Locking-free finite elements for some unilateral crack problems in the linearized elasticity and applications

Z. Belhachmi

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Goal

- Efficient discretisations for problems in nonsmooth domains, e.g. cracked domains.
- Locking-free discretisations for nearly incompressible elasticity in unilateral contact mechanics

Basic idea

- First ingredient: Mixed(-hybrid) variational formulation.
- Second ingredient: Extend (smooth) domain formulation.

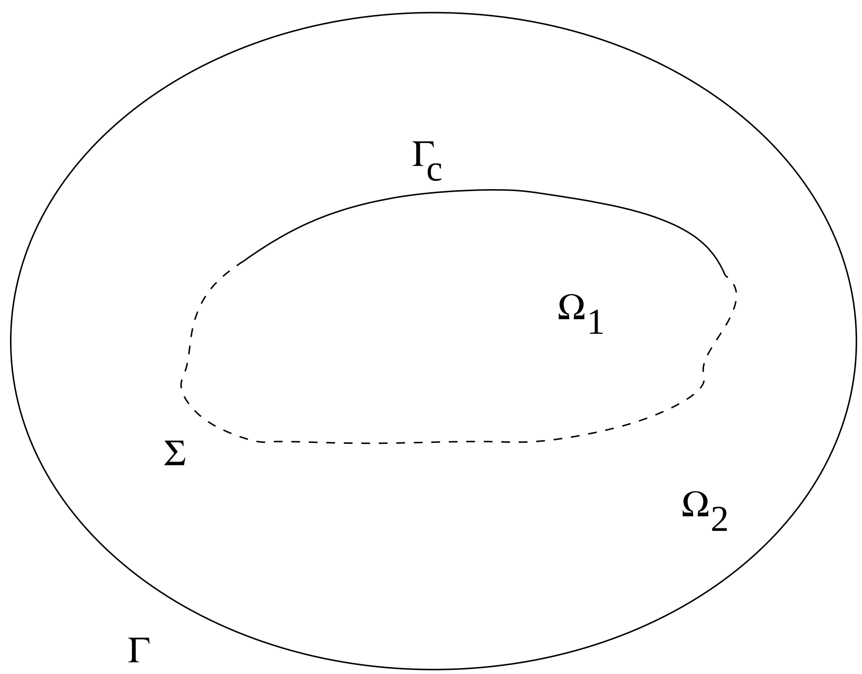
Some applications

- Shape optimisation.
- Geometric inverse problems

J-M. Sac Épée. S. Tahir (LMAM, University of Metz).

Outline

1. The linearized elasticity case
   - Formulation with the symmetry of the constraint tensor
   - Formulation without symmetry: modified Hellinger-Reisner formulation
2. Finite elements discretisations
   - The PEERS element
   - BDMS element
3. Error analysis
4. Experiments
5. Application: Computing the topological derivative in nonsmooth domains
6. Summary and outlook



Domain with a crack

Formulation with the symmetry of σ

- **The equations for the linearized elasticity:**

$$
\begin{aligned}
\sigma = C^{-1}\epsilon(u) \quad & \text{in } \Omega_c \\
-div\ \sigma = f \quad & \text{in } \Omega_c \\
u = 0 \quad & \text{on } \Gamma_d \\
\sigma \cdot \nu = g \quad & \text{on } \Gamma_N, \\
[u]\,\nu \geq 0,\ \ [\sigma_\nu] = 0,\ \ \sigma_\nu\,[u\cdot\nu] = 0 \quad & \text{on } \Gamma_C, \\
\sigma_\nu \leq 0,\ \ \sigma_t = 0 \quad & \text{on} \Gamma_C^\pm
\end{aligned}
$$

- Modelling the friction?

Unilateral crack problem: the elasticity system

- $$\sigma_\nu = (\sigma\nu)\nu,\ \sigma_t = \sigma\nu - \sigma_\nu\nu$$
  $$\epsilon = \tfrac{1}{2}(\nabla u + \nabla u^T),$$
- $$\sigma = C^{-1}\epsilon(u) = 2\mu\epsilon + \lambda tr(\epsilon)I$$
  $$2\mu = \tfrac{E}{1+\nu},\ \lambda = \tfrac{E\nu}{(1+\nu)(1-2\nu)}$$
- nearly incompressible material: $\lambda \to +\infty(\nu \to \frac{1}{2})$
  purely displacement formulation not good $\longrightarrow$ **numerical locking**

