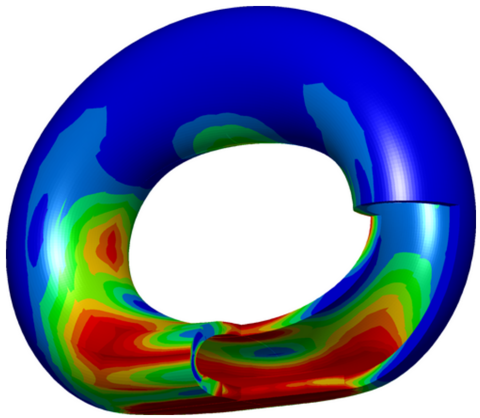
A class of well-posed new approximations for constrained second order hyperbolic equations, application to contact problems



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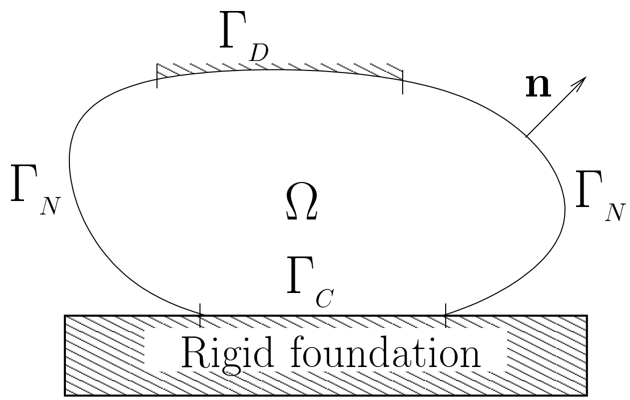
Outline

- An Elastodynamic frictionless contact problem.
- Space semi-discretization and ill-posedness.
- Some existent schemes.
- The mass redistribution method.
- An abstract hyperbolic problem
- The new semi-discretization methods and well-posedness.
- Numerical experiments.

Elastodynamic frictionless contact problem



$$\rho\ddot{u} - \text{div}\,\boldsymbol{\sigma}(u) = f, \qquad \text{in }]0,T] \times \Omega, \tag{1}$$

$$\boldsymbol{\sigma}(u) = \mathcal{A}\boldsymbol{\varepsilon}(u), \qquad \text{in }]0,T] \times \Omega, \tag{2}$$

$$\boldsymbol{\sigma}(u)\mathrm{n} = g, \qquad \text{on }]0,T] \times \Gamma_N, \tag{3}$$

$$u = 0, \qquad \text{on }]0,T] \times \Gamma_D, \tag{4}$$

$$u(0,x) = u_0(x), \;\; \dot{u}(0,x) = u_1(x), \text{ in } \Omega. \tag{5}$$

$$\text{Contact condition on } \Gamma_C : \;\; u_N \leq 0, \; \boldsymbol{\sigma}_N(u) \leq 0, \; u_N \boldsymbol{\sigma}_N(u) = 0, \;\; \boldsymbol{\sigma}_T(u) = 0. \tag{6}$$

Hybrid weak formulation

$$
\begin{cases}
\text{Find } u:[0,T]\longrightarrow V, \lambda_N:[0,T]\longrightarrow X_N' \text{ and } \lambda_T:[0,T]\longrightarrow X_T' \text{ satisfying} \\
\langle \ddot{u}, v\rangle_{V',V} + a(u,v) = l(v) + \langle \lambda_N, v_N\rangle_{X_N', X_N}, \quad \forall\, v \in V. \\
\lambda_N \in \Lambda_N, \quad \langle \mu_N - \lambda_N, u_N\rangle_{X_N', X_N} \geq 0, \quad \forall \mu_N \in \Lambda_N, \\
u(0) = u_0, \quad \dot{u}(0) = u_1.
\end{cases}
$$

where

$$
a(u,v) = \int_\Omega \mathcal{A}\varepsilon(u) : \varepsilon(v), \quad l(v) = \int_\Omega f.v dx + \int_{\Gamma_N} g.v d\Gamma,
$$

$$
X_N = \{v_N|_{\Gamma_C} : v \in V\}, \quad X_T = \{v_T|_{\Gamma_C} : v \in V\}, \quad K_N = \{v_N \in X_N : v_N \leq 0\},
$$

$$
\Lambda_N = -K_N^* = \{\mu_N \in X_N' : \langle \mu_N, v_N\rangle_{X_N', X_N} \geq 0, \; \forall\, v_N \in X_N, v_N \leq 0\},
$$

Space semi-discretization

A finite Element discretization of the hybrid problem if a choice of

- $V^h \subset V$ a finite element space. The discretization of the trace spaces are then defined as

$$X_N^h = \{v_N^h : v^h \in V^h\},$$

- $X_N'^h \subset X_N' \cap L^2(\Gamma_C)$, a finite element space. (a direct discretization of Duvaut-Lions weak formulation is equivalent to the choice $X_N'^h = X_N^h$)
- $\Lambda_N^h \subset X_N'^h$ the discrete convexe of admissible stresses on Γ_C. Generally, the choice $\Lambda_N^h = X_N'^h \cap \Lambda_N$ do not lead to an exploitable discretization.

$$\left\{\begin{array}{l} \text{Find } u^h : [0,T] \longrightarrow V^h, \lambda_N^h : [0,T] \longrightarrow X_N'^h \text{ and } \lambda_T^h : [0,T] \longrightarrow X_T'^h \text{ satisfying} \\ \langle \ddot{u}^h, v\rangle_{V',V} + a(u^h, v^h) = l(v^h) + \langle \lambda_N^h, v_N^h\rangle_{X_N',X_N} + \langle \lambda_T^h, v_T^h\rangle_{X_T',X_T}, \quad \forall\, v^h \in V^h, \\ \lambda_N^h \in \Lambda_N^h, \;\; \langle \mu_N^h - \lambda_N^h, u_N^h\rangle_{X_N',X_N} \geq 0, \;\; \forall \mu_N^h \in \Lambda_N^h, \\ u^h(0) = u_0^h, \;\; \dot{u}^h(0) = u_1^h. \end{array}\right.$$

Space semi-discretization

This leads to the following “measure differential inclusion”

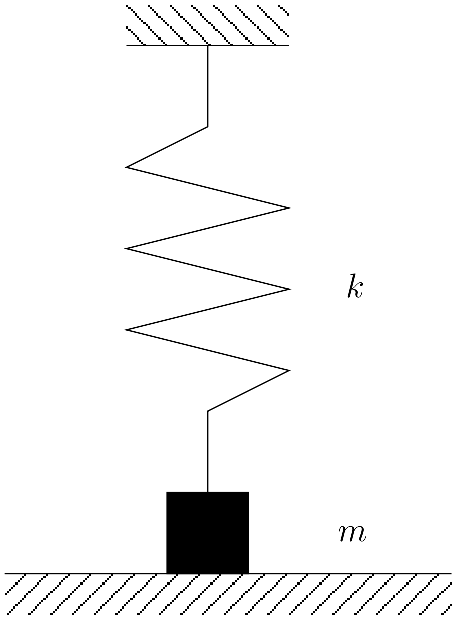
$$\begin{cases} M\ddot{U} + KU = l + B_N^T \lambda_N, \\ -B_N U \in N_{\bar{\Lambda}_N^h}(\lambda_N), \\ u(0) = u_0, \ \ \dot{u}(0) = u_1. \end{cases}$$

where M is the mass matrix, K the rigidity matrix, $U \in I\!R^{N_d}$, the expression of $B_N, \bar{\Lambda}_N^h$ depends on the discretization of Λ_N and the normals cones are to be taken relatively to the euclidean scalar product.

There is no uniqueness result for this kind of measure differential inclusion. It is, in fact, easy to find an infinite number of solutions, even without the friction condition.

For a particular choice of the discretization of Λ_N, one can obtain a nodal contact and friction condition.

Illustration of the non-uniqueness



One dof system:

Equation for the vertical motion:

$$\begin{cases} m\ddot{u} + ku = -F_N \\ F_N = (F_N - ru_N)_-, \ \ \forall r > 0, \\ u(0) = u_0, \ \ \dot{u}(0) = u_1. \end{cases}$$

With initial condition $u_0 = 1$ and $u_1 = 0$ the solution is

$$\begin{aligned} u(t) &= \cos(\sqrt{k/m}\, t), \quad \text{for } t < \frac{\pi}{2}\sqrt{m/k}, \\ u(t) &= a\cos(-\sqrt{k/m}\, t), \quad \text{for } \frac{\pi}{2}\sqrt{m/k} < t < \frac{3\pi}{2}\sqrt{m/k}, \end{aligned}$$

for any $a \geq 0$. The equation is not well-posed ! So the space semi-discretization of the dynamic contact problem is not well-posed neither.

