Fictitious domain methods for numerical realization of unilateral problems

J.H.

(jointly with T. Kozubek and R. Kučera)

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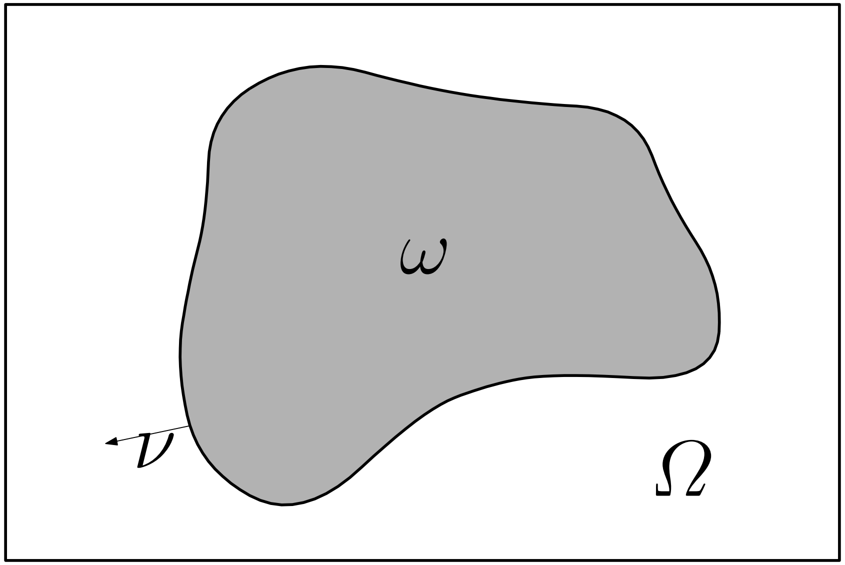
Outline

- Fictitious domain methods - main idea
- Application FDM to unilateral problems
- Semismooth Newton method
- Numerical examples

Fictitious domain methods - main idea



$$(\mathcal{P}(\omega)) \quad u(\omega)$$

$$\Downarrow$$

$$(\mathcal{P}(\Omega)) \quad u(\Omega)$$

Requirement: $u(\Omega)\big|_{\omega} = u(\omega).$

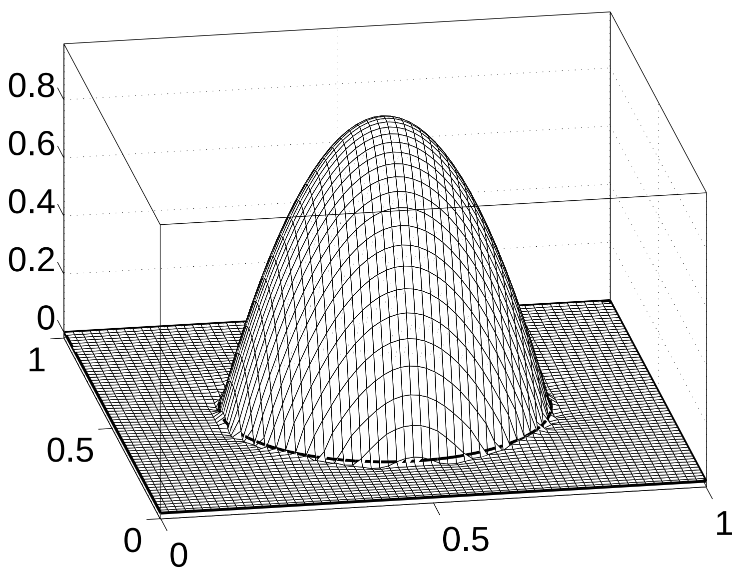
FDM for elliptic equations (Dirichlet, Neumann b.v.p.)

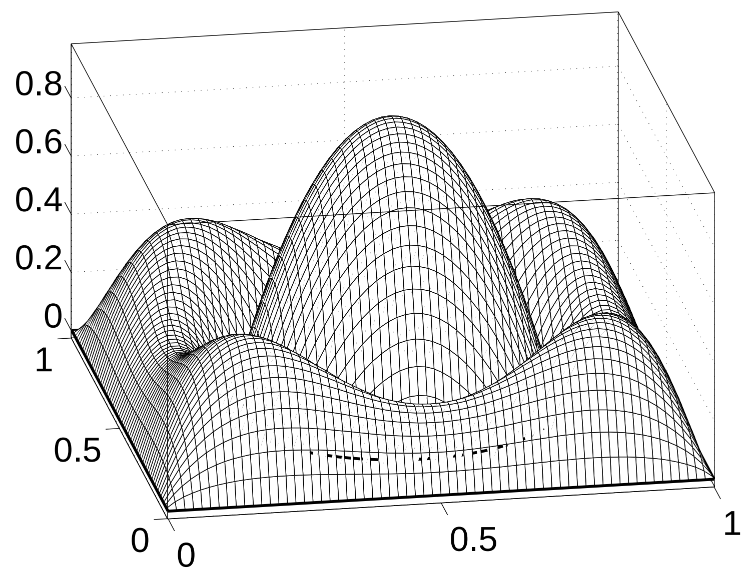
- distributed Lagrange multipliers (Glowinski, Maitre, J.H., . . .),
- boundary Lagrange multipliers (Glowinski, Girault, T. Kozubek, J.H., . . .).

