A velocity-based time-stepping scheme for vibro-impact problems

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I - Description of the dynamics

We consider a mechanical system with a finite number of degrees of freedom.
The unconstrained dynamics is given by

$$M(q)\ddot{q} = f(t, q, \dot{q}).$$

We assume that the system is submitted to perfect unilateral constraints described by

$$q(t) \in K = \{q \in \mathbb{R}^d;\ g_\alpha(q) \geq 0\ \forall \alpha \in \{1, \dots, \nu\}\}, \quad \nu \geq 1.$$

Adding the reaction force due to the constraints, we obtain

$$(1) \qquad M(q)\ddot{q} = f(t, q, \dot{q}) + \mu, \quad \mu \in -N_K(q), \quad \mathrm{Supp}(\mu) \subset \{t;\ q(t) \in \partial K\}$$

with

$$N_K(q) = \Big\{\sum_{\alpha=1}^{\nu} \lambda_\alpha \nabla g_\alpha(q),\ \lambda_\alpha \leq 0 \text{ if } g_\alpha(q) \leq 0,\ \lambda_\alpha = 0 \text{ otherwise}\Big\}.$$

The transmission of the velocity at impacts is modelled by Newton's law

$$(2) \qquad \dot{q}^+(t) = -e\dot{q}^-(t) + (1+e)\mathrm{proj}_{q(t)}\big(T_K(q(t)), \dot{q}^-(t)\big)$$

where $e \in [0,1]$ is a restitution coefficient and

$$T_K(q) = V(q) = \big\{w \in \mathbb{R}^d; \nabla g_\alpha(q) \cdot w \geq 0 \text{ if } g_\alpha(q) \leq 0\big\}.$$

Following J.J. Moreau's approach, (1)-(2) can be replaced by a Measure Differential Inclusion

$$(\mathrm{MDI}) \qquad f(t,q,\dot{q}) - M(q)\ddot{q} \in N_{V(q)}\left(\frac{\dot{q}^+ + e\dot{q}^-}{1+e}\right).$$

Reference: J.J. Moreau, "Unilateral contact and dry friction in finite freedom dynamics", 1988.

More precisely we consider the following Cauchy problem:

Problem (P) Let $(q_0, u_0) \in K \times V(q_0)$ be admissible initial data.
Find $\tau > 0$ and $u \in BV\big([0,\tau]; \mathbb{R}^d\big)$ such that u and the function q defined by

$$q(t) = q_0 + \int_0^t u(s)\,ds \qquad \forall t \in [0,\tau]$$

satisfy:

$$u^+(0) = u_0, \qquad u(t) = \frac{u^+(t) + e u^-(t)}{1+e} \qquad \forall t \in (0,\tau),$$

and

$$f\big(t,q,u\big)dt - M(q)du \in N_{V(q)}(u)$$

with

$$N_{V(q)}(u) = \begin{cases} \big\{y \in \mathbb{R}^d;\ y \cdot (x-u) \le 0\ \forall x \in V(q)\big\} \text{ if } u \in V(q), \\ \emptyset \text{ otherwise.} \end{cases}$$

II - Description of the scheme

For a given time-step $h > 0$, we define the approximate positions and velocities by

$$q_{h,0} = q_0, \quad u_{h,0} = u_0$$

and for all $i \geq 0$

$$q_{h,i+1} = q_{h,i} + h u_{h,i}$$

$$f\left(t_{h,i+1}, q_{h,i+1}, u_{h,i+1}\right) - M(q_{h,i+1})\left(\frac{u_{h,i+1} - u_{h,i}}{h}\right) \in N_{V(q_{h,i+1})}\left(\frac{u_{h,i+1} + e u_{h,i}}{1+e}\right).$$

Using the definition of $N_{V(q)}(\cdot)$, we can rewrite it as

$$u_{h,i+1} = -e u_{h,i} + (1+e)\mathrm{proj}_{q_{h,i+1}}\left(V(q_{h,i+1}), u_{h,i} + \frac{h}{1+e} M^{-1}(q_{h,i+1}) f\left(t_{h,i+1}, q_{h,i+1}, u_{h,i+1}\right)\right).$$

Reference: J.J. Moreau, "Dynamique de systèmes à liaisons unilatérales avec frottement sec éventuel, essais numériques", 1986.

Properties

• If $q_{h,i+1} \in \mathrm{Int}(K)$, we obtain simply

$$\frac{q_{h,i+2} - 2q_{h,i+1} + q_{h,i}}{h^2} = M^{-1}(q_{h,i+1}) f\left(t_{h,i+1}, q_{h,i+1}, u_{h,i+1}\right)$$

which is a discretization of the unconstrained dynamics.

• The constraint is satisfied at the velocity level at each time step in the following sense

$$\frac{u_{h,i+1} + e u_{h,i}}{1+e} \in V(q_{h,i+1}).$$

• The ponderation between $u_{h,i+1}$ and $u_{h,i}$ leads to a correct reflection of the velocity at impacts.

