Numerical Verification of Conjectures *à la Stark* in the Abelian Case

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Conjectures *à la Stark*

Let K be a totally real field and let N/K be a finite abelian extension of number fields with conductor $\mathfrak{f} = \mathfrak{f}_0\mathfrak{f}_\infty$ and Galois group G. Let S be a finite set of places of K such that $S_\infty \cup \{\mathfrak{p} \mid \mathfrak{f}_0\} \subset S$.

$$\left(\begin{array}{c} \text{Values of transcendentals functions} \\ \text{associated to } N/K \text{ and to } S \\ \text{at } s = 0 \text{ or } s = 1 \end{array} \right) \sim \left(\begin{array}{c} \text{Regulator} \\ \text{of some} \\ S\text{-units in } N \end{array} \right)$$

The Abelian Rank One Stark Conjecture

The Brumer-Stark Conjecture

Solomon's Conjecture

The Abelian Rank One Stark Conjecture

Assume that S contains a totally split place v (also that $|S| > 2$). Then there exists a v-unit $\varepsilon \in N$ such that

$$\text{(a)} \quad \log|\sigma(\varepsilon)|_w = -m\,\zeta'_S(0,\sigma), \ \forall\sigma \in G$$

$$\text{(b)} \quad N(\varepsilon^{1/m})/K \text{ is an abelian extension}$$

where $m = \mathrm{Card}(W_N)$ and w is a fixed place of N dividing v.

Numerical verification: take v real (so $m = 2$), compute approximations of the values of $\zeta'_S(0,\sigma)$ and construct the minimal polynomial of $\varepsilon \in U_N$ over K using the formula

$$\sigma(\varepsilon) = e^{-2\,\zeta'_S(0,\sigma)}$$

for all $\sigma \in G$.

An example

Base field		Extension N/K	
$K = \mathbb{Q}(\sqrt{1093})$		Conductor	$\mathfrak{f} = \mathfrak{p}_3 v'$
Discriminant	$d_K = 1093$	with	$3\mathcal{O}_K = \mathfrak{p}_3\mathfrak{q}_3$
ring of integers	$\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$	and	$v'(\sqrt{1093}) > 0$
with	$\omega = (1+\sqrt{1093})/2$	Galois group	$G \simeq C_{10}$
Class group	$\mathrm{Cl}_K \simeq C_5$	Set of places	$S = \{v, v', \mathfrak{p}_3\}$

Then the numbers $e^{-2\zeta_S'(0,\sigma)}$ (with $\sigma \in G$) are the *roots* of the following polynomial which defines N over K

$$X^{10} + (-32\omega - 507)X^9 + (801\omega + 12858)X^8 + (-6575\omega - 105364)X^7 + (22986\omega + 368523)X^6 + (-35264\omega - 565234)X^5 + (22986\omega + 368523)X^4 + (-6575\omega - 105364)X^3 + (801\omega + 12858)X^2 + (-32\omega - 507)X + 1$$

The Brumer-Stark Conjecture

Assume N is totally complex and define

$$\gamma = m \sum_{\sigma \in G} \zeta_S(0,\sigma)\sigma^{-1} \in \mathbb{Z}[G]$$

Then, for any fractional ideal $\mathfrak{A}$ of N, there exists $\varepsilon_{\mathfrak{A}} \in N^{\times}$ such that

(a) $\mathfrak{A}^{\gamma} = (\varepsilon_{\mathfrak{A}})$

(b) $|\varepsilon_{\mathfrak{A}}|_w = 1, \ \forall w \mid \infty$

(c) $N(\varepsilon_{\mathfrak{A}}^{1/m})/K$ is an abelian extension

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Numerical verification: compute γ and test the conjecture for ideals $\mathfrak{A}$ generating $\mathrm{Cl}(N)$ over $\mathbb{Z}[G]$: compute a generator α of $\mathfrak{A}^{\gamma}$, find a unit u such that $\varepsilon := u\alpha$ satisfies (b) and check if (c) holds.

