# The Silence of the Powersets

## WQO-BQO: What is up?

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Joint work with Sergio Abriola, Simon Halfon,
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- Tight bounds for the ordinal invariants of $\mathcal{P}_{\text{fin}}(X)$.
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♣ **There will be ordinals**

And I will not give definitions.

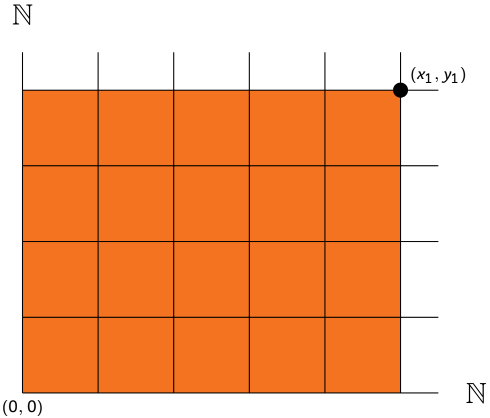
# Slow paced walk in the garden of ordinal invariants

## My favourite well-founded set

# It's $\mathbb{N} \times \mathbb{N}$ with the componentwise ordering

## Decreasing sequence $(x_i, y_i)_{i \in \mathbb{N}}$

For all $i \in \mathbb{N}$, $x_{i+1} \leq x_i$, $y_{i+1} \leq y_i$, and $(x_{i+1}, y_{i+1}) \neq (x_i, y_i)$.



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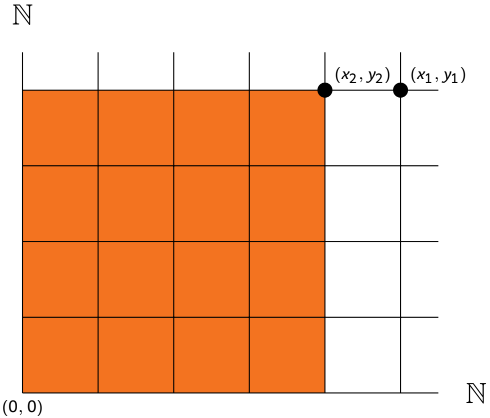

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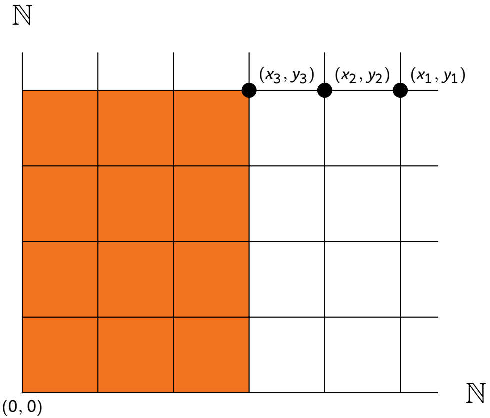

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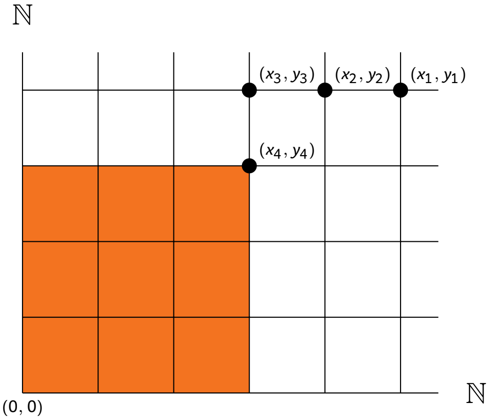

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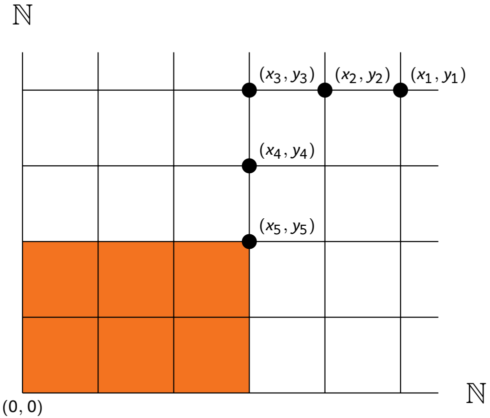

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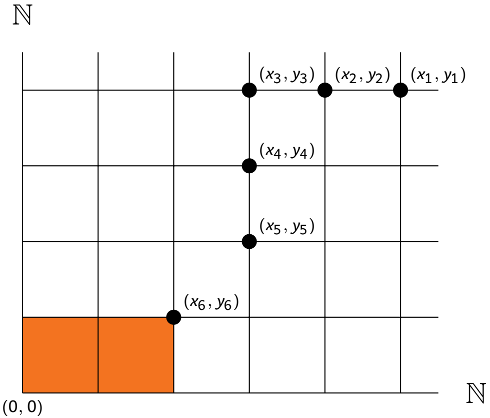