Variational formulation

- $$X(\Omega_c) = \{\sigma \in L^2(\Omega_c;\mathbb{R}^{2\times 2}_{\text{sym}}),\ div\ \sigma \in L^2\Big(\Omega_c;\mathbb{R}^2\Big), \sigma\nu = 0 \text{ on } \Gamma_N\}$$
  $$K(\Omega_c) = \{\sigma \in X(\Omega_c), [\sigma\nu] = 0 \text{ on } \Gamma_c, \sigma_\tau = 0 \text{ on } \Gamma_c, \sigma_\nu \leq 0 \text{ on } \Gamma_c\}$$
  - $$\begin{cases} a(\sigma, \tau - \sigma) + b(\tau - \sigma, u) \geq 0, \quad \tau \in K(\Omega_c) \\ b(\sigma, v) = -L(v), \quad (v, \eta) \in V(\Omega_c) \end{cases}$$
- $$\begin{cases} a(\sigma,\tau) = (C^{-1}\sigma,\tau) = \displaystyle\int_{\Omega_c} C^{-1}\sigma : \tau\ dx \\ b(\sigma,u) = (div\ \sigma, u) = \displaystyle\int_{\Omega_c} div\ \sigma \cdot u\ dx \\ L(v) = (f,v) = \displaystyle\int_{\Omega_c} f \cdot v\ dx \end{cases}$$
- Existence, uniqueness and stability wrt data follow from the coercivity of $a(.,.)$ and the Brezzi-Babuska inf-sup condition on $b(.,.)$.

Extended variational formulation

- The smooth domain formulation is obtained by extending σ and $\mathbf{u}$ to the whole domain $\Omega = \Omega_c \cup \Gamma_c$. Resulting problem is obtained with Ω_c replaced by Ω, and

$$\mathbf{K} = \{\mathbf{q} \in \mathbf{X}(\Omega),\ \sigma_\tau = 0,\ \sigma_\nu \leq 0,\ \text{on}\, \Gamma_c\}\,.$$

- **theorem** the extended problem admits a unique solution $(\sigma, \mathbf{u})$ s.t

$$\|\sigma\| + \|\mathbf{u}\| \leq c\|\mathbf{f}\|.$$

- equilibrium equation and the Hook's law in the sense of distributions in Ω_c.

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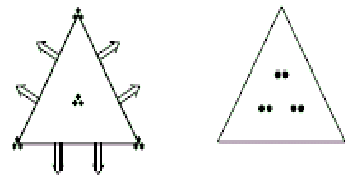
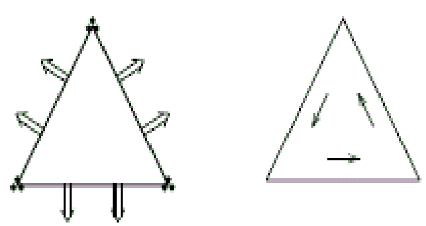
Arnold-Winther element

- **Difficulty: the inf-sup condition**
- for any $T \in T_h$, set

$$\Sigma_T = P_2(T, \mathbb{R}^{2\times 2}_{\mathsf{sym}}) + \left\{\tau \in P_3(T, \mathbb{R}^{2\times 2}_{\mathsf{sym}});\ \mathsf{div}\,\tau = 0\right\}$$
$$= \left\{\tau \in P_3(T, \mathbb{R}^{2\times 2}_{\mathsf{sym}});\ \mathsf{div}\,\tau \in P_1(T)^2\right\} .$$

$$X_h = \{\sigma \in X;\ \sigma_T \in X_T\}$$
$$V_h = \left\{v_h \in (\mathcal{C}(\overline{\Omega}))^2;\ v_{h,|T} \in (P_1(T))^2,\ \forall T \in \mathcal{T}_h\right\}.$$





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- **The equations for the linearized elasticity:**

$$\begin{aligned}
C\sigma - \nabla u + \gamma &= 0 \quad \text{in } \Omega_c \\
\sigma - \sigma^T &= 0 \quad \text{in } \Omega_c \\
-div\ \sigma &= f \quad \text{in } \Omega_c \\
u &= 0 \quad \text{on } \Gamma_d \\
\sigma \cdot \nu &= g \quad \text{on } \Gamma_N, \\
[u]\,\nu \geq 0,\ \ [\sigma_\nu] = 0,\ \ \sigma_\nu\,[u \cdot \nu] &= 0 \quad \text{on } \Gamma_C, \\
\sigma_\nu \leq 0,\ \ \sigma_t &= 0 \quad \text{on} \Gamma_C^{\pm}
\end{aligned}$$

$$\gamma = rot(u)$$

Variational formulation

- $$X(\Omega_c) = \{\sigma \in L^2(\Omega_c;\mathbb{R}^{2\times 2}),\ div\ \sigma \in L^2\Big(\Omega_c;\mathbb{R}^2\Big), \sigma\nu = 0 \text{ on } \Gamma_N\}$$
$$K(\Omega_c) = \{\sigma \in X(\Omega_c), [\sigma\nu] = 0 \text{ on } \Gamma_c, \sigma_\tau = 0 \text{ on } \Gamma_c, \sigma_\nu \leq 0 \text{ on } \Gamma_c\}$$
$$W(\Omega_c) = L^2(\Omega_c, M^{2\times 2}_{skew})$$

- $$\begin{cases} a(\sigma, \tau - \sigma) + b(\tau - \sigma; u, \gamma) \geq 0, \quad \tau \in K(\Omega_c) \\ b(\sigma; v, \eta) = -L(v), \quad (v, \eta) \in V(\Omega_c) \times W(\Omega_c) \end{cases}$$

- $$\begin{cases} a(\sigma, \tau) = (C^{-1}\sigma, \tau) = \int_{\Omega_c} C^{-1}\sigma : \tau \ dx \\ b(\sigma; u, \gamma) = (div\ \sigma, u) + (\sigma, \gamma) = \int_{\Omega_c} div\ \sigma \cdot u\ dx + \int_{\Omega_c} \sigma : \gamma\ dx \\ L(v) = (f, v) = \int_{\Omega_c} f \cdot v\ dx \end{cases}$$

- Existence, uniqueness and stability wrt data follow from the coercivity of $a(.,.)$ and the Brezzi-Babuska inf-sup condition on $b(.,.)$.