FDM advantages and disadvantages

Advantages: easy realization of $(\mathcal{P}(\Omega))$ (uniform partitions, efficient fast solvers, ...).

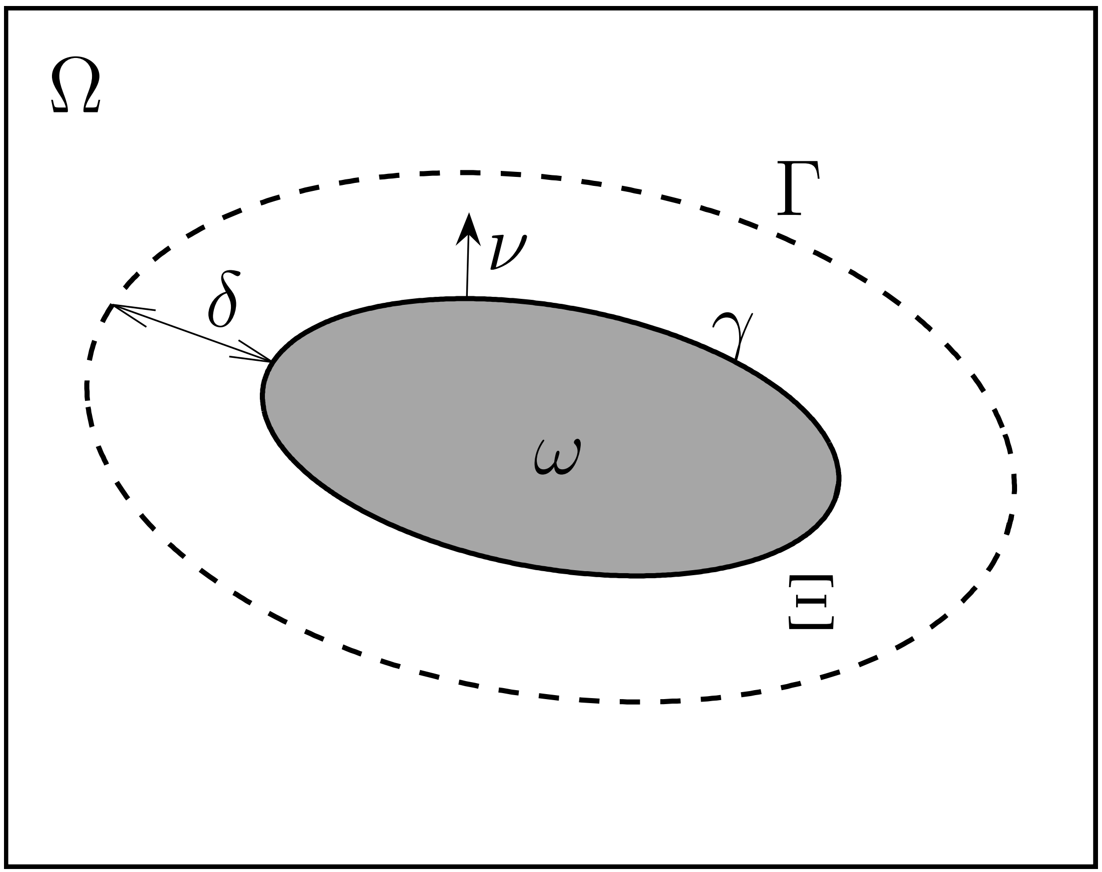
Disadvantages: more unknowns, problems with local refinements and lower regularity of $u(\Omega)$ $(u(\Omega) \in H^{3/2-\varepsilon}(\Omega), \varepsilon > 0)$, typical convergence rate in $H^1(\Omega)$ for non-fitted meshes: $\mathcal{O}(h^{1/2}),\ h \to 0+$.





Smooth version of FDM

Smooth version of FDM (T. Kozubek, R. Kučera, G. Peichl, J.H.)



Fictitious domain formulation of unilateral problems

Problem $(\mathcal{P}(\omega))$

$$-\Delta u + u = f \quad \text{in } \omega,$$
$$u \geq g,\ \frac{\partial u}{\partial n_\gamma} \geq 0,\ \frac{\partial u}{\partial n_\gamma}(u-g) = 0 \quad \text{on } \gamma,$$

$$\omega \subset \mathbb{R}^2,\ \gamma := \partial\omega,\ f \in L^2_{loc}(\mathbb{R}^2),\ g \in H^{1/2}(\gamma).$$

$$\Updownarrow$$

Find $u \in K$ *such that*

$$(u, v-u)_{1,\omega} \geq (f, v-u)_{0,\omega} \quad \forall v \in K,$$

$$K = \{v \in H^1(\omega) |\ v \geq g \text{ a.e. on } \gamma\}.$$

Equivalent formulation of $(\mathcal{P}(\omega))$

Find $u \in H^1(\omega)$ *such that*

$$(u,v)_{1,\omega} = (f,v)_{0,\omega} + \langle \frac{\partial u}{\partial n_\gamma}, v\rangle_\gamma \quad \forall v \in H^1(\omega),$$

$$\frac{\partial u}{\partial n_\gamma} \in H_+^{-1/2}(\gamma),$$

$$\langle \mu - \frac{\partial u}{\partial n_\gamma}, u - g\rangle_\gamma \geq 0 \quad \forall \mu \in H_+^{-1/2}(\gamma).$$