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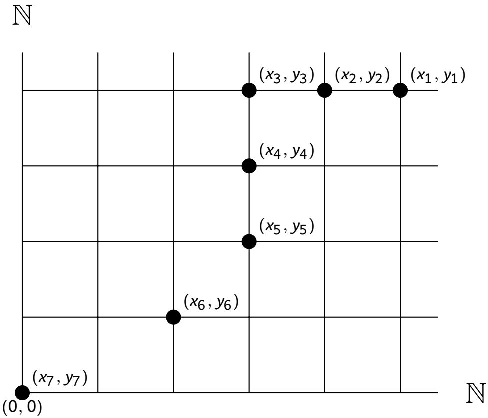

# How long can I play this game?

♣ **Definition via descent equations**

$$\mathsf{h}(S) \triangleq \sup\{\mathsf{h}(S{\downarrow_{<x}}) + 1 \colon x \in S\}$$

## ♣ Definition via descent equations

$$h(S) \triangleq \sup\{h(S{\downarrow_{<x}}) + 1 : x \in S\}$$

## Some examples

- $h(\mathbb{N}) = \omega$
- $h(\mathbb{N} \times \mathbb{N} \times \mathbb{N}) = \omega$
- $h(\mathbb{N} \sqcup \mathbb{N}) = \omega$
- $h(\mathbb{N} + \mathbb{N}) = \omega \cdot 2$

# Slow paced walk in the garden of ordinal invariants

## Well-quasi-orders and their ordinal invariants

# Variations

### ♣ Well quasi-orders

- No infinite bad sequence: $o(X)$ [de Jongh *and* Parikh, 1977]

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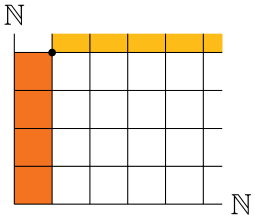

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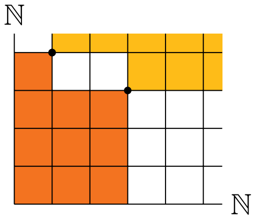

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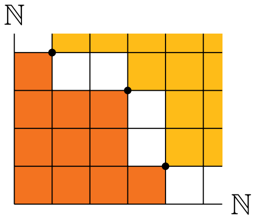

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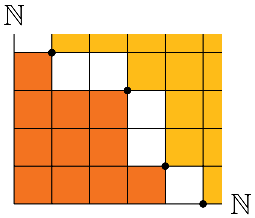

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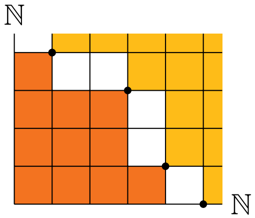


## Theorem: Kříž and Thomas [1990]

$w, h \leq o \leq w \otimes h$ (will never be defined)

# Applications

♦ **To control the runtime of algorithms**

- Schmitz and Schnoebelen [2011]
- Figueira, Figueira, Schmitz, and Schnoebelen [2011]
- Haddad, Schmitz, and Schnoebelen [2012]
- Abriola and Figueira [2014]
- Abriola, Figueira, and Senno [2015]
- Schmitz [2019]
- Balasubramanian [2020]

For instance, “length function theorems”.

# Slow paced walk in the garden of ordinal invariants

## How to train your ordinal invariant skills

# A plethora of nice compositional theorems

♣ de Jongh and Parikh [1977], Schmidt [1981], Abraham and Bonnet [1999], Van der Meeren, Rathjen, and Weiermann [2015], Džamonja, Schmitz, and Schnoebelen [2020]

| Space | M.O.T. | Height | Width |
|---|---|---|---|
| $\alpha$ | $\alpha$ | $\alpha$ | 1 |
| $A \sqcup B$ | $\oplus$ | max | $\oplus$ |
| $A + B$ | $+$ | $+$ | max |
| $A \times B$ | $\otimes$ | $\hat{\oplus}$ | ? |
| $A \cdot B$ | $\cdot$ | $\cdot$ | $\odot$ |
| $\mathrm{M}^{\diamond}(A)$ | $\omega^{\widehat{\mathrm{o}(A)}}$ | $\mathrm{h}^{\star}(A)$ | $\mathrm{o}(\mathrm{M}^{\diamond}(A))$ |
| $A^{<\omega}$ | $\omega^{\omega^{(\mathrm{o}(X)^{\pm})}}$ | $\mathrm{h}^{\star}(A)$ | $\mathrm{o}(A^{<\omega})$ |
| $\mathcal{P}_{\mathrm{fin}}(A)$ | ? | ? | ? |

Assuming $\mathrm{o}(A) = \omega^{\alpha} > 1$ to simplify...

