On the active control of crack growth in elastic media

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1. Introduction - Problem statement

$\Omega \subset \mathbb{R}^2$: elastic structure, fixed on $\Gamma_0 \subset \partial\Omega$, submitted to a normal load such that

$$\boldsymbol{f}\mathcal{X}_{\Gamma_f} + \boldsymbol{h}\mathcal{X}_{\Gamma_h} \in (L^2(\partial\Omega\backslash\Gamma_0))^2$$

where

$$\boldsymbol{f} \in (L^2(\Gamma_f))^2, \quad \boldsymbol{h} \in (L^2(\Gamma_h))^2, \quad \Gamma_f, \Gamma_h \subset \partial\Omega\backslash\Gamma_0, \quad \Gamma_f \cap \Gamma_h = \emptyset.$$

The domain Ω contains a crack γ of extremity $\boldsymbol{F}$, unloaded and free. The corresponding displacement field $\boldsymbol{u} = (u_1, u_2)$ lies in the convex set

$$\mathbf{K} = \{\boldsymbol{v} \in (H^1_{\Gamma_0}(\Omega))^2, [\boldsymbol{v}\cdot\boldsymbol{\nu}] \leq 0 \text{ on } \gamma\} \quad \text{where} \quad H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0\},$$

and minimizes at equilibrium the energy $J(., \gamma)$ on $\mathbf{K}$:

$$J(\boldsymbol{v}, \gamma) = \frac{1}{2}\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{v})\,\nabla\boldsymbol{v})dx - \int_{\Gamma_f} \boldsymbol{f}\cdot\boldsymbol{v}\,d\sigma - \int_{\Gamma_h} \boldsymbol{h}\cdot\boldsymbol{v}\,d\sigma. \tag{1}$$

The field $\boldsymbol{u}$ solving (1) satisfies (2):

$$\begin{cases} -\,\boldsymbol{div}\ \boldsymbol{\sigma}(\boldsymbol{u}) = 0, \qquad \boldsymbol{\sigma}(\boldsymbol{u}) \equiv \mathbb{A}\,\boldsymbol{\varepsilon}(\boldsymbol{u}), \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) \equiv (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)/2 \quad \text{in} \quad \Omega, \\ \boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_0 \subset \partial\Omega, \qquad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{f}\mathcal{X}_{\Gamma_f} + \boldsymbol{h}\mathcal{X}_{\Gamma_h} \quad \text{on} \quad \partial\Omega\backslash\Gamma_0, \\ [u_\nu] \equiv [\boldsymbol{u}\cdot\boldsymbol{\nu}] \leq 0, \quad \sigma_\nu \equiv (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu})\cdot\boldsymbol{\nu} \leq 0, \quad [u_\nu]\sigma_\nu = 0, \quad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu} = 0 \quad \text{on} \quad \gamma. \end{cases} \tag{2}$$

Notation:

$$\Gamma = \partial\Omega \backslash (\Gamma_0 \cup \Gamma_f \cup \gamma).$$

In order to reduce the energy release rate g (see hereafter), one may act on the boundary load $\boldsymbol{h}\mathcal{X}_{\Gamma_h}$.

1. In this respect, assuming fixed the main load $\boldsymbol{f}$ and its support Γ_f, we consider, for any L (in $[0,1]$), the following nonlinear problem:

$$(P_{\Gamma_h}) : \quad \inf_{\mathcal{X}_{\Gamma_h} \in \mathcal{X}_L} g(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h}); \quad \mathcal{X}_L = \{\mathcal{X} \in L^\infty(\Gamma, \{0,1\}), \|\mathcal{X}\|_{L^1(\Gamma)} = L\|\mathcal{X}_\Gamma\|_{L^1(\Gamma)}\}.$$

For any fixed $\boldsymbol{h}$ in $(L^2(\Gamma_h))^2$, (P_{Γ_h}) is an optimal design problem which consists in finding the optimal distribution of the support $\Gamma_h \subset \Gamma$ of the additional load $\boldsymbol{h}$. Remark that the support Γ_h may *a priori* be composed of several disjoint components.

2. On the other hand, the support $\Gamma_h \subset \Gamma$ of the additional force being fixed, one may also consider the following problem: for any L (in $[0,1]$)

$$(P_h) : \inf_{\boldsymbol{h} \in (L^2_L(\Gamma_h))^2} g(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h}); \quad (L^2_L(\Gamma_h))^2 = \{\boldsymbol{h} \in (L^2(\Gamma_h))^2, \|\boldsymbol{h}\|_{(L^2(\Gamma_h))^2} = L\|\boldsymbol{f}\|_{(L^2(\Gamma_f))^2}\},$$

which consists in optimizing the amplitude of $\boldsymbol{h}$ in order to reduce g and therefore preventing the crack growth.

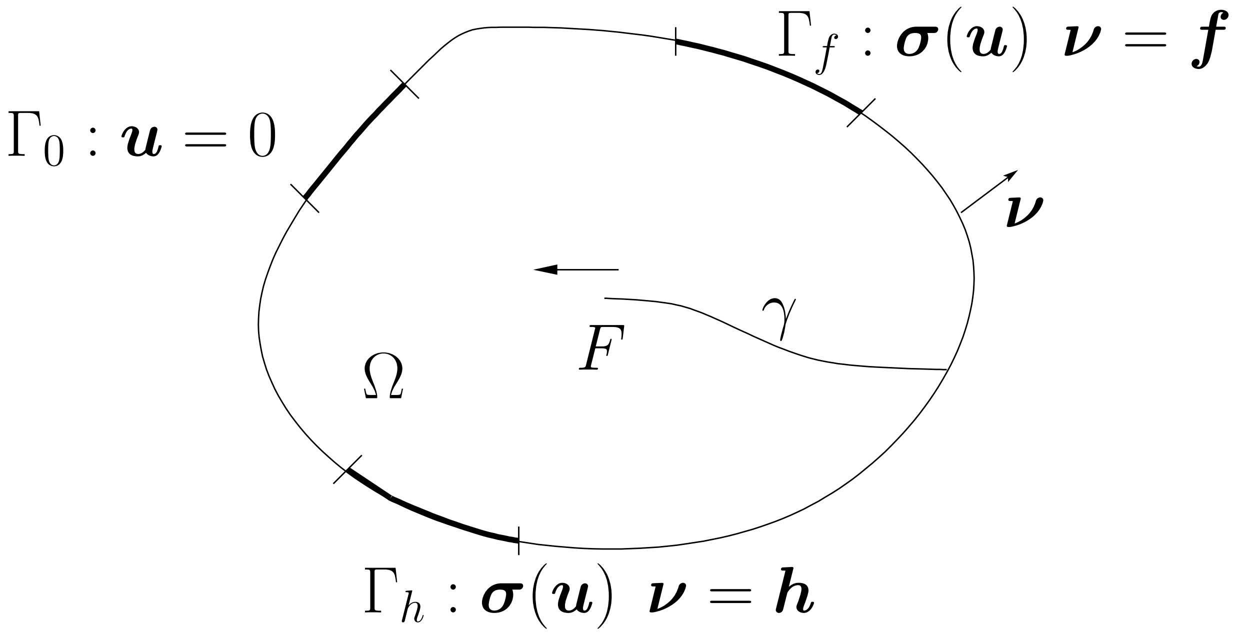


Illustration of the problem (P_{Γ_h}): Optimization of the location of Γ_h, the support of the extra load $\boldsymbol{h}$ in order to minimize the energy release rate.

