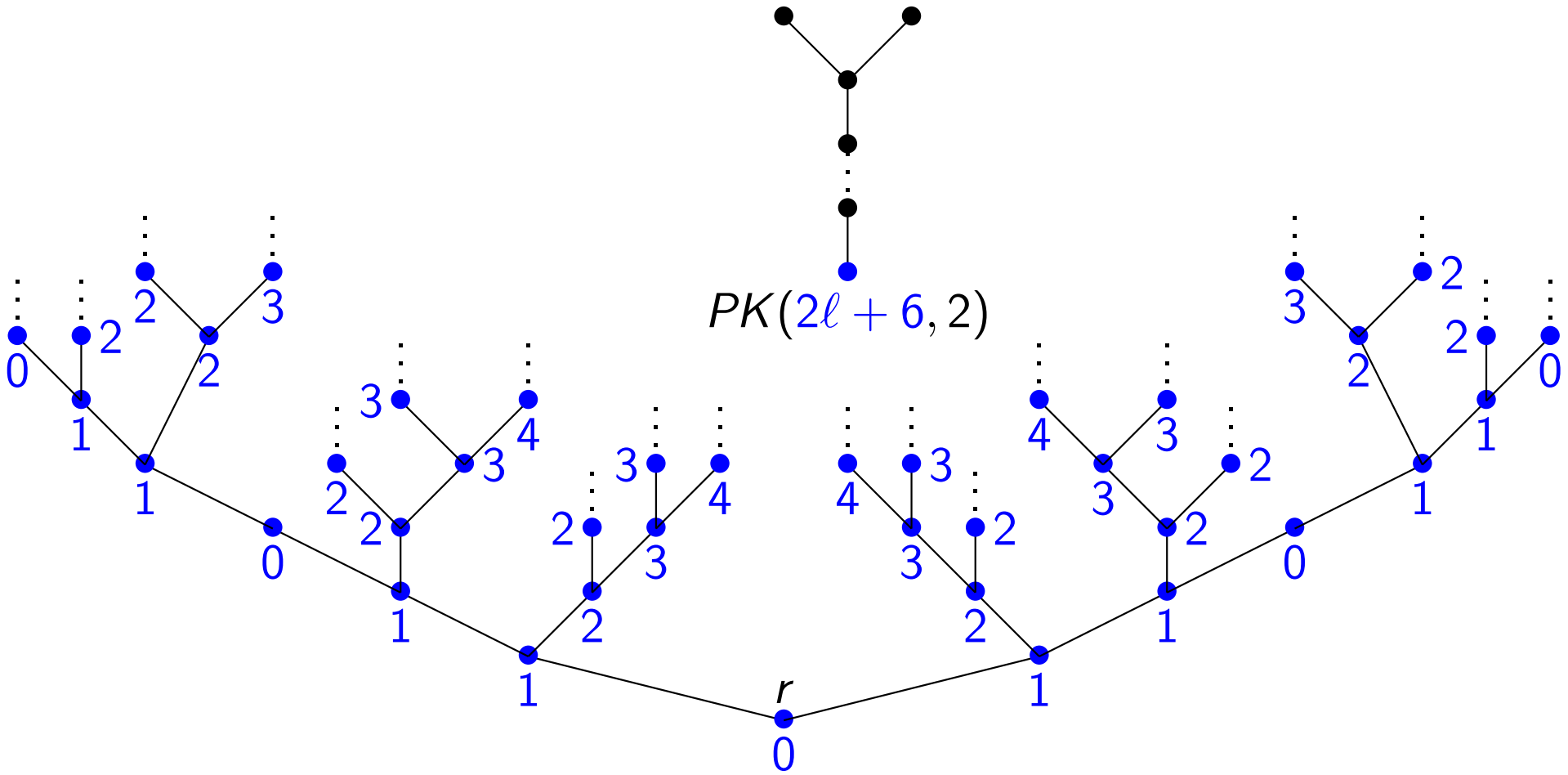
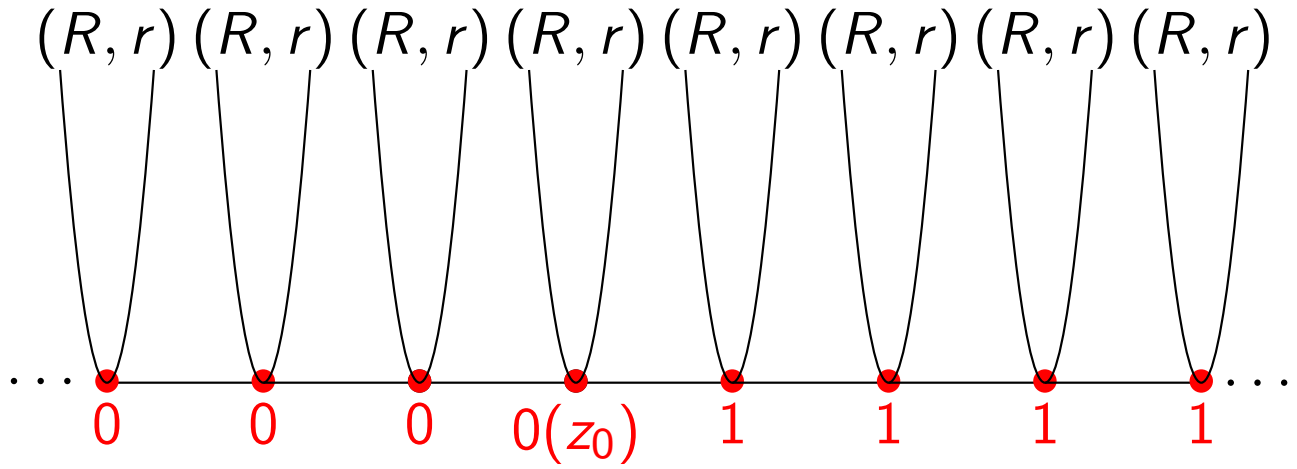
