Primal-dual active set strategies for plastic contact problems

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Outline

1. **Formulation of continuous problem**
   - Strong and weak form
   - Projection operators
2. **Discretization and numerical solution**
   - Nonlinear complementarity functions
   - Semismooth Newton method
   - Primal-dual active set strategy
3. **Numerical results**
4. **Extension: Overlapping domain decomposition**
5. **Summary and conclusions**

Problem setting

J2-plasticity for small deformations with linear hardening

$\boldsymbol{u}:$	$\Omega \times [0,T] \to \mathbb{R}^d$	Displacement field
$\boldsymbol{\sigma}:$	$\Omega \times [0,T] \to \mathbb{R}^{d\times d}$	Stress field
$\boldsymbol{\varepsilon}^p:$	$\Omega \times [0,T] \to \mathbb{R}^{d\times d}$	Plastic strain
$\alpha:$	$\Omega \times [0,T] \to \mathbb{R}$	Hard. variable
$\mu^{pl}:$	$\Omega \times [0,T] \to \mathbb{R}$	Plastic multiplier
$\boldsymbol{\lambda}:$	$\Gamma_c \times [0,T] \to \mathbb{R}^d$	Contact stress

$$\mathrm{div}(\boldsymbol{\sigma}) = \boldsymbol{f}, \quad \Omega \times [0,T]\text{, plus BC}$$ Mechanical equilibrium

$$\boldsymbol{\sigma} = \mathbf{C}^{el}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p),$$ Constitutive law

$$\dot{\boldsymbol{\varepsilon}}^p = \mu^{pl}\partial_{\boldsymbol{\sigma}}\phi^{pl}(\boldsymbol{\sigma},\alpha),$$

$$\mu^{pl} \geq 0, \qquad \phi^{pl} \leq 0, \qquad \mu^{pl}\phi^{pl} = 0,$$ Plasticity constraints

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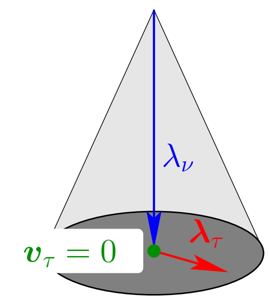
Mechanical equilibrium

Constitutive law

$$\begin{aligned} \mu^{pl} &\geq 0, \\ \phi^{pl} &\leq 0, \\ \mu^{pl}\phi^{pl} &= 0, \end{aligned}$$

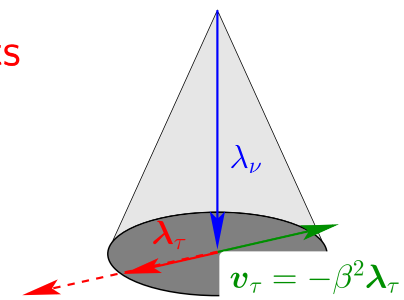
Plasticity constraints



$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{\nu} - g &\leq 0, \quad \Gamma_c \times [0,T] \\ \lambda_\nu &\geq 0, \\ \lambda_\nu(\boldsymbol{u}\cdot\boldsymbol{\nu} - g) &= 0, \\ \|\boldsymbol{\lambda}_\tau\| - \mathfrak{F}|\lambda_\nu| &\leq 0, \\ \dot{\boldsymbol{u}}_\tau + \mu^{co}\boldsymbol{\lambda}_\tau &= 0, \\ \mu^{co}(\|\boldsymbol{\lambda}_\tau\| - \mathfrak{F}|\lambda_\nu|) &= 0 \end{aligned}$$

Non-penetration constraints

Coulomb friction



Plasticity vs. contact, strong form

J2-plasticity for small deformations with linear isotropic/kinematic hardening

$$\begin{array}{ll} \boldsymbol{u}: & \text{Primal variable} \\ \boldsymbol{\varepsilon}^p, \alpha: & \text{Inner variables} \end{array}$$

$$\begin{aligned} \operatorname{div}(\boldsymbol{\sigma}) &= \boldsymbol{f}, && \text{in } \Omega \\ \boldsymbol{\sigma} &:= \mathbb{C}^{el}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^p), && \text{in } \Omega \\ \boldsymbol{\eta} &:= \operatorname{dev}(\boldsymbol{\sigma}) - \tfrac{2}{3}K\boldsymbol{\varepsilon}^p, && \text{in } \Omega \end{aligned}$$

Contact problem for small deformations with Coulomb or Tresca friction

$$\begin{array}{ll} \boldsymbol{u}: & \text{Primal variable} \\ \boldsymbol{\lambda} = \lambda_\nu \nu + \boldsymbol{\lambda}_\tau: & \text{Dual variables} \end{array}$$

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$$\begin{aligned} Y(\alpha) &:= \sigma_0 + H\alpha, && \text{in } \Omega \\ \phi^{pl}(\boldsymbol{\eta}, \alpha) &:= \|\boldsymbol{\eta}\| - Y(\alpha), && \text{in } \Omega \\ \dot{\boldsymbol{\varepsilon}}^p &= \mu^{pl}\partial_{\boldsymbol{\eta}}\phi^{pl}(\boldsymbol{\eta}, \alpha), && \text{in } \Omega \end{aligned}$$

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Y(\alpha) &:= \sigma_0 + H\alpha, && \text{in } \Omega \\
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\mu^{pl} &\geq 0, && \text{in } \Omega \\
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\dot{\alpha} &= \mu^{pl}, && \text{in } \Omega
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\end{aligned}$$

Plasticity vs. contact, projections

Elastic region for relative deviatoric stress $\boldsymbol{\eta}$:

$$\mathbb{E} := \{(\boldsymbol{\tau}, \beta) : \phi^{pl}(\boldsymbol{\tau}, \beta) \leq 0\},$$

Admissible stresses:

$$\mathbb{M}(\lambda_\nu) := \{(\boldsymbol{\mu}_\tau, \mu_\nu) : \phi^{co}(\boldsymbol{\mu}_\tau, \lambda_\nu) \leq 0, \quad \mu_\nu \geq 0\},$$

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$$P_{p,\alpha}(\boldsymbol{\eta}) := \min\left(1, \frac{Y(\alpha)}{\|\boldsymbol{\eta}\|}\right)\boldsymbol{\eta},$$

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$$P_{\tau,\lambda_\nu}(\boldsymbol{\lambda}_\tau) := \min\left(1, \frac{Y(\lambda_\nu)}{\|\boldsymbol{\lambda}_\tau\|}\right)\boldsymbol{\lambda}_\tau,$$

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Trial values ($c_p > 0$):

$$\boldsymbol{\eta}^{tr} := \boldsymbol{\eta} + c_p\left(\boldsymbol{\varepsilon}^p - \boldsymbol{\varepsilon}^p_{old}\right),$$

Special choice $c_p = (2\mu + \frac{2}{3}K)$:

$$\boldsymbol{\eta}^{tr} = 2\mu\,\mathrm{dev}(\boldsymbol{\varepsilon}(\boldsymbol{u})) - c_p\boldsymbol{\varepsilon}^p_{old},$$

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Trial values ($c_\tau, c_\nu > 0$):

$$\boldsymbol{\lambda}^{tr}_\tau := \boldsymbol{\lambda}_\tau + c_\tau\left(\boldsymbol{u}_\tau - \boldsymbol{u}_{\tau,old}\right),$$

$$\lambda^{tr}_\nu := \lambda_\nu + c_\nu(\boldsymbol{u}_\nu - g).$$

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KKT conditions are equivalent to

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Quasi-variational inequality

Find $\boldsymbol{u} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$, $(\boldsymbol{\eta}, \alpha) \in \mathbb{E}$, $\boldsymbol{\lambda} \in \mathbb{M}(\lambda_\nu)$

equilibrium $\quad a(\boldsymbol{u}, \boldsymbol{v}) + b^{pl}(\epsilon(\boldsymbol{v}), \boldsymbol{\varepsilon}^p) + b^{co}(\boldsymbol{v}, \boldsymbol{\lambda}) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega = 0, \qquad \boldsymbol{v} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$

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plasticity	$-b^{pl}(\dot{\boldsymbol{\varepsilon}}^p, (C^{el})^{-1}(\boldsymbol{\tau}-\boldsymbol{\eta})) - b^{hr}(\dot\alpha, \beta-\alpha) \le 0,$	$(\boldsymbol{\tau},\beta) \in \mathbb{E}$
friction	$b^{co}(\dot{\boldsymbol{u}}_\tau, \boldsymbol{\mu}-\boldsymbol{\lambda}) \le 0,$	$\boldsymbol{\mu} \in \mathbb{M}(\lambda_\nu)$
non-penetration	$b^{co}(u_\nu\boldsymbol{\nu}, \boldsymbol{\mu}-\boldsymbol{\lambda}) - \langle g\boldsymbol{\nu}, \boldsymbol{\mu}-\boldsymbol{\lambda}\rangle_{\Gamma_c} \le 0,$	$\boldsymbol{\mu} \in \mathbb{M}(\lambda_\nu)$

with

$$
\begin{aligned}
\mathbb{M}(\lambda_\nu) &:= \{(\boldsymbol{\mu}_\tau, \mu_\nu) : \phi^{co}(\boldsymbol{\mu}_\tau, \lambda_\nu) \le 0, \mu_\nu \ge 0\},\\
\mathbb{E} &:= \{(\boldsymbol{\tau},\beta) : \phi^{pl}(\boldsymbol{\tau},\beta) \le 0\},\\
a(\boldsymbol{u},\boldsymbol{v}) &:= \int_\Omega C^{el}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}),\\
b^{pl}(\boldsymbol{\varepsilon},\boldsymbol{\tau}) &:= -\int_\Omega C^{el}\boldsymbol{\varepsilon} : \boldsymbol{\tau},\\
b^{hr}(\alpha,\beta) &:= \int_\Omega H\alpha\beta,\\
b^{co}(\boldsymbol{v},\boldsymbol{\mu}) &:= \int_{\Gamma_C} \boldsymbol{v}\cdot\boldsymbol{\mu}
\end{aligned}
$$

From inequality to equality

Algebraic inequality formulation in finite element basis

Equilibrium	$A_h \boldsymbol{u}_h - B_h^{pl} \boldsymbol{\varepsilon}_h^p + B_h^{co} \boldsymbol{\lambda}_h = \boldsymbol{f}_h,$
Plasticity	discrete KKT
Contact & friction	discrete KKT