Assumption: $\frac{\partial u}{\partial n_\gamma} \in L^2_+(\gamma)$. Then

$$(\mu - \frac{\partial u}{\partial n_\gamma}, u-g)_\gamma \geq 0 \,\forall \mu \in L^2_+(\gamma) \Leftrightarrow \frac{\partial u}{\partial n_\gamma} = P(\frac{\partial u}{\partial n_\gamma} - \rho(u-g)),\ \rho > 0.$$

$P:\ L^2(\gamma) \to L^2_+(\gamma)\ \ldots$ projection mapping

Fictitious domain formulation of $\big(\mathcal{P}(\omega)\big)$

Problem $(\hat{\mathcal{P}}(\Gamma))$

Find $(\hat{u}, \lambda) \in H_0^1(\Omega) \times H^{-1/2}(\Gamma)$ *such that*

$$(\hat{u}, v)_{1,\Omega} = (f, v)_{0,\Omega} + \langle \lambda, v \rangle_\Gamma \quad \forall v \in H_0^1(\Omega),$$

$$\frac{\partial}{\partial n_\gamma}\hat{u}(\omega) = P\Big(\frac{\partial}{\partial n_\gamma}\hat{u}(\omega) - \rho(\hat{u}(\omega) - g)\Big), \quad \rho > 0$$

where $\hat{u}(\omega) := \hat{u}_{|_\omega}$ and $\Gamma \subset \Omega$ is a closed curve surrounding ω

(i**)** $\gamma = \Gamma$; **(**ii**)** $\operatorname{dist}(\gamma, \Gamma) > 0$.

Lemma 1 Let $(\hat{u}, \lambda)$ be a solution of $(\hat{\mathcal{P}}(\Gamma))$. Then $\hat{u}(\omega)$ solves $\big(\mathcal{P}(\omega)\big)$.

Optimal control formulation of $(\hat{\mathcal{P}}(\Gamma))$

State problem

$$(\hat{u}(\mu), v)_{1,\Omega} = (f, v)_{0,\Omega} + \langle \mu, v\rangle_\Gamma \quad \forall v \in H_0^1(\Omega)$$

$\mu \in H^{-1/2}(\Gamma)$ ... control variable

Cost functional

$$\Psi(\mu) = \frac{1}{2}\Big\| \frac{\partial}{\partial n_\gamma}(\hat{u}(\mu)_{|\omega}) - P(\frac{\partial}{\partial n_\gamma}(\hat{u}(\mu)_{|\omega}) - \rho(\hat{u}(\mu)_{|\omega} - g))\Big\|_{0,\gamma}^2$$

$(\hat{u}, \lambda)$ solves $(\hat{\mathcal{P}}(\Gamma)) \Leftrightarrow \Psi(\lambda) = 0.$

Existence analysis

(*i*) $\gamma = \Gamma$

Lemma 2 Problem $(\hat{\mathcal{P}}(\gamma))$ has a unique solution.

(ii) $\operatorname{dist}(\gamma, \Gamma) > 0$: *approximate controlability property*

$\lambda \in H^{-1/2}(\Gamma)$ given;

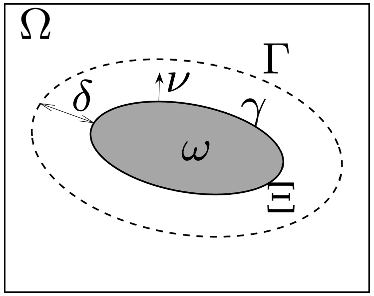
Problem $(\mathcal{P}(\lambda))$

Find $u(\lambda) \in H^1(\Xi)$ *such that*

$$(u(\lambda), v)_{1,\Xi} = (f, v)_{0,\Xi} + \langle \lambda, v \rangle_\Gamma \; \forall v \in H^1_0(\Xi),$$

$$u(\lambda) = 0 \quad \text{on } \partial\Omega,$$

$$u(\lambda) = u \quad \text{on } \gamma,$$



$$\Xi = \Omega \setminus \overline{\omega}$$

where $u \in K$ solves $(\mathcal{P}(\omega))$.

Existence analysis

$$\Phi : H^{-1/2}(\Gamma) \mapsto H^{-1/2}(\gamma); \quad \Phi(\lambda) = \frac{\partial u(\lambda)}{\partial n_\gamma} \in H^{-1/2}(\gamma)$$

Lemma 3 The image $\Phi(H^{-1/2}(\Gamma))$ is dense in $H^{-1/2}(\gamma)$.