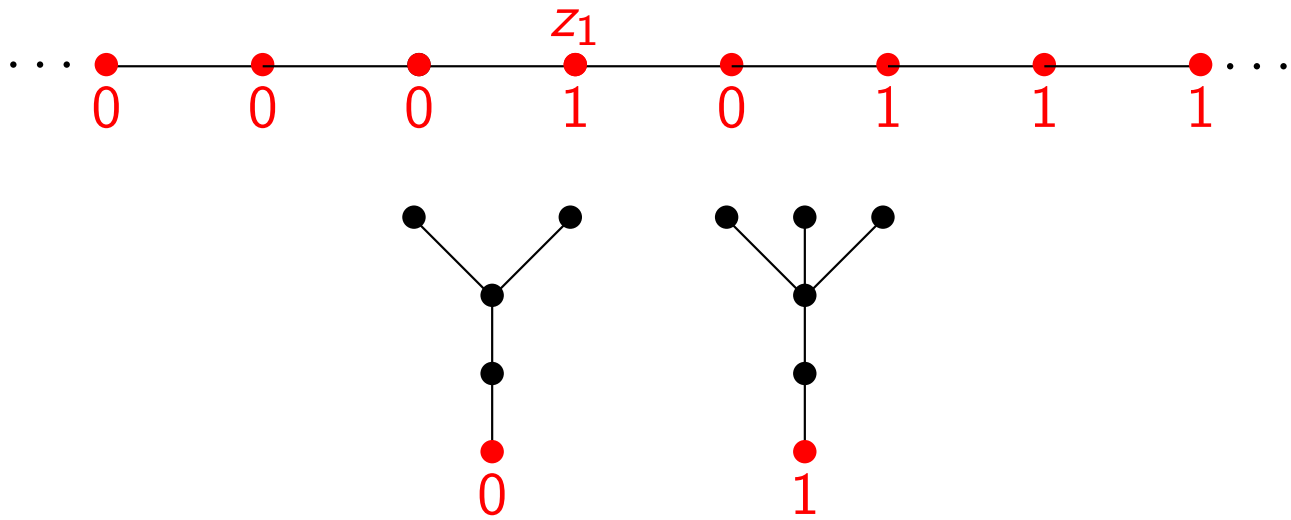