# Pop Quizz

## Computing invariants

| Space | o | h | w |
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**Advertisement**

For topological spaces $(X, \tau)$, *an analogue of* $\circ$ exists and satisfies the *same* equations!

Breathe

# The powerset: *a* necessary evil

## But you might think it should not be

# The finitary powerset *as a* quotient

## Definition of the ordering

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**A function** letters: $X^{<\omega} \to \mathcal{P}_{\text{fin}}(X)$

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- letters is surejctive
- letters is monotone
- Tells us that $\mathcal{P}_{\mathsf{fin}}(X)$ is wqo...
- ... does not provide ordinal bounds!

## Powerset and lexicographic sum

$$\mathcal{P}_{\mathsf{fin}}(A + B) = \mathcal{P}_{\mathsf{fin}}(A) + (\mathcal{P}_{\mathsf{fin}}(B) \setminus \{\emptyset\})$$

# Powerset and usual operations

**Powerset and lexicographic sum**

$\mathcal{P}_{\text{fin}}(A + B) = \mathcal{P}_{\text{fin}}(A) + (\mathcal{P}_{\text{fin}}(B) \setminus \{\emptyset\})$

**Powerset and disjoint union**

$\mathcal{P}_{\text{fin}}(A \sqcup B) = \mathcal{P}_{\text{fin}}(A) \times \mathcal{P}_{\text{fin}}(B)$

# An example of the non-functionnality

## Recall...

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## ♦ Whoopsie...

| Space | o | h | w |
|---|---|---|---|
| $(\omega + \omega) \sqcup (\omega + \omega)$ | $\omega \cdot 4$ | $\omega \cdot 2$ | $2$ |
| $(\omega \sqcup \omega) + (\omega \sqcup \omega)$ | $\omega \cdot 4$ | $\omega \cdot 2$ | $2$ |
| $\mathcal{P}_{\text{fin}}((\omega + \omega) \sqcup (\omega + \omega))$ | $\omega^2 \cdot 4$ | $\omega^2 \cdot 3$ | $\omega \cdot 3$ |
| $\mathcal{P}_{\text{fin}}((\omega \sqcup \omega) + (\omega \sqcup \omega))$ | $\omega^2 \cdot 2$ | $\omega^2 \cdot 2$ | $\omega$ |

# The powerset: *a* necessary evil

## Powersets *a*nd maps

## Invariant preserving maps

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♣ **Reflections (extend relations)**

Whenever: $f(x) \leq_Y f(y) \implies x \leq_X y$.

# Preservation of ordinal invariants

## ♣ How to use these maps

| | $\mathrm{o}(X)$ | $\mathrm{h}(X)$ | $\mathrm{w}(X)$ |
|---|---|---|---|
| $A \leq_{\mathrm{st}} B$ | $\mathrm{o}(A) \leq \mathrm{o}(B)$ | $\mathrm{h}(A) \leq \mathrm{h}(B)$ | $\mathrm{w}(A) \leq \mathrm{w}(B)$ |
| $A \leq_{\mathrm{aug}} B$ | $\mathrm{o}(A) \geq \mathrm{o}(B)$ | | $\mathrm{w}(A) \geq \mathrm{w}(B)$ |
| $A \hookrightarrow B$ | $\mathrm{o}(A) \leq \mathrm{o}(B)$ | | $\mathrm{w}(A) \leq \mathrm{w}(B)$ |

# Preservation of ordinal invariants

♣ **How to use these maps**

| | o($X$) | h($X$) | w($X$) |
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♦ **The last line is for upper bounds!**

# The finitary powerset functor

**Lifting functions $f\colon X \to Y$.**

$\mathcal{P}_{\mathrm{fin}}(f)\colon \mathcal{P}_{\mathrm{fin}}(X) \to \mathcal{P}_{\mathrm{fin}}(Y)$ maps $S$ to $\{f(s)\colon s \in S\}$.

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1. $f$ is a substructure map $\implies \mathcal{P}_{\text{fin}}(f)$ too!
2. $f$ is an augmentation map $\implies \mathcal{P}_{\text{fin}}(f)$ too!
3. $f$ is a reflection map ... Noooo.

# Upper and lower bounds

## Trivial bounds

# What we will prove

- **What are the trivial bounds here?**

| Lower Bound | Invariant | Upper Bound |
|---|---|---|
| $1 + \mathrm{o}(X)$ | $\mathrm{o}(\mathcal{P}_{\mathrm{fin}}(X))$ | $2^{\mathrm{o}(X)}$ |
| $1 + \mathrm{h}(X)$ | $\mathrm{h}(\mathcal{P}_{\mathrm{fin}}(X))$ | $2^{\mathrm{h}(X)}$ |
| $2^{\mathrm{w}(X)}$ | $\mathrm{w}(\mathcal{P}_{\mathrm{fin}}(X))$ | none |

- **All are tight (and given for limit ordinals)**

# Using our maps

**Easy1:** $1 + o(X) \leq o(\mathcal{P}_{\text{fin}}(X))$**!**

$1 + o(X) \leq_{\text{st}} \mathcal{P}_{\text{fin}}(X)$ via $1 \mapsto \emptyset$, $x \mapsto \{x\}$.

