MATHEMATICS OF COMPUTATION
Volume 00, Number 0, Pages 000–000
S 0025-5718(XX)0000-0

CORRIGENDUM TO THE PAPER "THE BRUMER-STARK CONJECTURE IN SOME FAMILIES OF EXTENSIONS OF SPECIFIED DEGREE"

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Barry Smith has found an error in the statement and proof of Lemma 2.5 in our paper [GRT] (Math. Comp. **73** (2004), 297-315). This Lemma concerns a cyclic Galois extension K/E of CM fields of odd prime degree p. Towards the end of the proof, it is claimed that every root of unity in E is a norm from K. Our reasoning for this has a gap (the local part of the argument does not work at places where the quadratic extension E/E^+ is split, where $F = E^+$ is the maximal totally real subfield of E), and the statement can indeed fail, as confirmed by a concrete example calculated by Barry Smith. We will first state a corrected version of the Lemma, and then we will explain how the *proof* of Proposition 2.2 has to be adapted. Fortunately the *statement* of this Proposition (please refer to [GRT]) need not be changed at all, and therefore the rest of the paper is unaffected. (The only other place where Lemma 2.5 is used is in the proof of Proposition 2.1, but there ζ_p is not in K, so the relevant case of the Lemma is case (i) below, where the formula remains the same.)

Lemma 2.5 (revised). Let K/E be as above and let H be the Galois group of K/E (so H is cyclic of odd prime order p). Then

(i) If either $\zeta_p \notin K$, or if $\zeta_p \in K$ and every root of unity in E is a norm from K (note that we may restrict ourselves to p-power roots of unity in this norm statement without changing anything, since $[K : E] = p$), then we have the same formula as in the original version of the Lemma, that is
$$|A_K^H| = |A_E| \cdot e^-(K/E).$$
(ii) In the case complementary to (i), we have the corrected formula
$$|A_K^H| = |A_E| \cdot e^-(K/E)/p.$$

Proof. For part (i), the old proof works. Part (ii) is exactly the case where the norm index $[\mu_E : N_{K/E}K^* \cap \mu_E]$ is not 1 (contrary to what was claimed in the old version). But then this index is exactly p since H is of order p. The same argument as before gives the formula, now with the p in the denominator. $\square$

We recall that the case $\zeta_p \notin K$ was called Case I in [GRT], and the complementary case $\zeta_p \in K$ was Case II. *We have to redo the proof of Proposition 2.2 in* [GRT] *when case* (*ii*) *of Lemma 2.5 (revised) applies*. For convenience, let us call this the "bad" case, and for clarity we repeat what it means: $\zeta_p \in K$ and there is some (p-power) root of unity in E that is not a norm from K. The rest of this corrigendum is concerned with the proof of Proposition 2.2 in the "bad" case.

As in [GRT], we distinguish three cases.

(a) Both $\mu_{p^\infty}(K)$ and $\mu_{p^\infty}(E)$ are of order p;
(b) Both $\mu_{p^\infty}(K)$ and $\mu_{p^\infty}(E)$ are of order $p^e > p$; and
(c) $|\mu_{p^\infty}(E)| = p^{e-1}$ and $|\mu_{p^\infty}(K)| = p^e$.

But by looking at the Galois action one sees that in case (c) the norm $\mu_K \to \mu_E$ is already onto, so this is not a "bad" case! Moreover, (a) can be subsumed into (b) by also allowing $e = 1$, so we just do (b) for all $e \geq 1$.

As before we have

$$|A_K| = p^{e+1} \cdot [\mathbb{Z}_p[\zeta_p] : (\bar{\alpha})] \cdot [\mathbb{Z}_p : (\theta(E/F))]$$

and

$$|A_E| = p^e \cdot [\mathbb{Z}_p : (\theta(E/F))].$$

Here α is some element with $(\sigma - 1)\alpha = \theta(K/F)$. Let us write simply θ for the latter. The module $(\sigma-1)A_K$ is an epimorphic image of A_K/A_K^H. Now by part (ii) of Lemma 2.5 (revised), and since the term $e^-(K/E)$ is a nontrivial power of p, we get that $|A_K^H|$ is a multiple of $|A_E|$, so A_K/A_K^H has order dividing $|A_K|/|A_E|$, which is $p[\mathbb{Z}_p[\zeta_p] : (\bar{\alpha})]$. (In [GRT] it says "smaller than" (i.e. properly dividing) instead of "dividing"; that is the only change so far.) Therefore $(\sigma - 1)A_K$ is annihilated by $(\sigma - 1)\alpha$ (formerly and erroneously by α; the extra $(\sigma - 1)$ now takes care of the factor p), and A_K is annihilated by $(\sigma - 1)^2\alpha = (\sigma - 1)\theta$. So for any ideal $\mathfrak{a}$ with class in A_K we have $\mathfrak{a}^{(\sigma-1)\theta} = (x)$ for some anti-unit x. We have to find a BS-annihilator for the ideal $\mathfrak{a}^{p^e\theta}$.

As $(x)^N$ is the unit ideal (recall $N = 1 + \sigma + \ldots + \sigma^{p-1}$), we know that x^N is a root of unity. But since we are in the "bad" case, this entails that x^N is the p-th power of another root of unity. Hence x^{p-N} is a p-th power.

We may find $\xi \in \mathbb{Z}[\sigma]$ with $p - N = \xi(\sigma - 1)$. Recalling from the setting in [GRT] that $N\theta = 0$, we therefore obtain

$$\mathfrak{a}^{p^e\theta} = \mathfrak{a}^{p^{e-1}(p-N)\theta} = (x^{p^{e-1}\xi}).$$

Let us put $y = x^{p^{e-1}\xi}$ and show that y is p^e-abelian over F. By the same arguments as in [GRT], it suffices for this that $y^{\sigma-1}$ is a p^e-th power in K. But indeed, $y^{\sigma-1} = x^{p^{e-1}\xi(\sigma-1)} = x^{p^{e-1}(p-N)}$, and we already observed that $x^{(p-N)}$ is a p-th power. So we have proved that Proposition 2.2 is also correct in the "bad" case, that is, the case where the old version of Lemma 2.5 fails.

This essentially concludes our corrigendum. Let us just add that in a way the argument has become esthetically more appealing, since there is no longer an extra margin in the verification of the p^e-abelian property, in contrast with the end of the first paragraph in the proof of Proposition 2.2 in [GRT], where we said "The punch line is that we still have a margin." Forget that punch line!

We take this opportunity to mention two very small bits in [GRT] which we also would like to modify for the sake of clarity and precision: (1) In the Remark just preceding Propositon 2.2 (page 305), second line, replace "is identical with" by "is included in". (2) In the statement of Proposition 2.2, read "in particular" instead of "in other words".

Acknowledgment: We would like to thank Barry Smith for bringing these issues to our attention and for some illuminating remarks about the sensitivity of the Brumer-Stark conjecture.

References

[GRT] Cornelius Greither, Xavier-François Roblot, Brett A. Tangedal, The Brumer-Stark conjecture in some families of extensions of specified degree, *Math. Comp.* **73** (2004), 297-315.

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