2. The energy release rate

Crack γ: rectilinear in the neighborhood of $\boldsymbol{F}$ and oriented along $\boldsymbol{e_1}$.

Field: $\boldsymbol{\psi} = (\psi_1(x_1, x_2), 0) \in \mathbf{W} \equiv \{\boldsymbol{\psi} \in (W^{1,\infty}(\Omega, \mathbb{R}))^2, \boldsymbol{\psi} = 0 \text{ on } \partial\Omega\backslash\gamma\}$.

Definition 1 (Energy release rate) *The derivative of the functional $-J(\boldsymbol{u}, \gamma)$ with respect to a variation of γ in the direction $\boldsymbol{\psi}$ is defined as the derivative at 0 of the function $\eta \rightarrow -J(\boldsymbol{u}, (Id + \eta\boldsymbol{\psi})(\gamma))$, i.e.*

$$J(\boldsymbol{u}, (Id + \eta\boldsymbol{\psi})(\gamma)) = J(\boldsymbol{u}, \gamma) + \eta \frac{\partial J(\boldsymbol{u}, \gamma)}{\partial \gamma} . \boldsymbol{\psi} + o(\eta^2).$$

In the sequel, we denote by $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h})$ this derivative. ■

Lemma 1 *The first derivative of $-J$ with respect to γ in the direction $\boldsymbol{\psi} = (\psi_1, 0) \in \mathbf{W}$ is given by*

$$\begin{aligned} g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h}) &= -\frac{1}{2} \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u}) \nabla \boldsymbol{u}) div\, \boldsymbol{\psi} dx + \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u}) \nabla \boldsymbol{u} \nabla \boldsymbol{\psi}) dx \\ &= -\frac{1}{2} \int_\Omega \sigma_{ij} u_{j,i} \psi_{1,1} dx + \int_\Omega \sigma_{ij} u_{j,1} \psi_{1,i} dx \end{aligned}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{f}, \mathcal{X}_{\Gamma_f}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h})$ is the solution of the elasticity problem. ■

Remark 1 *A simple choice for* $\boldsymbol{\psi} = (\psi_1, 0)$ *is given by the radial function*

$$\psi_1(\boldsymbol{x}) = \zeta(dist(\boldsymbol{x}, \boldsymbol{F})), \quad \forall \boldsymbol{x} \in \Omega,$$

where the function $\zeta \in C^1(\mathbb{R}^+; [0,1])$ *is defined as follows:*

$$\zeta(r) = \begin{cases} 1 & r \leq r_1 \\ \dfrac{(r - r_2)^2(3r_1 - r_2 - 2r)}{(r_1 - r_2)^3} & r_1 \leq r \leq r_2 \\ 0 & r \geq r_2 \end{cases}$$

with $0 < r_1 < r_2 < dist(\partial\Omega \backslash \gamma, \boldsymbol{F}) = \inf_{\boldsymbol{x} \in \partial\Omega\backslash\gamma} dist(\boldsymbol{x}, \boldsymbol{F})$.

3. Well-posedness and relaxation of the problems

Optimal location problem (P_{Γ_h}) : ill-posed in general.

PROPOSITION 1 *Let $\boldsymbol{h} \neq 0$ be fixed in $(L^2(\Gamma))^2$. If Γ_h is composed of a finite number of disjoint components, then problem (P_{Γ_h}) admits at least a solution.* ■

Relaxation of (P_{Γ_h}) :

$$(RP_{\Gamma_h}) : \quad \inf_{s \in S_L} g_{\psi}(\boldsymbol{u}, \boldsymbol{h}, s); \quad S_L = \{s \in L^{\infty}(\Gamma, [0,1]), \|s\|_{L^1(\Gamma)} = L\|\mathcal{X}_{\Gamma}\|_{L^1(\Gamma)}\}$$

where L (in $[0,1]$) is the real parameter which appears in the definition of (P_{Γ_h}), and $\boldsymbol{u}$ the solution of the elasticity problem where

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{f}\mathcal{X}_{\Gamma_f} + s(\boldsymbol{x})\boldsymbol{h}\mathcal{X}_{\Gamma_h} \quad \text{on} \quad \partial\Omega\backslash\Gamma_0, \qquad (\text{ instead of } \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{f}\mathcal{X}_{\Gamma_f} + \boldsymbol{h}\mathcal{X}_{\Gamma_h}),$$

THEOREM 1

- *The problem (RP_{Γ_h}) is well-posed;*
- *The minimum of (RP_{Γ_h}) equals the infimum of (P_{Γ_h}).*

PROPOSITION 2 *Let $\Gamma_h \neq \emptyset$ be fixed in Γ and $L \in [0,1]$. The problem (P_h) admits at least a solution in $(L^2_L(\Gamma_h))^2$.* ■

4. Derivative of $g_{\boldsymbol{\psi}}$ with respect to s and $\boldsymbol{h}$

For a fixed field $\boldsymbol{\psi} = (\psi_1, 0)$, find the expression of the derivatives of $g_{\boldsymbol{\psi}}$ with respect to the variation of $s \in L^\infty(\Gamma, [0,1])$ and $\boldsymbol{h} \in (L^2(\Gamma_h))^2$. Penalization of the contact with a function denoted g (!).

A weak solution $\boldsymbol{u} \in (H^1_{\Gamma_0}(\Omega))^2$ is characterized by the following formulation

$$\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u})\,\nabla \boldsymbol{v})dx + \epsilon^{-1}\int_\gamma \nabla g([\boldsymbol{u}])\cdot[\boldsymbol{v}]d\sigma = \int_{\Gamma_f} \boldsymbol{f}\cdot\boldsymbol{v}\,d\sigma + \int_{\Gamma_h} \boldsymbol{h}\cdot\boldsymbol{v}\,d\sigma, \quad \forall \boldsymbol{v} \in (H^1_{\Gamma_0}(\Omega))^2.$$

Perturbation of s: $s^\eta = s + \eta s_1 \in L^\infty(\Gamma, [0,1])$.

$$\frac{\partial g_{\boldsymbol{\psi}}(\boldsymbol{u}(s), \boldsymbol{h}, s)}{\partial s}\cdot s_1 = \lim_{\eta\to 0}\frac{g_{\boldsymbol{\psi}}(\boldsymbol{u}(s^\eta), \boldsymbol{h}, s^\eta) - g_{\boldsymbol{\psi}}(\boldsymbol{u}(s), \boldsymbol{h}, s)}{\eta}.$$

THEOREM 2 *The first variation of $g_{\boldsymbol{\psi}}$ with respect to s in the direction $s_1 \in L^\infty(\Gamma, [0,1])$ is*

$$\frac{\partial g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s)}{\partial s}\cdot s_1 = -\int_\Gamma s_1(\boldsymbol{x})\boldsymbol{h}\cdot\boldsymbol{p}\,d\sigma, \quad \forall s_1 \in L^\infty(\Gamma, [0,1])$$

where $\boldsymbol{p} \in (H^1_{\Gamma_0}(\Omega))^2$ is solution of the following (weak) adjoint problem:

$$\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{p})\,\nabla\boldsymbol{\phi})dx - \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u})\,\nabla\boldsymbol{\phi})div\,\boldsymbol{\psi}dx + \int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{\phi})\,\nabla\boldsymbol{u}\,\nabla\boldsymbol{\psi})dx$$
$$+\int_\Omega Tr(\boldsymbol{\sigma}(\boldsymbol{u})\,\nabla\boldsymbol{\phi}\,\nabla\boldsymbol{\psi})dx + \epsilon^{-1}\int_\gamma \nabla(\nabla g([\boldsymbol{u}])\cdot[\boldsymbol{\phi}])\cdot[\boldsymbol{p}]\,d\sigma = 0$$