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**Easy2:** $1 + \mathrm{h}(X) \leq \mathrm{h}(\mathcal{P}_{\mathrm{fin}}(X))$**!**

Same proof!

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♣ **Easy3:** $2^{\mathrm{w}(X)} \leq \mathrm{h}(\mathcal{P}_{\mathrm{fin}}(X))$**?**

...

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Same proof!

♣ **Easy3:** $2^{\mathrm{w}(X)} \leq \mathrm{h}(\mathcal{P}_{\mathrm{fin}}(X))$**?**

...

♦ **Easy3: unbounded**

Take $X_\alpha = \omega \sqcup \alpha$. Then, $\mathcal{P}_{\mathrm{fin}}(X) \equiv \omega \times (1 + \alpha)$ has width $\alpha$. However, $\mathrm{w}(X_\alpha) = 2$.

# Upper and lower bounds

## Upper bound on the maximal order type

# Proof scheme

## ♦ In general

- By induction on the invariants of $X$
- Start with the descent equation
  - When the ordinal invariant of $X$ is finite: "combinatorics"
  - When the ordinal invariant of $X$ is a successor: "decomposition"
  - When the ordinal invariant of $X$ is a limit: "continuity"

# A splitting lemma

## Splitting over the maximal order type

Let $(X, \leq)$ be a wqo such that $\mathrm{o}(X) = \alpha + \beta$ for some ordinals $\alpha, \beta$. Then there exists a partition $X = X_\alpha \uplus X_\beta$ such that: $\mathrm{o}(X_\alpha) = \alpha$, $\mathrm{o}(X_\beta) = \beta$, and $X_\alpha + X_\beta \geq_{\mathsf{aug}} X \geq_{\mathsf{aug}} X_\alpha \sqcup X_\beta$.

## Splitting over the maximal order type

Let $(X, \leq)$ be a wqo such that $\mathrm{o}(X) = \alpha + \beta$ for some ordinals $\alpha, \beta$. Then there exists a partition $X = X_\alpha \uplus X_\beta$ such that: $\mathrm{o}(X_\alpha) = \alpha$, $\mathrm{o}(X_\beta) = \beta$, and $X_\alpha + X_\beta \geq_{\mathrm{aug}} X \geq_{\mathrm{aug}} X_\alpha \sqcup X_\beta$.

# A splitting lemma

## Splitting over the maximal order type

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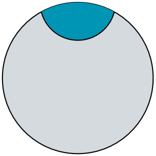
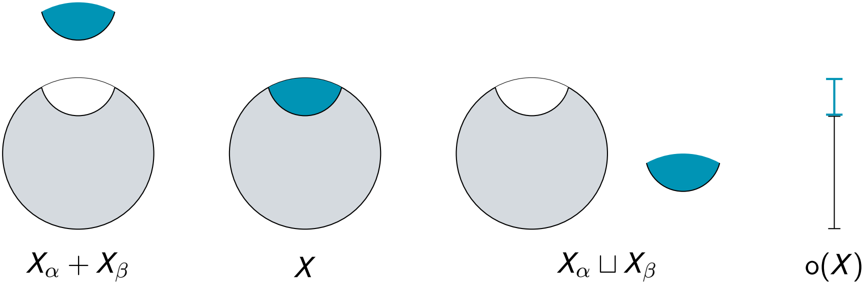



# A splitting lemma

## Splitting over the maximal order type

Let $(X, \leq)$ be a wqo such that $\mathrm{o}(X) = \alpha + \beta$ for some ordinals $\alpha, \beta$. Then there exists a partition $X = X_\alpha \uplus X_\beta$ such that: $\mathrm{o}(X_\alpha) = \alpha$, $\mathrm{o}(X_\beta) = \beta$, and $X_\alpha + X_\beta \geq_{\mathrm{aug}} X \geq_{\mathrm{aug}} X_\alpha \sqcup X_\beta$.



# The result

**Theorem**

For all wqo $X$, $\mathrm{o}(\mathcal{P}_{\mathrm{fin}}(X)) \leq 2^{\mathrm{o}(X)}$.

# The result

**Theorem**

For all wqo $X$, $o(\mathcal{P}_{\text{fin}}(X)) \leq 2^{o(X)}$.