Theorem 1 For every $\epsilon > 0$ there exist: $\lambda_\epsilon \in H^{-1/2}(\Gamma)$, $\hat{u}_\epsilon \in H_0^1(\Omega)$ and $\delta_\epsilon \in H^{-1/2}(\gamma)$ satisfying

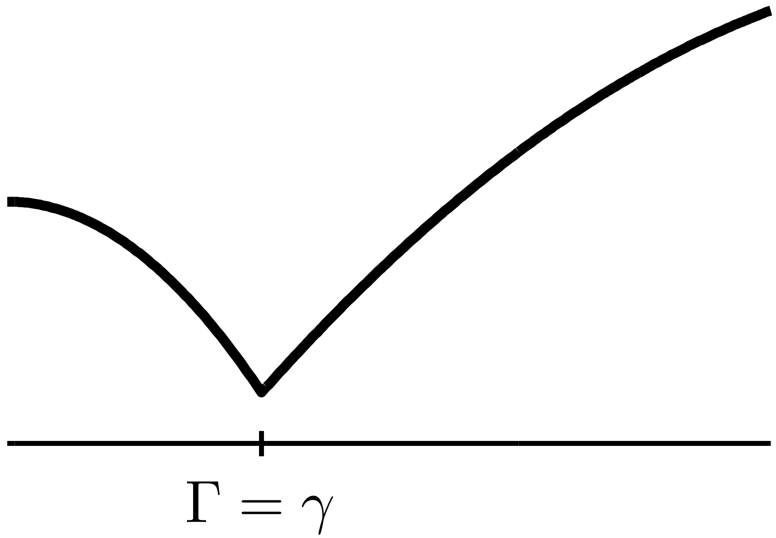
$$(\hat{u}_\epsilon, v)_{1,\Omega} = (f, v)_{0,\Omega} + \langle \lambda_\epsilon, v \rangle_\Gamma + \langle \delta_\epsilon, v \rangle_\gamma \quad \forall v \in H_0^1(\Omega),$$

$$\frac{\partial \hat{u}_\epsilon(\omega)}{\partial n_\gamma} = P(\frac{\partial \hat{u}_\epsilon(\omega)}{\partial n_\gamma} - \rho(\hat{u}_\epsilon(\omega) - g)) \text{ on } \gamma, \ \rho > 0,$$

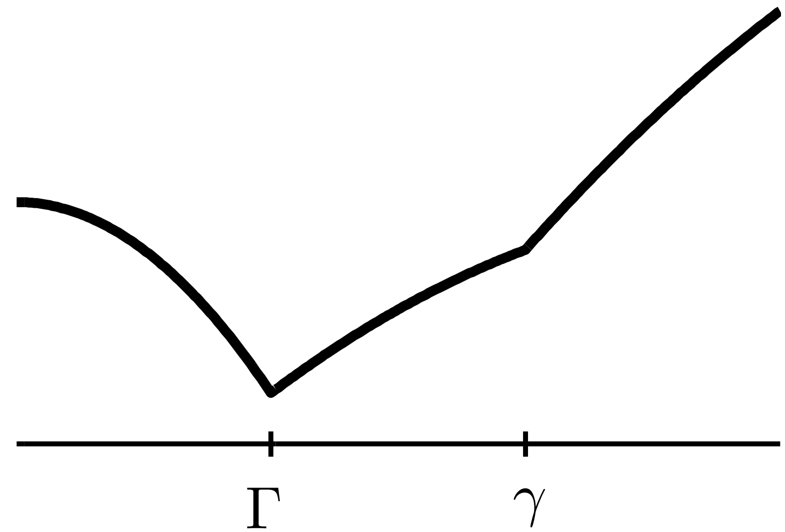
$$\|\delta_\epsilon\|_{-1/2,\gamma} \leq \epsilon$$

where $\hat{u}_\epsilon(\omega) := \hat{u}_{\epsilon|_\omega}$.

Singularity of the solution



(i) Non-smooth variant



(ii) "Smooth" variant

Optimal control formulation

$$\gamma = \Gamma \Rightarrow \min_{L^2(\Gamma)} \Psi(\mu) = 0 \qquad \mathrm{dist}\,(\gamma, \Gamma) > 0 \Rightarrow \inf_{L^2(\Gamma)} \Psi(\mu) = 0$$

Discretization

$$H_0^1(\Omega) \sim V_h, \quad L^2(\gamma) \sim \Lambda_H(\gamma), \quad L^2(\Gamma) \sim \Lambda_H(\Gamma)$$

$$\dim V_h = n, \quad \dim \Lambda_H(\gamma) = \dim \Lambda_H(\Gamma) = m$$

Problem $(\hat{\mathcal{P}}(\Gamma))_h^H$

Find $(\hat{u}_h, \lambda_H) \in V_h \times \Lambda_H(\Gamma)$ *such that*

$$(\hat{u}_h, v_h)_{1,\Omega} = (f, v_h)_{0,\Omega} + (\lambda_H, v_h)_{0,\Gamma} \quad \forall v_h \in V_h,$$

$$\delta_H \hat{u}_h = P(\delta_H \hat{u}_h - \rho(\tau_H \hat{u}_h - g_H)), \quad \rho > 0$$

$$\frac{\partial \hat{u}_{h|_\omega}}{\partial n_\gamma} \sim \delta_H \hat{u}_h, \ \ \hat{u}_{h|_\gamma} \sim \tau_H \hat{u}_h, \ \ g \sim g_H$$

Matrix form of $(\hat{\mathcal{P}}(\Gamma))_h^H$

$$V_h = \{\varphi_j\}_{j=1}^n, \;\; \Lambda_H(\gamma) = \{\psi_i\}_{i=1}^m, \;\; \Lambda_H(\Gamma) = \{\widetilde{\psi}_i\}_{i=1}^m$$

(A1) $\{\psi_i\}_{i=1}^m$ is orthonormal in $L^2(\gamma)$;