for all $\boldsymbol{\phi} \in (H^1_{\Gamma_0}(\Omega))^2$. ■

Remark 2 *If the crack is oriented along the axis* $(O, \boldsymbol{e_1})$ *and if* λ, μ *are the Lamé coefficients, then* $\boldsymbol{p} = (p_1, p_2)$ *is formally solution of the following equations:*

$$\begin{cases} -\sigma_{ij,i}(\boldsymbol{p}) + (\sigma_{ij}(\boldsymbol{u})\psi_{1,1})_{,i} - (\sigma_{ij}(\boldsymbol{u})\psi_{1,i})_{,1} \\ \qquad\qquad -\lambda(u_{i,1}\psi_{1,i})_{,j} - \mu((u_{i,1}\psi_{1,j})_{,i} + (u_{j,1}\psi_{1,i})_{,i}) = 0 \quad in \quad \Omega, \\ \boldsymbol{p} = 0 \quad on \quad \Gamma_0, \\ \sigma_{12}(\boldsymbol{p}) = \mu u_{1,2}\psi_{1,1} + \epsilon^{-1}(g_{,11}([\boldsymbol{u}])[p_1] + g_{,12}([\boldsymbol{u}])[p_2]) \quad on \quad \gamma, \\ \sigma_{22}(\boldsymbol{p}) = (\lambda + 2\mu)u_{2,2}\psi_{1,1} + \epsilon^{-1}(g_{,12}([\boldsymbol{u}])[p_1] + g_{,22}([\boldsymbol{u}])[p_2]) \quad on \quad \gamma, \\ \boldsymbol{\sigma}(\boldsymbol{p})\boldsymbol{\nu} = 0 \quad on \quad \partial\Omega \backslash (\Gamma_0 \cup \gamma). \end{cases}$$

Similarly, assuming Γ_h fixed in Γ, we obtain the first derivative of $g_{\boldsymbol{\psi}}$ with respect to $\boldsymbol{h}$:

THEOREM 3 *The first derivative of* $g_{\boldsymbol{\psi}}$ *with respect to* $\boldsymbol{h}$ *in the direction* $\boldsymbol{h_1}$ *is given by*

$$\frac{\partial g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h})}{\partial \boldsymbol{h}} \cdot \boldsymbol{h_1} = -\int_{\Gamma_h} \boldsymbol{h_1} \cdot \boldsymbol{p} \, d\sigma, \quad \forall \boldsymbol{h_1} \in (L^2(\Gamma_h))^2$$

where $\boldsymbol{p}$ *is the solution of the adjoint problem.* ∎

5. Descent algorithms

Descent algorithm for (RP_{Γ_h})

Descent direction: $s_1 = \boldsymbol{h} \cdot \boldsymbol{p}$

Size restriction on s: $\|s\|_{L^1(\Gamma)} = L|\Gamma|$: we introduce a Lagrange multiplier λ and a new cost function:

$$g_{\boldsymbol{\psi},\lambda}(\boldsymbol{u}, \boldsymbol{h}, s) = g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s) + \lambda(\|s\|_{L^1(\Gamma)} - L|\Gamma|), \quad \forall s \in L^\infty(\Gamma, [0,1])$$

leading to

$$\frac{\partial g_{\boldsymbol{\psi},\lambda}(\boldsymbol{u}, \boldsymbol{h}, s)}{\partial s} \cdot s_1 = -\int_\Gamma s_1(\boldsymbol{x}) \boldsymbol{h} \cdot \boldsymbol{p} \, d\sigma + \lambda \int_\Gamma s_1(\boldsymbol{x}) d\sigma$$

and to the descent direction

$$s_1 = \boldsymbol{h} \cdot \boldsymbol{p} - \lambda \quad \text{on} \quad \Gamma.$$

Consequently, for any function $\eta_s \in L^\infty(\Gamma, \mathbb{R}^+)$ with $\|\eta_s\|_{L^1(\Gamma)}$ small enough, we have $g_{\boldsymbol{\psi},\lambda}(\boldsymbol{u}, \boldsymbol{h}, s + \eta_s s_1) \leq g_{\boldsymbol{\psi},\lambda}(\boldsymbol{u}, \boldsymbol{h}, s)$. The multiplier λ is then determined so that, for any function $\eta_s \in L^\infty(\Gamma, \mathbb{R}^+)$, $\|s + \eta_s s_1\|_{L^1(\Gamma)} = L|\Gamma|$, leading to

$$\lambda = \frac{(\int_\Gamma s(\boldsymbol{x}) d\sigma - L|\Gamma|) + \int_\Gamma \eta_s(\boldsymbol{x}) \boldsymbol{h} \cdot \boldsymbol{p} \, d\sigma}{\int_\Gamma \eta_s(\boldsymbol{x}) d\sigma}.$$

At last, the function η_s is chosen so that $s + \eta_s s_1 \in [0,1]$, for all $\boldsymbol{x} \in \Gamma$. A simple and efficient choice consists in taking $\eta_s(\boldsymbol{x}) = \varepsilon s(\boldsymbol{x})(1 - s(\boldsymbol{x}))$ for all $\boldsymbol{x} \in \Gamma$ where ε is a small positive parameter.

Consequently, the descent algorithm to solve numerically the relaxed problem (RP_{Γ_h}) may be structured as follows. Let $\Omega \subset \mathbb{R}^2$, Γ_0, Γ_f in $\partial\Omega$, $\boldsymbol{f} \in (L^2(\Gamma_f))^2$, $\boldsymbol{h} \in (L^2(\Gamma_h))^2$, $L \in (0,1)$ and $\varepsilon < 1$, $\varepsilon_1 << 1$ be given ;

- Initialization of the density $s^{(0)} \in L^\infty(\Gamma; (0,1))$;
- For $k \geq 0$, iterate until convergence (i.e. $|g_{\psi,\lambda}(\boldsymbol{u}, \boldsymbol{h}, s^{(k+1)}) - g_{\psi,\lambda}(\boldsymbol{u}, \boldsymbol{h}, s^{(k)})| \leq \varepsilon_1 |g_{\psi,\lambda}(\boldsymbol{u}, \boldsymbol{h}, s^{(0)})|$) as follows:
  - Computation of the solution $\boldsymbol{u}(s^{(k)})$ of the elasticity problem and then the solution $\boldsymbol{p}(s^{(k)})$ of the adjoint problem, both corresponding to $s = s^{(k)}$.
  - Computation of the descent direction $s_1^{(k)} = \boldsymbol{h} \cdot \boldsymbol{p}(s^{(k)}) - \lambda^{(k)}$.
  - Update the density $s^{(k)}$ in Γ:

$$s^{(k+1)} = s^{(k)} + \varepsilon s^{(k)}(1 - s^{(k)}) s_1^{(k)}, \tag{3}$$

    with $\varepsilon \in \mathbb{R}^+$ small enough in order to ensure the decrease of the cost function and $s^{(k+1)} \in L^\infty(\Gamma, [0,1])$.

