The extended finite element method with integral matching

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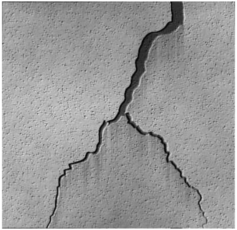
Fracture Mechanics and *FEM*

- Crack mechanisms in various industrial situations (aeronautics, civil engineering,...)
- *FEM* is the standard tool for numerical simulations in an industrial framework



- *FEM* has serious difficulties solving fracture problems ....

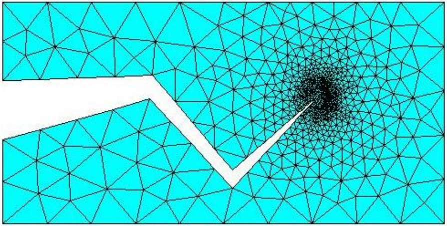
Fracture Mechanics and *FEM*

Presence of a crack =

Discontinuity + Singularity

FE mesh adapted to the geometry of the crack line

Local mesh refining near the crack tip



Fracture Mechanics and *FEM*

Presence of a crack =

Discontinuity	+	Singularity
FE mesh adapted to the geometry of the crack line		Local mesh refining near the crack tip

Limits of *FEM* in fracture mechanics

Crack propagation simulation incremental process

$\Longrightarrow$ Successive updates of the mesh :

- hight computational cost
- additional errors

Enriched *FEM*

To overcome the drawbacks of standard *FEM* :

Enriched finite element methods

Global displacement field

$$u_h = w_h + e_h$$

Enriched discrete space

$$V_h = W_h + E_h$$

- W_h is a standard *FE* space
- E_h is a space of enrichment functions

The enrichment procedure...

- uses informations about the solution
- is localized by means of a *unity partition method*

XFEM : eXtended Finite Element Method

[*Moës* et al., 1999]

The Aim of this work

- To analyse advantages and limits of the standard *XFEM* method
- A non-conformal *XFEM* type method (with integral matching)
- Numerical experiments : an optimal rate of convergence
- A mathematical error estimate

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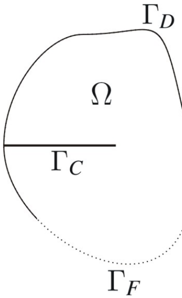
Cracked solid

Ω *cracked* plane domain

Isotropic linear elasticity

$$\begin{cases} \sigma(u) = D\,\varepsilon(u) & \text{on } \Omega \\ -\operatorname{div}\sigma(u) = g & \text{on } \Omega \\ u = d & \text{on } \Gamma_D \\ \sigma(u)n = f & \text{on } \Gamma_F \\ \sigma(u)n = 0 & \text{on } \Gamma_C \end{cases}$$



Variational formulation

Find u such that

$$\begin{cases} u \in V \\ a(u, v-u) = L(v-u) \quad \forall v \in V \end{cases}$$

where :

$$V = \{u : u \in \boldsymbol{H}^1(\Omega) \ : \ u = d \ \text{ on } \ \Gamma_D\}$$

and

$$\begin{aligned} a(u,v) &= \int_\Omega \sigma(u) : \varepsilon(v)\, dx \\ \sigma(u) &= \lambda\, tr\varepsilon(u)\, I + 2\mu\, \varepsilon(u) \\ \boldsymbol{X}(\Omega) &= \{v = (v_i) \ : \ v_i \in X(\Omega; \mathbb{R}^2)\} \end{aligned}$$

Asymptotic displacement at the crack tip

The displacement field :

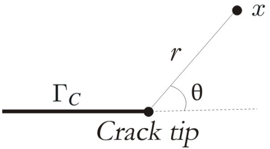
$$\begin{cases} u = u_R + u_S \\ u_S = K_I \, u_I + K_{II} \, u_{II} \end{cases}$$

where :

$K_I, \quad K_{II} \in \mathbb{R}$ (the *stress intensity factors*)

$u_R \in \boldsymbol{H}^{2+\epsilon}(\Omega)$ for a fixed $\epsilon > 0$ (the regular part)