- $as(\gamma) = \gamma_{2,1} - \gamma_{1,2}$
- **Extended formulation**

$$\Omega_c \longrightarrow \Omega$$

- **Note**
  - Contact conditions to be understood in a weak sense.
  - equilibrium equation in the sense of distributions in Ω_c.

Hybrid formulation

- Set $M = H^{\frac{1}{2}}_{00}(\Gamma_c)^2$,

$$M_+ = \left\{\mu \in H^{\frac{1}{2}}_{00}(\Gamma_c);\ \mu \geq 0 \text{.a.e.}\right\}.$$

-

$$\begin{cases} a(\sigma, \tau - \sigma) + b(\tau - \sigma; u, \gamma) + d_t(\tau, \lambda_t) + d_n(\tau, \lambda_n) = 0, \quad \tau \in X \\ b(\sigma; v, \eta) = -L(v), \quad (v, \eta) \in V(\Omega_c) \times W(\Omega_c) \\ d_t(\sigma, \mu_t) = 0, \quad \forall \mu_t = (\mu_1, \mu_2) \in M, \quad \mu_i \cdot \nu_i = 0 \\ d_n(\sigma, \mu_n - \lambda_n) \leq 0, \quad \forall \mu_n \in M_+ \end{cases}$$

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$$\begin{cases} d_t(\tau, \mu_t) = << \tau_t, \mu_t >>_{\frac{1}{2},00,\Gamma_c} \\ d_n(\tau, \mu_n) = < \tau_\nu, \mu_n >_{\frac{1}{2},00,\Gamma_c} \end{cases}$$

- Existence, uniqueness and stability wrt data hold

PEERS element

- for any $T \in T_h$, set $A = \{rot\ b_T\}.$, $B_0(T) = \{\tau/(\tau_{i1}, \tau_{i2}) \in A;\ i = 1, 2\}$ and $X_T = RT_0(T)^2 \oplus B_0(T)$,

$$X_h = \{\sigma \in X;\ \sigma_T \in X_T\}$$

$$V_h = \{v_h \in L^2(\Omega)^2;\ v_{h|T} \in (P_0(T))^2,\ \forall T \in \mathcal{T}_h\}$$
$$\mathbf{W}_h = \left\{\gamma_h \in W \cap C^0(\overline{\Omega}, M^{2\times2}_{skew}); \gamma_{h|T} \in P_1(T; M^{2\times2}_{skew}),\ \forall T \in \mathcal{T}_h\right\}$$

- $$W_h^0(\Gamma_c) = \{\mu_h,\ \mu_{h|t_i} \in \mathbb{P}_0(t_i),\quad 0 \leq i \leq I-1\}$$

$$M_h = (W_h^0)^2$$

$$M_h^+ = \{\mu_h \in W_h^0,\ \mu_h \geq 0,\}$$

Lower order BDM element

- for any $T \in T_h$, set

$$R_T = \{v;\ v(x,y) = (a,b) + c(-y,x), a,b,c \in \mathbb{R}\}$$

$$X_h = \{\sigma \in X;\ \sigma = \sigma^1 + \sigma^2 + \sigma^3, \sigma^1_T \in P_1^{2,2}(T)$$
$$\sigma^2_T \in b_T \nabla R_T, \sigma^3_T \in B_0(T), T \in T_h\}$$

$$V_h = \{v_h \in (C^0(\bar{\Omega}))^2;\ v_{h|T} \in R_T,\ \forall T \in \mathcal{T}_h\}$$

- $$W_h^1(\Gamma_c) = \{\mu_h,\ \mu_{h|t_i} \in \mathbb{P}_1(t_i),\quad 1 \leq i \leq I-2,$$
$$\mu_{h|t_e} \in \mathbb{P}_0(t_e), e = 0, I-1\}$$

$$M_h = (W_h^1)^2$$

$$M_h^+ = \{\mu_h \in W_h^1,\ \int_{\Gamma_c} \mu_h \psi_h \geq 0, \psi_h \in W_h^1\}$$

Result:

- Each discrete problem, resp. hybrid problem, admits a unique solution $(\sigma_h, u_h, \gamma_h)$, resp. $(\sigma_h, u_h, \gamma_h, \lambda_{th}, \lambda_{nh})$

