A BRUMER-STARK CONJECTURE FOR NON-ABELIAN GALOIS EXTENSIONS

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Abstract. Let K/k be an abelian extension of number fields. The Brumer-Stark conjecture predicts that a group ring element constructed from special values of L-functions associated to K/k annihilates the ideal class group of K. Moreover it specifies that the generators obtained have special properties. The aim of this article is to state and study a generalization of this conjecture to non-abelian Galois extensions that is, in spirit, very similar to the original conjecture.

1. Introduction

The Brumer-Stark conjecture was first stated by Tate [21] and applies to abelian extensions of number fields. It combines a conjecture of Brumer and ideas coming from conjectures of Stark. Let K/k be an abelian extension. The main ingredient of the conjecture is a certain group-ring element in $\mathbb{Z}[\mathrm{Gal}(K/k)]$, called the Brumer-Stickelberger element, constructed from the values at $s = 0$ of the L-functions associated to the extension K/k. The Brumer part of the conjecture states that the Brumer-Stickelberger element annihilates the class group of the field K. The Stark part of the conjecture predicts that the principal ideals obtained in this way admit generators satisfying special properties. A very nice reference for the Brumer-Stark conjecture, and Stark conjectures in general, is the book of Tate [22], see also [4] and [6]. The aim of this article is to generalize the Brumer-Stark conjecture to Galois non-abelian extensions.

The plan of this paper is the following. In the second section, we state the Brumer-Stark conjecture, some of its properties and say a few words about its current status. To avoid confusion in the setting of this paper, we will call this conjecture the abelian Brumer-Stark conjecture and will call the conjecture that we propose the Galois Brumer-Stark conjecture. The third section is devoted to the generalization of the Brumer-Stickelberger element to the Galois case. There, we rely on an earlier work of Hayes [13] that constructs this generalization and studies its properties. We show that it also satisfies additional properties very similar to the abelian case. It is known that the Brumer-Stickelberger element is rational and a suitable denominator is known in the abelian case. We make a first conjecture, called the Integrality Conjecture, on a suitable denominator for this element in the general case. This conjecture is part of our generalization of the abelian Brumer-Stark conjecture. The next section introduces the notion of strong central extensions. These extensions play a fundamental role in our generalization. The Galois Brumer-Stark conjecture is stated in Section 5 and we study its properties in Section 6 with the generalization of the properties of the abelian Brumer-Stark conjecture in view. The last section is devoted to the study of the conjecture in the special case where the Galois group of the extension contains an abelian normal subgroup of prime index. In this setting, we prove that the abelian Brumer-Stark conjecture implies the Galois Brumer-Stark conjecture.

Different generalizations to the non-abelian case of the Brumer conjecture and Brumer-Stark conjecture are stated by Nickel [14] (see also the work of Burns [2]). In an appendix at the end of the paper, we state the weak version of Nickel's non-abelian Brumer-Stark conjecture and compare it with our conjecture.

Note. Many of the results of this article are extracted from the PhD thesis [7] of the first named author or generalizations of results contained in this thesis.

Convention. We denote the action of elements of Galois groups on elements, ideals, etc., using the exponent notation with the convention that they act on the left, that is $\alpha^{\sigma\gamma} = (\alpha^\gamma)^\sigma$.

2. The abelian Brumer-Stark conjecture

In this section, we state the abelian Brumer-Stark conjecture and review some of its properties. Let K/k be an abelian extension of number fields. Denote by G its Galois group. Fix S a finite set of places of k containing the infinite places of k and the finite places of k that ramify in K/k. To simplify the exposition, we assume from now on that the cardinality of S is at least two.[1] The interested reader can refer to [22, IV§6] for the statement of the conjecture when $|S| = 1$. To a character χ of G is associated the S-truncated Hecke L-function of χ defined for $\mathrm{Re}(s) > 1$ by

$$L_{K/k,S}(s,\chi) := \prod_{\mathfrak{p}\notin S} (1 - \chi(\sigma_\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})^{-1}$$