$\alpha \in \mathbb{N}$: $|\mathcal{P}_{\text{fin}}(X)| \leq 2^\alpha$ ✓

$\alpha + n$: $o(\mathcal{P}_{\text{fin}}(X)) \leq o(\mathcal{P}_{\text{fin}}(X_\alpha)) \otimes o(\mathcal{P}_{\text{fin}}(X_n)) \leq 2^{\alpha+n}$ ✓

$\alpha$ **limit:**

$$\begin{aligned}
\mathcal{P}_{\text{fin}}(X)_{\not\supseteq_{\mathcal{H}} S} &= \{T \in \mathcal{P}_{\text{fin}}(X) \mid S \not\subseteq_{\mathcal{H}} T\} \\
&= \{T \in \mathcal{P}_{\text{fin}}(X) \mid \exists x \in S, T \in \mathcal{P}_{\text{fin}}(X_{\not\geq x})\} \\
&= \bigcup_{x \in S} \mathcal{P}_{\text{fin}}(X_{\not\geq x}) \\
&\geq_{\text{aug}} \bigsqcup_{x \in S} \mathcal{P}_{\text{fin}}(X_{\not\geq x})
\end{aligned}$$ ✓

# Upper and lower bounds

## Upper bound on the height

# A splitting lemma

**Splitting over the height**

Let $X$ be a wqo such that $\mathsf{h}(X) = \alpha + n$ with $\alpha$ limit and $n \geq 1$. There exists a partition $X = X_1 \uplus X_2$ such that $\mathsf{h}(X_1) = \alpha$, $\mathsf{h}(X_2) = n$, and $\mathcal{P}_{\mathsf{fin}}(X) \leq_{\mathsf{st}} \mathcal{P}_{\mathsf{fin}}(X_1) \cdot \mathcal{P}_{\mathsf{fin}}(X_2)$.

# A splitting lemma

**Splitting over the height**

Let $X$ be a wqo such that $\mathsf{h}(X) = \alpha + n$ with $\alpha$ limit and $n \geq 1$. There exists a partition $X = X_1 \uplus X_2$ such that $\mathsf{h}(X_1) = \alpha$, $\mathsf{h}(X_2) = n$, and $\mathcal{P}_{\mathsf{fin}}(X) \leq_{\mathsf{st}} \mathcal{P}_{\mathsf{fin}}(X_1) \cdot \mathcal{P}_{\mathsf{fin}}(X_2)$.

♣ **Roughly the same proof as before**

# The result

**Theorem**

For all wqo $X$, there exists $m \in \mathbb{N}$, $\mathsf{h}(\mathcal{P}_{\mathsf{fin}}(X)) \leq 2^{\mathsf{h}(X)+m}$.

Furthermore, $m = 0$ when $\mathsf{h}(X)$ is a limit ordinal.

### Theorem

For all wqo $X$, there exists $m \in \mathbb{N}$, $\mathsf{h}(\mathcal{P}_{\mathsf{fin}}(X)) \leq 2^{\mathsf{h}(X)+m}$.

Furthermore, $m = 0$ when $\mathsf{h}(X)$ is a limit ordinal.

- $\alpha \in \mathbb{N}$**:** $|\mathcal{P}_{\mathsf{fin}}(X)| \leq 2^{\alpha}$ ✓
- $\alpha + n$**:** $\mathsf{h}(\mathcal{P}_{\mathsf{fin}}(X)) \leq \mathsf{h}(\mathcal{P}_{\mathsf{fin}}(X_\alpha)) \cdot \mathsf{h}(\mathcal{P}_{\mathsf{fin}}(X_n)) \leq 2^{\alpha+m}$ ✓

$\alpha$ **limit:**

$$\begin{aligned} \mathcal{P}_{\mathsf{fin}}(X)_{\sqsubset_{\mathcal{H}} S} &= \{T \in \mathcal{P}_{\mathsf{fin}}(X) \colon T \sqsubset_{\mathcal{H}} S\} \\ &= \{T \in \mathcal{P}_{\mathsf{fin}}(X) \colon T \subseteq {\downarrow} S\} \\ &\leq_{\mathsf{st}} \mathcal{P}_{\mathsf{fin}}({\downarrow} S) \qquad ✓ \end{aligned}$$

# Upper and lower bounds

## Lower bound of the width

# A splitting lemma

**Splitting over the width**

Let $X$ be a wqo such that $\mathrm{w}(X) = \alpha + n$ with $n \geq 0$. There exists $Y$ of width $\alpha$ such that $Y \sqcup \Gamma_n \leq_{\mathrm{st}} X$.

# A splitting lemma

### Splitting over the width

Let $X$ be a wqo such that $\mathrm{w}(X) = \alpha + n$ with $n \geq 0$. There exists $Y$ of width $\alpha$ such that $Y \sqcup \Gamma_n \leq_{\mathrm{st}} X$.

### ♣ Remark for the continuity

If $\beta < \mathrm{w}(X)$, then $X$ has a substructure $Y$ and $\beta \leq \mathrm{w}(Y) < \mathrm{w}(X)$.