Descent algorithm for (P_h)

Problem (P_h) is solved in a similar way. In order to ensure $\boldsymbol{h} \in (L^2_L(\Gamma_h))^2$, we introduce the new cost function:

$$g_{\psi,\lambda}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h}) = g_{\psi}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h}) + \lambda(\|\boldsymbol{h}\|^2_{(L^2(\Gamma_h))^2} - L^2\|\boldsymbol{f}\|^2_{(L^2(\Gamma_f))^2}).$$

So

$$\frac{\partial g_{\psi,\lambda}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h})}{\partial \boldsymbol{h}} \cdot \boldsymbol{h_1} = \int_{\Gamma_h} \boldsymbol{h_1} \cdot (-\boldsymbol{p} + 2\lambda \boldsymbol{h})\, d\sigma, \quad \forall \boldsymbol{h_1} \in (L^2(\Gamma_h))^2$$

leading to the descent direction $\boldsymbol{h_1} = (\boldsymbol{p} - 2\lambda\boldsymbol{h})$ so that for any $\varepsilon > 0$ small enough, $g_{\psi,\lambda}(\boldsymbol{u}, \boldsymbol{h} + \varepsilon(\boldsymbol{p} - 2\lambda\boldsymbol{h}), \mathcal{X}_{\Gamma_h}) \leq g_{\psi,\lambda}(\boldsymbol{u}, \boldsymbol{h}, \mathcal{X}_{\Gamma_h})$. At last, the multiplier λ is determined so that $\boldsymbol{h} + \varepsilon(\boldsymbol{p} - 2\lambda\boldsymbol{h}) \in (L^2_L(\Gamma_h))^2$; λ is then solution of the polynomial equation of order two:

$$\begin{aligned} 4\varepsilon\|\boldsymbol{h}\|^2_{(L^2(\Gamma_h))^2}\lambda^2 - 4\left(\int_{\Gamma_h} \boldsymbol{p}\cdot\boldsymbol{h}d\sigma + \|\boldsymbol{h}\|^2_{(L^2(\Gamma_h))^2}\right)\lambda \\ - \varepsilon^{-1}(L^2\|\boldsymbol{f}\|^2_{(L^2(\Gamma_f))^2} - \|\boldsymbol{h}\|^2_{(L^2(\Gamma_h))^2}) + 2\int_{\Gamma_h} \boldsymbol{h}\cdot\boldsymbol{p}d\sigma + \varepsilon\|\boldsymbol{p}\|^2_{(L^2(\Gamma_h))^2} = 0. \end{aligned} \tag{4}$$

Observe that the two roots are real if $\varepsilon > 0$ is small enough. The algorithm is then similar to the algorithm of the previous section, (3) being replaced by

$$\boldsymbol{h^{(k+1)}} = \boldsymbol{h^{(k)}} + \varepsilon(\boldsymbol{p}(\boldsymbol{h^{(k)}}) - 2\lambda^{(k)}\boldsymbol{h^{(k)}})$$

where $\lambda^{(k)}$ solves (4).

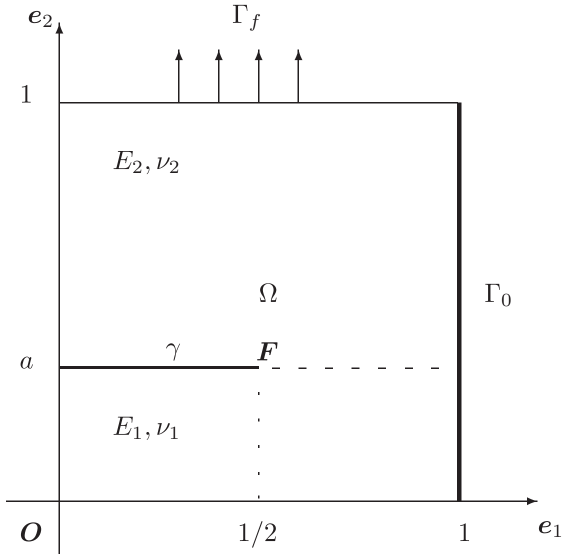
6. Numerical experiments

$\Omega = (0,1)^2$, fixed on $\Gamma_0 = \{1\} \times [0,1]$ with a crack $\gamma = [0, 0.5] \times \{a\}$, $(a \in (0,1))$, and submitted to the load $\boldsymbol{f} = (f_1, f_2) = (0, 10^6 N/m)$ on $\Gamma_f = [0.3, 0.6] \times \{1\}$.

Lower part of Ω (i.e. $[0,1] \times [0,a]$) with a Young modulus E_1 and a Poisson ratio ν_1

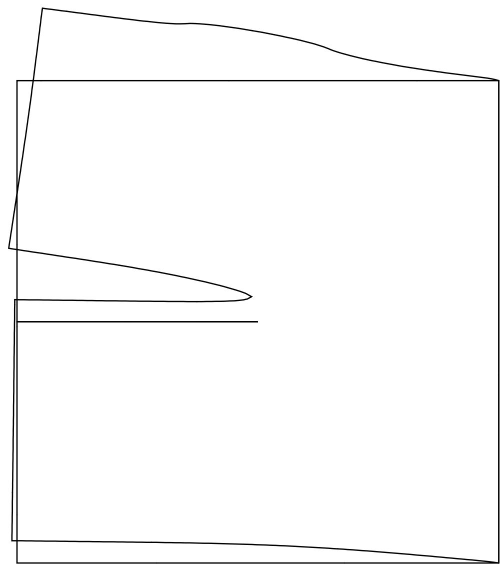
Upper part of Ω (i.e. $[0,1] \times [a,1]$) with a Young modulus E_2 and a Poisson ratio ν_2.

Standard P_1 finite elements, $h = 1/100$, amplification of 2×10^4 (deformation).



Setting of the problem

Young modulus of $2 \times 10^{11} Pa$, Poisson ratio of 0.3 and a centered crack ($a = 0.5$).



Initial and deformed configurations of Ω without additional extra force (i.e., $\boldsymbol{\mathcal{X}}_{\Gamma_h} = 0$) and $a = 0.5$: $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, 0) \approx 1.147 N/m$ ($r_1 = 0.1$ and $r_2 = 0.4$).

6.1 Problem (RP_{Γ_h})

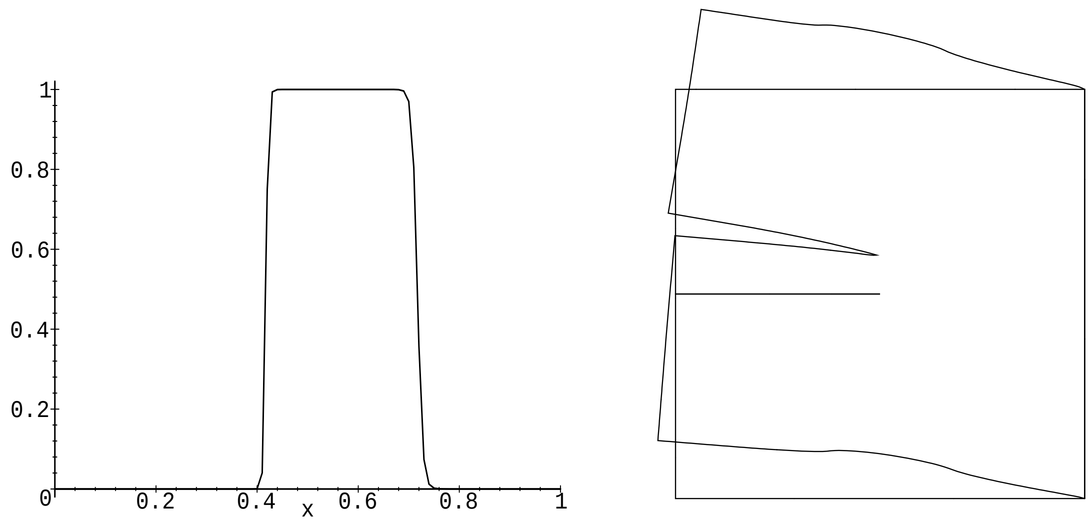
$\Gamma = [0,1] \times \{0\}$ (i.e., the lower edge of the structure), $\boldsymbol{h} = (0, h_2)$ with $h_2 = 10^6 N/m$ and $L = 0.3$ so that $\int_\Gamma s(\boldsymbol{x}) h_2 d\sigma = \int_{\Gamma_f} f_2 d\sigma$. Initialization with the constant density function $s^{(0)} \equiv L$ in Γ.