Impact law with a restitution coefficient

In order to select a particular solution of the previous problem, at the impact time, it is necessary to discribe the impact law and to introduce the concept of restitution coefficient. This was formalized by J.J. Moreau.

It consist to select the solution such that the inertia corresponding to normal velocity is partially absorbed by the impact. The restitution coefficient is a coefficient between 0 and 1 coresponding to the ratio of conserved intertia.

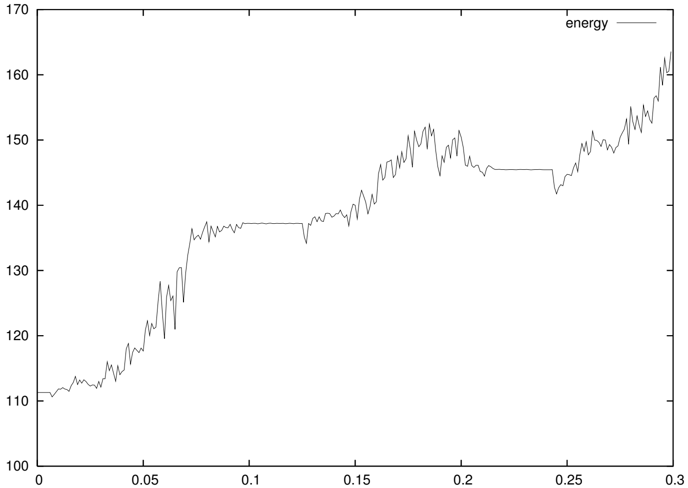
This theory is well adapted to rigid body but has no clear extension for deformable bodies.

Paoli Schatzman scheme

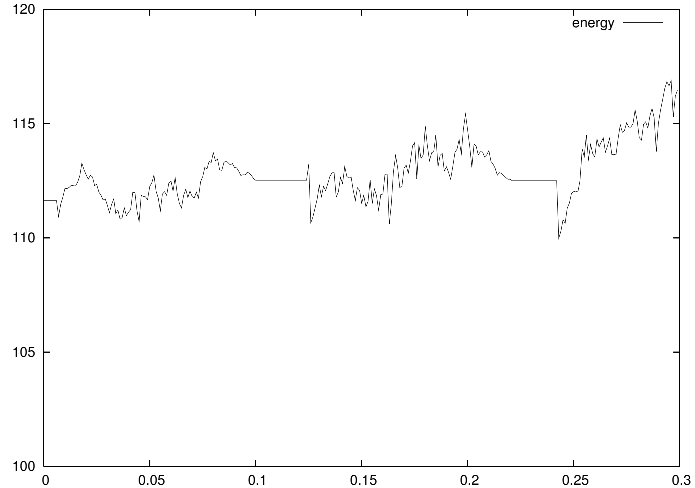
On the equation $M\ddot{U} + KU = f$ with some unilateral constraints $U.N_i \leq 0$ for some orthogonals N_i.

$$U^{n+1} = -eU^{n-1} + (1+e)P_K\left(\frac{2U^n - (1-e)U^{n-1} + \Delta t^2 F^n}{1+e}\right),$$

with $F^n = M^{-1}(f - KU^n)$, P_K the orthogonal projection on admissible displacements and e the restitution coefficient.



energy, $\Delta t = 0.0001$



energy, $\Delta t = 0.00001$

- As it is an explicit scheme, a smaller time step is necessary.
- There is some oscillations on the effective contact area (cf animation).
- The effective restitution coefficient is equal to e only in the case of a diagonal mass matrix.

An energy conserving scheme

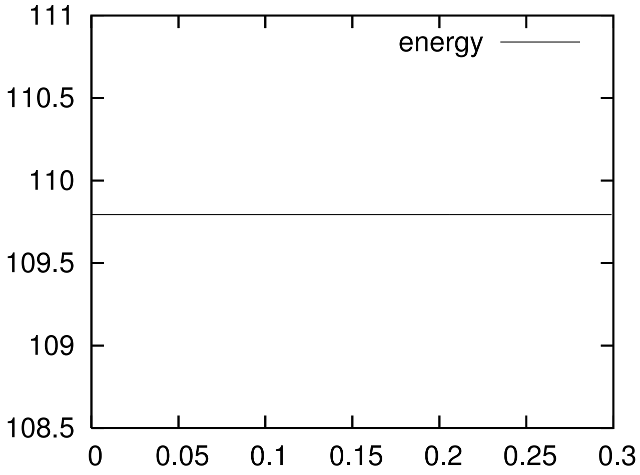
With an appropriate choice of Λ^h_N and a Lagrange finite element method, the contact condition can be written $\tilde{\lambda}^i_N \leq 0,\ \ U.N_i \leq 0,\ \ (\tilde{\lambda}^i_N)(U.N_i) = 0$, where on each finite element node in potential contact $\tilde{\lambda}^i_N$ and $U.N_i$ are the equivalent contact force and the normal displacement respectively. The idea is to replace this expression of the contact condition with the following equivalent expression in terms of normal velocity:

$$\begin{cases} U.N_i < 0 \implies \tilde{\lambda}^i_N = 0, \\ U.N_i \geq 0 \implies \dot{U}.N_i \leq 0,\ \ \tilde{\lambda}^i_N \leq 0,\ \ (\dot{U}.N_i)(\tilde{\lambda}^i_N) = 0. \end{cases}$$

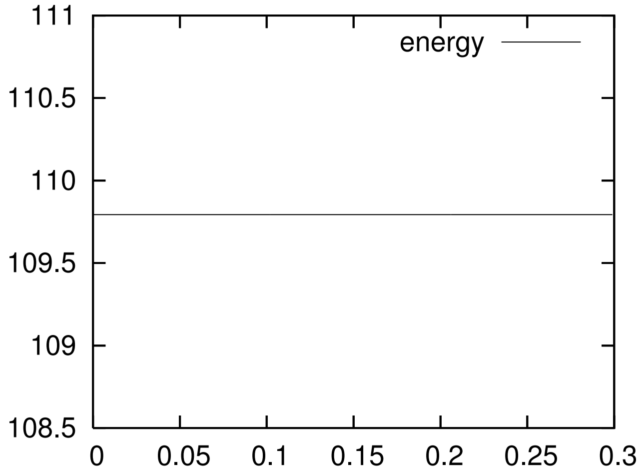
The proposed scheme is based on a midpoind scheme for the elastodynamic part and a central difference scheme for the contact condition. It is strictly energy conserving. Of course a nodal friction condition can be added, and this is stable when a central difference scheme is also used for the friction condition. The expression for the frictionless problem is:

$$\begin{cases} U^0 \text{ and } V^0 \text{ given }, U^1 = U^0 + \Delta t V^0 + \Delta t z(\Delta t) \quad \text{with } \lim\limits_{\Delta t \to 0} z(\Delta t) = 0. \\ \left.\begin{array}{l} M\left(\dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2}\right) + K\left(\dfrac{U^{n+1} + 2U^n + U^{n-1}}{4}\right) = L + \sum\limits_i \tilde{\lambda}^{i,n}_N N_i, \\ V^n = (U^{n+1} - U^{n-1})/2\Delta t, \\ U^n.N_i < 0 \implies \tilde{\lambda}^{i,n}_N = 0, \\ U^n.N_i \geq 0 \implies V^n.N_i \leq 0,\ \ \tilde{\lambda}^{i,n}_N \leq 0,\ \ (V^n.N_i)(\tilde{\lambda}^{i,n}_N) = 0. \end{array}\right\} \quad n \geq 1. \end{cases}$$

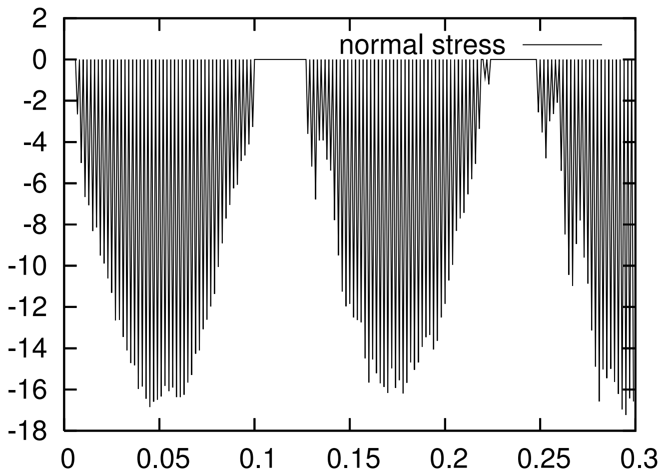
An energy conserving scheme, numerical tests