### ♦ Needed: Džamonja et al. [2020], Vialard [2022]

$\mathrm{w}(A \times B) \geq \mathrm{w}(B) \cdot \mathrm{o}(A)$ whenever $\mathrm{w}(B)$ is additively indecomposable.

# The result

**Theorem**

For all wqo $X$, either $\mathrm{w}(X) = k < \omega$ and $\mathrm{w}(\mathcal{P}_{\mathrm{fin}}(X)) \geq \mathrm{w}(\mathcal{P}_{\mathrm{fin}}(\Gamma_k))$, or $\mathrm{w}(\mathcal{P}_{\mathrm{fin}}(X)) \geq 2^{\mathrm{w}(X)}$.

# The result

**Theorem**

For all wqo $X$, either $\mathrm{w}(X) = k < \omega$ and $\mathrm{w}(\mathcal{P}_{\mathsf{fin}}(X)) \geq \mathrm{w}(\mathcal{P}_{\mathsf{fin}}(\Gamma_k))$, or $\mathrm{w}(\mathcal{P}_{\mathsf{fin}}(X)) \geq 2^{\mathrm{w}(X)}$.

$\alpha \in \mathbb{N}$**:** $\Gamma_k \leq_{\mathsf{st}} X$. ✓

$\alpha + n$**:** $2^{\alpha+n} \leq \mathrm{w}(\mathcal{P}_{\mathsf{fin}}(Y) \times \mathcal{P}_{\mathsf{fin}}(\Gamma_n)) \leq \mathrm{w}(\mathcal{P}_{\mathsf{fin}}(X))$. ✓

$\alpha$ **limit:** $\forall \beta < \alpha, \exists Y_\beta \leq_{\mathsf{st}} X, \beta \leq \mathrm{w}(Y) < \alpha$. ✓

# Blowing the powerset up

## Tightness of the easy bounds

# Lower bounds are tight!

Search $X_\alpha$ such that $f(X_\alpha) = \alpha$ and $f(\mathcal{P}_{\text{fin}}(X_\alpha))$ reaches the bound.

**♣ For the easy bounds**

$\mathcal{P}_{\text{fin}}(\alpha) = 1 + \alpha$.

**Reminder: we already did the width**

$X_\alpha = \omega \sqcup \alpha$

# Blowing the powerset up

## Exploding width *and* maximal order type

## Family 1

### Bad Family

- $\mathcal{H}_{\omega^0} = \Gamma_1$,
- $\mathcal{H}_{\omega^1} = \sum_{n\in\omega} \Gamma_n$,
- For $\alpha > 0$, $\mathcal{H}_{\omega^{\omega^\alpha}} = \sum_{\beta<\omega^\alpha} \mathcal{H}_{\omega^\beta}$
- And for $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ in CNF, $\mathcal{H}_{\omega^{\omega^{\alpha_1}+\cdots+\omega^{\alpha_n}}} = \mathcal{H}_{\omega^{\omega^{\alpha_1}}} \cdot \cdots \cdot \mathcal{H}_{\omega^{\omega^{\alpha_n}}}$ where $\cdot$ designates the direct (or lexicographic) product.

# Family 1

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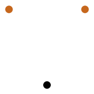
## ♣ Mini correction for simple sums

$\mathcal{H}_{\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}} = \sqcup_{i=1}^{n} \mathcal{H}_{\omega^{\alpha_i}}$.

# Draw *a* picture here

## Draw a picture here



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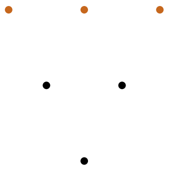

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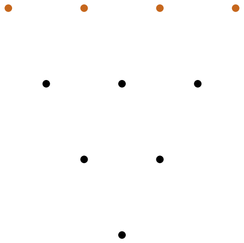

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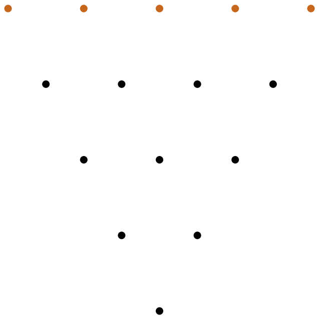

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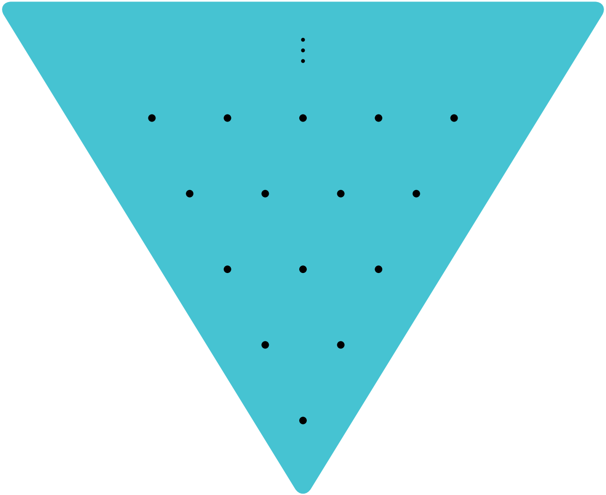