$u_S \in \boldsymbol{H}^{3/2-\eta}(\Omega)$ for a fixed $\eta > 0$ (the singular part)



Asymptotic displacement at the crack tip

Opening mode and shear mode :

$$u_I = \frac{1}{E}\sqrt{\frac{r}{2\pi}}(1+\nu)\begin{pmatrix} \cos\frac{\theta}{2}(\delta - \cos\theta) \\ \sin\frac{\theta}{2}(\delta - \cos\theta) \end{pmatrix}$$

$$u_{II} = \frac{1}{E}\sqrt{\frac{r}{2\pi}}(1+\nu)\begin{pmatrix} \sin\frac{\theta}{2}(\delta + 2 + \cos\theta) \\ \sin\frac{\theta}{2}(\delta - 2 + \cos\theta) \end{pmatrix}$$

where :

$$\delta = 3 - 4\nu \quad \text{in plane stresses}$$

[*Grisvard*, 1992]

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XFEM

XFEM : eXtended Finite Element Method

[*Moës* et al., 1999]

- A *FE* mesh independent of the crack
- An enrichment using a Heaviside function to capture the discontinuity
- An enrichment with non-smooth functions generating the asymptotic displacement at the crack tip
- The enrichment of the *FE* basis is localized using shape functions

Using *XFEM*...

$\Longrightarrow$ No longer remeshing or local refining

$\Longrightarrow$ Reduced computational time, improved accuracy

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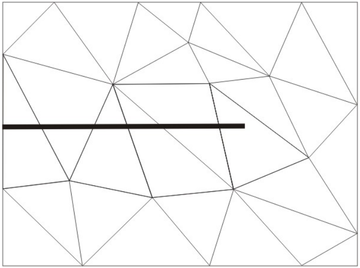
Enrichment strategy

- Mesh independent of the crack geometry



- Enrichment with a jump fct :

$$H(x) = \begin{cases} +1 & \text{above } \Gamma_C \\ -1 & \text{below} \end{cases}$$

- Enrichment with some singular functions :

$$F = \{F_j(x)\}_j = \{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\}$$

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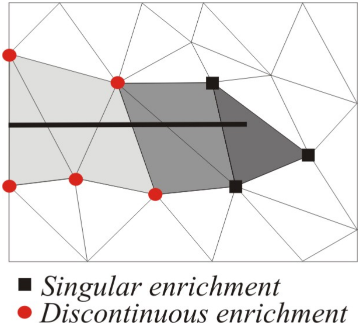

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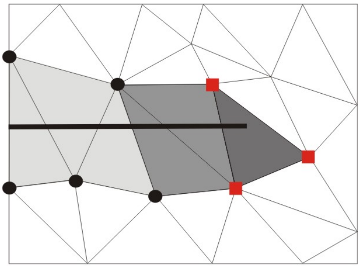

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■ *Singular enrichment*
● *Discontinuous enrichment*

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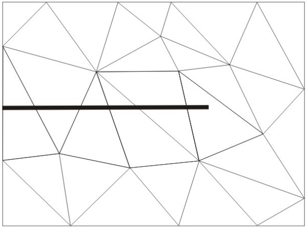
The *XFEM* problem

- The *XFEM* space

$$V_h = \left\{ v_h = \sum_{i\in I} a_i \, \varphi_i + \sum_{i\in I_H} b_i \, H \, \psi_i + \sum_{i\in I_F} \sum_j c_{ij} \, F_j \, \psi_i \ : \ a_i, b_i, c_{ij} \in \mathbb{R}^2 \right\}$$

φ_i : P_k *FE* basis functions

ψ_i : P_1 shape functions (*partition of unity*)



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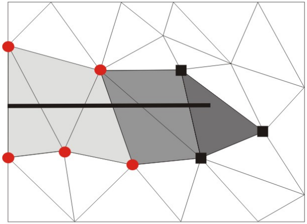

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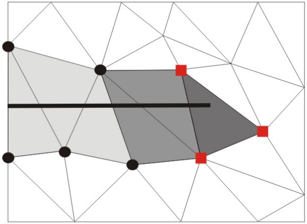