(A2) $\sigma = P\mu, \; \mu = \sum_{i=1}^m \mu_i\psi_i \Leftrightarrow \; \sigma_i = \max\{0, \mu_i\}, \; \sigma = \sum_{i=1}^m \sigma_i\psi_i$

$$g_H := \sum_{i=1}^m g_i\psi_i, \quad g_i = (\psi_i, g)_{0,\gamma},$$
$$\tau_H\hat{u}_h := \sum_{i=1}^m \mu_i\psi_i, \quad \mu_i = (\psi_i, \hat{u}_h)_{0,\gamma},$$

$$\nabla\hat{u}_h \sim \widetilde{\nabla}\hat{u}_h \in V_h, \quad \widetilde{\nabla}\hat{u}_h := \sum_{j=1}^n (\widetilde{\nabla}\hat{u}_h)_j\varphi_j,$$
$$(\widetilde{\nabla}\hat{u}_h)_j = \sum_{k\in\mathcal{K}} \alpha^{(k)}u_{j+k}, \; \alpha^{(k)} \in \mathbb{R}^2$$

$\delta_H\hat{u}_h := \sum_{i=1}^m \sigma_i\psi_i,$ where $\sigma_i = (\psi_i, \widetilde{\nabla}\hat{u}_h \cdot n_\gamma)_{0,\gamma}.$

Matrix form of $(\hat{\mathcal{P}}(\Gamma))_h^H$

> *Find* $(\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m$ *such that*
> $A\mathbf{u} = \mathbf{f} + B_\Gamma^\top \boldsymbol{\lambda},$
> $C_\gamma \mathbf{u} = \max\{0, C_\gamma \mathbf{u} - \rho(B_\gamma \mathbf{u} - \mathbf{g})\}.$

$A \in \mathbb{R}^{n\times n}$ $\ldots$ stiffness matrix,

$$B_\Gamma, B_\gamma \in \mathbb{R}^{m\times n} \quad \ldots \quad B_\gamma = (b_{\gamma,ij}),\ b_{\gamma,ij} = (\psi_i, \varphi_j)_{0,\gamma},$$
$$i = 1, \ldots, m,\ j = 1, \ldots, n,$$

$$C_\gamma \in \mathbb{R}^{m\times n} \quad \ldots \quad C_\gamma = (c_{\gamma,ij}),\ c_{\gamma,ij} = (\psi_i, \widetilde{\nabla}\varphi_j \cdot n_\gamma)_{0,\gamma},$$
$$i = 1, \ldots, m,\ j = 1, \ldots, n.$$

Piecewise smooth Newton method

$F : \mathbb{R}^{n+m} \mapsto \mathbb{R}^{n+m}$

$$F(\mathbf{y}) := \begin{pmatrix} A\mathbf{u} - B_\Gamma^\top \boldsymbol{\lambda} - \mathbf{f} \\ G(\mathbf{u}) \end{pmatrix}, \quad \mathbf{y} := \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix},$$

$$G(\mathbf{u}) := C_\gamma \mathbf{u} - \max\{0, C_\gamma \mathbf{u} - \rho(B_\gamma \mathbf{u} - \mathbf{g})\}.$$

$$F(\mathbf{y}) = 0$$

Piecewise smooth Newton method

$\mathbf{y}^0 \in \mathbb{R}^{n+m}$ given, $\quad k = 0;$
Find $\mathbf{d}^k \in \mathbb{R}^{n+m}$: $F(\mathbf{y}^k) + F^0(\mathbf{y}^k)\mathbf{d}^k = 0;$
set $\mathbf{y}^{k+1} = \mathbf{y}^k + \mathbf{d}^k,\ k := k+1,$
until stopping criterion.

$$F^o(\mathbf{y}) = \begin{pmatrix} A & -B_\Gamma^\top \\ G^o(\mathbf{u}) & 0 \end{pmatrix},$$

$$G_i^o(\mathbf{u}) = C_{\gamma,i} - s(C_{\gamma,i}\mathbf{u} - \rho(B_{\gamma,i}\mathbf{u} - g_i))\,(C_{\gamma,i} - \rho B_{\gamma,i}),$$

$$i = 1, \ldots, m$$

$$s : \mathbb{R} \mapsto \mathbb{R}, \quad s(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Active set strategy

$$\mathcal{I}(\mathbf{u}) := \{i \in \mathcal{M} : C_{\gamma,i}\mathbf{u} - \rho(B_{\gamma,i}\mathbf{u} - g_i) \leq 0\},$$
$$\mathcal{A}(\mathbf{u}) := \{1, 2, \ldots, m\} \setminus \mathcal{I}(\mathbf{u}).$$

$$G_i^o(\mathbf{u}) = \begin{cases} C_{\gamma,i}, & i \in \mathcal{I}(\mathbf{u}), \\ \rho B_{\gamma,i}, & i \in \mathcal{A}(\mathbf{u}). \end{cases}$$

Each iterative step leads to a non-symmetric saddle-point system

$$\begin{pmatrix} A & B_1^\top \\ B_2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{h} \end{pmatrix}$$

Projected Schur complement method (J.H., Kozubek, Kučera, Peichl, 2007).