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Inequality constraints are equivalent to $C_h := (C^{pl}, C^{co}) = \mathbf{0}$ with

$$C^{pl}(\boldsymbol{u}_h, (\boldsymbol{\varepsilon}_h^p, \alpha_h)) := \big(\boldsymbol{\eta}_h - P_{p,\alpha}(\boldsymbol{\eta}_h^{tr})\big)$$

$$C^{co}(\boldsymbol{u}_h, \boldsymbol{\lambda}_h) := \begin{pmatrix} (\boldsymbol{\lambda}_h)_\nu - P_\nu((\boldsymbol{\lambda}_h)_\nu^{tr}) \\ \big((\boldsymbol{\lambda}_h)_\tau - P_{\tau,\lambda_\nu^{tr}}((\boldsymbol{\lambda}_h)_\tau^{tr})\big) \end{pmatrix}$$

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Equivalent nonlinear system:

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Inequality constraints are equivalent to $C_h := (C^{pl}, C^{co}) = \mathbf{0}$ with

$$C^{pl}(\boldsymbol{u}_h, (\boldsymbol{\varepsilon}_h^p, \alpha_h)) := \max\{\|\boldsymbol{\eta}_h^{tr}\|, Y(\alpha)\} \left(\boldsymbol{\eta}_h - P_{p,\alpha}(\boldsymbol{\eta}_h^{tr})\right)$$

$$C^{co}(\boldsymbol{u}_h, \boldsymbol{\lambda}_h) := \begin{pmatrix} (\boldsymbol{\lambda}_h)_\nu - P_\nu((\boldsymbol{\lambda}_h)_\nu^{tr}) \\ \max\{\|(\boldsymbol{\lambda}_h)_\tau^{tr}\|, Y((\boldsymbol{\lambda}_h)_\nu^{tr})\} \left((\boldsymbol{\lambda}_h)_\tau - P_{\tau,\lambda_\nu^{tr}}((\boldsymbol{\lambda}_h)_\tau^{tr})\right) \end{pmatrix}$$

Equivalent nonlinear system:

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Additional factor leads to different Newton scheme

$\longrightarrow$ better stability properties ?

Semismooth Newton method

Alart/Curnier [88,91], Hintermüller et al [02], Christensen/Pang [98], Christensen [02]

$$
\begin{aligned}
A_h \delta \boldsymbol{u}_h^{(k)} - B_h^{pl} \delta \boldsymbol{\varepsilon}_h^{p,(k)} + B_h^{co} \delta \boldsymbol{\lambda}_h^{(k)} &= \boldsymbol{f}_h - A_h \boldsymbol{u}_h^{(k)} + B_h^{pl} \boldsymbol{\varepsilon}_h^{p,(k)} - B_h^{co} \boldsymbol{\lambda}_h^{(k)}, \\
\partial_{\boldsymbol{u}} C_h^{pl,(k)} \delta \boldsymbol{u}_h^{(k)} + \partial_{(\boldsymbol{\varepsilon}_h^p, \alpha_h)} C_h^{pl,(k)} \delta(\boldsymbol{\varepsilon}_h^{p,(k)}, \alpha_h^{(k)}) &= -C_h^{pl}(\boldsymbol{u}_h^{(k)}, (\boldsymbol{\varepsilon}_h^{p,(k)}, \alpha_h^{(k)})), \\
\partial_{\boldsymbol{u}} C_h^{co,(k)} \delta \boldsymbol{u}_h^{(k)} + \partial_{\boldsymbol{\lambda}} C_h^{co,(k)} \delta \boldsymbol{\lambda}_h^{(k)} &= -C_h^{co}(\boldsymbol{u}_h^{(k)}, \boldsymbol{\lambda}^{(k)}).
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$$\begin{aligned} A_h \delta \boldsymbol{u}_h^{(k)} - B_h^{pl} \delta \boldsymbol{\varepsilon}_h^{p,(k)} + B_h^{co} \delta \boldsymbol{\lambda}_h^{(k)} &= \boldsymbol{f}_h - A_h \boldsymbol{u}_h^{(k)} + B_h^{pl} \boldsymbol{\varepsilon}_h^{p,(k)} - B_h^{co} \boldsymbol{\lambda}_h^{(k)}, \\ \partial_{\boldsymbol{u}} C_h^{pl,(k)} \delta \boldsymbol{u}_h^{(k)} + \partial_{(\boldsymbol{\varepsilon}_h^p, \alpha_h)} C_h^{pl,(k)} \delta(\boldsymbol{\varepsilon}_h^{p,(k)}, \alpha_h^{(k)}) &= -C_h^{pl}(\boldsymbol{u}_h^{(k)}, (\boldsymbol{\varepsilon}_h^{p,(k)}, \alpha_h^{(k)})), \\ \partial_{\boldsymbol{u}} C_h^{co,(k)} \delta \boldsymbol{u}_h^{(k)} + \partial_{\boldsymbol{\lambda}} C_h^{co,(k)} \delta \boldsymbol{\lambda}_h^{(k)} &= -C_h^{co}(\boldsymbol{u}_h^{(k)}, \boldsymbol{\lambda}^{(k)}). \end{aligned}$$

Introduction of **active sets:** (index $\cdot^{(k)}$ omitted)

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$$A_h \delta \boldsymbol{u}_h^{(k)} - B_h^{pl} \delta \boldsymbol{\varepsilon}_h^{p,(k)} + B_h^{co} \delta \boldsymbol{\lambda}_h^{(k)} = \boldsymbol{f}_h - A_h \boldsymbol{u}_h^{(k)} + B_h^{pl} \boldsymbol{\varepsilon}_h^{p,(k)} - B_h^{co} \boldsymbol{\lambda}_h^{(k)},$$

$$\partial_{\boldsymbol{u}} C_h^{pl,(k)} \delta \boldsymbol{u}_h^{(k)} + \partial_{(\boldsymbol{\varepsilon}_h^p, \alpha_h)} C_h^{pl,(k)} \delta(\boldsymbol{\varepsilon}_h^{p,(k)}, \alpha_h^{(k)}) = -C_h^{pl}(\boldsymbol{u}_h^{(k)}, (\boldsymbol{\varepsilon}_h^{p,(k)}, \alpha_h^{(k)})),$$

$$\partial_{\boldsymbol{u}} C_h^{co,(k)} \delta \boldsymbol{u}_h^{(k)} + \partial_{\boldsymbol{\lambda}} C_h^{co,(k)} \delta \boldsymbol{\lambda}_h^{(k)} = -C_h^{co}(\boldsymbol{u}_h^{(k)}, \boldsymbol{\lambda}^{(k)}).$$

Introduction of **active sets:** (index $\cdot^{(k)}$ omitted)

plasticity: elementwise ($\mathcal{E}_h$)

$$\mathcal{A}_\alpha := \{ j \in \mathcal{E}_h : \Delta\alpha_j := \alpha_j - \alpha_{j,old} > 0 \},$$

$$\mathcal{I}_\alpha := \{ j \in \mathcal{E}_h : \Delta\alpha_j \le 0 \},$$

$$\mathcal{A}_p := \left\{ j \in \mathcal{E}_h : \phi^{pl}(\boldsymbol{\eta}_j^{tr}, \alpha_j) > 0 \right\},$$

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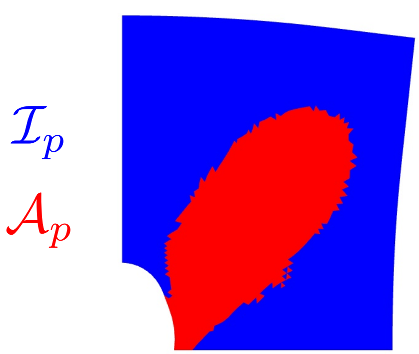


Semismooth Newton method

Alart/Curnier [88,91], Hintermüller et al [02], Christensen/Pang [98], Christensen [02]

$$\begin{aligned} A_h\delta\boldsymbol{u}_h^{(k)} - B_h^{pl}\delta\boldsymbol{\varepsilon}_h^{p,(k)} + B_h^{co}\delta\boldsymbol{\lambda}_h^{(k)} &= \boldsymbol{f}_h - A_h\boldsymbol{u}_h^{(k)} + B_h^{pl}\boldsymbol{\varepsilon}_h^{p,(k)} - B_h^{co}\boldsymbol{\lambda}_h^{(k)}, \\ \partial_{\boldsymbol{u}} C_h^{pl,(k)}\delta\boldsymbol{u}_h^{(k)} + \partial_{(\boldsymbol{\varepsilon}_h^p,\alpha_h)} C_h^{pl,(k)}\delta(\boldsymbol{\varepsilon}_h^{p,(k)}, \alpha_h^{(k)}) &= -C_h^{pl}(\boldsymbol{u}_h^{(k)}, (\boldsymbol{\varepsilon}_h^{p,(k)}, \alpha_h^{(k)})), \\ \partial_{\boldsymbol{u}} C_h^{co,(k)}\delta\boldsymbol{u}_h^{(k)} + \partial_{\boldsymbol{\lambda}} C_h^{co,(k)}\delta\boldsymbol{\lambda}_h^{(k)} &= -C_h^{co}(\boldsymbol{u}_h^{(k)}, \boldsymbol{\lambda}^{(k)}). \end{aligned}$$

Introduction of **active sets:** (index $\cdot^{(k)}$ omitted)

plasticity: elementwise ($\mathcal{E}_h$)

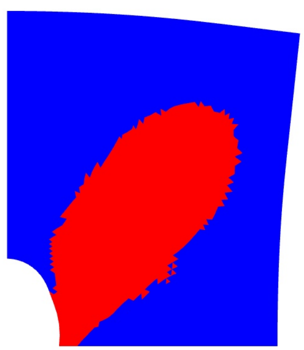
$$\begin{aligned} \mathcal{A}_\alpha &:= \{j \in \mathcal{E}_h : \Delta\alpha_j := \alpha_j - \alpha_{j,old} > 0\}, \\ \mathcal{I}_\alpha &:= \{j \in \mathcal{E}_h : \Delta\alpha_j \le 0\}, \\ \mathcal{A}_p &:= \{j \in \mathcal{E}_h : \phi^{pl}(\boldsymbol{\eta}_j^{tr}, \alpha_j) > 0\}, \\ \mathcal{I}_p &:= \{j \in \mathcal{E}_h : \phi^{pl}(\boldsymbol{\eta}_j^{tr}, \alpha_j) \le 0\}. \end{aligned}$$

contact: nodewise ($\mathcal{N}_h$)

$$\begin{aligned} \mathcal{A}_n &:= \{j \in \mathcal{N}_h : \lambda_{j,n}^{tr} > 0\}, \\ \mathcal{I}_n &:= \{j \in \mathcal{N}_h : \lambda_{j,n}^{tr} \le 0\}, \\ \mathcal{A}_t &:= \{j \in \mathcal{N}_h : \phi^{co}(\boldsymbol{\lambda}_{j,t}^{tr}, \lambda_{j,n}^{tr}) > 0\}, \\ \mathcal{I}_t &:= \{j \in \mathcal{N}_h : \phi^{co}(\boldsymbol{\lambda}_{j,t}^{tr}, \lambda_{j,n}^{tr}) \le 0\}. \end{aligned}$$