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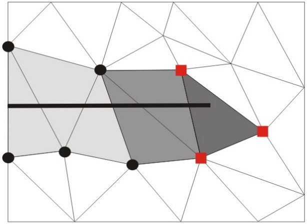
φ_i : P_k *FE* basis functions

ψ_i : P_1 shape functions (*partition of unity*)

- Discrete variational formulation

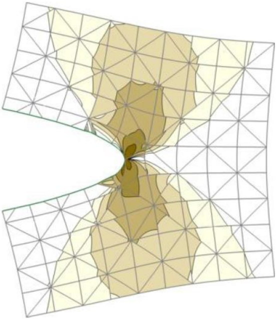
$$\begin{cases} \text{Find } \ u_h \in V_h \ \text{ such that} \\ a(u_h, v_h - u_h) = L(v_h - u_h) \quad \forall v_h \in V_h \end{cases}$$



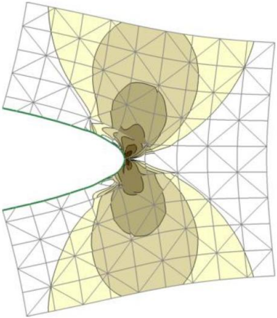
■ *Singular enrichment*
● *Discontinuous enrichment*

Convergence of standard *XFEM*

Nonhomogeneous Dirichlet condition (Mode I exact solution).

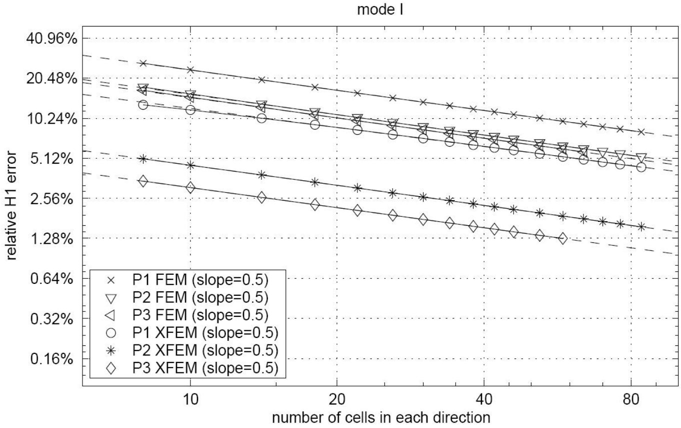


Von Mises, $P2$



Von Mises, $P3$

Convergence of standard *XFEM*



Convergence of standard *XFEM*

- The *XFEM* energy norm error is lower than the *FEM* error
- The *XFEM* convergence is in $\sqrt{h}$ for P_1 finite elements

  $\Longrightarrow$ The rate of convergence is not improved with respect to *FEM* [*Stazi* et al., 2003]
- *XFEM* is localy non-unisolvent

  (there exists two linear relations between the different $\psi_i\ F_j$)

Orientation

The enrichment area vanishes when $h \to 0$.

$\Longrightarrow$ Enrich all the d.o.f. in some area independent of h around the crack tip

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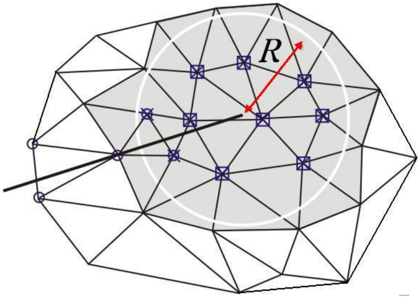
Surface enrichment

- *XFEM* with fixed enrichment area

  I_F is the set of the nodes lying in a surface S independent of h, say $S = B(x_0, R)$

$$V_h = \left\{ v_h = \sum_{i \in I} a_i \, \varphi_i + \sum_{i \in I_H} b_i \, H\varphi_i + \sum_{i \in I_F} \sum_j c_{ij} \, F_j \, \psi_i \right\}$$



Surface enrichment

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Numerical tests

- ▷ Optimal convergence
- ▷ High computational cost
- ▷ Non-unisolvence $\longrightarrow$ ill-conditionned system

[*Béchet* et al., 2005] [*Laborde* et al., 2005]

Globalized enrichment

- *XFEM* with a gathering of the d.o.f.