- **theorem** Let $(\sigma; u, \gamma, \lambda = ((\lambda_{t1}, \lambda_{t2}), \lambda_n))$ be the solution of the hybrid problem. Suppose that $u_{|_{\Omega^1}} \in H^2(\Omega^1),\ u_{|_{\Omega^2}} \in H^2(\Omega^2)$, and that $\sigma_{|_{\Omega^1}} \in H^1(\Omega^1),\ \sigma_{|_{\Omega^2}} \in H^1(\Omega^2)$. Let $(\sigma_h; u_h, \gamma_h, \lambda_h) = ((\lambda_{ht1}, \lambda_{ht2}), \lambda_{hn}))$ be the solution of the PEERS-based discrete problem

$$\begin{aligned} \| & \sigma - \sigma_h \|_{L^2(\Omega,\mathbb{R}^{2\times 2})} + \| u - u_h \|_V + \| \gamma - \gamma_h \|_W + \\ + & \| \lambda_t - \lambda_{ht} \|_{L^2(\Gamma_c,\mathbb{R}^2)} + \| \lambda_n - \lambda_{hn} \|_{L^2(\Gamma_c)} \leq C(u,\sigma) h^{\frac{1}{2}}. \end{aligned}$$

theorem Let $(\sigma; u, \gamma, \lambda_t = ((\lambda_{t1}, \lambda_{t2}), \lambda_n))$ be the solution of Problem (1.12). Suppose that $u_{|_{\Omega^1}} \in H^2(\Omega^1),\ u_{|_{\Omega^2}} \in H^2(\Omega^2)$, and that $\sigma_{|_{\Omega^1}}\ (resp.\ \gamma_{|_{\Omega^1}}\ ) \in H^1(\Omega^1, \mathbb{R}^{2,2}),\ \sigma_{|_{\Omega^2}}\ (resp.\ \gamma_{|_{\Omega^2}}\ ) \in H^1(\Omega^2, \mathbb{R}^{2,2})$. Let $(\sigma_h; u_h, \gamma_h, \lambda_{ht}) = ((\lambda_{ht1}, \lambda_{ht2}), \lambda_{hn}))$ be the solution of the BDM-based discrete Problem

$$\begin{aligned} \| & \sigma - \sigma_h \|_0 + \| u - u_h \|_V + \| \gamma - \gamma_h \|_0 + \\ + & \| \lambda_t - \lambda_{ht} \|_{H_{00}^{\frac{1}{2}}(\Gamma_c,\mathbb{R}^2)} + \| \lambda_n - \lambda_{hn} \|_{H_{00}^{\frac{1}{2}}(\Gamma_c)} \leq C(u,\sigma) h^{\frac{3}{4}}. \end{aligned}$$

Constant scalar $C(u,\sigma)$ is linear depending on $\| u_{|_{\Omega^l}} \|_{H^2(\Omega^l)}$, $\| \sigma_{|_{\Omega^l}} \|_{H^1(\Omega^l)^4}$ and $\| div\ \sigma_{|_{\Omega^l}} \|_{H^1(\Omega^l)^2} \quad l = 1, 2.$

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Experiments

Implementation

- Uzawa algorithm
- Elimination of the constraint
- A modified algorithm
  - Solve

$$\min_{\Phi\geq 0}\left(\frac{1}{2}{}^t\Phi^tL\mathbf{K}^{-1}L\Phi - {}^t\Phi^tL\mathbf{K}^{-1}\mathbf{F} + \frac{1}{2}{}^t\mathbf{F}\mathbf{K}^{-1}\mathbf{F}\right)$$

  - $\Phi = S\Lambda$, solve

$$\mathbf{U} = \mathbf{K}^{-1}(\mathbf{F} - L\,S\Lambda)$$

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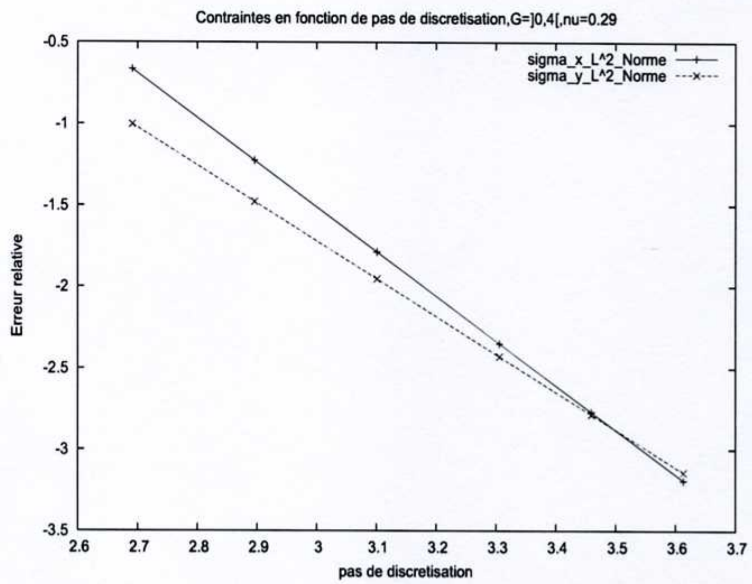

FIG. 3.5 – La norme L^2 de l'erreur relative des contraintes en fonction du pas de discrétisation.