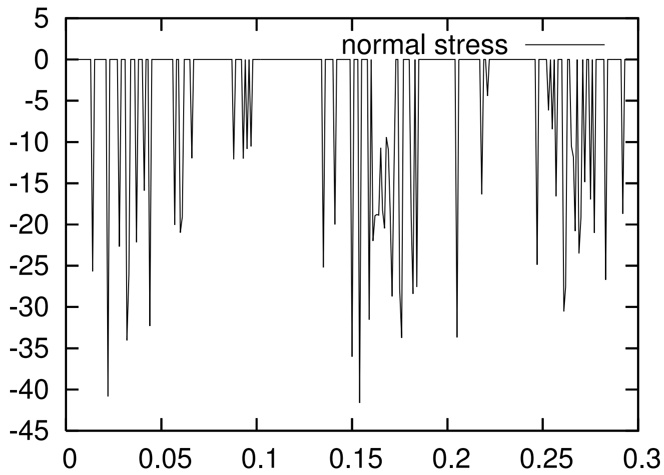
energy, $\Delta t = 0.001$



energy, $\Delta t = 0.0001$



contact stress, $\Delta t = 0.001$



contact stress, $\Delta t = 0.0001$

Other energy conserving schemes

In the context of large deformations, two energy conserving time integration schemes have been proposed:

- Laursen and Chawla 1997 : Also with a contact in terms of velocity. Allows a small interpenetration.
- Laursen and Love 2002 : Gonzales scheme (midpoint scheme with an additional term for the nonlinear elastic law) and a post modification of the normal and tangential velocity on the contact boundary.

Mass Redistribution Method

The ill-posedness of the space semi-discretized problem comes from the fact that contact nodes have an inertia. The idea is to build a modified mass matrix, equivalent in the sens that toatl mass, gravity center and moments of inertia are not changed and such that there is no inertia for the contact nodes. i.e. if

$$U.N_i, i \in I_c$$

represents the normal displacements on the contact boundary, then the modified mass matrix should satisfy

$$N_i^T M N_j = 0, \;\; \forall i, j \in I_c.$$

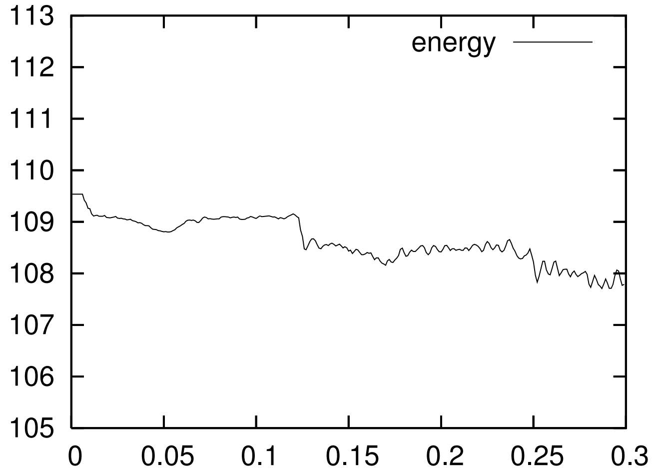
We proved in this context that

- the well-posedness of the space semi-discretized problem is recovered, and it represents a Lipschitz o.d.e. in time with a Lipschitz solution,
- the space semi-discretized problem is energy conserving.

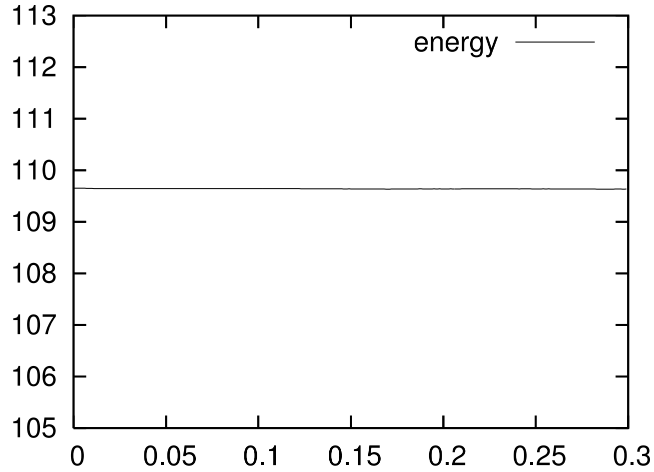
So that any time integration scheme compatible with the elastodynamic part is convergent for this problem. When the time step goes to zero, the discretized solution tends to energy conservation.

Mass Redistribution Method, numerical tests

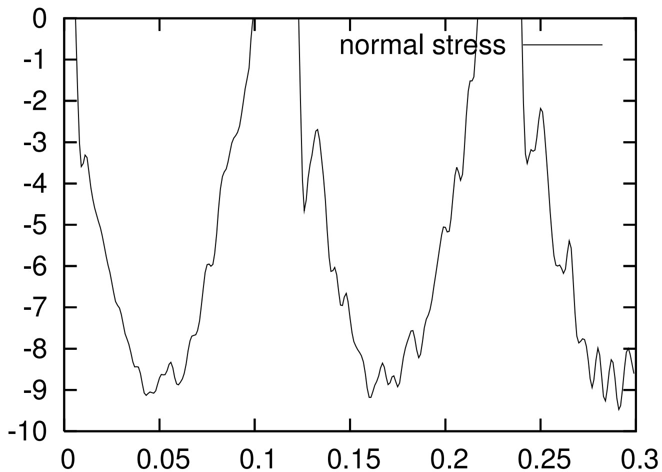
With a Newmark scheme.



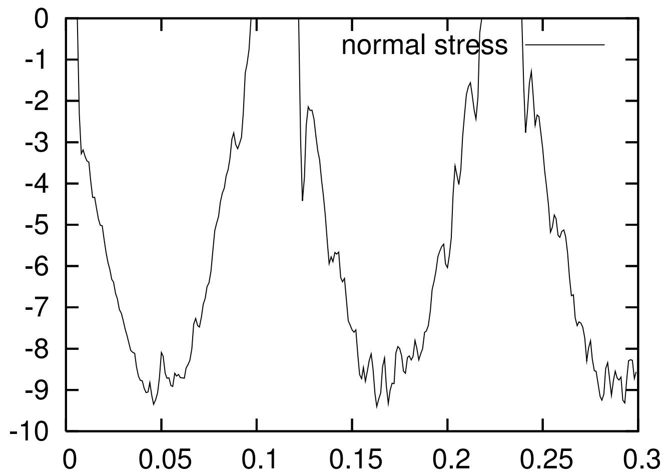
energy, $\Delta t = 0.001$



energy, $\Delta t = 0.0001$



contact stress, $\Delta t = 0.001$



contact stress, $\Delta t = 0.0001$

An abstract hyperbolic equation

$$\begin{cases} \text{Find } u : [0,T] \to K \text{ such that} \\ \dfrac{\partial^2 u}{\partial t^2}(t) - Au(t) \in f - N_K(u(t)) \quad \text{for a.e. } t \in]0,T], \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = v_0, \end{cases} \tag{7}$$

where W be a Hilbert $W \subset H = L^2(\Omega) \subset W'$, with compact and continuous inclusions, $A : W \to W'$ a linear elliptic continuous operator, and $K \subset W$ a closed convex. An equivalent variational inequality:

$$\begin{cases} \text{Find } u : [0,T] \to K \text{ such that for a.e. } t \in]0,T], \\ \langle \dfrac{\partial^2 u}{\partial t^2}(t), w - u(t) \rangle_{W',W} + a(u(t), w - u(t)) \geq l(w - u(t)) \;\; \forall w \in K, \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = v_0, \end{cases} \tag{8}$$

where

$$a(u,v) = \langle Au, v \rangle_{W',W}, \quad l(v) = \langle f, v \rangle_{W',W}.$$

On energy conservation

The terminology “variationnal inequality” is used here in the sense that Problem (7) derives from the conservation of the energy functional

$$J(t) = \frac{1}{2}\int_\Omega (\frac{\partial u}{\partial t}(t))^2 dx + \frac{1}{2}a(u(t),u(t)) - l(u(t)) + I_K(u(t)),$$

where $I_K(u(t))$ is the convex indicator function of K. However, it is generally not possible to prove that each solution to Problem (8) is energy conserving, due to the weak regularity involved.