An example

Extension N/K		Top field	
$K = \mathbb{Q}(\sqrt{23})$		$N = \mathbb{Q}(\theta)$	
Conductor	$\mathfrak{f} = \mathfrak{p}_2^7 v\, v'$	with $\theta^8 + 16\theta^6 + 86\theta^4 + 176\theta^2 + 98 = 0$	
with	$2\mathcal{O}_K = \mathfrak{p}_2^2$	Discriminant	$d_N = 2^{27}23^4$
Ray class group	$\mathrm{Cl}_K(\mathfrak{f}) \simeq C_4 \times C_2^3$	Class group	$\mathrm{Cl}_N \simeq C_{26}$
Congruence group	$(\mathrm{Cl}_K(\mathfrak{f}) : \mathcal{H}) = 4$	Galois group	$G = \langle \sigma \rangle \simeq C_4$
Set of places	$S = \{v, v', \mathfrak{p}_2\}$	$\mathbb{Z}[G]$-generator	$\mathrm{Cl}_N = \langle \mathfrak{P}_7 \rangle_{\mathbb{Z}[G]}$

Compute $\gamma = 8 + 12\sigma - 8\sigma - 12\sigma^2 = 4(2+3\sigma)(1-\sigma)$. Then $\mathfrak{P}_7^\gamma = (\alpha)$ with

$$\alpha = \tfrac{1}{13841287201}\big(342647708\theta^7 - 29342815536\theta^6 + 12834211116\theta^5 - 363823380016\theta^4 + 88834152264\theta^3 - 1218060670888\theta^2 + 141302660240\theta - 92400986335$$

and $\mathcal{N}_{N/N^{\sigma^2}}(\alpha) = 1$ and $X^2 - \alpha^{\sigma-1} = (X-\alpha_1)(X-\alpha_2)$ in $N[X]$. The same is true with γ replaced by $\gamma/2$ and α replaced by $\alpha^{1/2}$ (but not with γ replaced with $\gamma/4$).

Solomon's Conjecture: Twisted Zeta Functions

Assume that $\mathfrak{f}_\infty$ is empty, $\mathfrak{f}_0 \neq \mathcal{O}_k$, and $N = K(\mathfrak{f})$ ($\mathfrak{f}$ not necessarily conductor).

Define $W_\mathfrak{f} = \{(\xi, \mathfrak{a}) \text{ with } \mathfrak{a} \lhd \mathcal{O}_k,\ \xi : \mathfrak{a} \to \mathbb{C}^\times \text{ and } \mathrm{Ann}_{\mathcal{O}_k}(\xi) = \mathfrak{f}\}$.

Say that $(\xi, \mathfrak{a}) \sim (\xi', \mathfrak{a}')$ if $\xi' = \xi \circ c$ and $\mathfrak{a} = c\mathfrak{a}'$ for some $c \in k^\times$.

Then $\mathfrak{W}_\mathfrak{f} = W_\mathfrak{f}/\sim$ is canonically isomorphic to $\mathrm{Cl}_K(\mathfrak{f})$.

Fix a finite set T of prime ideals disjoint from S, and for $\mathfrak{w} \in \mathfrak{W}_\mathfrak{f}$ choose $(\xi, \mathfrak{a}) \in \mathfrak{w}$ such that $(\mathfrak{a}, T) = 1$ and define for $\Re(s) > 1$

$$Z_T(s, \mathfrak{w}) = \sum_{\substack{\alpha \in \mathfrak{a}/U_K(\mathfrak{f}) \\ (\alpha, T)=1}} \xi(\alpha)\, |\mathcal{N}\mathfrak{a}^{-1}\alpha|^{-s} \quad \text{and} \quad \Phi_T(s) = \sum_{\mathfrak{c} \in \mathrm{Cl}_K(\mathfrak{f})} Z_T(s, \mathfrak{w}_\mathfrak{c}) \sigma_\mathfrak{c}^{-1}$$

$\Phi_\emptyset(s)$ extends to an holomorphic function on $\mathbb{C}[G]$ and for any odd prime p with $(p, \mathfrak{f}) = 1$ there exists a p-adic function $\Phi_p(s)$ on $\mathbb{Z}_p[G]$ interpolating the values of $\Phi_{T_p}(m)$ at negative integers m congruent to 1 modulo $p - 1$ where $T_p = \{\mathfrak{p} \mid p\}$.