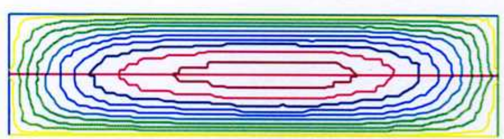

FIG. 3.6 – Isovaleurs de u_1

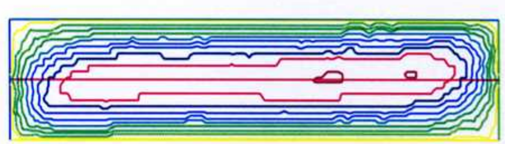

FIG. 3.7 – Isovaleurs de u_2.

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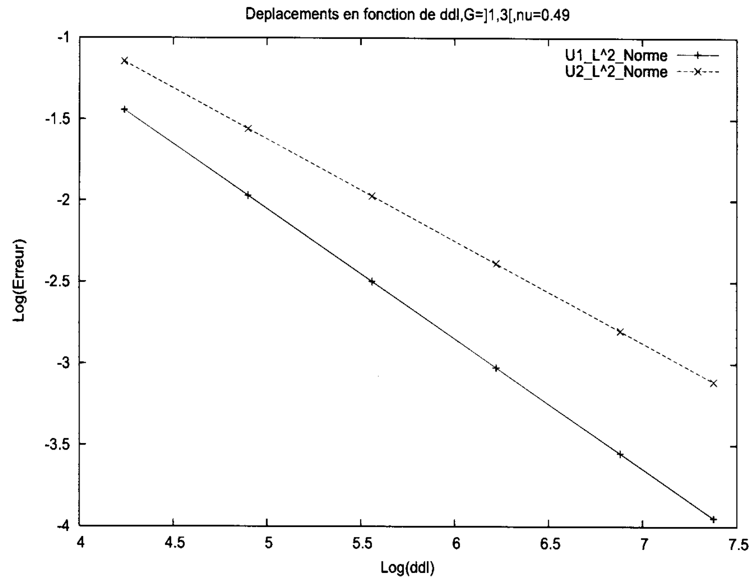

FIG. 3.12 – La norme L^2 de l'erreur relative des déplacements en fonction des ddl.

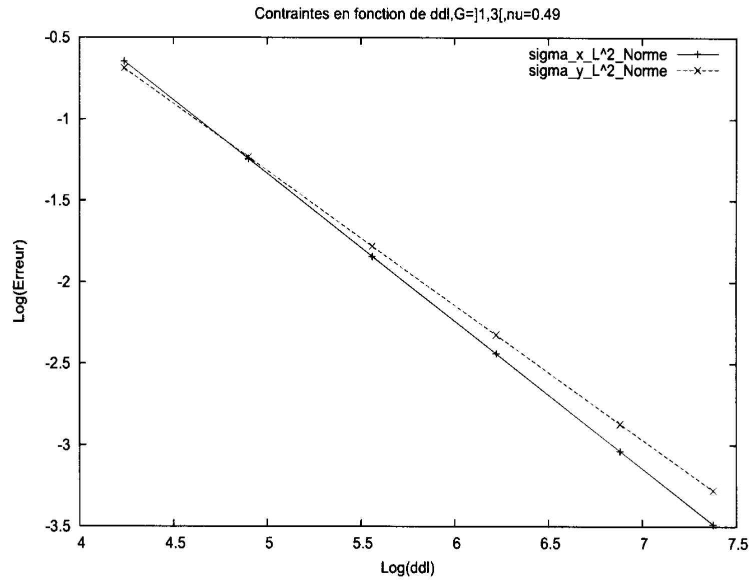

FIG. 3.13 – La norme L^2 de l'erreur relative des contraintes en fonction des ddl.

- Computing the topological derivative
  (J. Sokolowski, K. Szulk)

$$J(\Omega_\epsilon) - J(\Omega) = f(\epsilon)g(x) + o(f(\epsilon))$$

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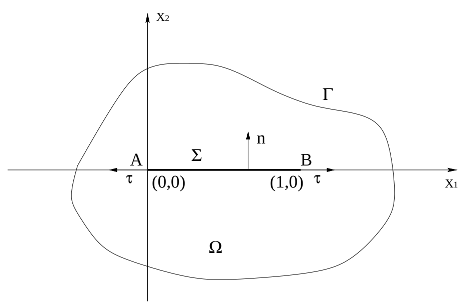
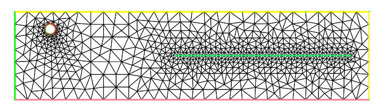
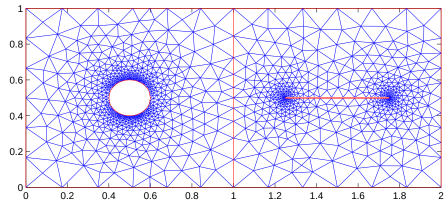
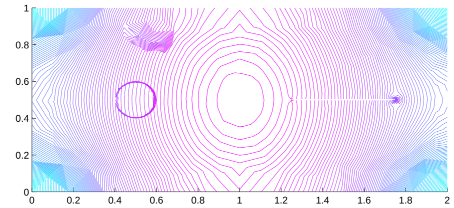
Experiments (2)