$\mathcal{T}_0(0)$ and $\mathcal{T}_1(0)$

$\mathcal{T}_0(0)$ on tp_0

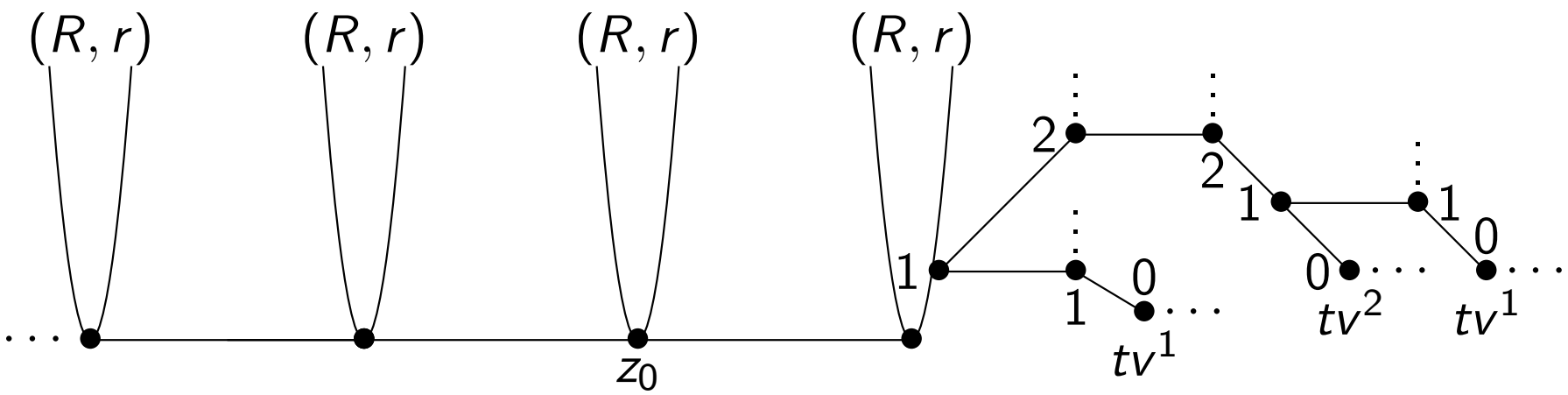


$\mathcal{T}_1(0)$ is similarly constructed on the following double ray (tp_1)



Target Vertex

target vertex tv^ℓ of height ℓ : a tree vertex $v \in \mathcal{T}_0(k)$ such that the label of the last consecutive pair $w, w' \in P_{z_0,v}$ is ℓ.



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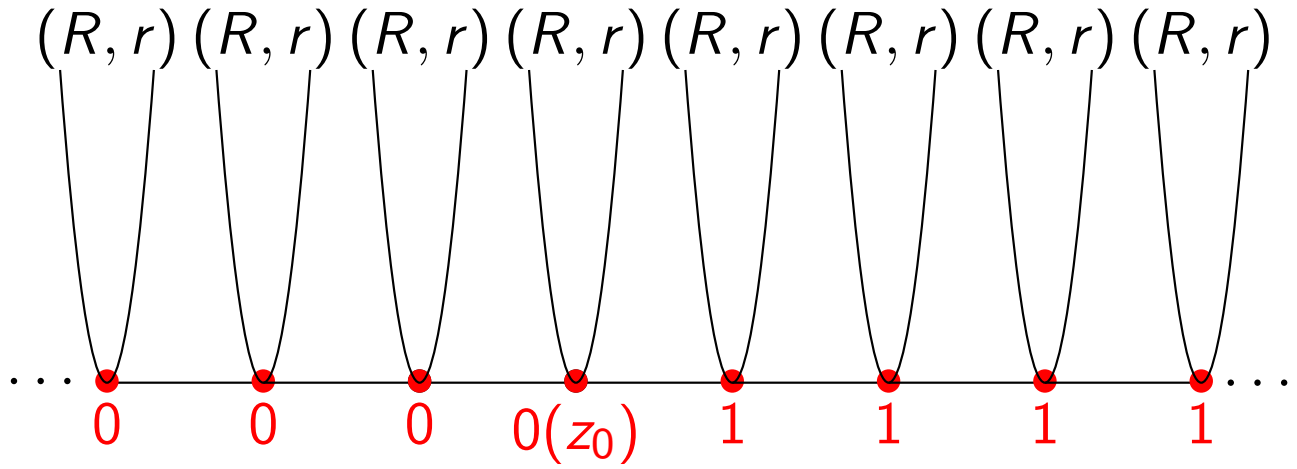