Solomon's Conjecture: S-units and Regulators

Let $r = [K : \mathbb{Q}]$ and fix $\iota_1, \ldots, \iota_r : N \hookrightarrow \mathbb{R}$ representatives of the action of G on $S_\infty(N)$ and for each prime p, fix $j_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and let $\iota_{i,p} = j_p \circ \iota_i : N \hookrightarrow \mathbb{C}_p$.

Define logarithmic maps λ_i, $\lambda_{i,p}$ from $U_S = U_{N,S(N)}$ to $\mathbb{R}[G]$, $\mathbb{C}_p[G]$ by

$$\lambda_i(u) = \sum_{\sigma \in G} \log |\iota_i \circ \sigma(u)|\, \sigma^{-1} \text{ and } \lambda_{i,p}(u) = \sum_{\sigma \in G} \log_p(\iota_{i,p} \circ \sigma(u))\, \sigma^{-1}$$

And define complex/p-adic regulators maps on $\bigwedge^r_{\mathbb{Q}[G]} \mathbb{Q}U_S$ by

$$R(u_1 \wedge \cdots \wedge u_r) = \det(\lambda_i(u_l))_{i,l} \quad \text{and} \quad R_p(u_1 \wedge \cdots \wedge u_r) = \det(\lambda_{i,p}(u_l))_{i,l}$$

For $\chi \in \hat{G}$, let $r(S, \chi) = \dim_{\mathbb{C}}(e_\chi\, \mathbb{C}U_S)$ then $r(S, \chi) = r + \mathrm{ord}_{s=1}(\chi(\Phi_\emptyset(s)))$.

Set $e_{S,>r} = |G| \sum_{r(S,\chi)>r} e_\chi \in \mathbb{Z}[G]$ and

$$\Lambda_{S,>r} = \ker\left(e_{S,>r} : \overline{\textstyle\bigwedge^r_{\mathbb{Z}[G]} U_S} \to \overline{\textstyle\bigwedge^r_{\mathbb{Z}[G]} U_S}\right)$$

Solomon's (Weak Refined Combined) Conjecture

There exists a unique $\eta_{\mathfrak{f}} \in \frac{1}{2}\mathbb{Z}[1/g]\Lambda_{S,>r}$ with $g = |G|$ such that

(a) $$\frac{2^r}{\sqrt{d_K}} R(\eta_{\mathfrak{f}}) = \Phi_{\emptyset}(1)$$

(b) $$\prod_{\mathfrak{p} \in T_p} \left(1 - \mathcal{N}\mathfrak{p}^{-1}\sigma_{\mathfrak{p}}\right) \frac{2^r}{j_p(\sqrt{d_K})} R_p(\eta_{\mathfrak{f}}) = \Phi_p(1), \ \forall p \text{ with } (p, \mathfrak{f}) = 1$$

(c) $$\eta_{\mathfrak{f}} \in \mathbb{Z}[1/g]\Lambda_{S,>r} \ \text{ if } \ \mathfrak{f} \neq \mathfrak{q}^l \iff r(S, \chi_0) > r$$

(d) $$\nu\, \eta_{\mathfrak{f}} \in \Lambda_{S,>r}, \ \forall \nu \in I(\mathbb{Z}[G])$$

Numerical verification: take $r = 2$, compute $\Phi_{\emptyset}(1)$ (using complex Hecke L-functions) and $\Lambda_{S,>r}$. Find a $\mathbb{Z}[G]$-generator γ of a subgroup of finite index d in $\Lambda_{S,>r}$ and $A \in \frac{1}{2d}\mathbb{Z}[1/g]$ such that $A\, R(\gamma) = \frac{\sqrt{d_K}}{4}\Phi_{\emptyset}(1)$, then set $\eta_{\mathfrak{f}} = A\gamma$. Take p split with $p - 1 \mid f$ and compute $\Phi_p(1)$ using Shintani's method to check (b).