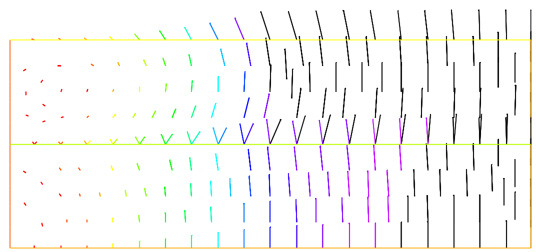


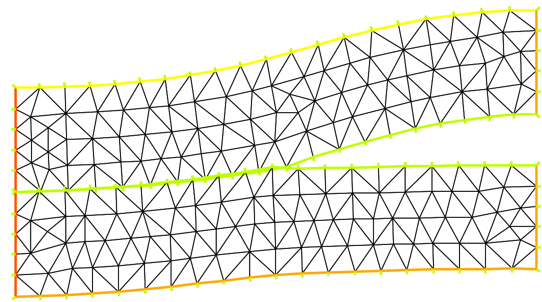






Experiments (3)





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Summary

- Efficient discretisations.
- flexibility for the triangulations
- (numerical) analysis in the framework of smooth (Lipschitz) domains

Thank you for your attention !

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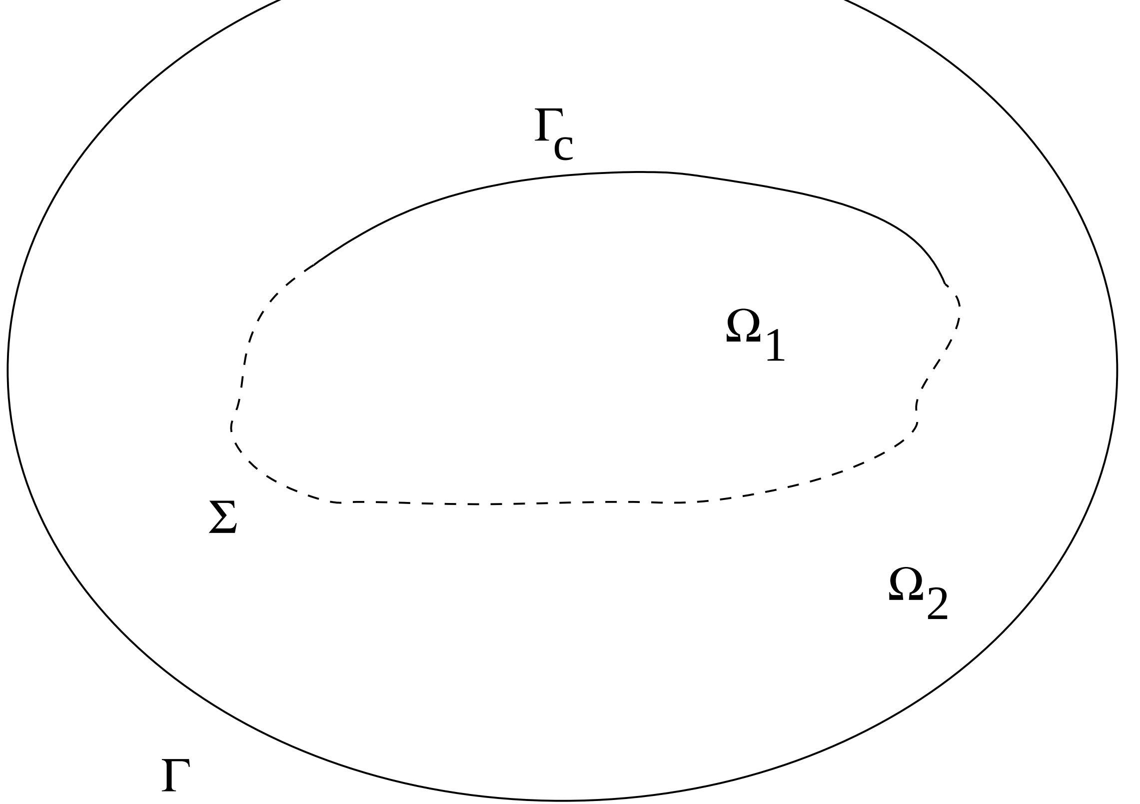

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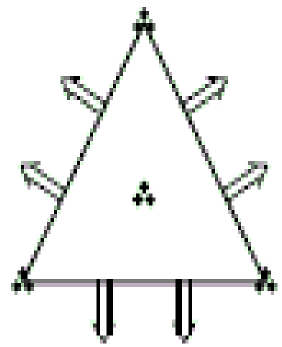