$$c_{ij} = c_j \qquad \sum_{i \in I_F} \psi_i = \chi$$

$$V_h = \left\{ v_h = \sum_{i \in I} a_i \, \varphi_i + \sum_{i \in I_H} b_i \, H \, \varphi_i + \sum_j c_j \, F_j \, \chi \right\}$$

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Numerical tests

- ▷ Negligible additional cost
- ▷ Improved condition number
- ▷ The rate of convergence decreases by $1/2$

[*Laborde, Pommier, Renard & Salaün*, 2005]

Cutoff *XFEM*

- *XFEM* with a cutoff function

$$\chi \text{ is a } C^2\text{- function s. t. : } \begin{cases} \chi(r) = 1 & \text{if } \ r < R_0, \\ 0 < \chi(r) < 1 & \text{if } \ R_0 < r < R_1, \\ \chi(r) = 0 & \text{if } \ R_1 < r. \end{cases}$$

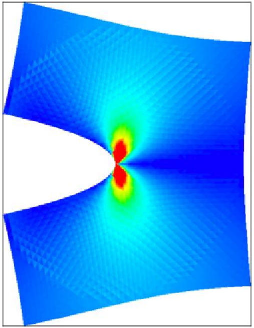
$$\mathcal{V}_h = \left\{ v_h : v_h = \sum_{i \in I} a_i \, \varphi_i + \sum_{i \in I_H} b_i \, H \, \varphi_i + \sum_j c_j \, F_j \, \chi \right\}$$

A mathematical result of optimal error estimate : [*Chahine* et al., 2006]

Numerical simulations

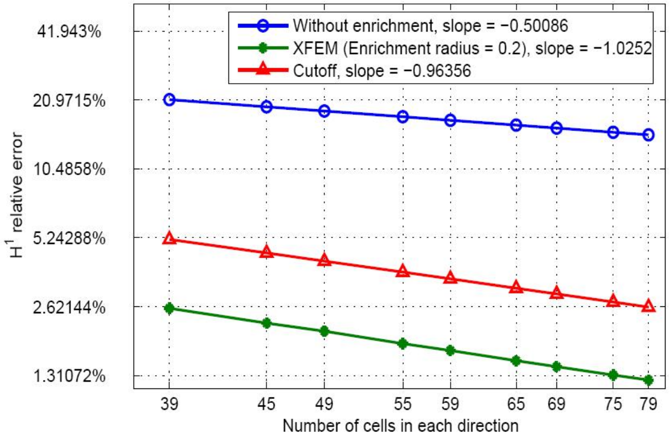
A regular mesh

Non-homogenuous boundary Dirichlet conditions (Mode I exact solution)