# Family 1

### Highly regular wqos

$\mathsf{h}(\mathcal{H}_{\omega^\alpha}) = \mathsf{o}(\mathcal{H}_{\omega^\alpha}) = \mathsf{w}(\mathcal{H}_{\omega^\alpha}) = \omega^\alpha$

♦ **Consequence:**

$\mathsf{o}(\mathcal{P}_{\mathsf{fin}}(\mathcal{H}_\alpha)) = \mathsf{w}(\mathcal{P}_{\mathsf{fin}}(\mathcal{H}_\alpha)) = 2^\alpha.$

# Blowing the powerset up

## Exploding height

### Simon's family

Define $\mathcal{P}_\alpha$ as follows:

- $\mathcal{P}_\omega = \omega$.
- If $\beta < \omega^\alpha$ limit, $\mathcal{P}_{\omega^\alpha+\beta} = (\mathcal{P}_{\omega^\alpha} + 1) \times \mathcal{P}_\beta$.
- $\mathcal{P}_{\omega^{\alpha+1}} = (\mathcal{P}_{\omega^\alpha} + 2)^{<\omega}$.
- If $\alpha$ limit, $\mathcal{P}_{\omega^\alpha} = \sum_{0<\beta<\alpha} \mathcal{P}_{\omega^\beta}$.

# Family 2

## Simon's family

Define $\mathcal{P}_\alpha$ as follows:

- $\mathcal{P}_\omega = \omega$.
- If $\beta < \omega^\alpha$ limit, $\mathcal{P}_{\omega^\alpha+\beta} = (\mathcal{P}_{\omega^\alpha} + 1) \times \mathcal{P}_\beta$.
- $\mathcal{P}_{\omega^{\alpha+1}} = (\mathcal{P}_{\omega^\alpha} + 2)^{<\omega}$.
- If $\alpha$ limit, $\mathcal{P}_{\omega^\alpha} = \sum_{0<\beta<\alpha} \mathcal{P}_{\omega^\beta}$.

## ♣ Mini correction for successors

$$\mathcal{P}_{\alpha+m} = \mathcal{P}_\alpha + \underbrace{\Gamma_2 + \cdots + \Gamma_2}_{m \text{ times}}.$$

# Family 2

♦ **Consequence: tight bounds**

$\mathsf{h}(\mathcal{P}_\alpha) = \alpha$ and $\mathsf{h}(\mathcal{P}_{\mathsf{fin}}(\mathcal{P}_\alpha)) = 2^\alpha$ when $\alpha$ is a limit ordinal.

# Blowing the powerset up

## A funny construction

# Can we make two parameters explode?

## Inspired by constructions from Hasegawa [2002]

- $\mathcal{W}_0 = \omega^{<\omega}$
- $\mathcal{W}_{\alpha+1} = \mathcal{W}_\alpha^{<\omega}$
- $\mathcal{W}_\alpha = \lim_{\beta<\alpha} \mathcal{W}_\beta$ (trust me: has a formal meaning)

# Can we make two parameters explode?

## Inspired by constructions from Hasegawa [2002]

- $\mathcal{W}_0 = \omega^{<\omega}$
- $\mathcal{W}_{\alpha+1} = \mathcal{W}_\alpha^{<\omega}$
- $\mathcal{W}_\alpha = \lim_{\beta<\alpha} \mathcal{W}_\beta$ (trust me: has a formal meaning)

## ♦ Invariants

$\mathsf{o}(\mathcal{W}_\alpha) = \mathsf{w}(\mathcal{W}_\alpha)$ goes to $\epsilon_0$, while $\mathsf{h}(\mathcal{W}_\alpha) = \omega$.

## ♣ Consequence

$\mathsf{h}(\mathcal{P}_{\mathsf{fin}}(\mathcal{W}_\alpha)) = \omega$ and $\mathsf{o}(\mathcal{P}_{\mathsf{fin}}(\mathcal{W}_\alpha))$ reaches the upper bound

# Final thoughts

## The yummy sandwich

# We often can compute the invariants

**♣ An easy case (used *a* lot in this talk!)**

If $\mathsf{w}(X) = \mathsf{o}(X)$, then it holds for $\mathsf{w}(\mathcal{P}_{\mathsf{fin}}(X)) = \mathsf{o}(\mathcal{P}_{\mathsf{fin}}(X)) = 2^{\mathsf{o}(X)}$.