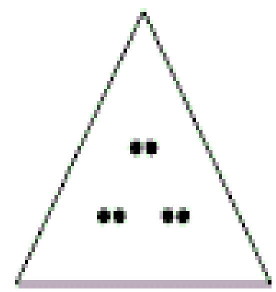

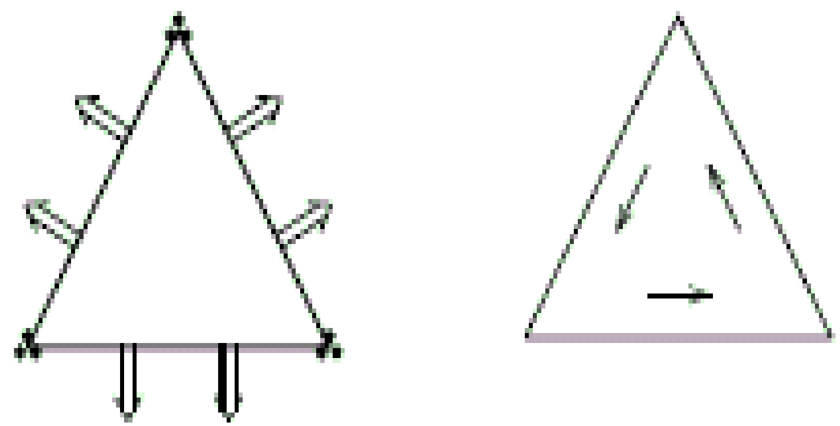

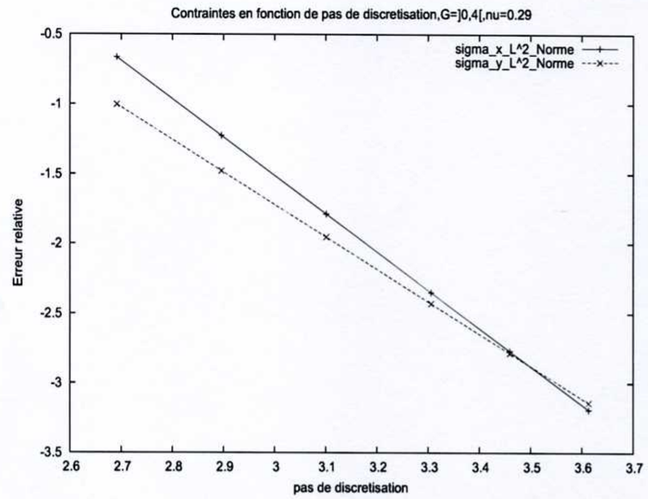

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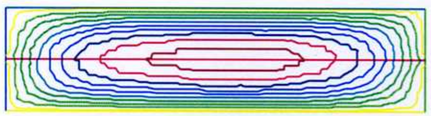


FIG. 3.6 – Isovaleurs de u_1

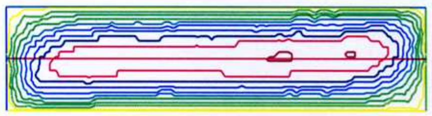


FIG. 3.7 – Isovaleurs de u_2.

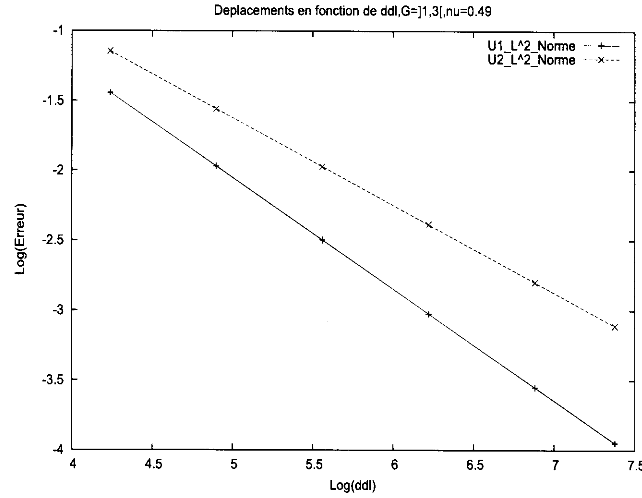


FIG. 3.12 – La norme L^2 de l'erreur relative des déplacements en fonction des ddl.

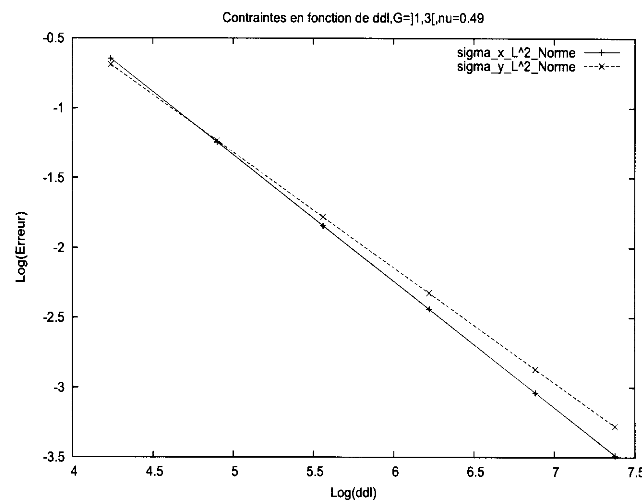
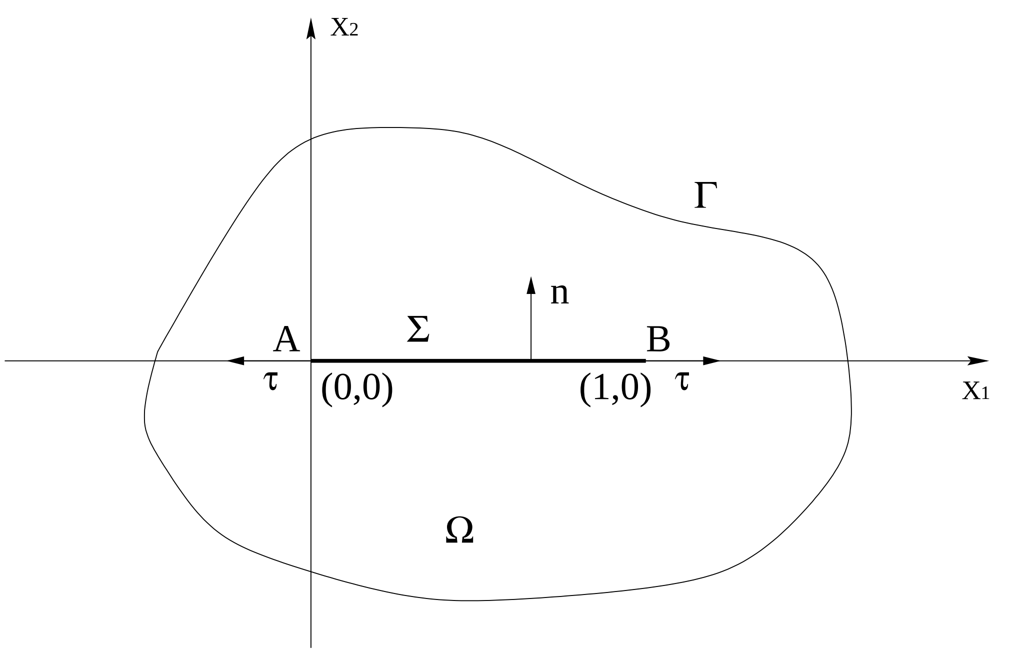