where $\mathfrak{p}$ runs through the prime ideals of k not in S, $\sigma_\mathfrak{p}$ is the Frobenius automorphism of $\mathfrak{p}$ in G, and $\mathcal{N}(\mathfrak{p})$ is the absolute norm of the ideal $\mathfrak{p}$. This function admits a meromorphic continuation to $\mathbb{C}$, which is in fact analytic if the character χ is non-trivial. A main object of the abelian Brumer-Stark conjecture is the Brumer-Stickelberger element. It is a relative analogue of the Stickelberger element of cyclotomic fields and is defined by the formula

$$\theta_{K/k,S} := \sum_{\chi\in\hat{G}} L_{K/k,S}(0,\chi)\, e_{\bar{\chi}} \in \mathbb{C}[G]$$

where $\hat{G}$ denotes the group of characters of G and, for $\chi \in \hat{G}$, e_χ is the associated idempotent. Another characterization of this element is that it is the only element in $\mathbb{C}[G]$ such that

$$\chi(\theta_{K/k,S}) = L_{K/k,S}(0,\bar{\chi})$$

for all character $\chi \in \hat{G}$. A third characterization of this element is in term of partial zeta functions. For $\sigma \in G$, the partial zeta function associated to g (and the extension K/k and the set S) is defined, for $\mathrm{Re}(s) > 1$, by

$$\zeta_{K/k,S}(s,\sigma) := \sum_{\substack{(\mathfrak{a},S)=1\\ \sigma_\mathfrak{a}=\sigma}} \mathcal{N}(\mathfrak{a})^{-1}$$

where $\mathfrak{a}$ runs through the integral ideals of k, not divisible by any prime ideal in S, and whose Artin symbol $\sigma_\mathfrak{a}$ in G is equal to σ. This function also admits meromorphic continuation to the complex plane and the partial zeta functions are related to Hecke L-functions by the formula

$$L_{K/k,S}(s,\chi) = \sum_{\sigma\in G} \zeta_{K/k,S}(s,\sigma)\chi(\sigma). \tag{1}$$

From this we deduce the third characterization of the Brumer-Stickelberger element

$$\theta_{K/k,S} = \sum_{g\in G} \zeta_{K/k,S}(0,\sigma)\sigma^{-1}. \tag{2}$$

It follows from works of Deligne and Ribet [8] (see also the works of Barsky [1] and Pi. Cassou-Noguès [5]) that, for any $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, the annihilator in $\mathbb{Z}[G]$ of the group μ_K of roots of unity in K, we have $\xi\,\theta_{K/k,S} \in \mathbb{Z}[G]$. In particular, if we let w_K denote the cardinality of μ_K, we have

$$w_K\theta_{K/k,S} \in \mathbb{Z}[G]. \tag{3}$$

[1]The only non-trivial case that we are excluding is when k is a complex quadratic field and K is a subfield of the Hilbert class field of k.

We need one last notation before stating the abelian Brumer-Stark conjecture. We say that a non-zero element α in K is an anti-unit if all its conjugates have absolute value equal to 1. The group of anti-units of K is denoted by K°.

Conjecture (The abelian Brumer-Stark conjecture $\mathbf{BS}(K/k,S)$)**.**
For any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{w_K\theta_{K/k,S}}$ is principal and admits a generator $\alpha \in K^\circ$ such that $K(\alpha^{1/w_K})/k$ is abelian.

Remark. The last assertion that $K(\alpha^{1/w_K})/k$ is abelian does not depend upon the choice of the w_K-th root of α since all these roots generate the same extension of K.

Remark. The Brumer conjecture states that the ideal $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))\,\theta_{K/k,S}$ of $\mathbb{Z}[G]$ annihilates the class group Cl_K of K. The Brumer-Stark conjecture implies the Brumer Conjecture.