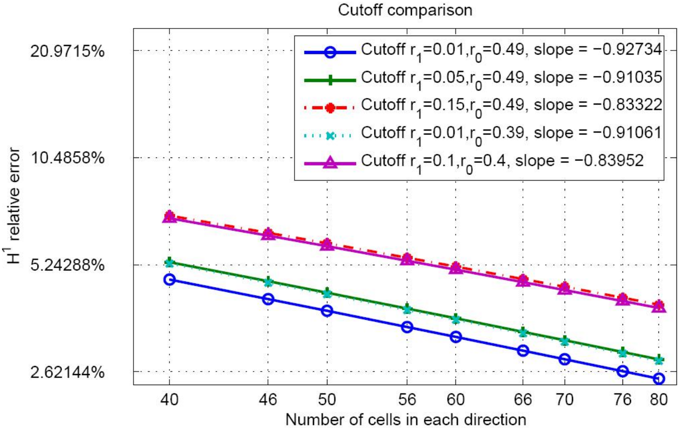


Von Mises

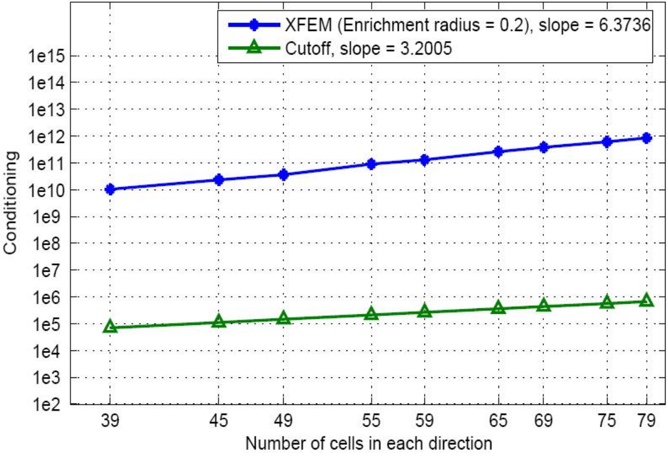
Convergence curves



Different cutoff functions



Condition number



[*Chahine et al.*, 2006]

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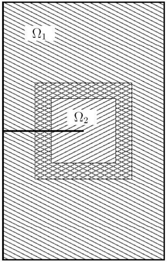
From *PUFEM* to a non-conformal *XFEM*

- *PUFEM*
  - ▷ Let Ω_1 and Ω_2 be two overlapping subdomains of Ω
  - ▷ Ω_2 is globaly enriched with the singular functions
  - ▷ A partition of unity (α_1, α_2) is associated to (Ω_1, Ω_2)

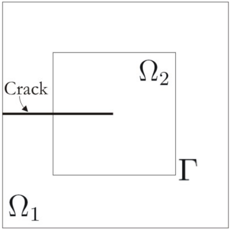


$$V_h = \{v_h = \alpha_1\, v_{h1} + \alpha_2\, v_{h2}\ :\ v_{h1} \in V_{h1},\ v_{h2} \in V_{h2}\}$$

- *Transition layer* : The width of $\Omega_1 \cap \Omega_2$ does not affect the (optimal) error estimate $\Longrightarrow$ Remove the transition layer and use a matching condition [*Laborde et al.*, 2005]

Hybrid formulation

$$\begin{cases} u = (u_1, u_2) \in \mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2, \ \lambda \in \mathcal{W} \\ a(u,v) = L(v) + \displaystyle\int_\Gamma \lambda \cdot [v] \quad \forall v \in \mathcal{V} \\ \displaystyle\int_\Gamma \mu \cdot [u] = 0 \quad \forall \mu \in \mathcal{W}, \end{cases}$$



$$a(u,v) = \sum_{k=1}^{2} a_k(u_k, v_k) = \sum_{k=1}^{2} \int_{\Omega_k} D\varepsilon(u_k) : \varepsilon(v_k) dx$$

$[u]$ is the *jump* of the displacement u at the interface Γ

$$\mathcal{V}_1 = \left\{ v_1 \in \boldsymbol{H}^1(\Omega_1) \ : \ v_1 = 0 \ \text{ on } \Gamma_D \right\}, \quad \mathcal{V}_2 = \boldsymbol{H}^1(\Omega_2)$$

$$\mathcal{W} = (\boldsymbol{H}^{1/2}(\Gamma))' = \boldsymbol{H}_{00}^{-1/2}(\Gamma)$$

Existence and uniqueness

Lemma

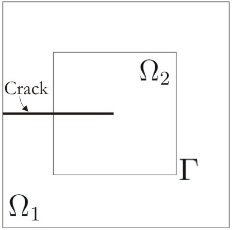
There exists a unique $(u, \lambda) \in \mathcal{V} \times \mathcal{W}$ *solution to the elasticity problem with integral matching .*
Moreover, the displacement field u *is solution of the global elasticity problem over* Ω *and the multiplier* λ *satisfies*

$$\lambda = \sigma(u)n \ \ on\ \Gamma,$$

where n *denotes the unit normal to* Γ *and* $\sigma(u) = D\varepsilon(u)$.