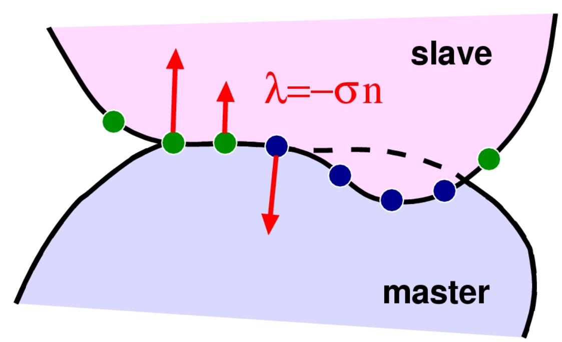
$\mathcal{I}_p$

$\mathcal{A}_p$



$\mathcal{I}_n$

$\mathcal{A}_n$



Primal-dual active set strategy: Plasticity

Local static condensation of plastic dof is possible:

$$\begin{pmatrix}\delta\alpha_j\\ \delta\boldsymbol{\varepsilon}^p_j\end{pmatrix} = \begin{cases} -\begin{pmatrix}\Delta\alpha_j\\ \Delta\boldsymbol{\varepsilon}^p_j\end{pmatrix} & j\in\mathcal{I}_p\cap\mathcal{I}_\alpha,\\ -\begin{pmatrix}\Delta\alpha_j\\ \left(1+\frac{H\Delta\alpha_j}{\sigma_y+H\alpha_j}\right)\Delta\boldsymbol{\varepsilon}^p_j\end{pmatrix} & j\in\mathcal{I}_p\cap\mathcal{A}_\alpha,\\ \begin{pmatrix}G_j\\ M_j\end{pmatrix} 2\mu\,\mathrm{dev}\,\boldsymbol{\varepsilon}(\delta\boldsymbol{u}) + \begin{pmatrix}g_j\\ \boldsymbol{m}_j\end{pmatrix} & j\in\mathcal{A}_p,\end{cases}$$

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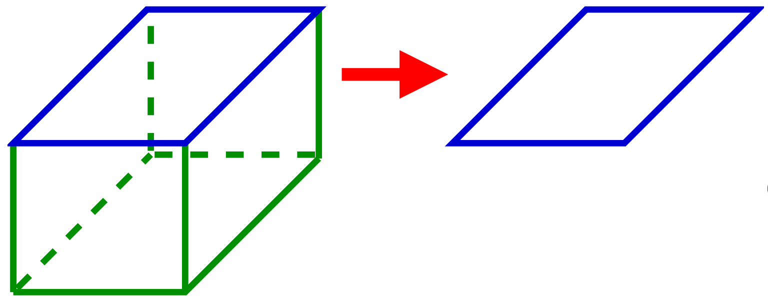
$\rightarrow$ System to be solved does not contain additional plastic dof.

Structure of NCP function C^{co}

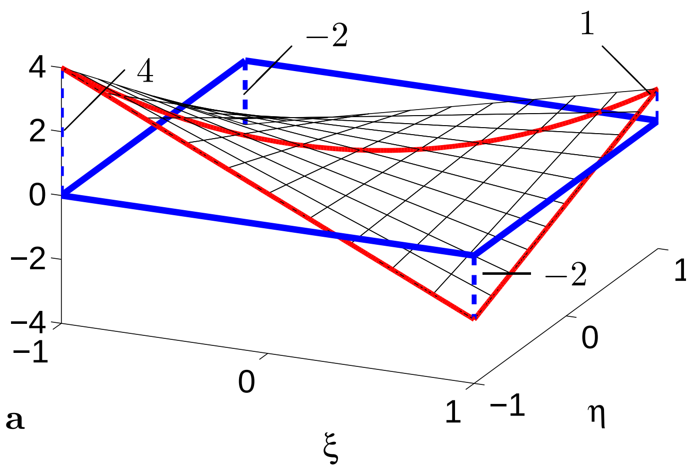
FE discretization:

Hexahedrals

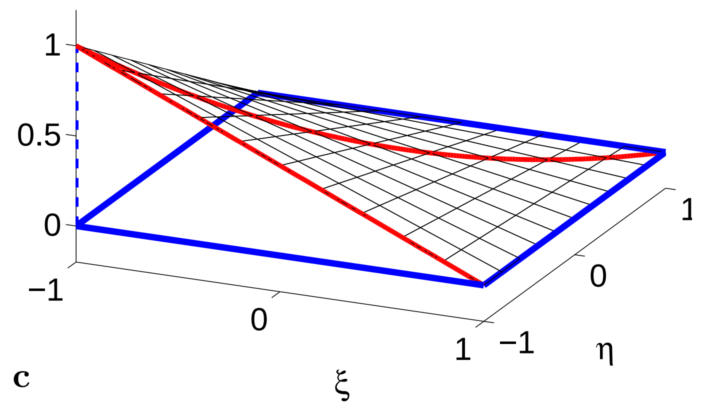


Interface:

Quadrilaterals



ψ_p, dual LM



φ_q, standard bilinear nodal shape function

Biorthogonality: $\mathbf{D}[p,q] = \int_{\square} \psi_p \varphi_q \, \mathbf{I}_3 = \delta_{pq} \int_{\square} \varphi_q \, \mathbf{I}_3 \;\Rightarrow\; \mathbf{D}$ diagonal

Complementary condition C^{co} decouples into **nodewise conditions** $C^{co}_j(\boldsymbol{u}_j, \boldsymbol{\lambda}_j)$.

Primal-dual active set strategy: Contact

Static condensation of contact LM is not always possible.

Primal-dual active set strategy: Contact

Static condensation of contact LM is not always possible.
But: Either primal or dual variable can be eliminated locally.

$$
\begin{aligned}
\delta\lambda_{j,n} &= -\lambda_{j,n} && j \in \mathcal{I}_n,\\
\delta u_{j,n} &= g_{j,n} - u_{j,n} && j \in \mathcal{A}_n,\\
\delta \boldsymbol{u}_{j,t} &= -\Delta \boldsymbol{u}_{j,t} && j \in \mathcal{I}_t \cap \mathcal{I}_n,\\
\delta \boldsymbol{u}_{j,t} &= -\tfrac{\mathfrak{F} c_t \Delta \boldsymbol{u}_{j,t}}{g_t + \mathfrak{F}\lambda_{j,n}^{tr}} \left(\delta\lambda_{j,n} + c_n \delta u_{j,n}\right) - \Delta \boldsymbol{u}_{j,t} && j \in \mathcal{I}_t \cap \mathcal{A}_n,\\
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Features: $\longrightarrow$ Matrix N_j is symmetric, N_j is not.

$\longrightarrow$ Safety factor $a_j := \min\left(1, \frac{Y_C(\lambda^{tr}_{j,n})}{\|\boldsymbol{\lambda}_{j,t}\|}\right)$ enhances stability.

$\longrightarrow$ System to be solved has size of displacement dof.

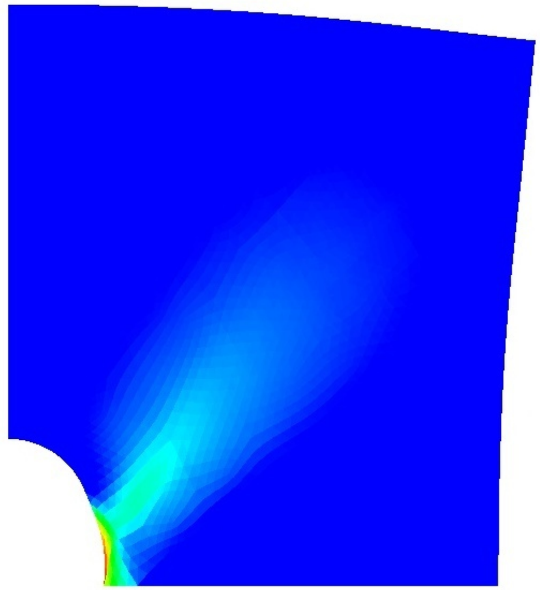
$\longrightarrow$ Lagrange multiplier can be computed in a post-process.

Plastic example: Plate with hole

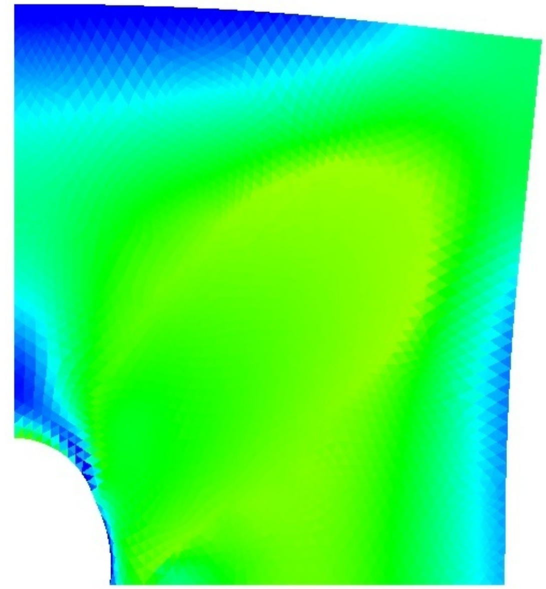
- Plate of length 2.0, circular hole of radius 0.2.
- On upper boundary, plate is pulled with force $f_t \cdot |0.5 - t|$ upwards for $t \in [0, 1]$ and $f_t = 1100$.
- Quasistatic computation with $\Delta t = \frac{1}{16}$.
- Material parameters

ν E	0.33 $6.9 \cdot 10^4$
σ_y H, K	$7.07 \cdot 10^2$ 0

- Active sets are updated in each Newton iteration, $c_p = 2\mu + a_0^{-2}K$.