Let v be a place in S and denote by $N_v := \sum_{\sigma\in D_v}\sigma \in \mathbb{Z}[G]$ the sum of all the elements in the decomposition group D_v of v in G. Then, one can prove, see [22, Chap. IV], that

$$N_v\,\theta_{K/k,S} = 0. \tag{4}$$

In particular, if the set S contains a place that is totally split in K/k, the Brumer-Stickelberger element is equal to 0 and the abelian Brumer-Stark conjecture is trivially true. Therefore, the conjecture is only meaningful when both k is totally real and K is totally complex.[2] In [21], Tate proves equivalent formulations of the conjecture that are very useful for its study. We will later on generalize this result to the non-abelian Galois case. For $\alpha \in K^\times$ and $\mathfrak{A}$ an integral ideal of K, we write $\alpha \equiv 1 \pmod{^*\,\mathfrak{A}}$ if $v_{\mathfrak{P}}(\alpha-1) \geq v_{\mathfrak{P}}(\mathfrak{A})$ for all prime ideals $\mathfrak{P}$ of K dividing $\mathfrak{A}$, where $v_{\mathfrak{P}}$ is the valuation associated to $\mathfrak{P}$. This is equivalent to the usual notion $\alpha \equiv 1 \pmod{\mathfrak{A}}$ when α is an algebraic integer.

Theorem 2.1 (Tate)**.** *Let $\mathfrak{A}$ be a fractional ideal of K. Then the following statements are equivalent.*

- *(i). There exists an anti-unit $\alpha \in K^\circ$ such that $\mathfrak{A}^{w_K\theta_{K/k,S}} = \alpha\mathcal{O}_K$ and $K(\alpha^{1/w_K})/k$ is abelian.*
- *(ii). There exist an extension L/K such that L/k is abelian and an anti-unit $\gamma \in L^\circ$ such that $(\mathfrak{A}\mathcal{O}_L)^{\theta_{K/k,S}} = \gamma\mathcal{O}_L$.*
- *(iii). For almost all prime ideals $\mathfrak{p}$ of k, there exists $\alpha_\mathfrak{p} \in K^\circ$ such that $\alpha_\mathfrak{p} \equiv 1 \pmod{^*\,\mathfrak{p}\mathcal{O}_K}$ and $\mathfrak{A}^{(\sigma_\mathfrak{p}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S}} = \alpha_\mathfrak{p}\mathcal{O}_K$ where $\sigma_\mathfrak{p}$ is the Frobenius automorphism of $\mathfrak{p}$ in G.*
- *(iv). There exist a family $(a_i)_{i\in I}$ of element of $\mathbb{Z}[G]$ generating $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ and a family $(\alpha_i)_{i\in I}$ of anti-units in K such that $\mathfrak{A}^{a_i\theta_{K/k,S}} = \alpha_i\mathcal{O}_K$ for all $i \in I$, and ${\alpha_i}^{a_j} = {\alpha_j}^{a_i}$ for all $i,j \in I$.*

Remark. Here and in the rest of the paper, when we say "for almost all prime ideals", we implicitly exclude the ramified primes; therefore the Frobenius automorphism is uniquely defined.

Remark. In part (ii), $(\mathfrak{A}\mathcal{O}_L)^{\theta_{K/k,S}}$ is defined by the formula $\big((\mathfrak{A}\mathcal{O}_L)^{n\theta_{K/k,S}}\big)^{1/n}$ where $n \geq 1$ is any integer such that $n\theta_{K/k,S} \in \mathbb{Z}[G]$. This is well-defined, when it exists, since the group of ideals of a number field is torsion-free.

Let $\mathfrak{A}$ be a fractional ideal of K. We say that $\mathbf{BS}(K/k,S;\mathfrak{A})$ holds if the ideal $\mathfrak{A}$ satisfies the equivalent conditions of Theorem 2.1. The conjecture $\mathbf{BS}(K/k,S)$ is thus the collection of the conjectures $\mathbf{BS}(K/k,S;\mathfrak{A})$ where $\mathfrak{A}$ ranges through the fractional ideals of K. In [21], Tate proves that the set of fractional ideals $\mathfrak{A}$ of K such that $\mathbf{BS}(K/k,S;\mathfrak{A})$ holds is a subgroup of the group of ideals of K, stable under the action of G, and that contains the principal ideals of K. In particular, $\mathbf{BS}(K/k,S)$ holds if the field K is principal. Now, let $\mathfrak{p}_0$ be a prime ideal of k not in S, then

$$\theta_{K/k,S\cup\{\mathfrak{p}_0\}} = (1-{\sigma_{\mathfrak{p}_0}}^{-1})\theta_{K/k,S}. \tag{5}$$

[2]Note that $K^\circ = \{\pm 1\}$ if K is not totally complex.