The new class of approximations

The goal of this section is to present well-posed space semi-discretizations of Problem (8). The strategy adopted is to use a Galerkin method with different approximations of u and of $v = \dfrac{\partial u}{\partial t}$. Let W^h and H^h be two finite dimensional vector subspaces of W and H respectively. Let $K^h \subset W$ be a closed convex nonempty approximation of K. The proposed approximation of Problem (8) is the following

$$\begin{cases} \text{Find } u^h : [0,T] \to K^h \text{ and } v^h : [0,T] \to H^h \text{ such that} \\ \displaystyle\int_\Omega \frac{\partial v^h}{\partial t}(w^h - u^h)dx + a(u^h, w^h - u^h) \geq l(w^h - u^h) \;\; \forall w^h \in K^h, \;\; \forall t \in]0,T], \\ \displaystyle\int_\Omega (v^h - \frac{\partial u^h}{\partial t})q^h dx = 0 \;\; \forall q^h \in H^h, \;\; \forall t \in]0,T], \\ u^h(0) = u_0^h, \quad v^h(0) = v_0^h. \end{cases} \tag{9}$$

Where $u_0^h \in K^h$ and $v_0^h \in H^h$ are some approximations of u_0 and v_0 respectively. Of course, when $H^h = W^h$ we recover a standard Galerkin approximation of Problem (8).

The new class of approximation, matrix expression

φ_i, $1 \le i \le N_W$ and ψ_i, $1 \le i \le N_H$ basis of W^h and H^h resp. Matrices A, B and C and vectors L, U and W defined such that

$$A_{i,j} = a(\varphi_i, \varphi_j), \quad B_{i,j} = \int_\Omega \psi_i \varphi_j dx, \quad C_{i,j} = \int_\Omega \psi_i \psi_j dx,$$

$$L_i = l(\varphi_i), \quad u^h = \sum_{i=1}^{N_W} U_i \varphi_i, \quad v^h = \sum_{i=1}^{N_H} V_i \psi_i,$$

the expression of Problem (9) in terms of vector and matrices is

$$\begin{cases} \text{Find } U : [0,T] \to \overline{K}^h \text{ and } V : [0,T] \to I\!R^{N_H} \text{ such that for a.e. } t \in]0,T], \\ (W - U(t))^T (B^T \dot{V}(t) + AU(t)) \ge (W - U(t))^T L, \quad \forall W \in \overline{K}^h, \\ CV(t) = B\dot{U}(t), \\ U(0) = U_0, \ V(0) = V_0. \end{cases} \tag{10}$$

where $\overline{K}^h = \{W \in I\!R^{N_W} : \sum_{i=1}^{N_W} W_i \varphi_i \in K^h\}$.

The new class of approximation, matrix expression

Since matrix C is always invertible, one has

$$V(t) = C^{-1} B\dot{U}(t),$$

and thus denoting

$$M = B^T C^{-1} B,$$

the unknown V can be eliminated and Problem (10) rewritten

$$\begin{cases} \text{Find } U : [0,T] \to \overline{K}^h \text{ such that} \\ (W - U(t))^T (M\ddot{U}(t) + AU(t)) \geq (W - U(t))^T L, \quad \forall W \in \overline{K}^h, \ \text{a.e. } t \in]0,T], \\ U(0) = U_0, \ \ B\dot{U}(0) = CV_0. \end{cases} \tag{11}$$

Sufficient condition for the well-posedness

Using a restrictive framework, we suppose that K^h is defined by

$$K^h = \{w^h \in W^h : g^i(w^h) \le \alpha^i, 1 \le i \le N_g\},$$

where $\alpha^i \in \mathbb{R}$ and $g^i : W^h \to \mathbb{R}$, $1 \le i \le N_g$ are some linear independent maps. Then

$$\overline{K}^h = \{W \in \mathbb{R}^{N_W} : (G^i)^T W \le \alpha_i, 1 \le i \le N_g\},$$

where $G^i \in \mathbb{R}^{N_W}$ are such that $g^i(w^h) = (G^i)^T W$, $1 \le i \le N_g$. We will also denote G the $N_W \times N_g$ matrix whose components are $G_{ij} = (G^i)_j$. Let us consider the subspace of W^h

$$F^h = \{w^h \in W^h : \int_\Omega w^h q^h = 0 \ \ \forall q^h \in H^h\}, \quad \text{and} \quad F = \{W \in \mathbb{R}^{N_W} : \sum_{i=1}^{N_W} W_i \varphi_i \in F^h\},$$

One has $F = \mathrm{Ker}(B)$. In this framework, let us consider the following condition

$$\inf_{\substack{Q \in \mathbb{R}^{N_g} \\ Q \ne 0}} \sup_{\substack{W \in F \\ W \ne 0}} \frac{Q^T G W}{|Q| \, |W|} > 0, \tag{12}$$

which is equivalent to the linear independance of maps g^i on F^h and also to G surjective on F.

Well-posedness result

Lemma 1 *If W^h, H^h and K^h satisfy the condition (12) then there exists F^c a sub-space of $I\!R^{N_W}$ such that $F^c \subset \mathrm{Ker}(G)$ and such that F and F^c are complementary sub-spaces.*

Theorem 1 *If W^h, H^h and K^h satisfy the condition (12) then Problem (11) admits a unique solution. Moreover, this solution is Lipschtiz-continuous with respect to t.*

Sketch of Proof. Using the decomposition $U = U_F + U_{F^c}$, $W = W_F + W_{F^c}$, with $U_F, W_F \in F$ and $U_{F^c}, W_{F^c} \in F^c$. The inequation of (11) can be written for $a.e.\ t \in\ ]0,T]$

$$\begin{aligned}(W_{F^c} - U_{F^c})^T (M\ddot{U}_{F^c} + AU_{F^c} + AU_F) + (W_F - U_F)^T (AU_{F^c} + AU_F) \\ \geq (W_{F^c} - U_{F^c})^T L + (W_F - U_F)^T L, \quad \forall W_F \in \overline{K}^h \cap F, \ \forall W_{F^c} \in F^c.\end{aligned} \tag{13}$$

Taking now $W_{F^c} = U_{F^c}$ one obtains

$$(W_F - U_F)^T AU_F \geq (W_F - U_F)^T (L - AU_{F^c}), \quad \forall W_F \in \overline{K}^h \cap F. \tag{14}$$

U_F depends Lipschitz-continuoulsy on U_{F^c}.

$U_{F^c}(t)$ verifies for $a.e.\ t \in\ ]0,T]$ the Lipschitz-continuous o.d.e.

$$W_{F^c}^T M\ddot{U}_{F^c} = W_{F^c}^T (L - AU_{F^c} - AU_F(U_{F^c}(t))) \quad \forall W_{F^c} \in F^c. \tag{15}$$

Energy conservation result

Introducing Lagrange multipliers the discrete problem can be re-written

$$\begin{cases} \text{Find } U : [0,T] \to \overline{K}^h \text{ and } \lambda^i : [0,T] \to I\!R, 1 \le i \le N_g \text{ such that for a.e. } t \in]0,T] \\ M\ddot{U}(t) + AU(t) = L + \sum_{i=1}^{N_g} \lambda^i(t) G^i, \\ \lambda^i(t) \le 0, \quad (G^i)^T U(t) - \alpha_i \le 0, \quad \lambda^i(t)((G^i)^T U(t) - \alpha_i) = 0, \quad 1 \le i \le N_g, \\ U(0) = U_0, \ \ B\dot{U}(0) = CV_0, \end{cases} \tag{16}$$

Proposition 1 *If W^h, H^h and K^h satisfy the condition (12) then the solution $U(t)$ to Problem (11) verifies the following persistency condition*

$$\lambda^i(t)(G^i)^T \dot{U}(t) = 0 \quad \forall t \in]0,T], \quad 1 \le i \le N_g.$$

Theorem 2 *If W^h, H^h and K^h satisfy the condition (12) then the solution $U(t)$ to Problem (11) is energy conserving in the sense that*

$$J^h(t) = \frac{1}{2}\dot{U}^T(t) M\dot{U}(t) + \frac{1}{2}U^T(t) AU(t) - U^T(t) L,$$

is constant with respect to t.