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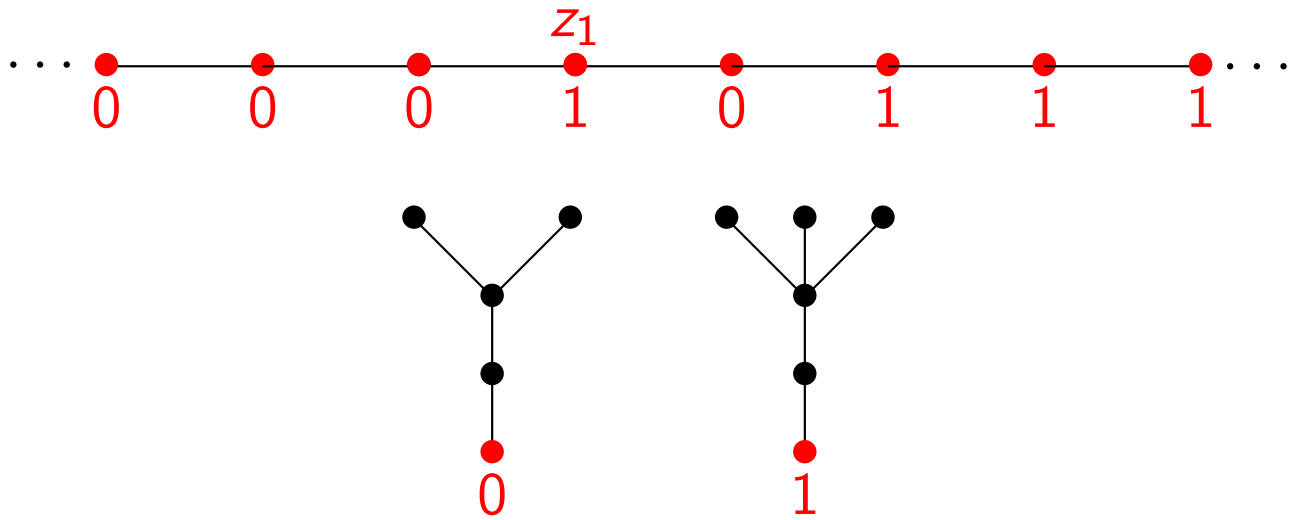