Example (bouncing ball): $d = 1$, $K = \mathbb{R}^+$, $M(q) \equiv 1$, $f \equiv 0$, $q_0 = 1$, $u_0 = -1$.
The solution of problem (P) is

$$\begin{cases} q(t) = 1 - t & \text{if } t \in [0,1], \\ q(t) = e(t-1) & \text{if } t \geq 1. \end{cases}$$

We obtain $q_{h,0} = 1$, $u_{h,0} = -1$ and for all $i \geq 0$

$$q_{h,i+1} = q_{h,i} + hu_{h,i}, \quad u_{h,i+1} = -eu_{h,i} + (1+e)\text{proj}_{q_{h,i+1}}\big(V(q_{h,i+1}), u_{h,i}\big)$$

with

$$\text{proj}_{q_{h,i+1}}\big(V(q_{h,i+1}), u_{h,i}\big) = \begin{cases} u_{h,i} & \text{if } q_{h,i+1} > 0, \\ \max(u_{h,i}, 0) & \text{if } q_{h,i+1} \leq 0. \end{cases}$$

Assume that $h \in (0,1)$. There exists $p \geq 1$ such that

$$p = \max\big\{k \geq 0;\ q_{h,i} > 0\ \forall i \in \{0, \dots, k\}\big\}.$$

Then

$$q_{h,i+1} = 1 - (i+1)h, \quad u_{h,i} = -1 \quad \forall i \in \{0, \dots, p\}$$

and $q_{h,p+1} \leq 0$. Thus $u_{h,p+1} = -eu_{h,p} + (1+e)\max(u_{h,p}, 0) = e$ and

$$q_{h,i} = 1 - (p+1)h + e(i-p-1)h, \quad u_{h,i} = e \quad \forall i \geq p+1.$$

III - Convergence results

References:

• single constraint case ($\nu = 1$), trivial mass matrix and $e = 0$: M. Monteiro Marques, "Differential inclusions in nonsmooth mechanical problems", 1993

• single constraint case, trivial mass matrix and $e \in [0, 1]$: M. Mabrouk, "A unified variational model for the dynamics of perfect unilateral contraints", 1998

New results:

• single constraint case, non trivial mass matrix and $e \in [0, 1]$ (joint work with R. Dzonou and M. Monteiro Marques)

• multi-constraint case, non trivial mass matrix, $e \in [0, 1]$

In the single constraint case we prove the convergence of the approximate positions and velocities under the following assumptions:

(H1) $f \in C^0\big([0,T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d\big)$ $(T > 0)$, s.t. f is locally Lispchitz continuous in its last two arguments,

(H2) $M \in C^1\big(\mathbb{R}^d; \mathcal{M}_{\mathrm{d*d,sdp}}(\mathbb{R})\big)$,

(H3) $g \in C^{1,1/2}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R})$ s.t. ∇g does not vanish in a neighbourhood of $\{q \in \mathbb{R}^d;\ g(q) = 0\}$.

Under assumptions (H1) and (H2) we can not expect a global existence result on $[0,T]$ for problem (P) but we have

Proposition (energy estimate): Let $R > \|u_0\|_{q_0}$. Then there exists $\tau(R) > 0$ s.t., for any solution (q,u) of problem (P) defined on $[0,\tau]$ (with $\tau \in (0,T]$), we have

$$\big\|u(t)\big\|_{q(t)} \leq R \quad \forall t \in \big[0, \min\big(\tau, \tau(R)\big)\big].$$

Reference: L.P. and M. Schatzman, "A numerical scheme for impact problems", 2002.

We define the sequence of approximate solutions $(q_h, u_h)_{h>0}$ by

$$u_h(t) = u_{h,i} \quad \text{if } t \in [t_{h,i}, t_{h,i+1}) \cap [0,T],$$
$$q_h(t) = q_0 + \int_0^t u_h(s)\,ds \quad \forall t \in [0,T].$$

So we prove

Theorem: Let $R > \|u_0\|_{q_0}$. Then there exists a subsequence of $(q_h, u_h)_{h>0}$, still denoted $(q_h, u_h)_{h>0}$, s.t.

$$q_h \rightarrow q \quad \text{in } C^0\Big([0, \min\big(T, \tau(R)\big)]; \mathbb{R}^d\Big)$$
$$u_h \rightarrow v \quad \text{pointwise in } \big[0, \min\big(T, \tau(R)\big)\big]$$

and (q, u) is a solution of problem (P) on $\big[0, \min\big(T, \tau(R)\big)\big]$, with

$$u(t) = \frac{v^+(t) + e v^-(t)}{1+e} \quad \forall t \in \big(0, \min\big(T, \tau(R)\big)\big).$$

Sketch of the proof

First step: local convergence result

• We establish uniform estimates for the discrete velocities on a time interval $[0, \tau]$, $\tau \in (0, T]$, by using Brouwer's fixed point theorem.