Equiv. plastic strain α



Effective stress σ_{eff}

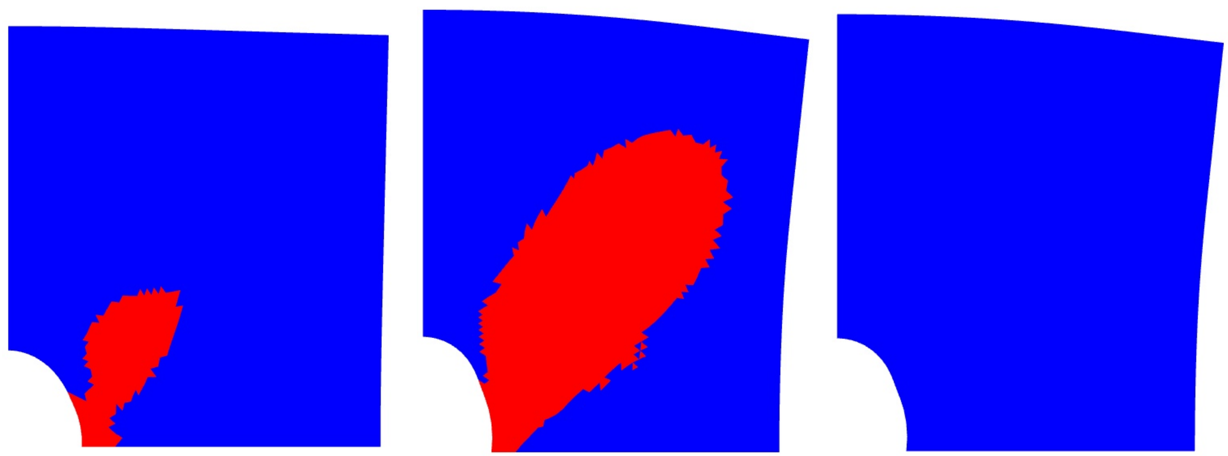
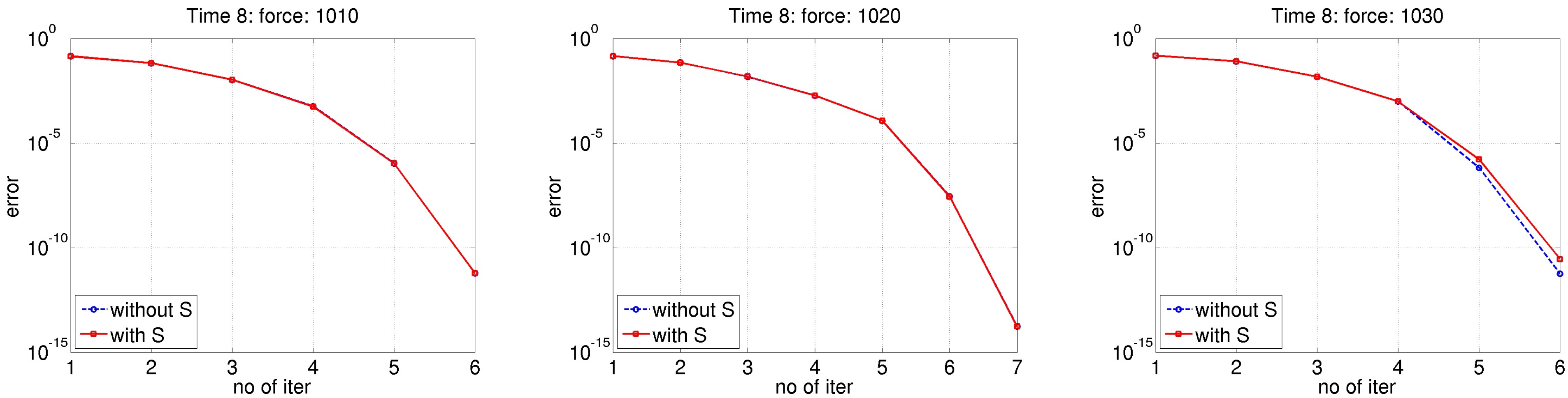
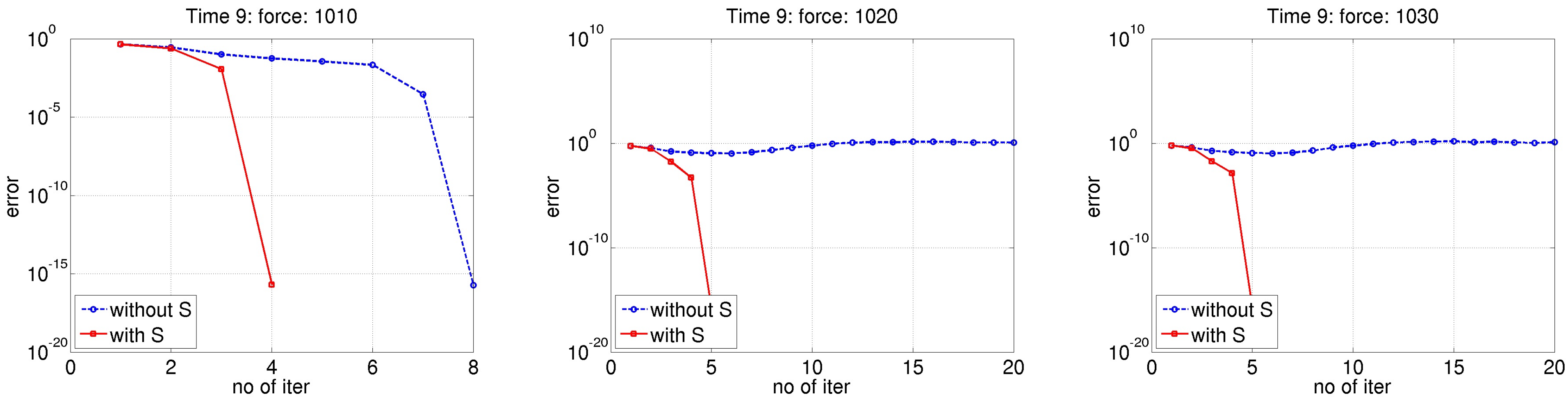


Plate with hole: Convergence

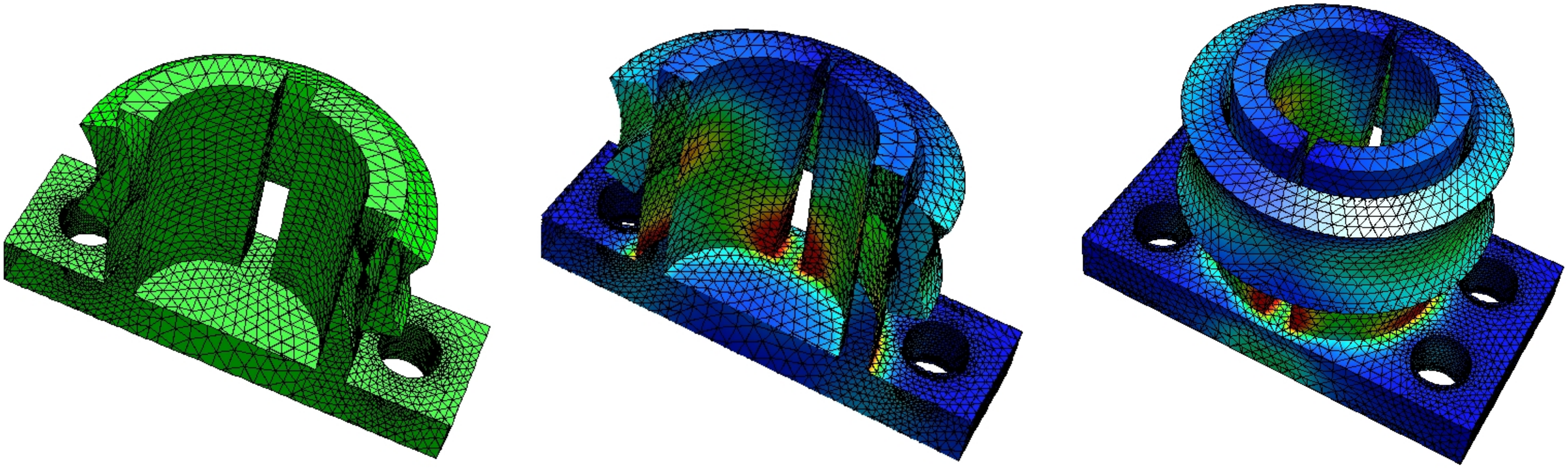
time step t_8, different forces f_t:

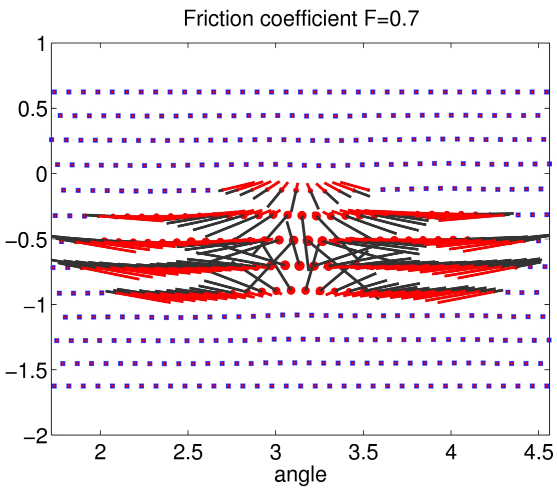


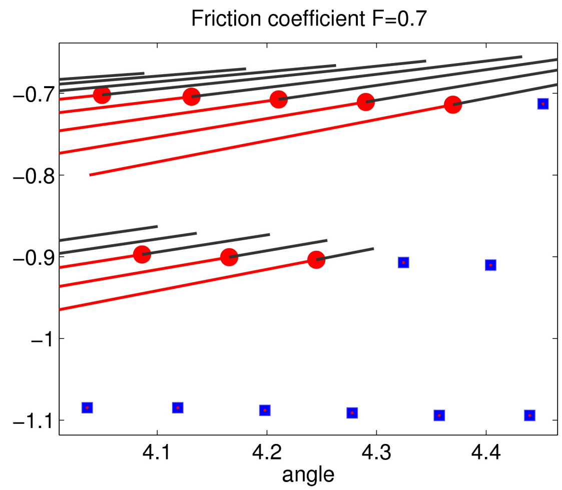
time step t_9, different forces f_t:

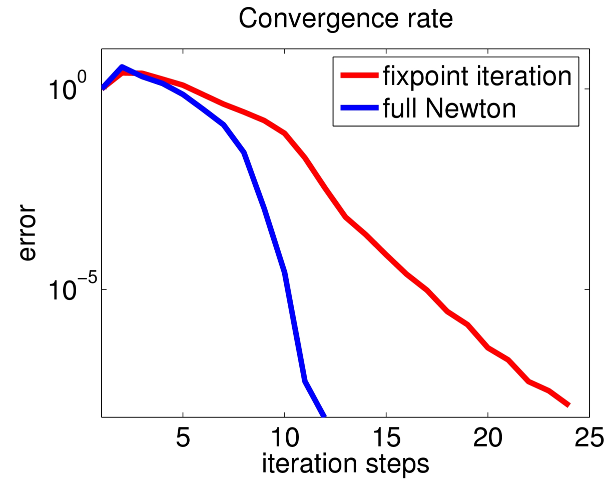


Contact example with Coulomb friction



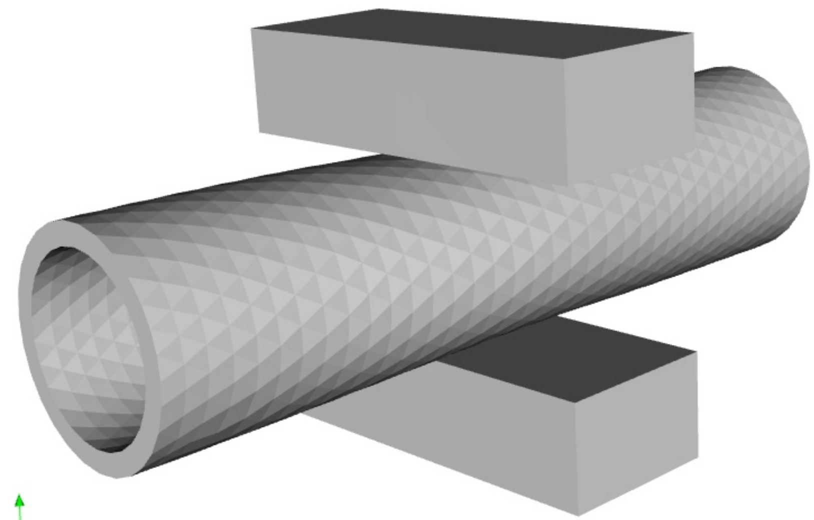






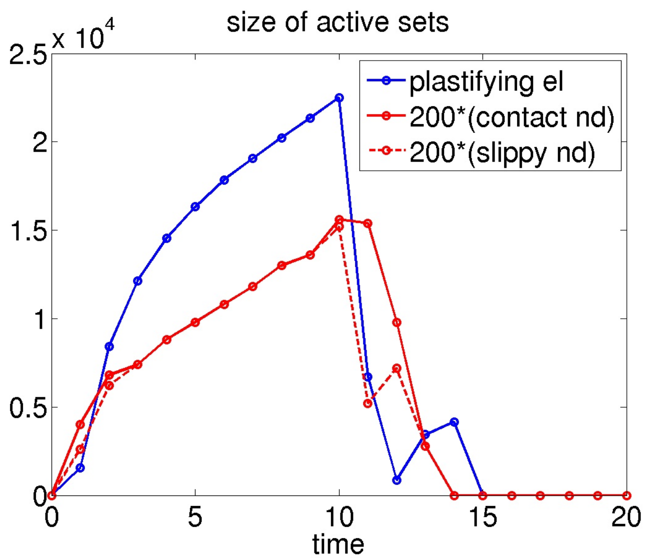
→ **Stabilized** Newton shows local superlinear and good global convergence

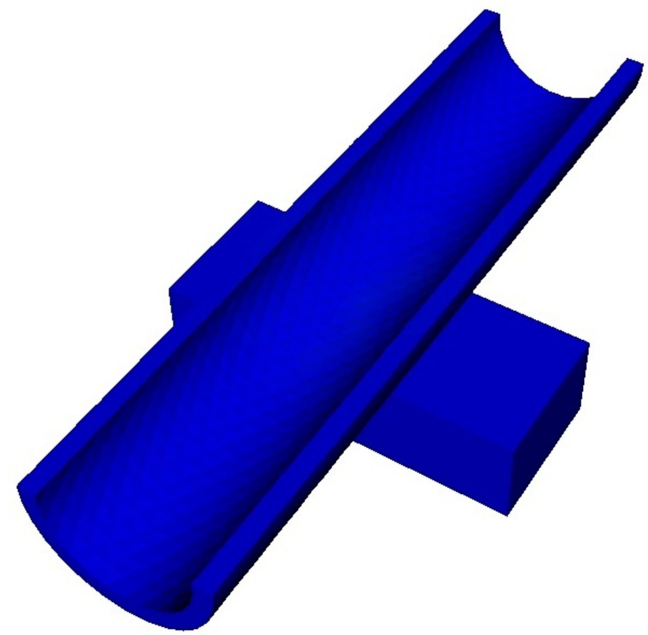
Combined example: Tube between plates



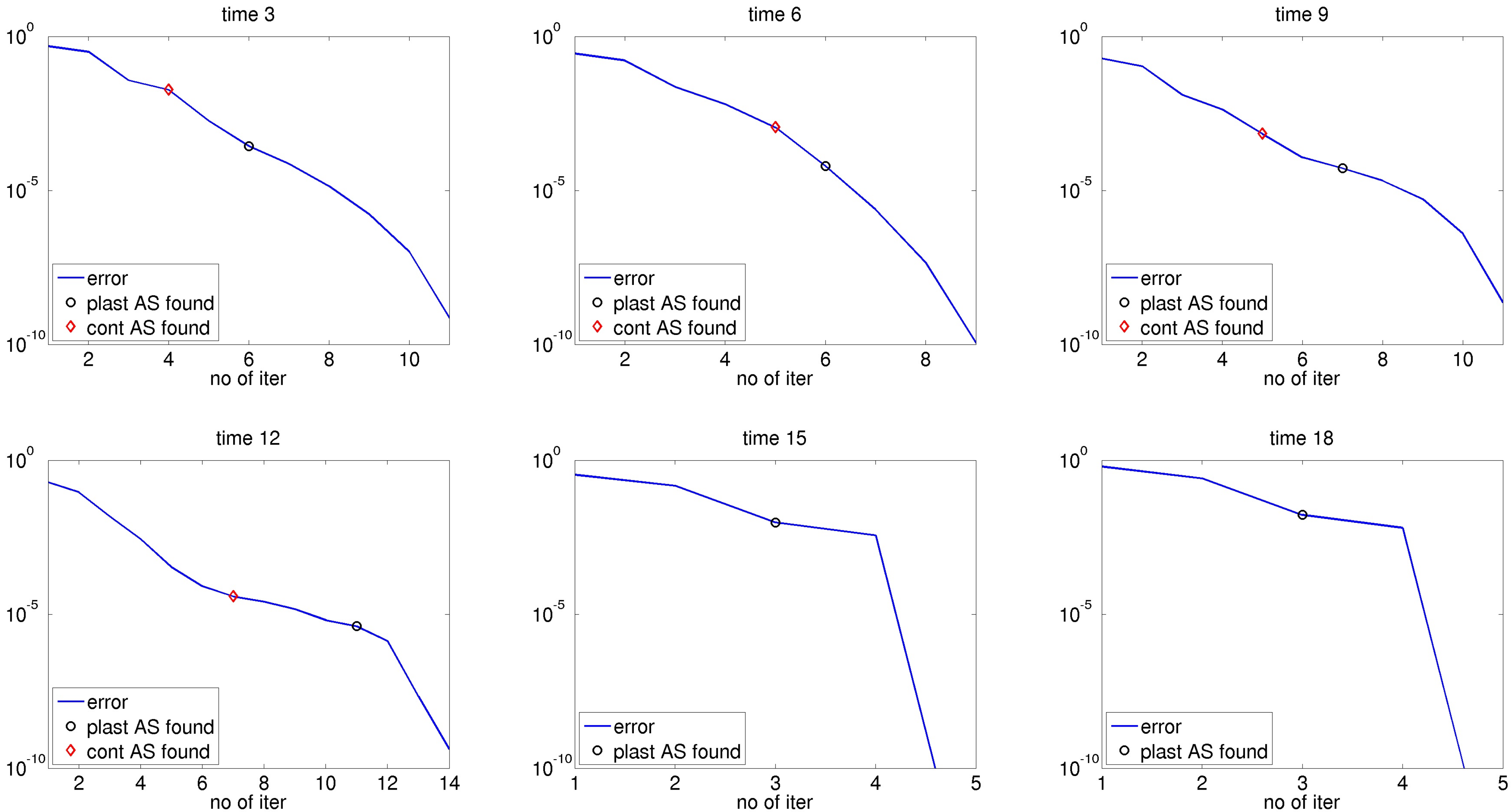
- Circular tube of length 4; fixed at one end, free otherwise.
- On upper and lower plate fixed displacement with $f_z(t) = 0.02 \cdot \min(t, 21 - t)$ at t-th load step, $t \in \{0, \ldots, 21\}$.
- All active sets updated in each Newton iteration.

	tube	plates
ν	0.3	0.3
E	$7 \cdot 10^3$	$7 \cdot 10^5$
σ_y	$2 \cdot 10^2$	10^3
H	$2 \cdot 10^3$	$2 \cdot 10^3$
K	10^3	10^3
g_t	0	0
$\mathfrak{F}$	0.1	0.1