Numerical examples

$H_0^1(\Omega) \sim H_{per}^1(\Omega)$;

$V_h \; \ldots \; Q_1$-elements

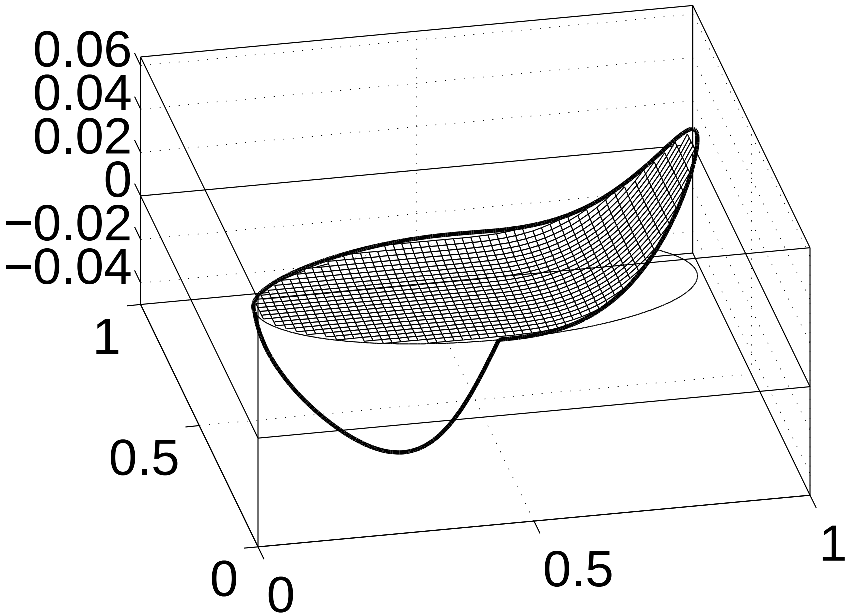
$\Lambda_H(\gamma)$, $\Lambda_H(\Gamma) \; \ldots \; P_0$-elements on polygonal approximations of γ and Γ, $H/h = |\log_2(h)|$

$\omega = \{(x, y) \in \mathbb{R}^2 | \; (x-0.5)^2/0.4^2 + (y-0.5)^2/0.2^2 < 1\}$

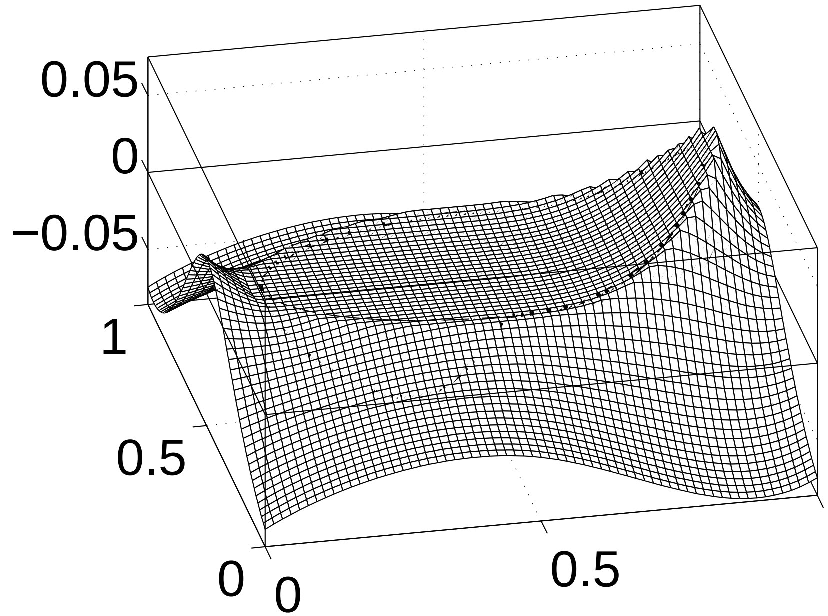
$\Omega = (0, 1) \times (0, 1)$

Example 1 (known solution)

$$u_{ex}(x,y) = ((x-0.5)^+)^3 + 0.5((y-0.5)^+)^3, \quad (x,y) \in \mathbb{R}^2$$



Graph of u_{ex}.



Computed solution $\hat{u}_h$.

Example 1 (known solution)

Step h	$n/m_{\mathcal{A}}/m_{\mathcal{I}}$	out./$\sum$inn. its.	C.time[s]	$\mathrm{Err}_{L^2(\omega)}$	$\mathrm{Err}_{H^1(\omega)}$	$\mathrm{Err}_{L^2(\gamma)}$
1/128	16641/24/10	5/24	0.31	4.1007e-2	3.3325e+0	1.7029e-2
1/256	66049/41/21	6/45	1.81	2.0074e-2	2.3310e+0	8.5208e-3
1/512	263169/77/33	7/69	13.07	9.7866e-3	1.6275e+0	4.2527e-3
1/1024	1050625/140/58	7/93	74.65	4.8435e-3	1.1449e+0	2.0830e-3
1/2048	4198401/261/99	7/115	432.6	2.2777e-3	7.8540e-1	1.0952e-3
1/4096	16785409/484/178	8/131	2328	1.1353e-3	5.5449e-1	7.2174e-4
Non-smooth variant		Convergence rates:		1.0374	0.5186	0.9346