Discrete formulation

$$\begin{cases} u_h = (u_{h1}, u_{h2}) \in \mathcal{V}_h = \mathcal{V}_{1h} \times \mathcal{V}_{2h},\ \lambda_h \in \mathcal{W}_h \\ a(u_h, v_h) = L(v_h) + b(v_h, \lambda_h) \quad \forall v^h \in \mathcal{V}_h \\ b(u_h, \mu_h) = 0 \quad \forall \mu_h \in \mathcal{W}_h \end{cases}$$



$$\mathcal{V}_{h1} = \left\{ v_{h1} = \sum_{i \in I(\Omega_1)} a_i\, \varphi_i + \sum_{i \in I_H(\Omega_1)} b_i\, H\, \varphi_i \,:\; a_i, b_i \in \mathbb{R}^2 \right\}$$

$$\mathcal{V}_{h2} = \left\{ v_{h2} = \sum_{i \in I(\Omega_2)} a_i\, \varphi_i + \sum_{i \in I_H(\Omega_2)} b_i\, H\varphi_i + \sum_j c_j\, F_j \,:\; a_i, b_i, c_j \in \mathbb{R}^2 \right\}.$$

$$\mathcal{W}_h = \{ \mu_h \in \boldsymbol{C}^0(\overline{\Gamma}) \;:\; \mu_h|_S \in \boldsymbol{P}_1,\ \forall S \in \mathcal{S}_h \} \quad \text{where} \quad \mathcal{S}^h \text{ subdivision of } \overline{\Gamma}.$$

Discrete formulation

Remark – The definition of $\mathcal{W}_h$

$$\mathcal{W}_h = \{\mu_h \in \boldsymbol{C}^0(\overline{\Gamma}) \ : \ \mu_h|_S \in \boldsymbol{P}_1, \ \forall S \in \mathcal{S}_h\}$$

does not contain any discontinuous enrichment
(so the discrete multipliers are continuous across the crack) !

Error estimate

Theorem

Let (u, λ) *be the solution to the integral matching continuous problem such that*

$$u_R = u - u_S \in \boldsymbol{H}^{2+\epsilon}(\Omega) \;\; \textit{and} \;\; \lambda \in \boldsymbol{H}^{1/2}(\Gamma),$$

then the solution (u_h, λ_h) *to the discrete hybrid problem satisfies*

$$\|u - u_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \leq Ch\left(\|u\|_{2+\epsilon,\Omega} + \|\lambda\|_{1/2,\Gamma}\right).$$

→ *Bypass the proof*

Proof

Steps of the proof

- An abstract error estimate : $\|u-u_h\|^2_{1,\Omega}+\|\lambda-\lambda_h\|^2_{-1/2,\Gamma}\leq$

$$\leq C\left\{\inf_{v_h\in\mathcal{V}_h}\|u-v_h\|^2_{1,\Omega}+\inf_{\mu_h\in\mathcal{W}_h}\|\lambda-\mu_h\|^2_{-1/2,\Gamma}\right\}$$

- Approximation of the (discontinuous) exact multiplplier :

$$\inf_{\varphi_h\in Y_h}\|\varphi-\varphi_h\|_{-1/2,\Gamma}\leq Ch\|\varphi\|_{1/2,\Gamma}$$

$$Y_h=\{\varphi_h\in C^0(\overline{\Gamma})\ :\ \varphi_h|_S\in P_1\ \forall S\in\mathcal{S}_h\}$$

- *XFEM* interpolation of the displacement field : [*Chahine* et al., 2007]

Proof

Steps of the proof

- An abstract error estimate : $\|u - u_h\|^2_{1,\Omega} + \|\lambda - \lambda_h\|^2_{-1/2,\Gamma} \leq$

$$\leq C \left\{ \inf_{v_h \in \mathcal{V}_h} \|u - v_h\|^2_{1,\Omega} + \inf_{\mu_h \in \mathcal{W}_h} \|\lambda - \mu_h\|^2_{-1/2,\Gamma} \right\}$$

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- *XFEM* interpolation of the displacement field : [*Chahine* et al., 2007]

Proof

Steps of the proof

- An abstract error estimate : $\|u - u_h\|^2_{1,\Omega} + \|\lambda - \lambda_h\|^2_{-1/2,\Gamma} \leq$

$$\leq C \left\{ \inf_{v_h \in \mathcal{V}_h} \|u - v_h\|^2_{1,\Omega} + \inf_{\mu_h \in \mathcal{W}_h} \|\lambda - \mu_h\|^2_{-1/2,\Gamma} \right\}$$