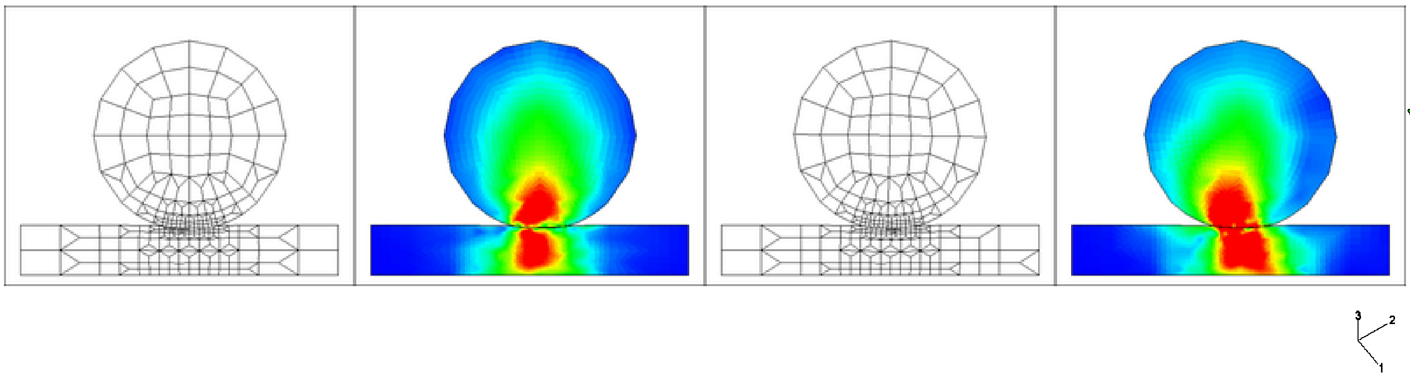


Tube between plates: Convergence



Extension: Overlapping domain decomposition

- Adaptive FEM: Frequent remeshing (Contact zone permanently moving)



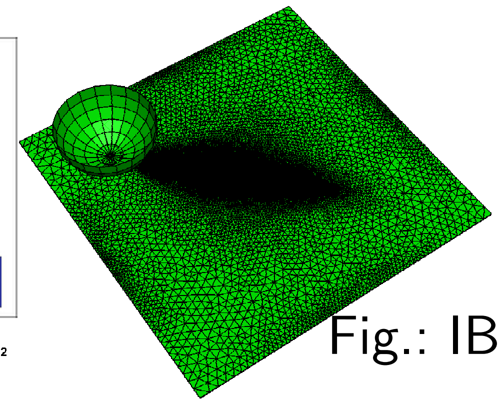
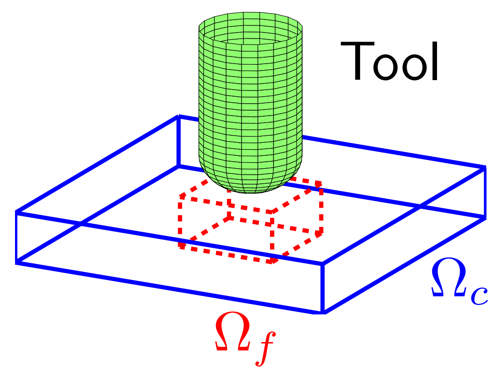


Fig.: IBF



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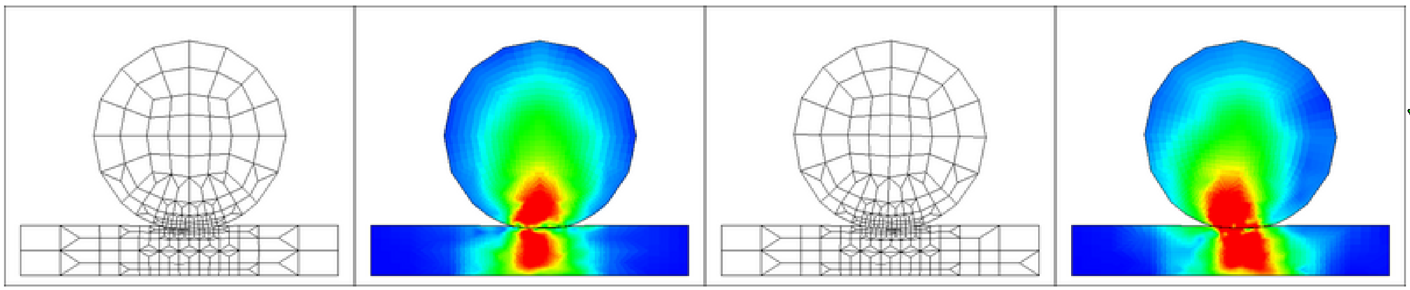


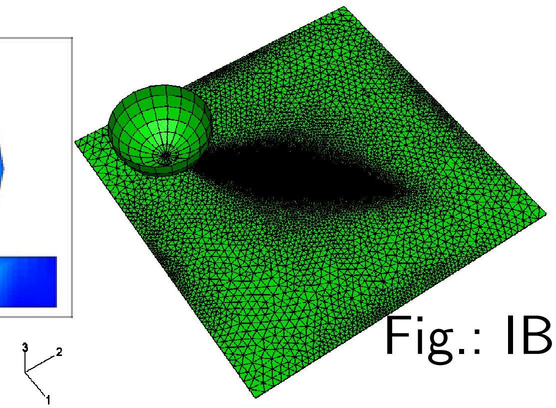
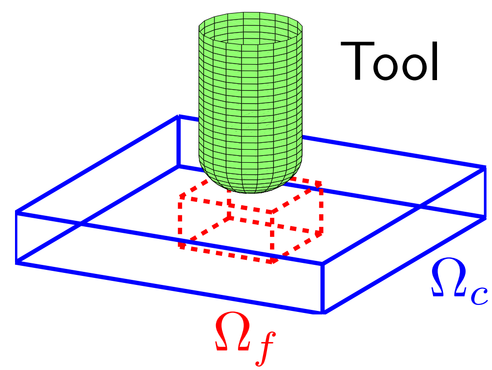


Fig.: IBF

- **Idea 1**: Fine grid Ω_f moving with forming zone

  $\longrightarrow$ Less coarse remeshings necessary

  $\longrightarrow$ dof saved, no “toward refining“



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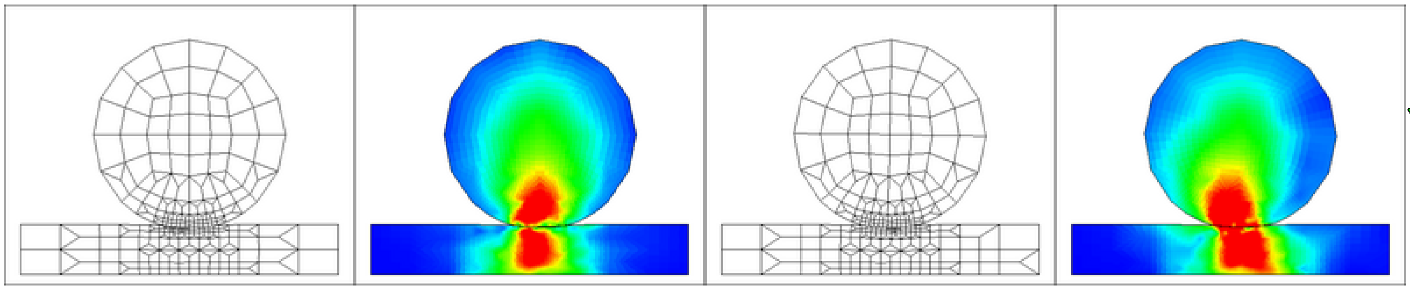

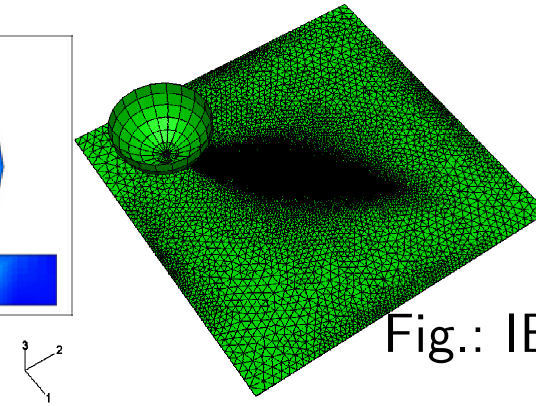

Fig.: IBF

- **Idea 1**: Fine grid Ω_f moving with forming zone
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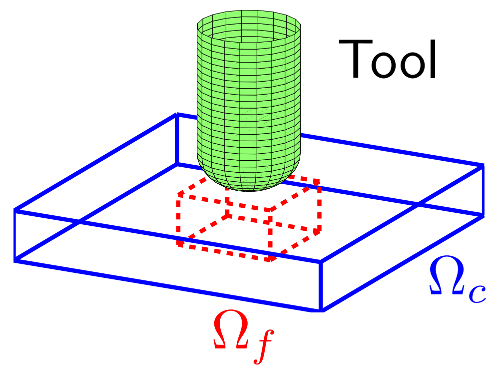
- **Idea 2**: Computation on Ω_c elastic only
  - $\longrightarrow$ Only assemblation of coarse elastic stiffness



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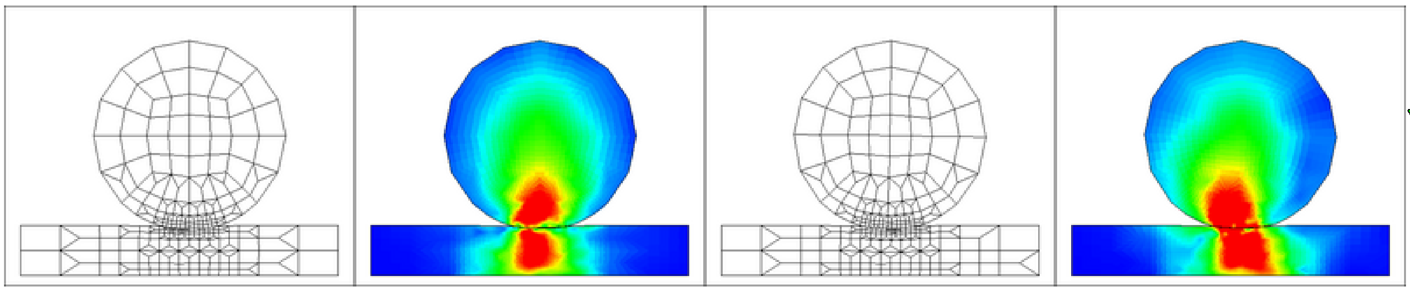