$E_1 = E_2 = 2 \times 10^{11} Pa$, $\nu_1 = \nu_2 = 0.3$ and $a = 0.5$.

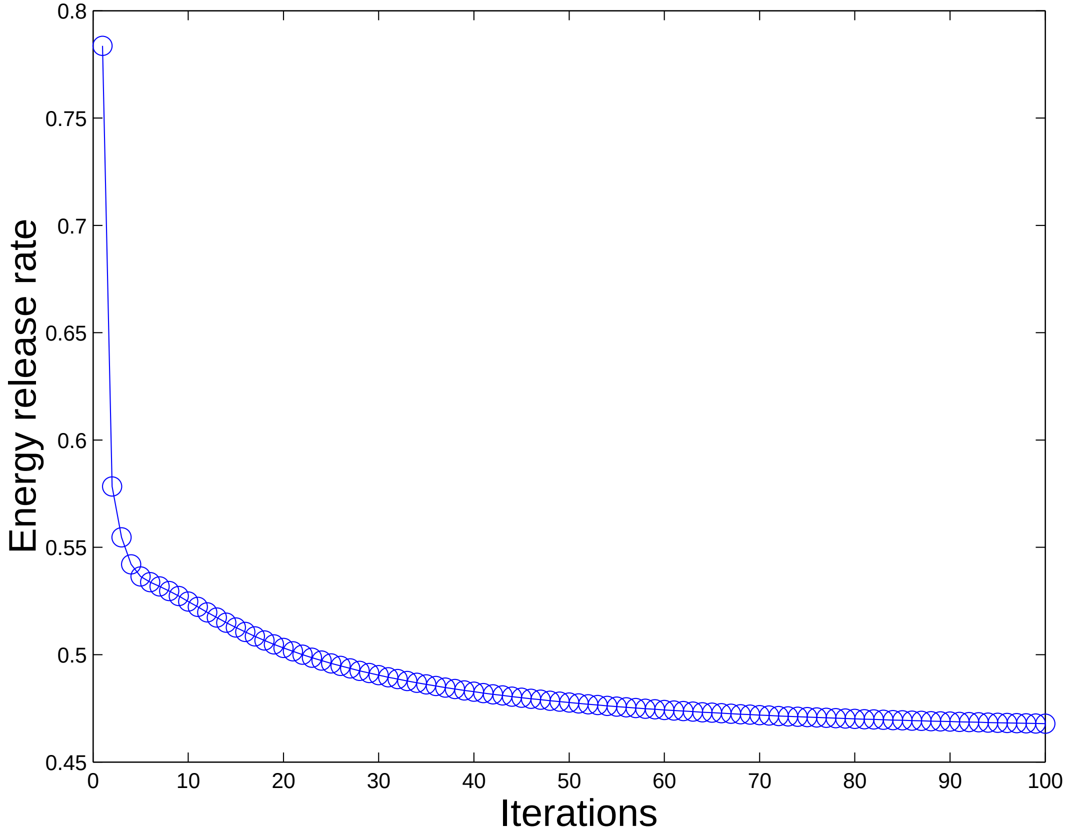
$g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s^{(0)}) \approx 0.7836 N/m$.

Symmetric density $s = \mathcal{X}_{[0.3,0.6]} \in S_L$: $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s) \approx 0.6203 N/m$.

Optimal density s^{opt} for which $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s^{opt}) \approx 0.4641 N/m$; $s^{opt} \approx \mathcal{X}_{[0.42,0.72]}$.



Resolution of (RP_{Γ_h}) - Optimal density s^{opt} (**Left**) and corresponding deformation (**Right**)- $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s^{opt}) \approx 0.4641 N/m$.

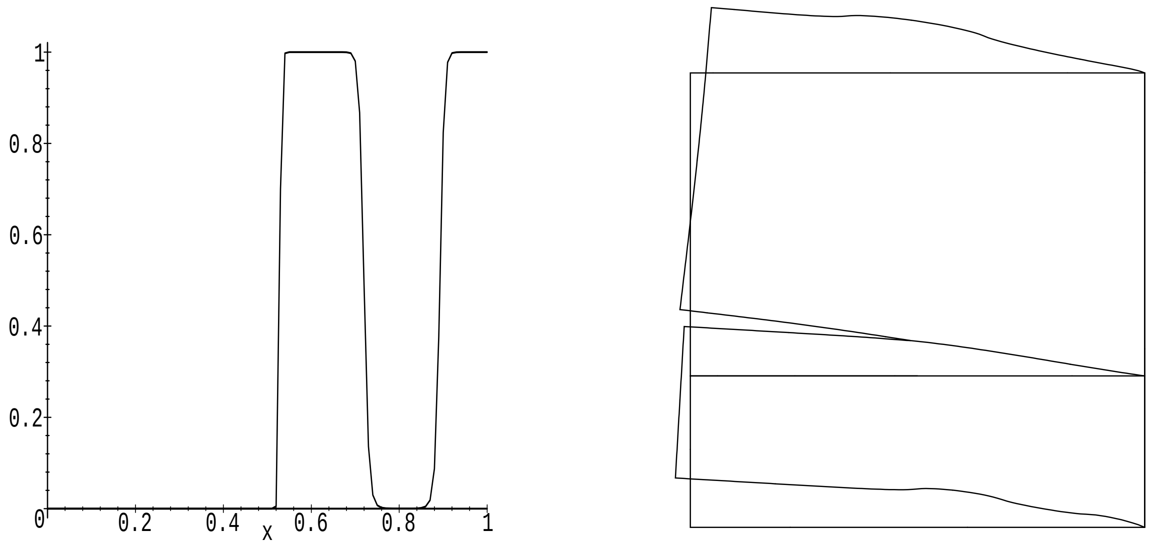


Evolution of $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s^{(k)})$ vs. $k \in [1, 100]$ obtained with $\varepsilon = 0.3$.

$E_1 = E_2 = 2 \times 10^{11} Pa$, $\nu_1 = \nu_2 = 0.3$ and $a = 1/3$.

Without additional force: $g_{\psi}(\boldsymbol{u}, \boldsymbol{h}, 0) \approx 0.3872 N/m$ (obtained with $r_1 = 0.1$ and $r_2 = 0.25$).