Then we prove that $(u_h)_{h>0}$ is bounded in $BV(0, \tau; \mathbb{R}^d)$.

• We pass to the limit by using Ascoli's and Helly's theorems: there exists a subsequence, still denoted $(q_h, u_h)_{h>0}$ s.t.

$$u_h \to v \quad \text{pointwise in } [0, \tau],$$

and

$$q_h \to q \quad \text{uniformly in } [0, \tau],$$

with

$$q(t) = q_0 + \int_0^t u(s)\, ds \quad \forall t \in [0, \tau],$$

and

$$u(t) = \frac{v^+(t) + e v^-(t)}{1 + e} \quad \forall t \in (0, \tau).$$

• We study the properties of the limit (q, u): we prove that

$$q(t) \in K \quad \forall t \in [0, \tau].$$

Then we establish that the MDI is satisfied on $J = \{t \in [0, \tau];\ u^+(t) = u^-(t)\}$ and that the velocity is correctly reversed on $[0, \tau] \setminus J$.

Second step: Convergence on $\left[0, \min\left(T, \tau(R)\right)\right]$, $R > \|u_0\|_{q_0}$.

We establish that, if $(q_h, u_h)_{h>0}$ converges to a solution (q, u) of problem (P) on $[0, \tau^*]$, with $\tau^* \in (0, T]$, we have

$$\limsup_{h \to 0} \sup\left\{\|u_{h,i}\|_{q_{h,i}};\ 0 \le t_{h,i} \le \min\left(\tau^*, \tau(R)\right)\right\}$$
$$\le \text{ess sup}\left\{\|u(t)\|_{q(t)};\ 0 \le t \le \min\left(\tau^*, \tau(R)\right)\right\} \le R$$

and we argue by contradiction.

In the multi-constraint case we replace assumption (H3) by

(H'3) for all $\alpha \in \{1, \dots, \nu\}$, $g_\alpha \in C^1(\mathbb{R}^d; \mathbb{R})$ s.t. ∇g_α is locally Lispchitz continuous and does not vanish in a neighbourhood of $\{q \in \mathbb{R}^d;\ g_\alpha(q) = 0\}$,

(H'4) $\big(\nabla g_\alpha(q)\big)_{\alpha \in J(q)}$, with $J(q) = \{\alpha;\ \ g_\alpha(q) = 0\}$, is linearly independent for all $q \in K$.

Moreover a new difficulty occurs: the motion is not continuous with respect to initial data.

Nevertheless continuity on data holds if

(H'5) for all $q \in K$, for all $(\alpha, \beta) \in J(q)^2$ s.t. $\alpha \neq \beta$

$$\begin{aligned} &\big(\nabla g_\alpha(q), M(q)^{-1} \nabla g_\beta(q)\big) \leq 0 \quad \text{if } e = 0, \\ &\big(\nabla g_\alpha(q), M(q)^{-1} \nabla g_\beta(q)\big) = 0 \quad \text{if } e \in (0, 1]. \end{aligned}$$

Reference: L.P., "Continuous dependence on data for vibro-impact problems", 2005.

Under these assumptions we prove once again the convergence of a subsequence of $(q_h, u_h)_{h>0}$ to a solution of problem (P).

The sketch of the proof is similar but now the main difficulty is the study of the reflection of the velocity at impacts.

Idea: If $u^+(t) \neq u^-(t)$ we prove first that

$$M\big(q(t)\big)\big(u^+(t) - u^-(t)\big) \in -N_K\big(q(t)\big)$$

i.e. there exists $(\mu_\alpha)_{1\leq\alpha\leq\nu}$ s.t.

$$u^+(t) - u^-(t) = \sum_{\alpha=1}^{\nu} \mu_\alpha M\big(q(t)\big)^{-1} \nabla g_\alpha\big(q(t)\big) \quad \text{with} \begin{cases} \mu_\alpha \geq 0 \text{ if } g_\alpha\big(q(t)\big) = 0, \\ \mu_\alpha = 0 \text{ otherwise.} \end{cases}$$

Then we observe that the impact law is satisfied iff

$$\nabla g_\alpha\big(q(t)\big) \cdot \big(u^+(t) + e u^-(t)\big) = 0 \quad \text{for all } \alpha \in \{1, \dots, \nu\} \text{ s.t. } \mu_\alpha > 0$$

which is proved by performing a precise study of the discrete velocities in a neighbourhood of t.

Other references about approximation of vibro-impact problems

• M.Schatzman, "A class of nonlinear differential equations of second order in time", *Non-linear Anal.,Theory, Methods and Applications*, 2(1978)355-373.

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• L. Paoli, "An existence result for vibrations with unilateral constraints: case of a non-smooth set of constraints", *Math. Models Methods Appl. Sci. (M3AS)*, 10-6(2001)815-831.

• M. Schatzman, "Penalty method for impact in generalized coordinates", *Phil. Trans. Roy. Soc. London A*, 359(2001)2429-2446.

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