An example

Extension N/K		Top field	
$K = \mathbb{Q}(\sqrt{37})$		$N = \mathbb{Q}(\theta)$	
Conductor	$\mathfrak{f} = 2\mathcal{O}_K$	with $\theta^6 - 3\theta^5 - 2\theta^4 + 9\theta^3 - 5\theta + 1 = 0$	
Galois group	$G = \langle\sigma\rangle \simeq C_3$	Discriminant	$d_N = 2^4 37^3$
Set of places	$S = \{v, v', 2\mathcal{O}_K\}$	Ring of integers	$\mathcal{O}_N = \mathbb{Z}[\theta]$

We have $r(S,\chi) = 2$ for all $\chi \in \hat{G}$ so $\Lambda_{S,>r} = \overline{\bigwedge^2_{\mathbb{Z}[G]} U_S}$. The element

$$\gamma = \left(\theta^5 - 2\theta^4 - 3\theta^3 + 5\theta^2 + 2\theta - 2\right) \wedge \left(\theta^3 - 2\theta^2 - 2\theta + 3\right)$$

generates $\Lambda_{S,>r}$ over $\mathbb{Z}[G]$. We compute

$$R(\gamma) \simeq 2.25967113346986198409 4 + 0.59733461270196572219 31(\sigma + \sigma^2)$$

$$4^{-1}\sqrt{37}\Phi_\emptyset(1) \simeq \left(0.53250095403296526985 42 - 1.12983556673493099204 7(\sigma + \sigma^2)\right)$$

so $AR(\gamma) = 4^{-1}\sqrt{37}\Phi_\emptyset(1)$ for $A = \frac{1}{2}(1 - \sigma - \sigma^2)$ and we set $\eta_\mathfrak{f} = A\gamma \in \frac{1}{2}\Lambda_{S,>r}$.

- For $p = 3$,
  $\Phi_3(1) \simeq 0.20202122200120202 20_3 + 0.00211222221211012 02_3(\sigma+\sigma^2)$ and

$$\left(1-\frac{\sigma_{\mathfrak{p}_3}}{3}\right)\left(1-\frac{\sigma_{\mathfrak{p}'_3}}{3}\right)\frac{4}{j_3(\sqrt{d_K})}R_3(\eta_{\mathfrak{f}}) = \Phi_3(1)$$

- For $p = 7$,
  $\Phi_7(1) \simeq 0.232034003422155306 1_7 + 0.624214462041162660 1_7(\sigma+\sigma^2)$ and

$$\left(1-\frac{\sigma_{\mathfrak{p}_7}}{7}\right)\left(1-\frac{\sigma_{\mathfrak{p}'_7}}{7}\right)\frac{4}{j_7(\sqrt{d_K})}R_7(\eta_{\mathfrak{f}}) = \Phi_7(1)$$

- For $p = 11$,
  $\Phi_{11}(1) \simeq 0.859AA8491A4592272_{11} + 0.593A1A1A496337044_{11}(\sigma+\sigma^2)$ and

$$\left(1-\frac{\sigma_{\mathfrak{p}_{11}}}{11}\right)\left(1-\frac{\sigma_{\mathfrak{p}'_{11}}}{11}\right)\frac{4}{j_{11}(\sqrt{d_K})}R_{11}(\eta_{\mathfrak{f}}) = \Phi_{11}(1)$$

And finally

$$(\sigma-1)\,\eta_{\mathfrak{f}} = (\sigma-1)\frac{1}{2}(1-\sigma-\sigma^2)\,\gamma = (\sigma-1)\,\gamma \in \Lambda_{S,>r}$$