A model problem

With $W = H^1(\Omega)$ and $K = \{w \in w : w \geq 0$ a.e. on $\Omega\}$ we consider the following problem:

$$\begin{cases} \text{Find } u : [0,T] \to K \text{ such that} \\ \dfrac{\partial^2 u}{\partial t^2}(t) - \Delta u(t) \in f - N_K(u(t)) \quad \text{for a.e. } t \in]0,T], \\ u = 0.05 \quad \text{on } \partial\Omega, \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = v_0, \end{cases}$$

This models for instance the dynamics of a membrane with an obstacle. Let $\mathcal{T}^h$ a regular mesh of Ω and W^h be the P_1+ finite element space

$$W^h = \{w^h \in \mathcal{C}^0(\Omega) : w^h = \sum_{a_i \in \mathcal{A}} w_i \varphi_i + \sum_{T \in \mathcal{T}^h} w_T \varphi_T\},$$

where $\mathcal{A}$ is the set of the interior vertex of the mesh, φ, $i \in \mathcal{A}$ are the piecewise affine function satisfying $\varphi(a_j) = \delta_{ij}$ and φ_T is a quadratic bubble function whose support is T. Let H^h be the P_0 finite element space

$$H^h = \{v^h \in L^2(\Omega) : v^h = \sum_{T \in \mathcal{T}^h} v_T \mathbb{1}_T\},$$

and finally, let K^h be defined as

$$K^h = \{w^h \in W^h : w^h(a_i) \geq 0 \text{ for all } a_i \text{ vertex of } \mathcal{T}^h\},$$

i.e. the constraint is prescribed at the vertex of the mesh. For this particular choice condition (12) is satisfied.

Numerical experiments

The numerical experiments are done on the problem described in the previous section, with

$$\Omega =]0,1[\times]0,1[, \quad \Gamma_D = \partial\Omega, \quad \Gamma_N = \emptyset, \quad f = -0.6.$$