- Approximation of the (discontinuous) exact multipliplier :

$$\inf_{\varphi_h \in Y_h} \|\varphi - \varphi_h\|_{-1/2,\Gamma} \leq Ch\|\varphi\|_{1/2,\Gamma}$$

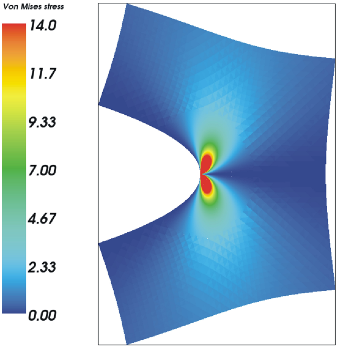
$$Y_h = \left\{ \varphi_h \in C^0(\overline{\Gamma}) \ : \ \varphi_h|_S \in P_1 \ \forall S \in \mathcal{S}_h \right\}$$

- *XFEM* interpolation of the displacement field : [*Chahine* et al., 2007]

Numerical simulations

A regular mesh

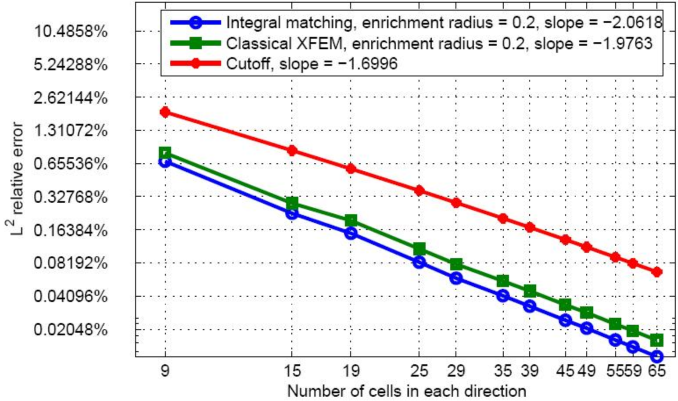
Non-homogenuous boundary Dirichlet conditions (Mode I exact solution)

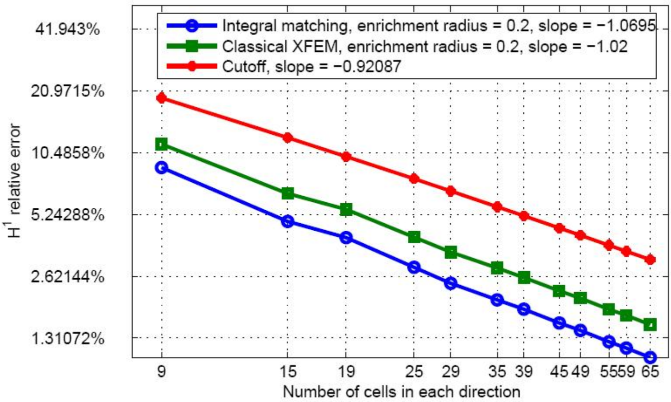


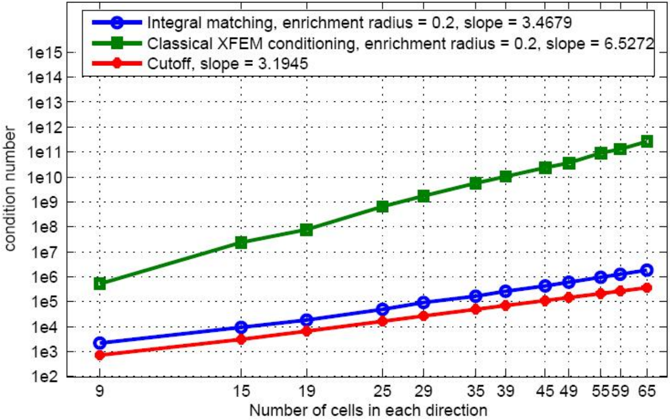
Von Mises

Number of degrees of freedom

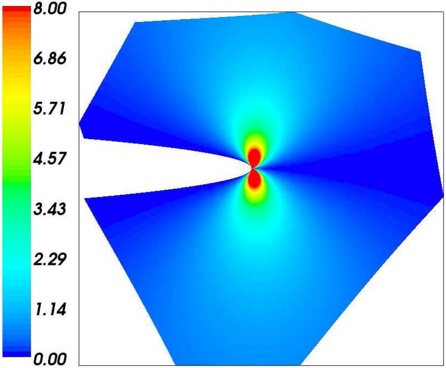
Number of cells in each direction	FEM	XFEM surface enrichment	XFEM cut-off	XFEM integral matching
40	3402	4962	3410	3508
60	7508	11014	7510	7656
80	13202	19578	13210	13404