FIG. 3.13 – La norme L^2 de l'erreur relative des contraintes en fonction des ddl.



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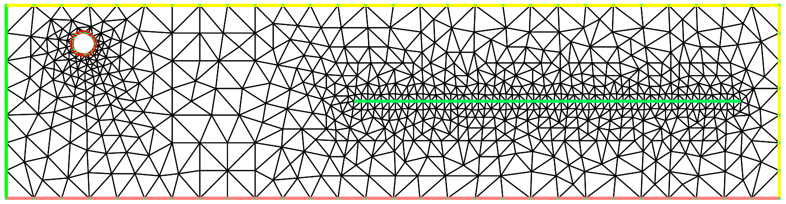

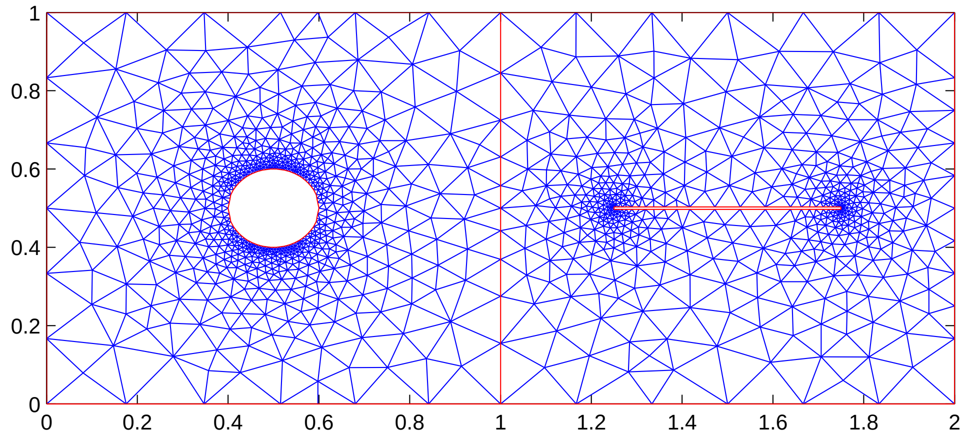
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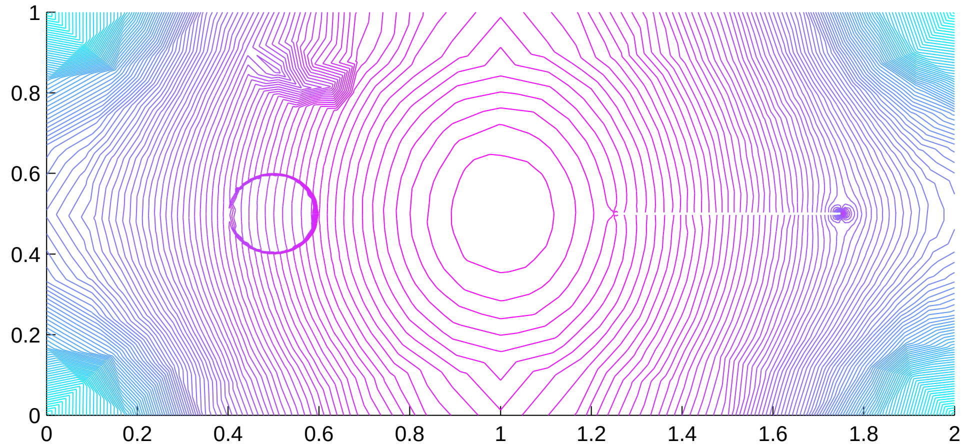
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