Step h	$n/m_{\mathcal{A}}/m_{\mathcal{I}}$	out./$\sum$inn. its.	C.time[s]	$\mathrm{Err}_{L^2(\omega)}$	$\mathrm{Err}_{H^1(\omega)}$	$\mathrm{Err}_{L^2(\gamma)}$
1/128	16641/26/8	5/48	0.47	3.2409e-4	2.9532e-1	5.0704e-4
1/256	66049/46/16	5/69	2.56	6.3196e-5	1.3041e-1	9.1074e-5
1/512	263169/81/29	5/112	20.89	1.5917e-5	6.5444e-2	2.6525e-5
1/1024	1050625/147/51	7/162	150.6	4.3527e-6	3.4223e-2	1.1771e-5
1/2048	4198401/270/90	6/190	674.1	1.3812e-6	1.9278e-2	5.1861e-6
1/4096	16785409/494/168	9/296	5000	8.0760e-7	1.4741e-2	2.2854e-6
Smooth variant		Convergence rates:		1.7617	0.8809	1.5012

Example 1 (known solution)

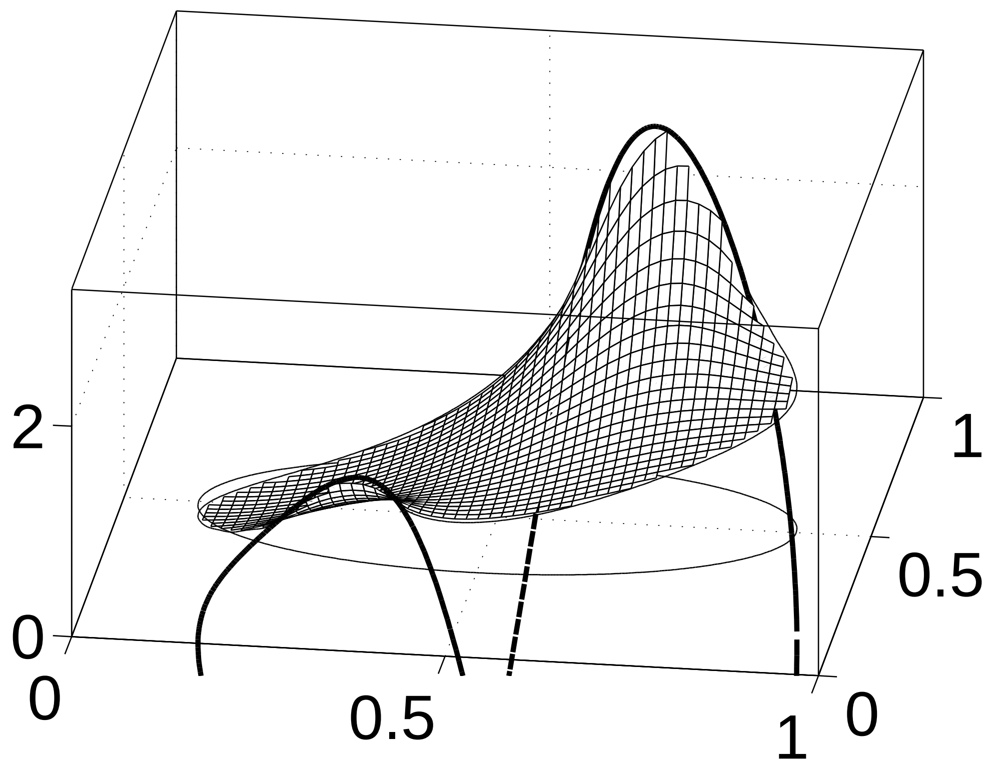
Dependance on $\delta := \operatorname{dist}(\gamma, \Gamma)$, $h = 1/256$.

Param. δ	$n/m_{\mathcal{A}}/m_{\mathcal{I}}$	Out./$\sum$inn. its.	C.time[s]	Err$_{L^2(\omega)}$	Err$_{H^1(\omega)}$	Err$_{L^2(\gamma)}$
$0h$	66049/41/21	6/45	1.81	2.0074e-2	2.3310e+0	8.5208e-3
$2h$	66049/44/18	5/42	1.68	1.6031e-3	6.5682e-1	1.4522e-3
$4h$	66049/46/16	5/56	2.06	8.8714e-5	1.5451e-1	1.6632e-4
$6h$	66049/46/16	5/69	2.56	6.3196e-5	1.3041e-1	9.1074e-5
$8h$	66049/46/16	5/101	3.62	6.0014e-5	1.2708e-1	5.1199e-5

Example 2 (without the absolute term)

$g(x, y) = 5\sin(2\varphi)(r^2 + r(\cos\varphi + \sin\varphi) + 0.5)^{1/2} - 1.5$ on γ,
where (φ, r) is the polar coordinate of the point $(x - 0.5, y - 0.5)$;