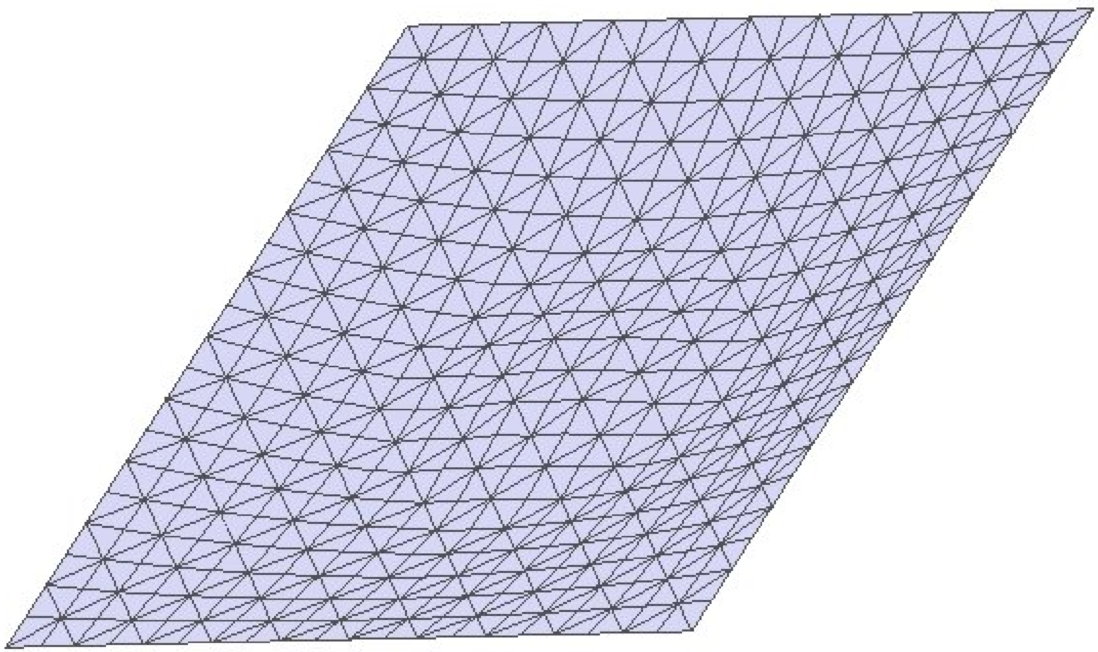
The initial condition is

$$u(0,x) = 0.02,$$

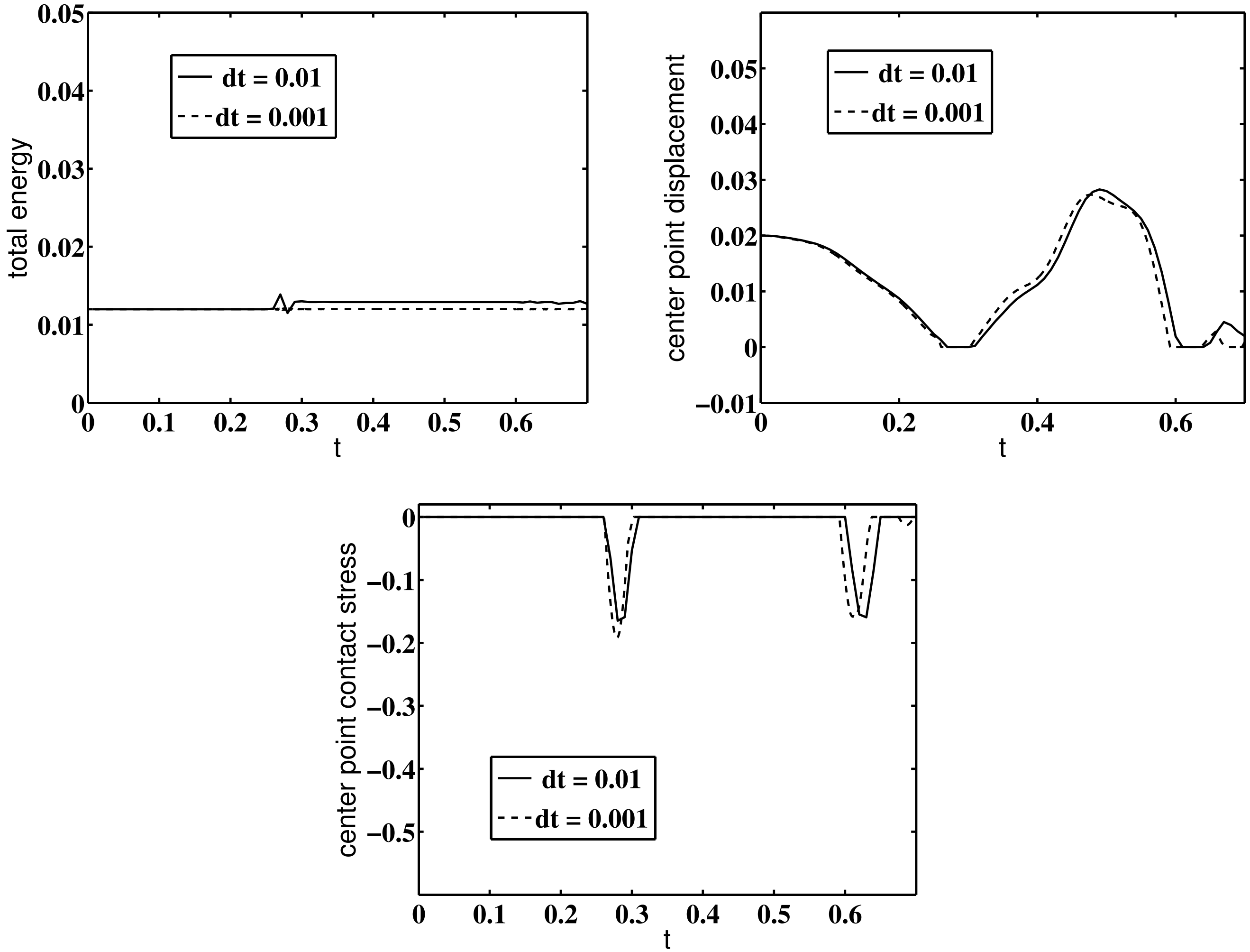
and the Dirichlet condition is

$$u(t,x) = 0.02, \quad x \in \Gamma_D.$$

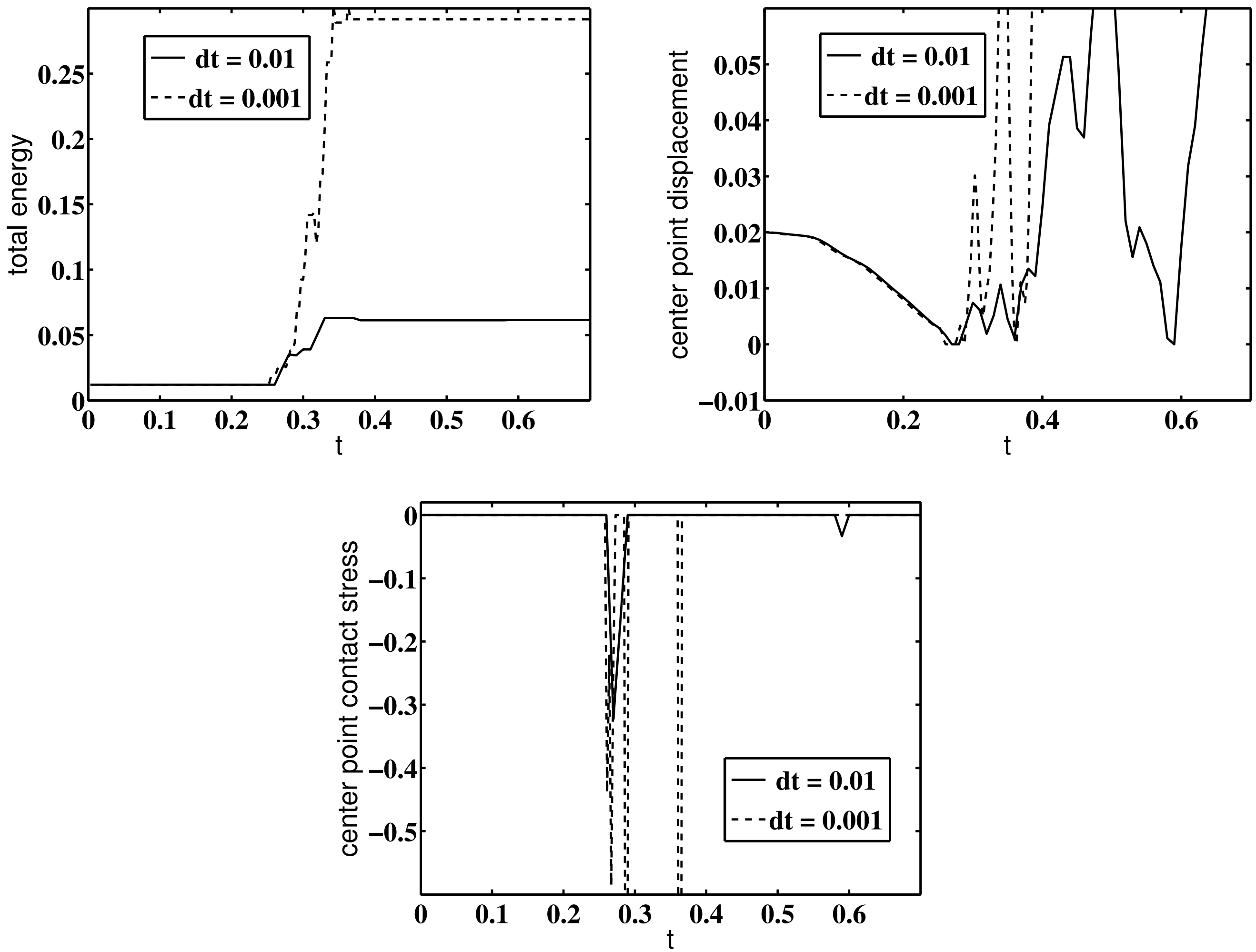
An example of computation, during the first impact ($h = 0.05$).



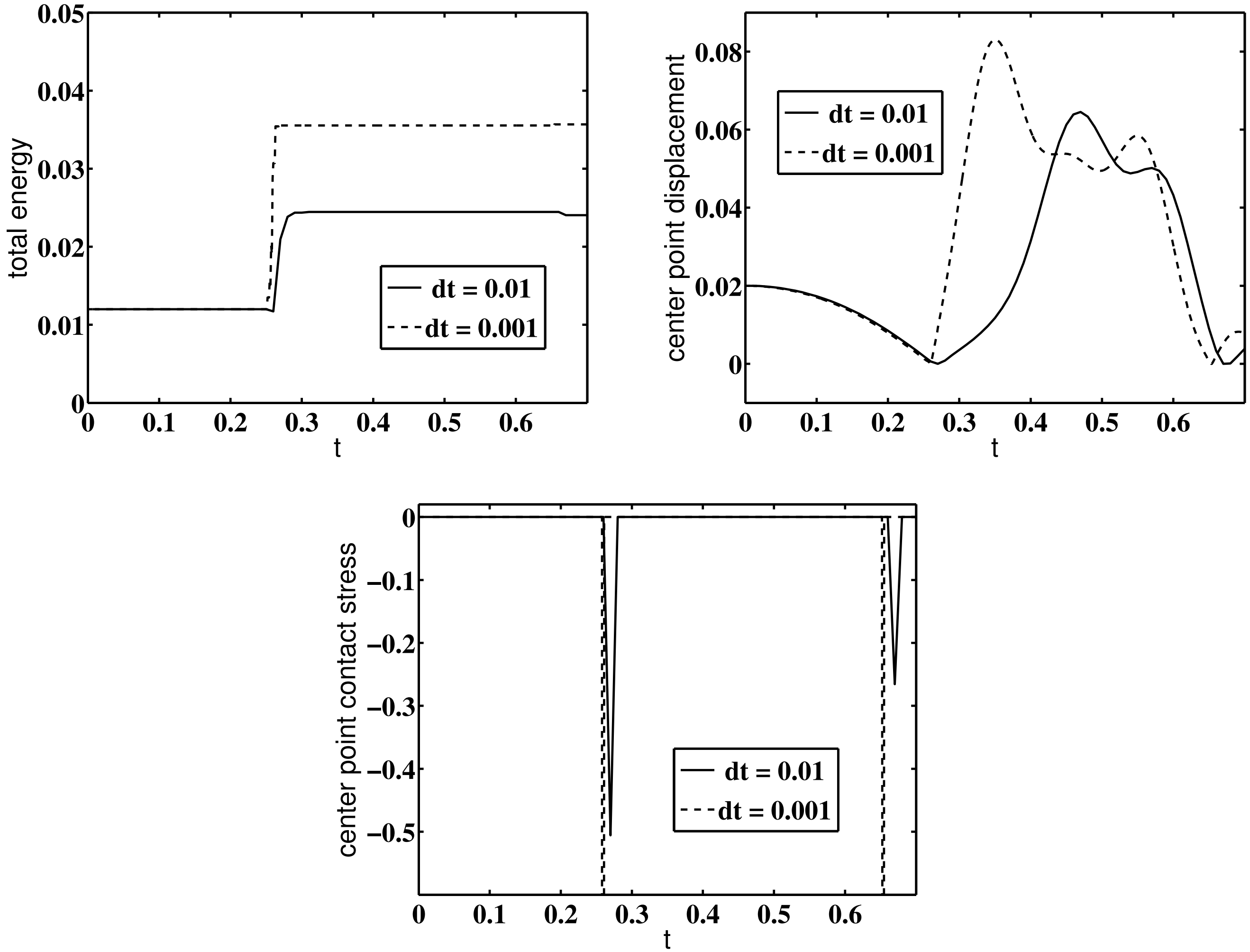
$P_1 + /P_0$ method, $h = 0.1$, midpoint scheme