Initial constant additional force $s^{(0)} = L = 0.3$ on Γ: the rate decreases from $0.5876 N/m$ to $0.1050 N/m$; $s^{opt} \approx \mathcal{X}_{[0.52,0.7]\cup[0.88,1]}$

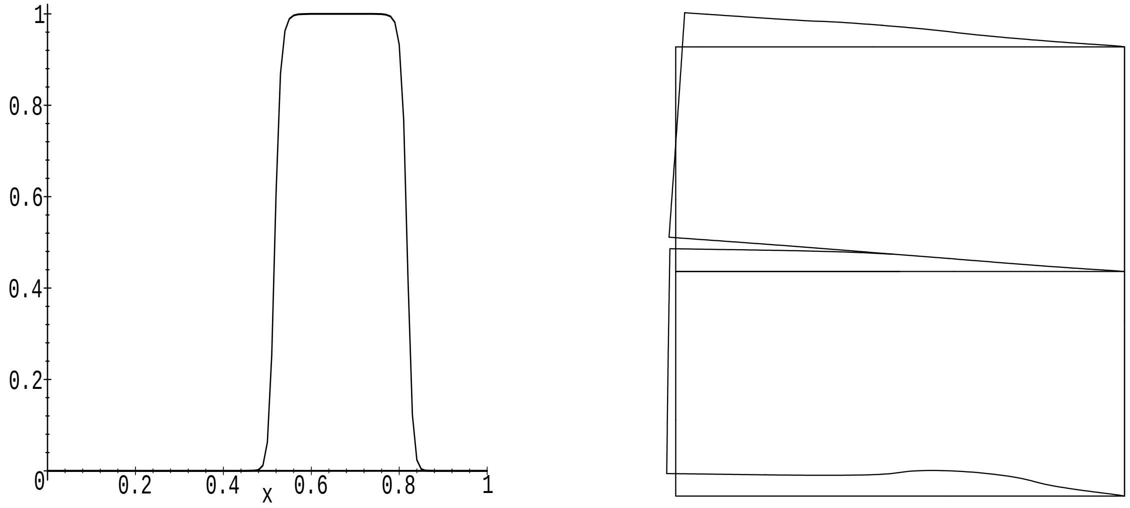


Resolution of (RP_{Γ_h}) - Optimal density s^{opt} (**Left**) and corresponding deformation (**Right**)- $g_{\psi}(\boldsymbol{u}, \boldsymbol{h}, s^{opt}) \approx 0.1050 N/m$.

$E_1 = 2 \times 10^{11} Pa$, $E_2 = 10^{12} Pa$, $\nu_1 = \nu_2 = 0.3$ and $a = 1/2$.

Without additional force, rate $= 0.2139 N/m$.

Choosing $s^{(0)} = L = 0.3$, the rate decreases from $0.2481 N/m$ to $0.0281 N/m$; $s^{opt} \approx \mathcal{X}_{[0.52,0.82]}$.



Resolution of (RP_{Γ_h}) - Optimal density s^{opt} (**Left**) and corresponding deformation (**Right**)-
$g_{\psi}(\boldsymbol{u}, \boldsymbol{h}, s^{opt}) \approx 0.0281 N/m$.

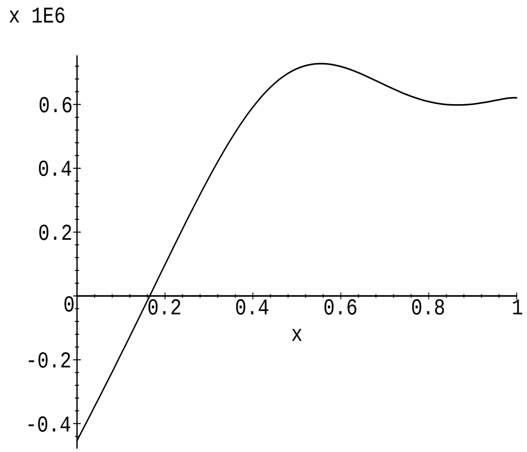
6.2 Problem (P_h)

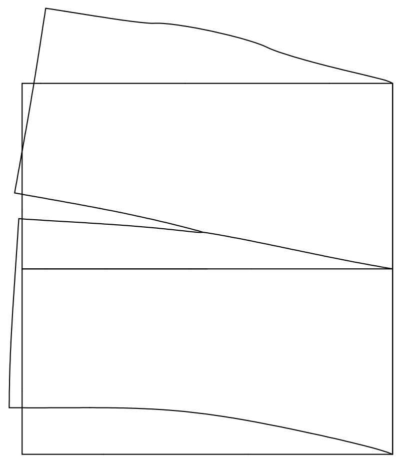
$\Gamma_h = [0,1] \times \{0\}$ and $\boldsymbol{h}$ is normal: $\boldsymbol{h} = (0, h_2)$.
We impose that the L^2-norm of the additional load $\boldsymbol{h}$ equals the L^2-norm of $\boldsymbol{f}$ $(L = 1)$.
The initial computations are achieved with a constant normal load on Γ_h, i.e.

$$h_2^{(0)} = |\Gamma_f|^{1/2} f_2 / |\Gamma_h|^{1/2} = \sqrt{0.3} f_2 \text{ for which } \boldsymbol{h}^{(0)} = (0, h_2^{(0)}) \in (L^2_{L=1}(\Gamma_h))^2$$

$E_1 = E_2 = 2 \times 10^{11} Pa$, $\nu_1 = \nu_2 = 0.3$ and $a = 1/2$.

Initial computation $g_\psi(\boldsymbol{u}, \boldsymbol{h}^{(0)}, \mathcal{X}_{\Gamma_h}) \approx 1.5927 N/m$.

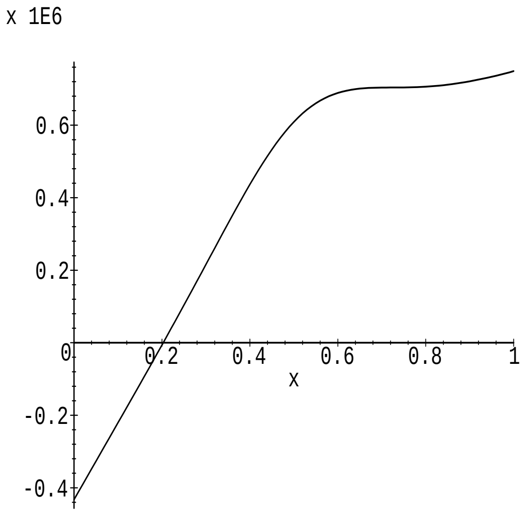


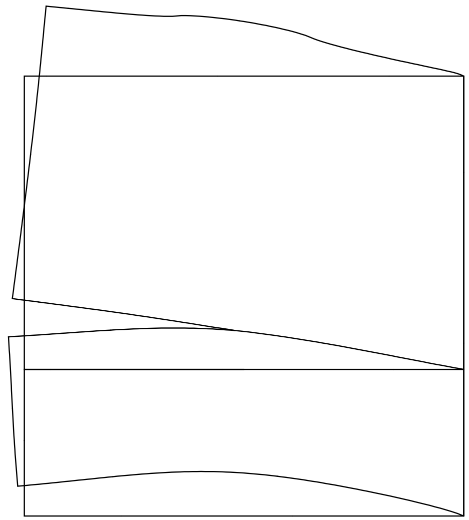


Resolution of (P_h) - Optimal density h_2^{opt} (**Left**) and corresponding deformation (**Right**)
$g_\psi(\boldsymbol{u}, \boldsymbol{h}^{opt}, \mathcal{X}_{\Gamma_h}) \approx 0.4328 N/m$.

$E_1 = E_2 = 2 \times 10^{11} Pa$, $\nu_1 = \nu_2 = 0.3$ and $a = 1/3$.