# We often can compute the invariants

♣ **An easy case (used a lot in this talk!)**

If $\mathrm{w}(X) = \mathrm{o}(X)$, then it holds for $\mathrm{w}(\mathcal{P}_{\mathrm{fin}}(X)) = \mathrm{o}(\mathcal{P}_{\mathrm{fin}}(X)) = 2^{\mathrm{o}(X)}$.

♦ **Is this a generic case? ...stay tuned!**

$$X := \alpha \mid X \cdot X \mid X + X \mid X \sqcup X \mid X \times X$$
$$Y := X^{<\omega} \mid Y^{<\omega} \mid \mathcal{P}_{\mathrm{fin}}(Y) \mid \mathrm{M}^{\diamond}(X) \mid Y \sqcup Y$$

# Final thoughts

## The ideal partner for the height

# We often can compute the invariants

## Nice computation

If $X = \mathsf{Idl}(Y)$ then $\mathsf{h}(\mathcal{P}_{\mathsf{fin}}(X)) = \mathsf{h}(\mathcal{P}_{\mathsf{inf}}(Y)) = 1 + \mathsf{o}(Y)$.

## ♣ Commutation

- $\mathsf{Idl}(\alpha) = \alpha + 1$;
- $\mathsf{Idl}(X \sqcup Y) = \mathsf{Idl}(X) \sqcup \mathsf{Idl}(Y)$;
- $\mathsf{Idl}(X \times Y) = \mathsf{Idl}(X) \times \mathsf{Idl}(Y)$;
- $\mathsf{Idl}(\mathsf{M}^{\diamond}_{\mathsf{n}}(X)) = \mathsf{M}^{\diamond}_{\mathsf{n}}(\mathsf{Idl}(X))$.

# A map that powerset likes

**Condensation:** $f\colon A \to B$ **(**$A \leq_{\text{cond}} B$**)**

- Surjective $\qquad f(A) = B$
- Monotone $\qquad f(\downarrow x) \subseteq \downarrow f(x)$
- $b \leq_B f(y) \implies \exists x \leq_A y, b = f(x).$ $\qquad f(\downarrow x) \supseteq \downarrow f(x)$

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- $b \leq_B f(y) \implies \exists x \leq_A y, b = f(x).$ $\qquad f(\downarrow x) \supseteq \downarrow f(x)$

**Main lemma**

If $B \leq_{\text{cond}} A$, then $\mathsf{h}(B) \leq \mathsf{h}(A)$.

**♣ Condensing tuples**

The function $\iota\colon X^n \to \mathsf{M}^{\diamond}_{\mathsf{n}}(X)$ mapping $(x_1, \ldots, x_n)$ to the multiset $\{\!| x_1, \ldots, x_n |\!\}$ is a condensation.

## ♦ Take back home

- The reasonable bounds on $\mathcal{P}_{\mathsf{fin}}$ are tight
- Lexicographic sum was used to reach upper bounds

## What's next?

$$(\mathsf{Idl}\ \&\ \leq_{\mathsf{cond}}) \setminus \{+\} = \heartsuit \quad .$$

## ♣ Variations...

- The (usual) powerset has similar properties?
- Abstract proofs assuming $F(X)_{<S} \subseteq F(X_{<\mathsf{support}(S)})$?

Thank you!

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# Cheatsheet

$$\alpha \mathbin{\hat{\oplus}} \beta \triangleq \sup\{\alpha' \oplus \beta' \colon \alpha' < \alpha, \beta' < \beta\}$$

$$\alpha^{\pm} \triangleq \begin{cases} \alpha - 1 & \text{if } \alpha \text{ is finite,} \\ \alpha + 1 & \text{if } \alpha = \epsilon + n \text{ with } \epsilon \text{ an } \epsilon\text{-number and } n \in \mathbb{N}, \\ \alpha & \text{otherwise.} \end{cases}$$

$$\mathrm{h}^{\star}(A) \triangleq \begin{cases} \mathrm{h}(A) \text{ if } \mathrm{h}(A) & \text{is additively indecomposable} \geq \omega \\ h(A) & \text{is additively indecomposable} \geq \omega \\ \mathrm{h}(A) \cdot \omega & \text{otherwise.} \end{cases}$$

$$\alpha^{\circ} \triangleq \begin{cases} \alpha + 1 & \text{if } \alpha = \epsilon + n, \\ \alpha & \text{otherwise.} \end{cases}$$

# Cheatsheet

$$\widehat{\alpha} \triangleq \omega^{\alpha_1^\circ} + \cdots + \omega^{\alpha_n^\circ} \text{ when } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

$$\alpha \odot 0 = 0 \quad \alpha \odot (\beta + 1) = (\alpha \odot \beta) \oplus \alpha \quad \alpha \odot \lambda = \sup_{\gamma<\lambda}(\alpha \odot \gamma + 1)$$