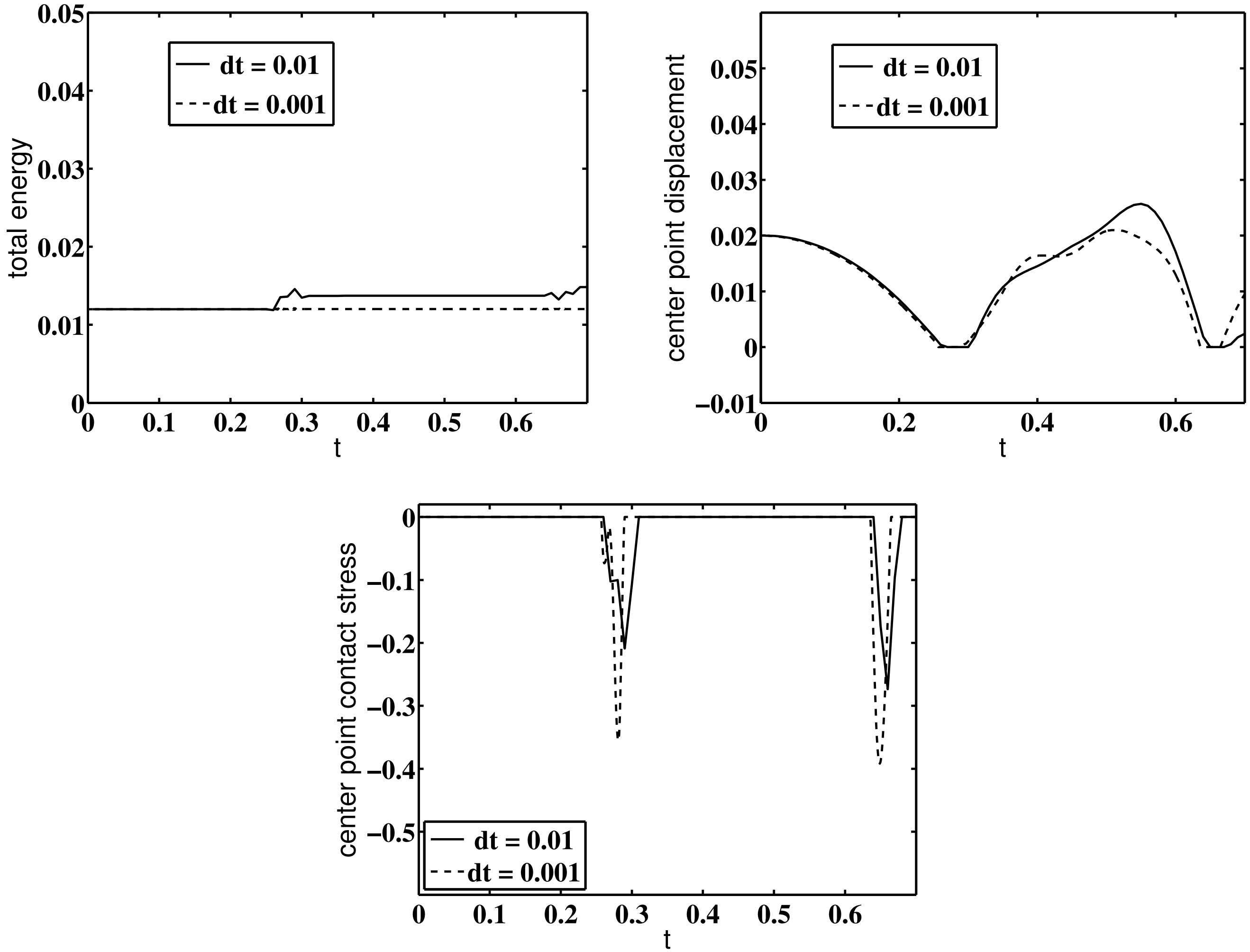
P_1/P_0 method, $h = 0.1$, midpoint scheme



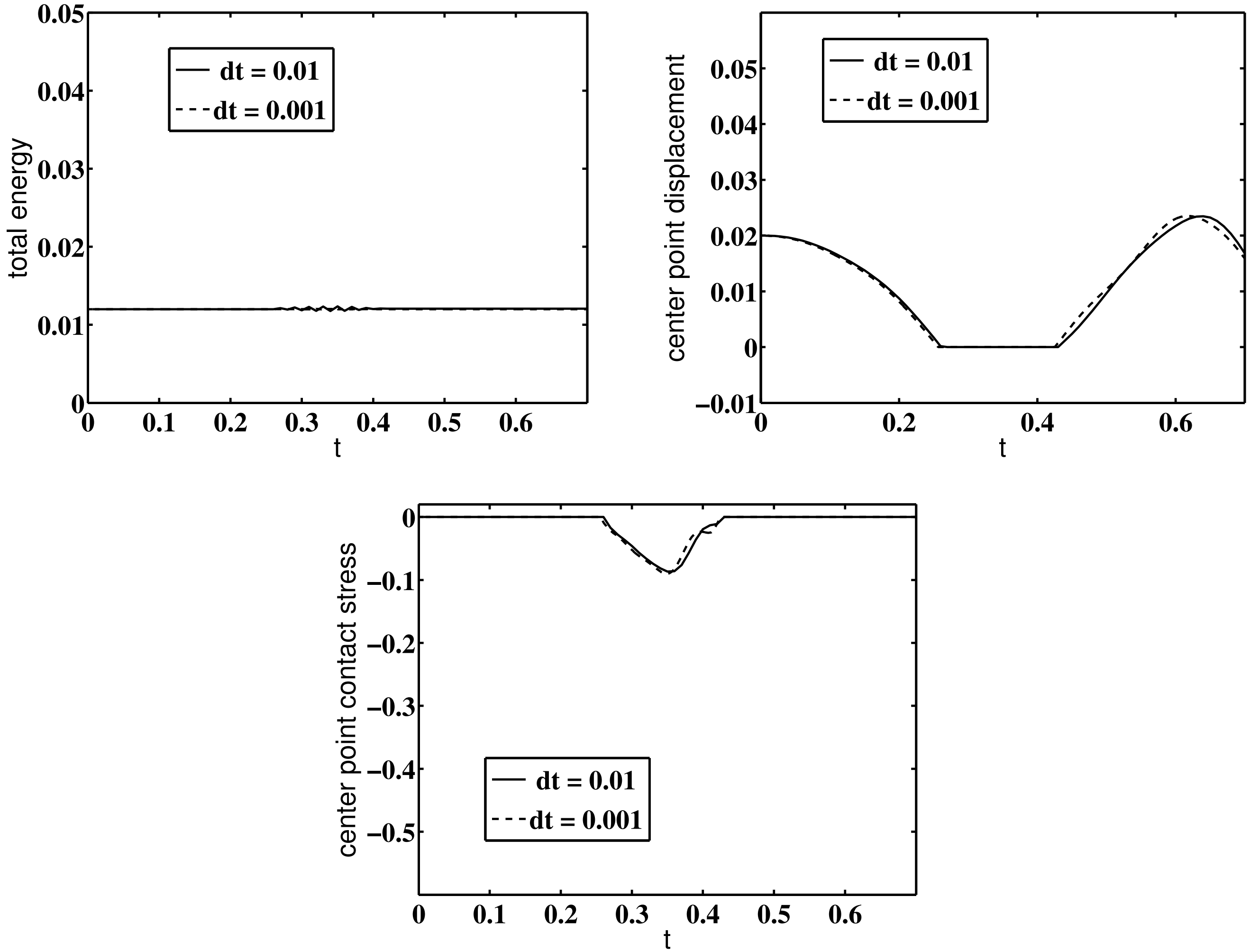
P_1/P_1 method, $h = 0.1$, midpoint scheme



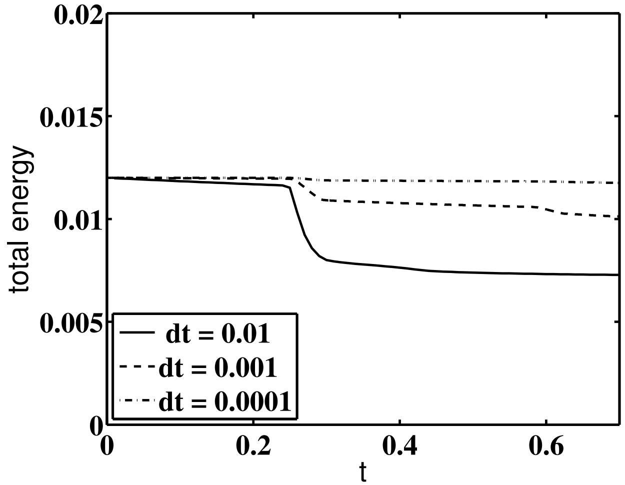
$P_1 + /P_1$ method, $h = 0.1$, midpoint scheme

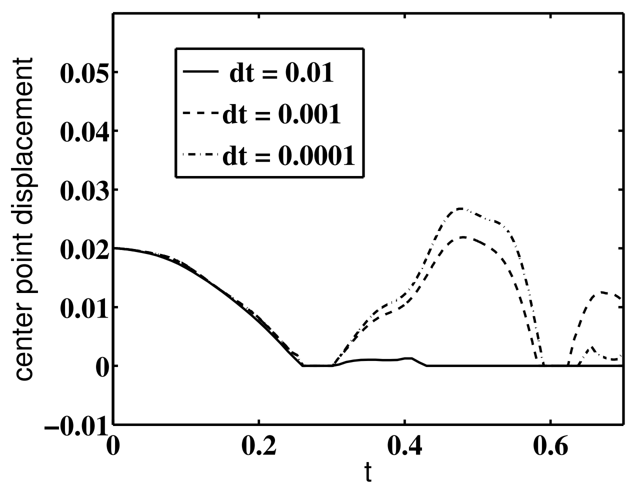


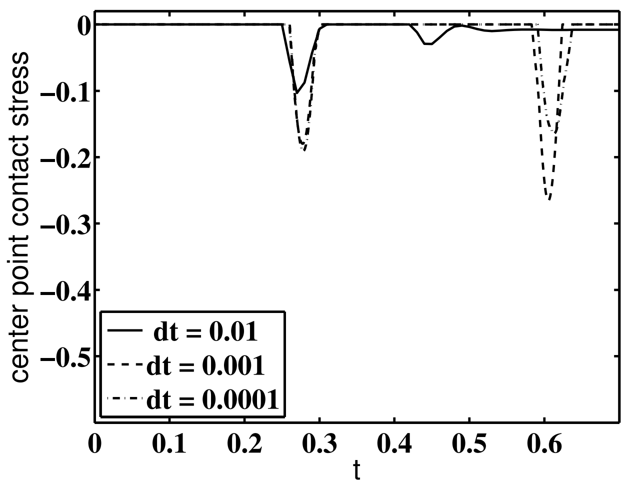
P_2/P_1 method, $h = 0.1$, midpoint scheme