The rate decreases from $1.1948 N/m$ to $0.0353 N/m$. (rate without additional force is $0.3872 N/m$)



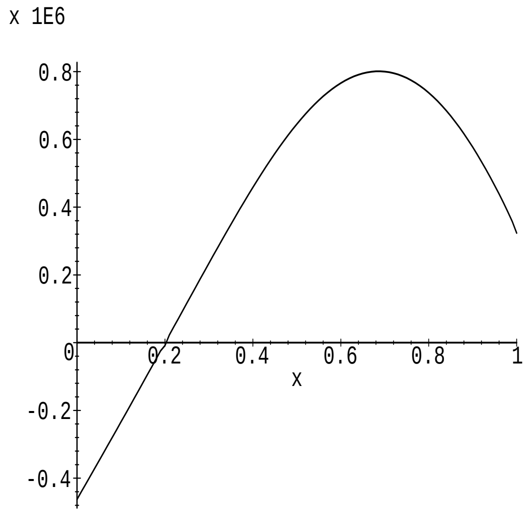


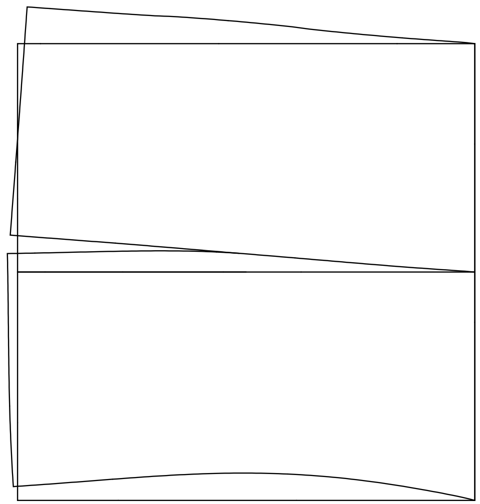
Resolution of (P_h) - Optimal density h_2^{opt} (**Left**) and corresponding deformation (**Right**) $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}^{opt}, \mathcal{X}_{\Gamma_h}) \approx 0.0353 N/m$.

$E_1 = 2 \times 10^{11} Pa$, $E_2 = 10^{12} Pa$, $\nu_1 = \nu_2 = 0.3$ and $a = 1/2$.

The rate decreases from $0.4799 N/m$ to $0.00679 N/m$.

Initial rate is $0.2139 N/m$.

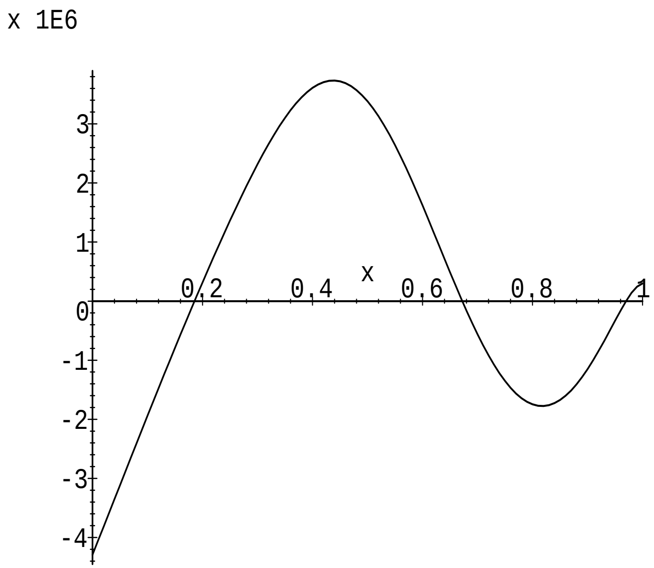


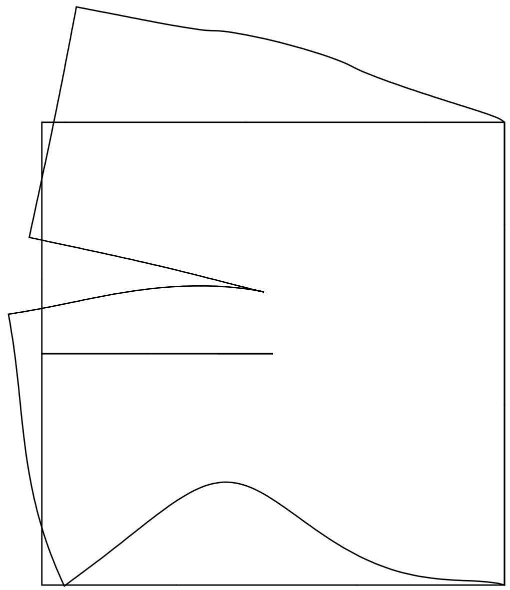


Resolution of (P_h) - Optimal density h_2^{opt} (**Left**) and corresponding deformation (**Right**)
$g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}^{opt}, \mathcal{X}_{\Gamma_h}) \approx 0.00679 N/m$.

$E_1 = E_2 = 2 \times 10^{11} Pa$, $\nu_1 = \nu_2 = 0.3$ and $a = 1/2$, without constraint on $\boldsymbol{h}$.

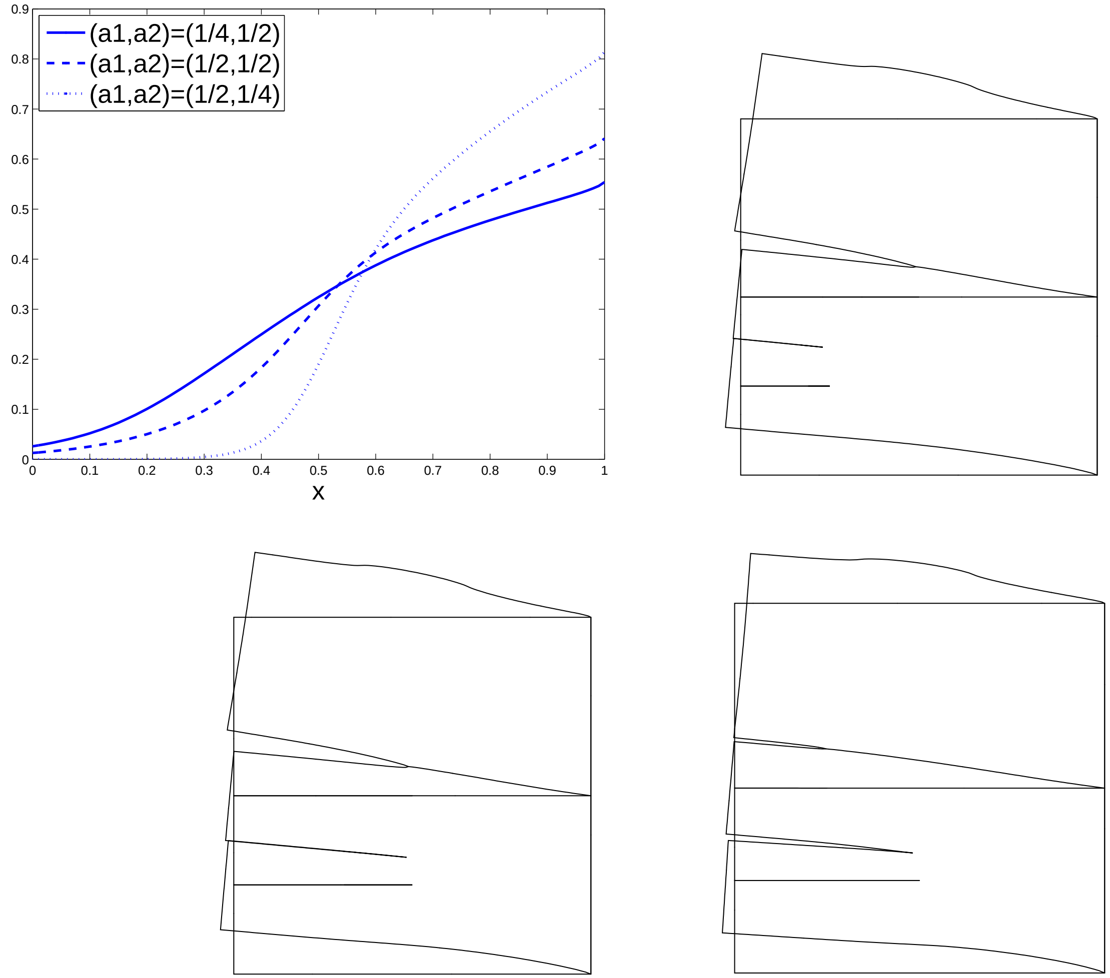
We obtain the value $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}^{opt}, \mathcal{X}_{\Gamma_h}) \approx 0.0383 N/m$ (corresponding to a reduction of order 30).