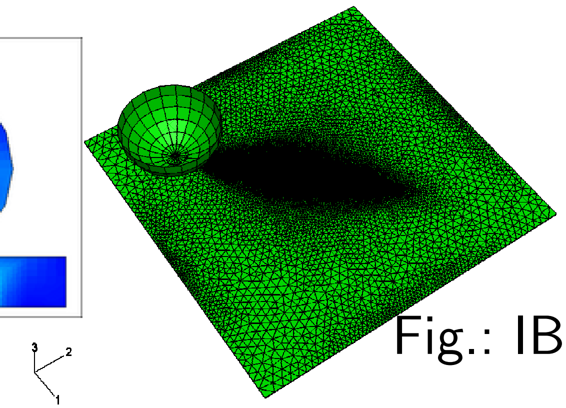

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- **Idea 1**: Fine grid Ω_f moving with forming zone

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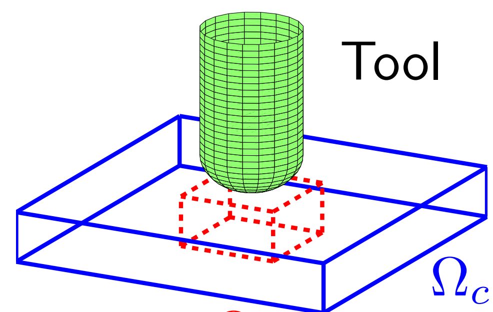
- **Idea 2**: Computation on Ω_c elastic only

  $\longrightarrow$ Only assemblation of coarse elastic stiffness

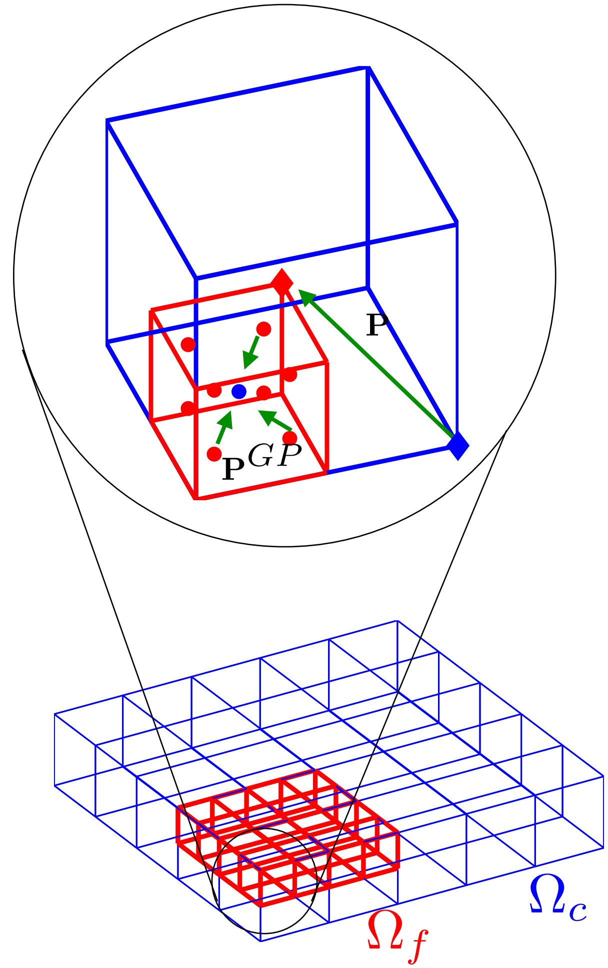
- **Idea 3**: Transfer of fine contact forces onto Ω_c

  $\longrightarrow$ No contact iteration on Ω_c necessary



ODDM algorithm

Main idea: Additive decomposition of displacement



ODDM algorithm

Main idea: Additive decomposition of displacement

Time loop

Displace fine grid

Coarse-fine iteration

Assemble $\boldsymbol{F}_H^{pl}(\mathbf{P}_H[\boldsymbol{\varepsilon}_h^p])$

Linear solve $\mathbf{A}_H \boldsymbol{U}_H = -\boldsymbol{F}_H^{pl} + \mathbf{B}_H^\top \mathbf{\Lambda}_h$

with $\mathbf{A}_H$ elastic stiffness

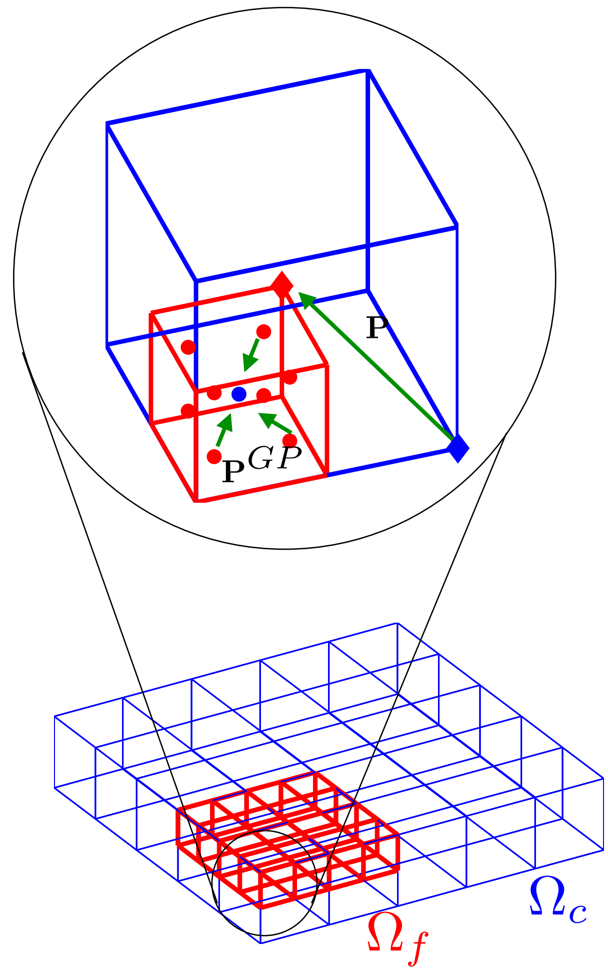
Nonlinear solve $\mathbf{B}_h \mathbf{\Lambda}_h - \boldsymbol{F}_h^{int}(\mathbf{P}_h[\boldsymbol{U}_H] + \boldsymbol{U}_h) = \mathbf{0}$

plus constraints

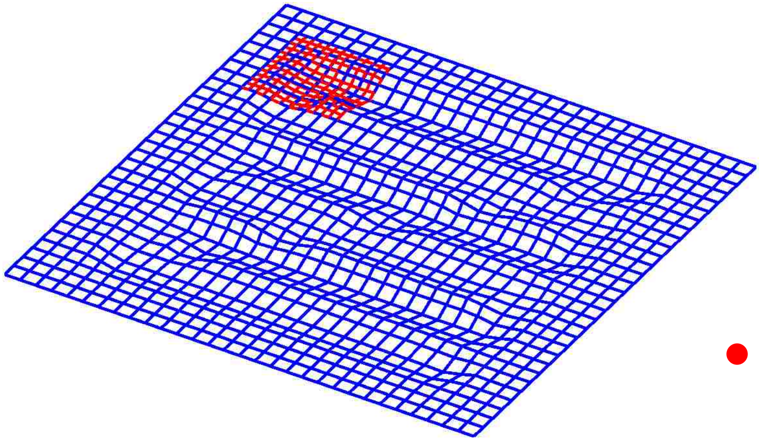
Compute and store $\boldsymbol{\varepsilon}_h^p$