Integral matching, enrichment radius = 0.2, slope = −2.0618
Classical XFEM, enrichment radius = 0.2, slope = −1.9763
Cutoff, slope = −1.6996
10.4858%
5.24288%
2.62144%
1.31072%
0.65536%
0.32768%
0.16384%
0.08192%
0.04096%
0.02048%
L2 relative error
9
15
19
25
29
35
39
45
49
55
59
65
Number of cells in each direction


Integral matching, enrichment radius = 0.2, slope = −1.0695
Classical XFEM, enrichment radius = 0.2, slope = −1.02
Cutoff, slope = −0.92087
41.943%
20.9715%
10.4858%
5.24288%
2.62144%
1.31072%
H^1 relative error
9
15
19
25
29
35
39
45
49
55
59
65
Number of cells in each direction


Integral matching, enrichment radius = 0.2, slope = 3.4679
Classical XFEM conditioning, enrichment radius = 0.2, slope = 6.5272
Cutoff, slope = 3.1945
condition number
1e15
1e14
1e13
1e12
1e11
1e10
1e9
1e8
1e7
1e6
1e5
1e4
1e3
1e2
9
15
19
25
29
35
39
45
49
55
59
65
Number of cells in each direction

Non-structured mesh



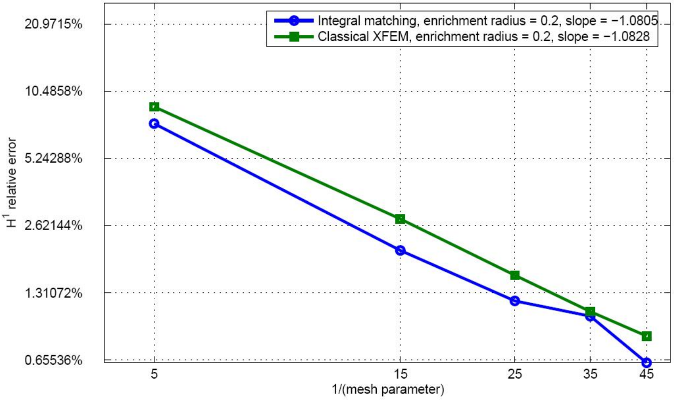
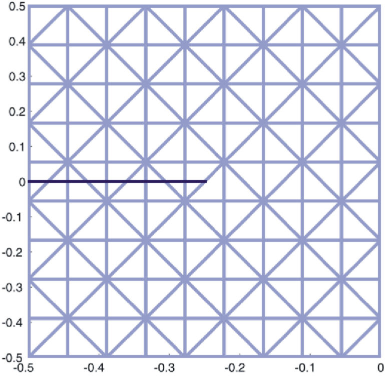

Integral matching, enrichment radius = 0.2, slope = −1.0805
Classical XFEM, enrichment radius = 0.2, slope = −1.0828
20.9715%
10.4858%
5.24288%
2.62144%
1.31072%
0.65536%
H^1 relative error
5
15
25
35
45
1/(mesh parameter)

Table of contents

Conclusions

- Essential ingredients of *XFEM* methods
  - Numerical integration
  - Level sets
- Some extensions of the method, applications and challenges
  - 3D fracture problems, dynamic crack propagation, geometric and constitutive nonlinearities
  - Adaptative mesh techniques
  - Implementation in industrial finite element codes
  - Plates and shells [*J. Lasry*]
  - More complex singularities and approximate enrichement [*E. Chahine*]



→ *Cutoff* → *Integral matching* → *Spider mode I* → *Spider bimaterial* → *RBXFEM*