Resolution of (P_h) without constraint on $\boldsymbol{h}$- Optimal density h_2^{opt} (**Left**) and corresponding deformation (**Right**) - $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}^{opt}, \mathcal{X}_{\Gamma_h}) \approx 0.0383 N/m$.

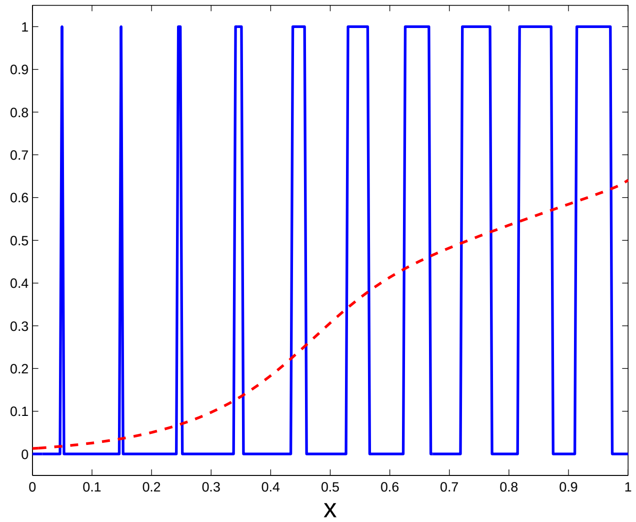
6.3 The case of two cracks



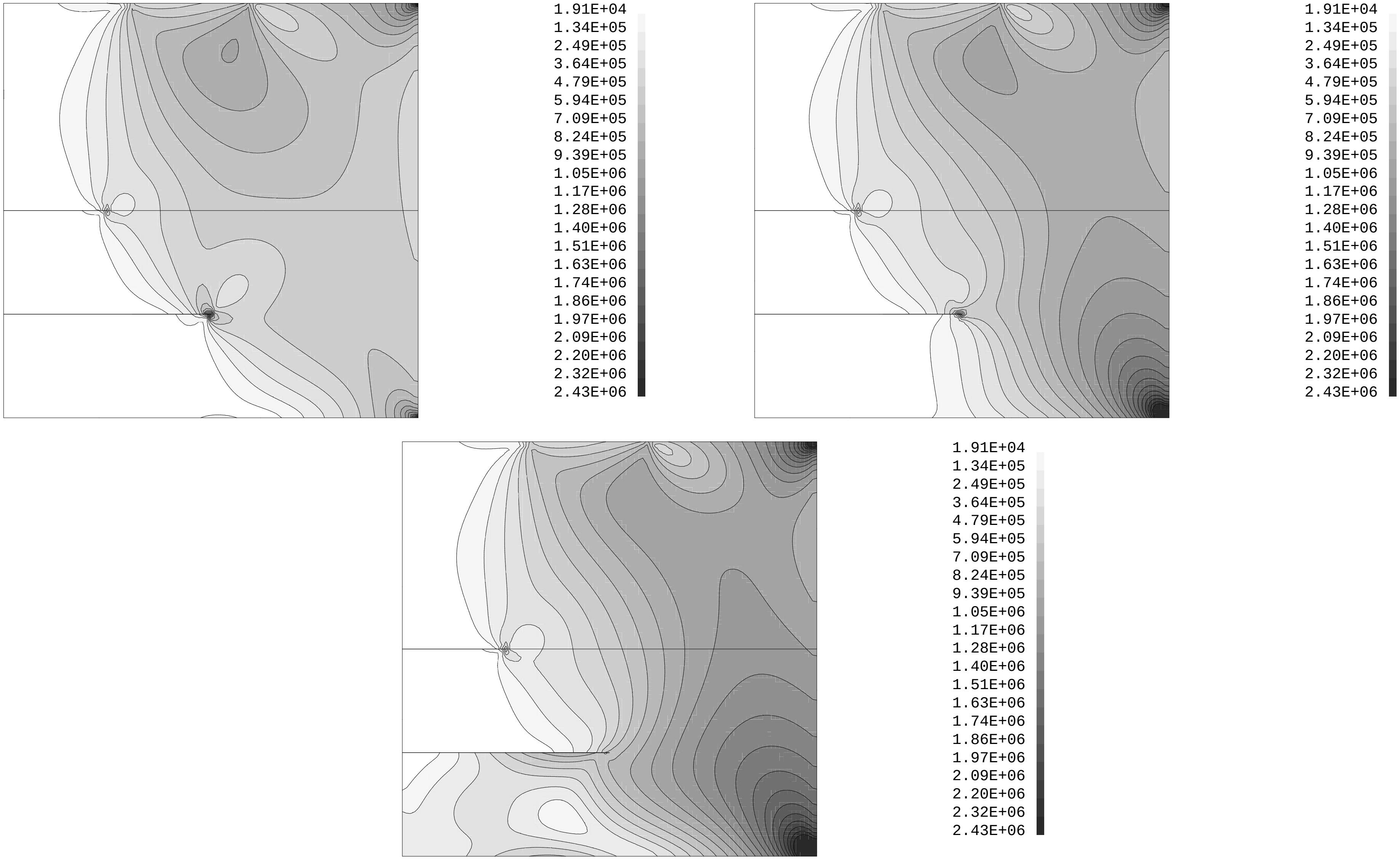
Resolution of (RP_{Γ_h}) - Limit densities (**top left**) and deformation for $(a_1, a_2) = (1/4, 1/2)$ (**top right**), $(a_1, a_2) = (1/2, 1/2)$ (**bottom left**) and $(a_1, a_2) = (1/2, 1/4)$ (**bottom right**).

$$(a_1,a_2)=(1/4,1/2): g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},0)=1.151,\ g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},s^{(0)})=0.861,\ g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},s^{lim})=0.582$$
$$(a_1,a_2)=(1/2,1/2): g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},0)=1.152,\ g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},s^{(0)})=1.49,\ g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},s^{lim})=0.668$$
$$(a_1,a_2)=(1/2,1/4): g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},0)=0.232,\ g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},s^{(0)})=0.461,\ g_{\boldsymbol{\psi}}(\boldsymbol{u},\boldsymbol{h},s^{lim})=0.102$$

Numerical values of the energy release rate (in N/m)



Penalization of the limit density s^{lim} in the case $(a_1,a_2)=(1/2,1/2)$ by a characteristic function $\mathcal{X}_{\Gamma_h}^{(10)}$.



Iso-values of the Von Mises stresses on Ω - **a)** without extra-force $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, 0) \approx 0.232N/m$ (**Top left**) - **b)** from (RP_{Γ_h}) $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}, s^{opt}) \approx 0.102N/m$ (**Top right**) and **c)** from (P_h) $g_{\boldsymbol{\psi}}(\boldsymbol{u}, \boldsymbol{h}^{opt}, \mathcal{X}_{\Gamma_h}) \approx 0.0556N/m$ (**Bottom**).