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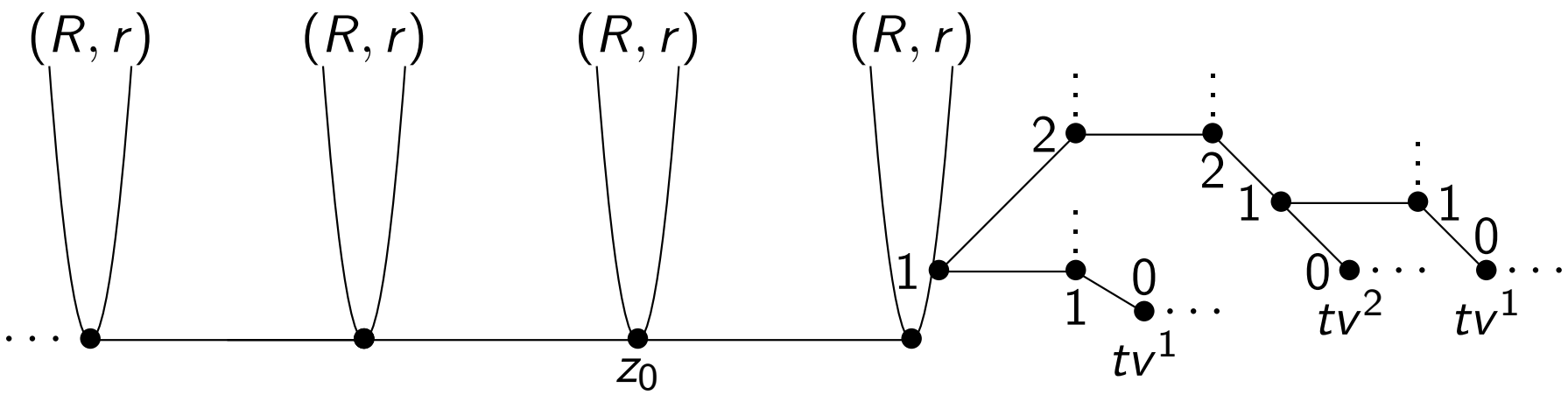
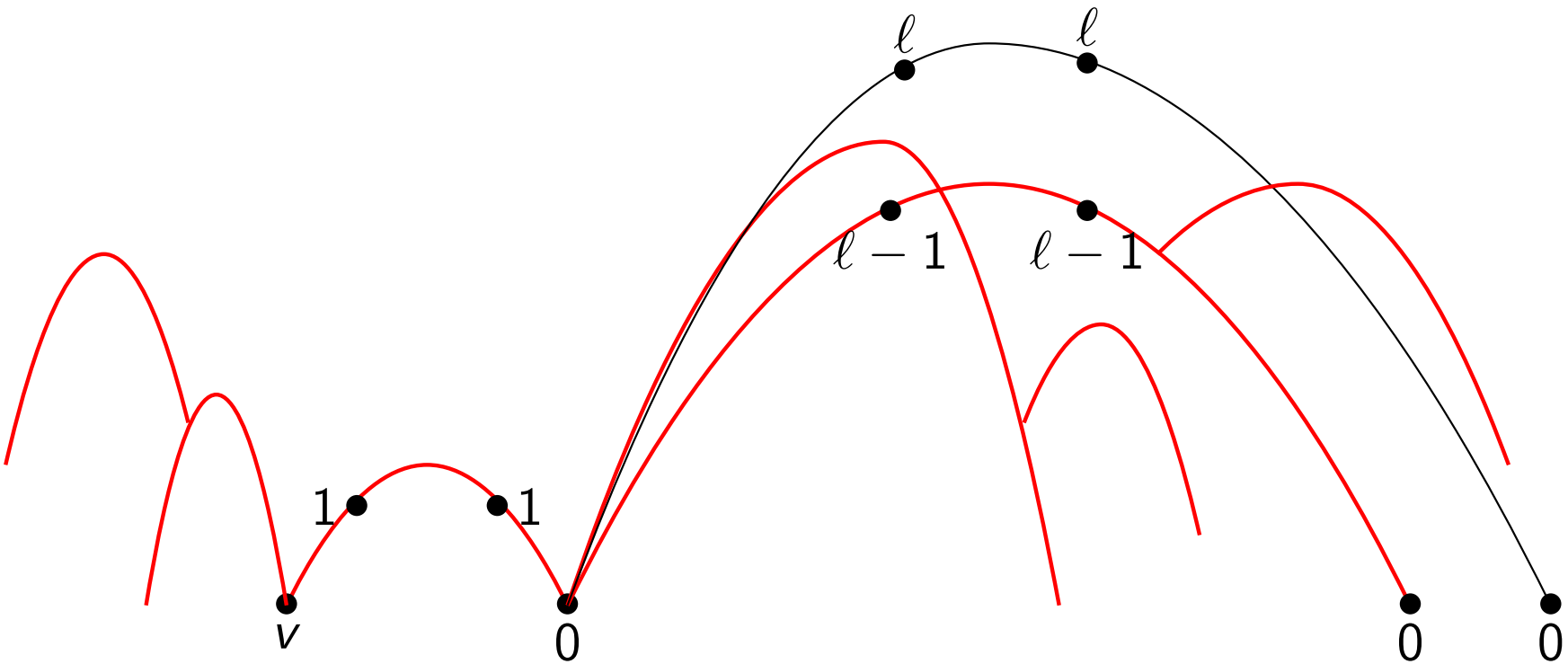