$f = -20$



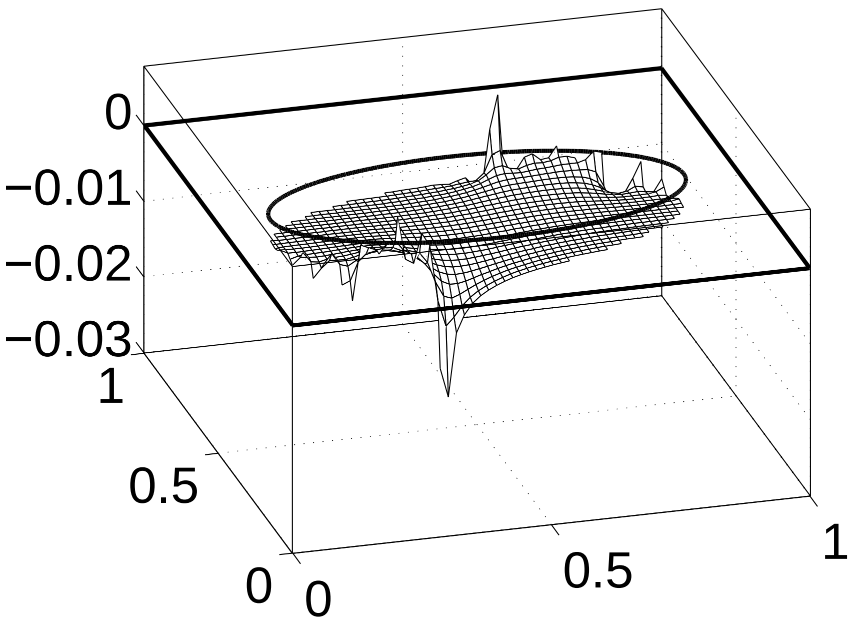
Reference solution u_{ref}.

Example 2 (without the absolute term)

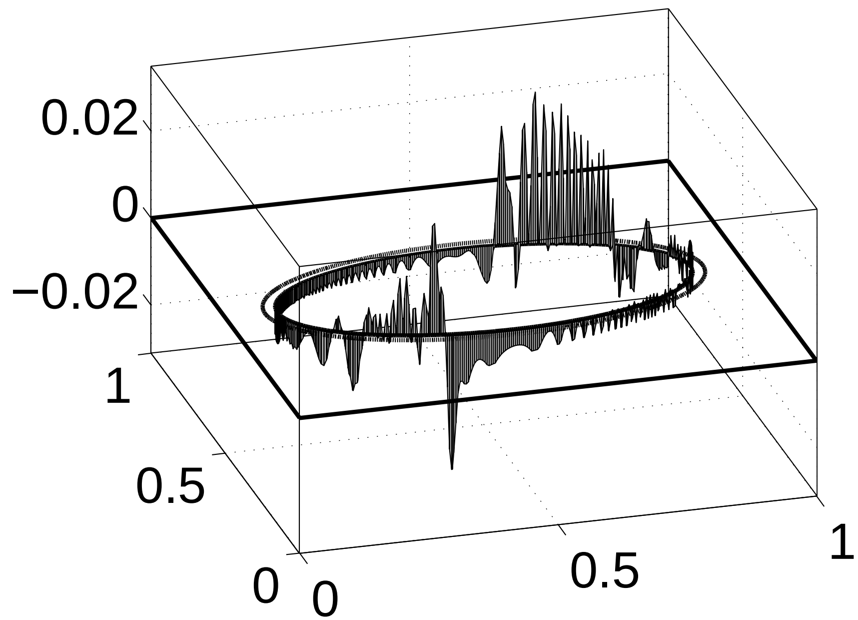
Step h	$n/m_{\mathcal{A}}/m_{\mathcal{I}}$	Out./$\sum$inn. its.	C.time[s]	$\mathrm{Err}_{L^2(\omega)}$	$\mathrm{Err}_{H^1(\omega)}$	$\mathrm{Err}_{L^2(\gamma)}$
1/128	16641/7/29	6/31	0.39	1.3521e-2	9.1807e-2	2.2136e-2
1/256	66049/13/49	6/54	2.3	7.0716e-3	6.0371e-2	6.9435e-3
1/512	263169/23/87	7/69	13.71	4.5768e-3	3.6564e-2	3.1013e-3
1/1024	1050625/41/157	9/127	106.3	2.7119e-3	3.8892e-2	1.6530e-3
1/2048	4198401/74/286	10/166	639.6	9.9011e-4	3.0425e-2	8.3333e-4
1/4096	16785409/136/526	11/241	4205	8.0521e-4	2.2513e-2	1.0264e-3
Non-smooth variant		Convergence rates:		0.8461	0.3719	0.9211

Step h	$n/m_{\mathcal{A}}/m_{\mathcal{J}}$	Out./$\sum$inn. its.	C.time[s]	$\mathrm{Err}_{L^2(\omega)}$	$\mathrm{Err}_{H^1(\omega)}$	$\mathrm{Err}_{L^2(\gamma)}$
1/128	16641/7/29	4/44	0.48	3.3419e-2	1.6955e-1	3.4271e-2
1/256	66049/14/48	5/86	3.58	6.5415e-3	8.8017e-2	7.9521e-3
1/512	263169/23/87	6/119	23.35	2.6697e-3	4.0197e-2	2.8341e-3
1/1024	1050625/41/157	7/221	174.9	3.2798e-4	8.3563e-3	4.7836e-4
1/2048	4198401/74/286	7/276	976.6	1.5904e-4	4.5718e-3	2.1104e-4
1/4096	16785409/136/526	9/442	7544	2.0672e-5	1.7129e-3	4.8686e-5
Smooth variant		Convergence rates:		2.0687	1.3775	1.8734

Numerical examples



Difference $\hat{u}_h - u_{ref}$ in ω.



Difference $\hat{u}_h - u_{ref}$ on γ.

Thank you for your attention.