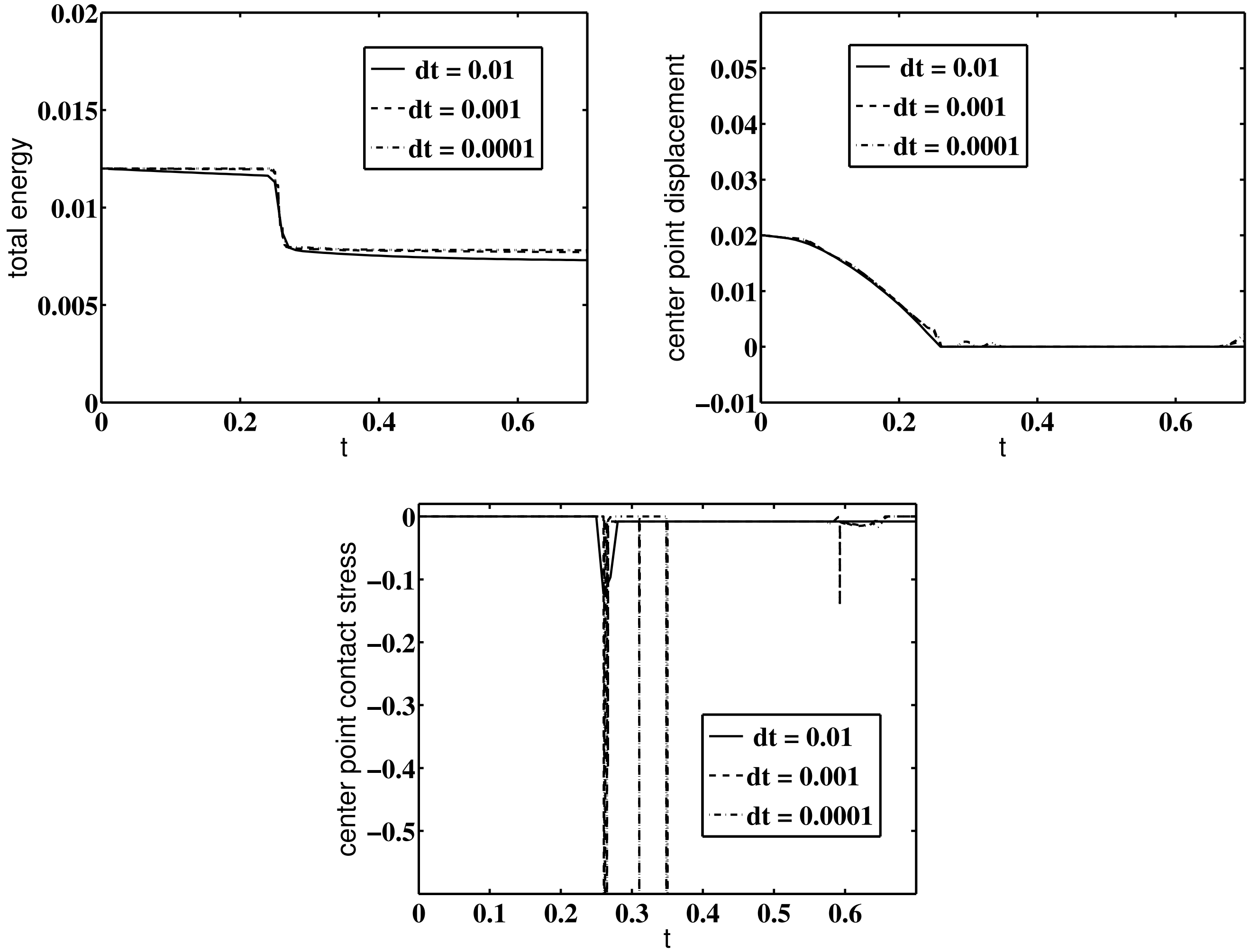
$P_1 + / P_0$ method, $h = 0.1$, backward Euler scheme



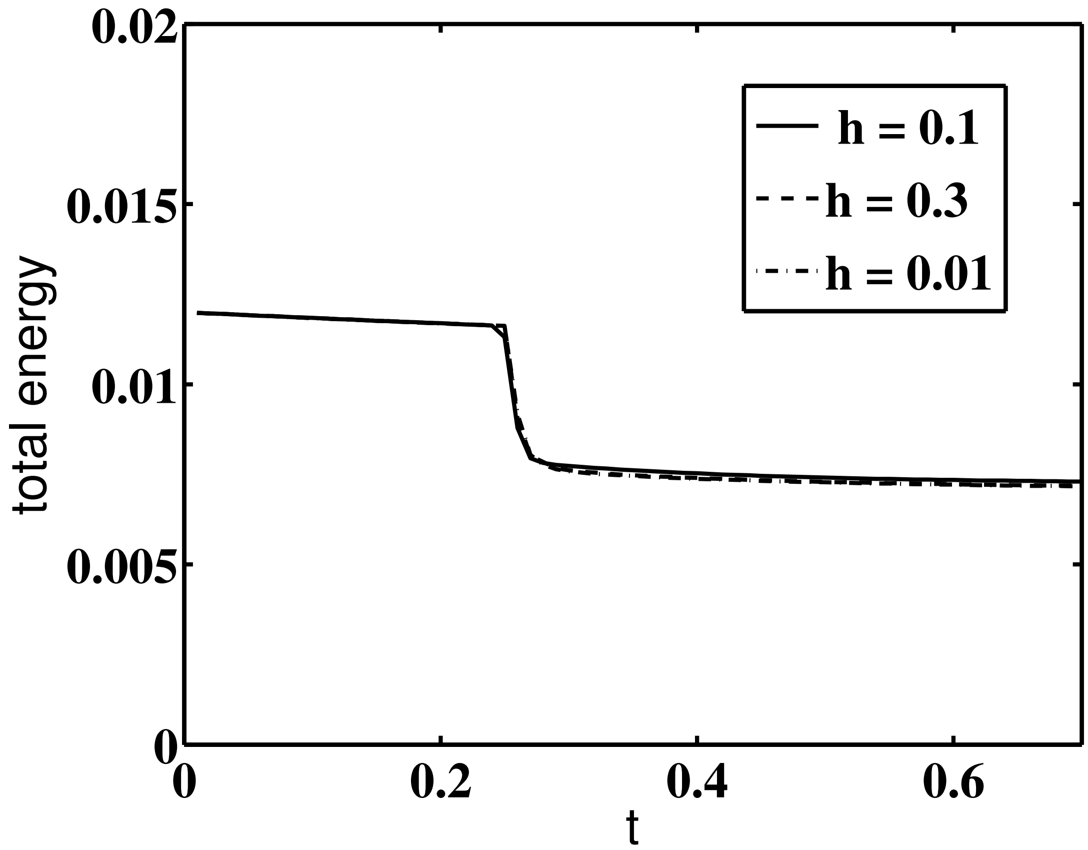




P_1/P_0 method, $h = 0.1$, backward Euler scheme



P_1/P_0 method, $\Delta t = 0.001$, backward Euler scheme



Concluding remarks

- The proposed methods are more general than the Mass Redistribution Method and have the same advantages. It is of arbitrary degree and can treat thin structures.
- Perspectives :
  - Develop an energy conserving full discretization.
  - Theoretical stability of mid-point scheme.
  - Treat more general convex of constraints.

- Predefined models : Linear and non-linear elasticity, small strain plasticity, Stokes, Navier-Stokes, Helmholtz, contact and friction conditions, plate models, cracks.
- Matrix library with linear solvers (Gmres, Bicgstab, Cg, ..) with preconditionners (ilu, ilut, ildlt, ildltt ...). Eigenvalues search. Interfaced with Lapack and SuperLU.
- Finite element methods P_K, Q_K, in arbitrary dimension $(1 \leq n \leq 16)$. Tensorial product of elements, hierarchic and composite elements, arbitrary degree geometric transformation, level sets.
- A set of integration methods (exact or not).
- Generic elementary matrices computation and generic assembly procedures.
- Matlab and Python interfaces.

Getfem++ is Freely available (J. Pommier, Y. Renard)

http://home.gna.org/getfem/

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