END coarse-fine iteration

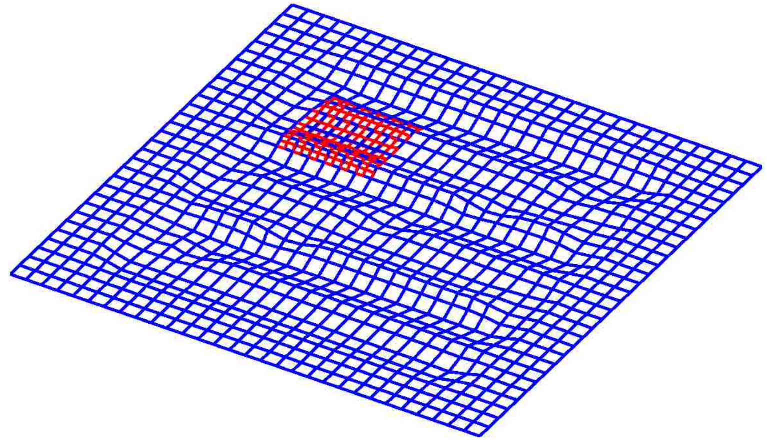
END time loop



ODDM example with contact



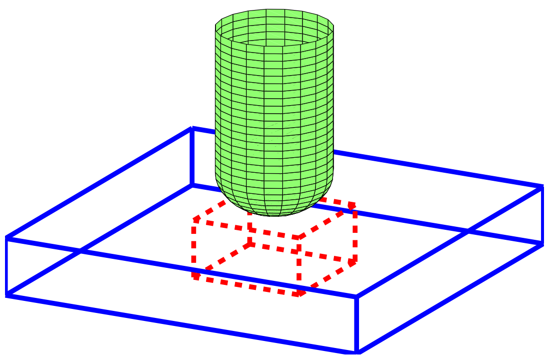
• 181

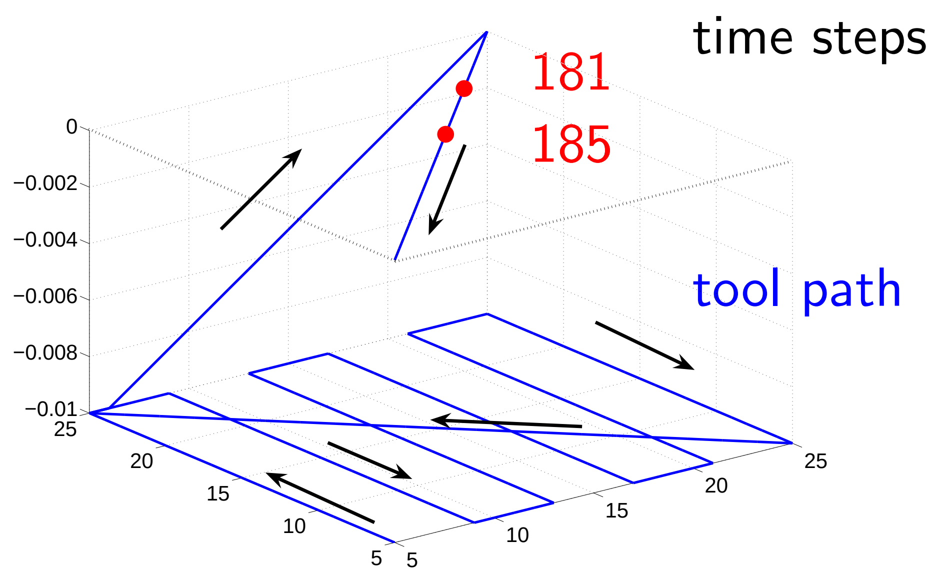


• 185

displ. exaggerated 200 ×

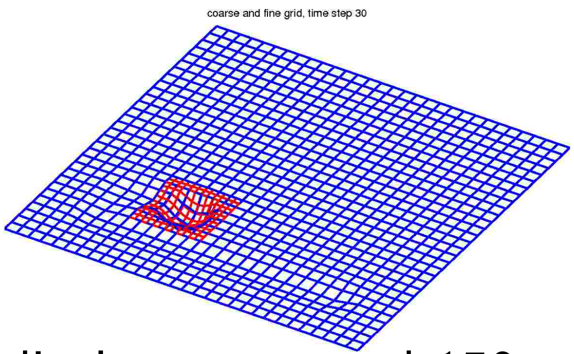
200 time steps





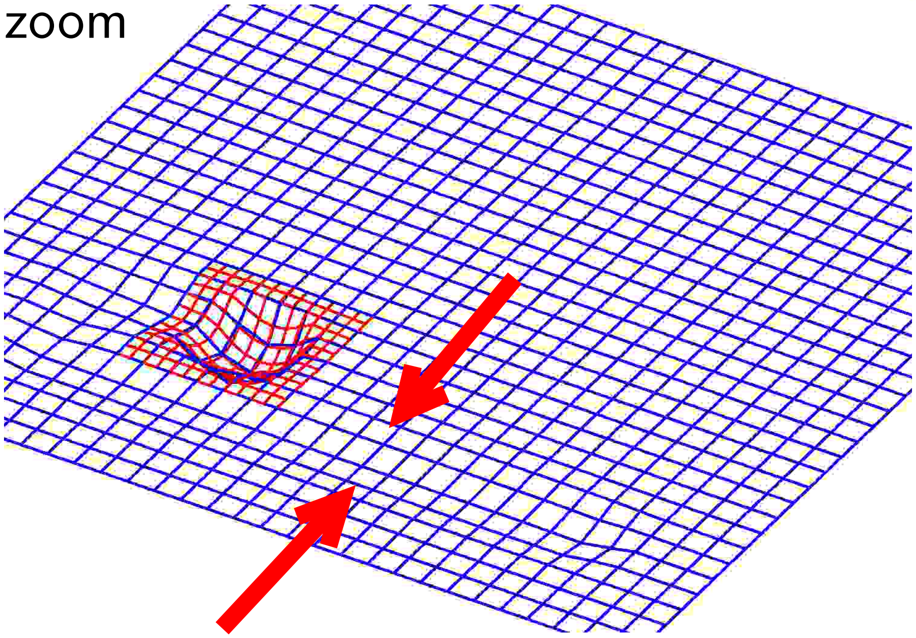
ODDM example with contact

time step 30

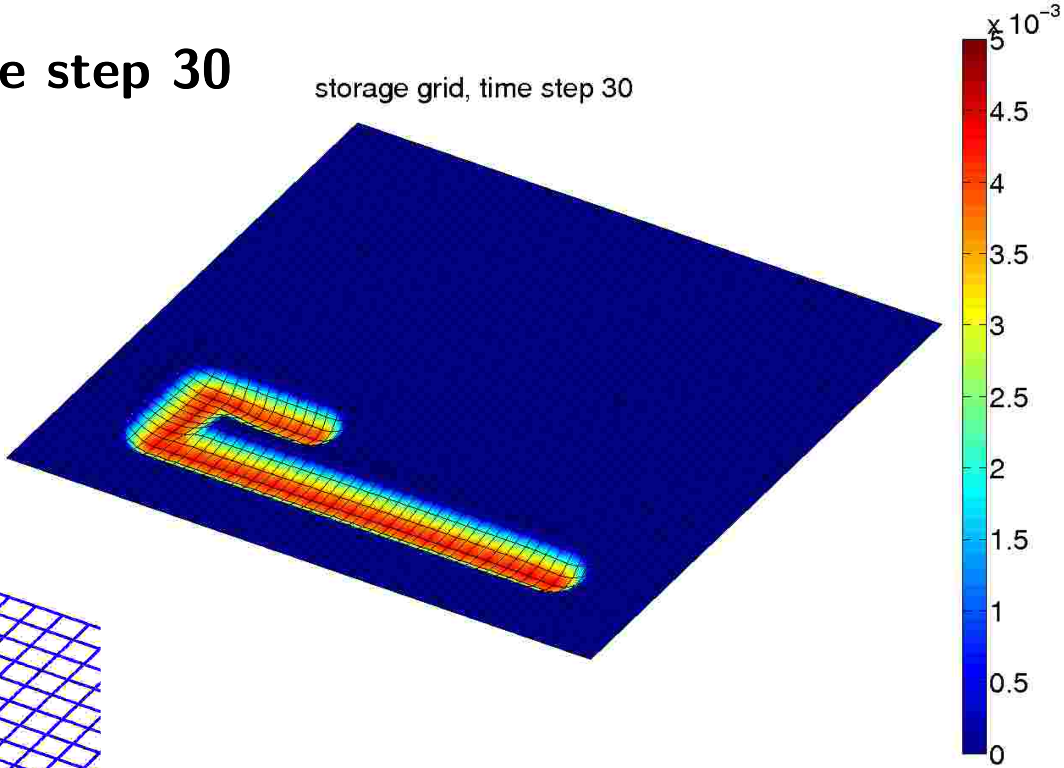


displ. exaggerated 150 ×

zoom



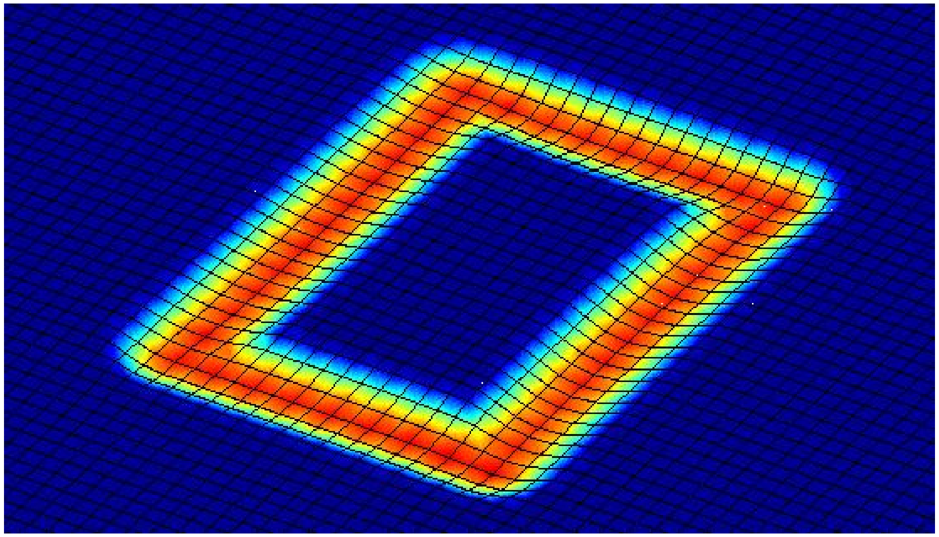
plastic deformation



displ. exaggerated 300 ×

α plotted

ODDM example – a different path

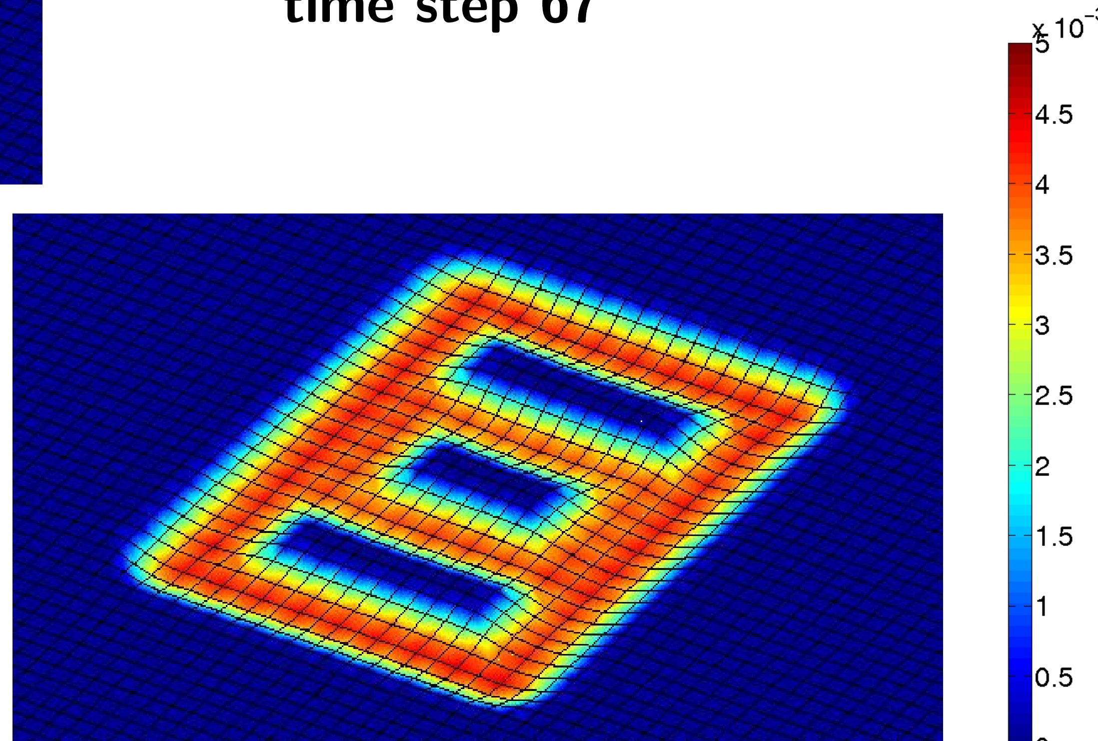


time step 67

time step 40

α plotted on

storage grid



Summary

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Thank you for your attention!