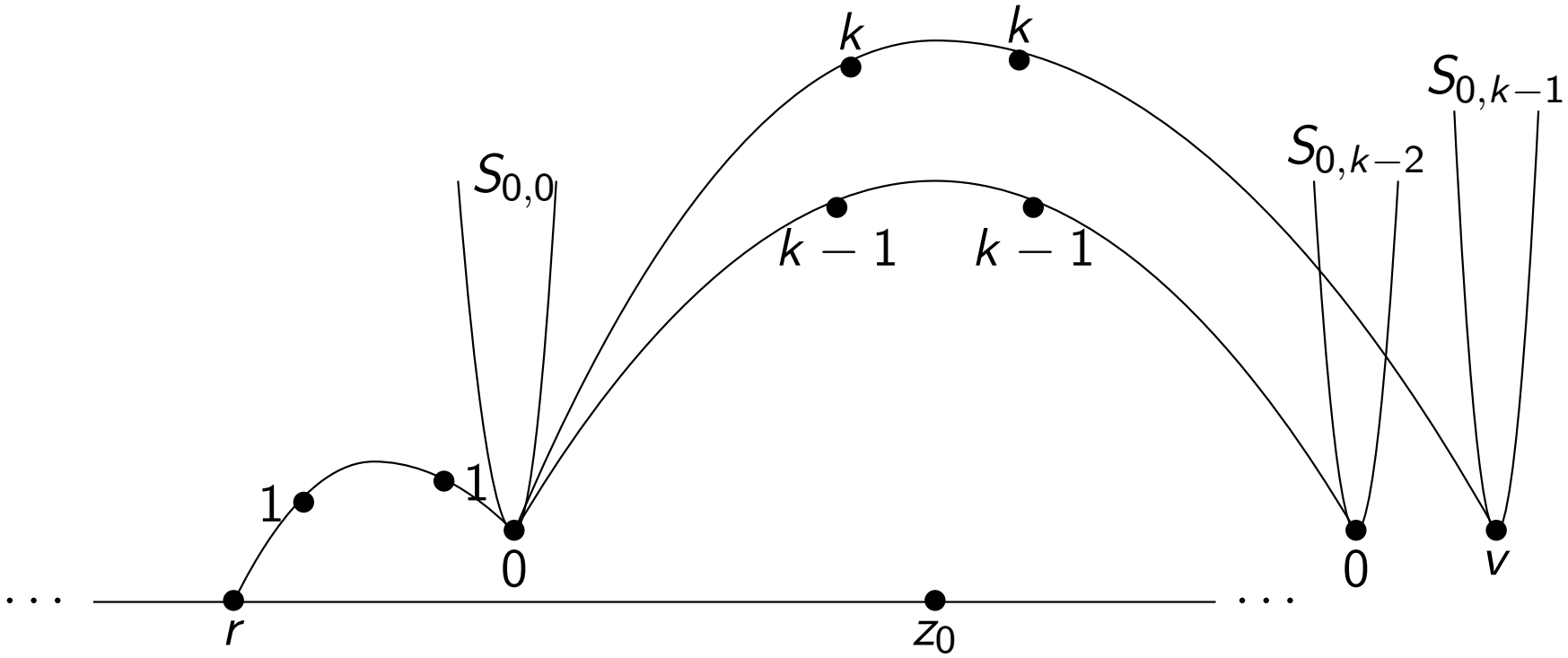
Crater

Let $v \in (R', r')$ be a target vertex of height ℓ.
$\mathcal{C}(v) = \{u \in \mathcal{T}_0(k) : ht_v(u) < \ell\}$.



$\mathcal{T}_0(k)$

Pick a target vertex $v \in \mathcal{T}_0(k-1)$ of height k



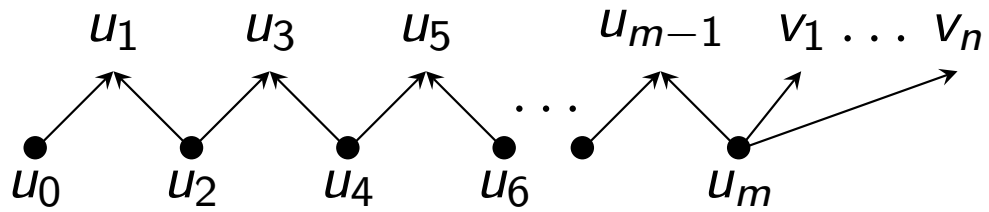
where $S_{0,0} \approx \mathcal{T}_0(0)$, $S_{0,k-2} \approx \mathcal{T}_0(k-2)$, $S_{0,k-1} \approx \mathcal{T}_0(k-1)$ and so on. More, $Sib(\mathcal{T}_0(k)) = \aleph_0$.

Main Property of $\mathcal{T}_0$ ($\mathcal{T}_1$)

Any non-isomorphic sibling of $\mathcal{T}_0(k)$ is almost equal to $\mathcal{T}_0(k)$, so has been embedded at some stage, the partial isomorphism extends to the whole structure. This implies that any sibling of $\mathcal{T}_0$ is isomorphic to either $\mathcal{T}_0$ or to $\mathcal{T}_1$

The following posets can be added to $\mathcal{T}_i$. Double rays in $\mathcal{T}_i$ can also be adapted by making them as infinite fences. Then, the siblings of each $\mathcal{T}_i$ are the same as trees (and relational structures) or as posets.

$PK(m, n)$



Conclusion

All conjectures of Bonato-Tardif, Tyomkyn and Thomassé are false.

Future Directions

Goal

Counting the number of siblings provides a good first insight into the siblings of a mathematical structure.
The real problem is to **fully understand** the **structure** of those **siblings**.

Thank You for Your Attention