It follows from this formula that the validity of $\mathbf{BS}(K/k, S)$ implies that of $\mathbf{BS}(K/k, S \cup \{\mathfrak{p}_0\})$. Therefore, the conjecture is true for any admissible set of places S if it is true for the minimal set that contains exactly the infinite places of k and the finite places that ramify in K/k. The validity of the abelian Brumer-Stark Conjecture is also preserved under change of extension as a consequence of part (ii) of Proposition 2.1. That is, if $K/K'/k$ is a tower of number fields, then the validity of $\mathbf{BS}(K/k, S)$ implies that of $\mathbf{BS}(K'/k, S)$. It also preserved under change of base, that is if $\mathbf{BS}(K/k, S)$ holds then so does $\mathbf{BS}(K/k', S')$ where $K/k'/k$ is a tower of number fields and S' denotes the set of places of k' above the places in k, see [12]. The following cases of the conjecture are proved by Tate (see [21] and [22]).

Theorem 2.2 (Tate)**.** *The abelian Brumer-Stark conjecture* $\mathbf{BS}(K/k, S)$ *is true in the following cases.*

- *The field k is the field $\mathbb{Q}$ of rational numbers.*[3]
- *The extension K/k is quadratic.*
- *The extension K/k is of degree 4 and is contained in a non-abelian Galois extension K/k_0 of degree 8.*

Sands proves the abelian Brumer-Stark conjecture when the group G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and, more generally, when the group G has exponent 2 with some additional technical hypothesis, see [19]. A local version of the conjecture is stated and is proved for some types of extensions of degree $2p$ (with p odd) and numerically studied in some others by Greither et al. in [11]. The local abelian Brumer-Stark conjecture at p holds for so-called "non-exceptional" primes p provided some appropriate Iwasawa μ-invariant vanishes by results of Nickel [15], and when S contains all the prime ideals above p and, again, some appropriate Iwasawa μ-invariant vanishes by results of Greither and Popescu [10]. Nickel shows in [14] that the local abelian Brumer-Stark conjecture outside of 2 is implied by the relevant special case of the Equivariant Tamagawa Number Conjecture (ETNC) plus some additional technical hypothesis. Since this special case of the ETNC was proved by Burns and Greither [3], this implies, in particular, the part outside of 2 of the abelian Brumer-Stark conjecture holds if K/k is a tame extension with K an abelian extension of $\mathbb{Q}$.

As mentioned in the introduction, generalizations to the non-abelian case of the Brumer-Stark conjecture (and also the Brumer conjecture) due to Nickel are stated in [14] (see also [2] for much more general conjectures due to Burns), we state these conjectures and study the links with our conjecture in an appendix at the end of this article.

3. The Galois Brumer-Stickelberger element

We assume from now on that the extension K/k is Galois, but not necessarily abelian. The set S still denotes a finite set of places of k containing the infinite places of k and the finite places that ramify in K/k. As in the abelian case, we assume also that S is cardinality at least 2. Note that the only non-trivial case we are excluding is when k is a complex quadratic field and K is an unramified extension of k. The first step in the generalization of the abelian Brumer-Stark conjecture is the construction of the Brumer-Stickelberger element associated to non-abelian Galois extensions. Fortunately, such a construction is provided by the work of Hayes [13]. We now review his construction and the first properties of the Brumer-Stickelberger element. Denote by $\hat{G}$ the set of irreducible characters of G. For $\chi \in \hat{G}$, let $L_{K/k,S}(s, \chi)$ denote the Artin L-function of χ with Euler factors at primes in S deleted. The Brumer-Stickelberger element is defined by

$$\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \chi)\, e_{\bar{\chi}} \tag{6}$$

[3] In this situation, it boils down to Stickelberger's theorem on cyclotomic sums.

where $e_\chi := \dfrac{\chi(1)}{|G|} \sum_{\sigma\in G} \chi(\sigma)\sigma^{-1}$ is the central idempotent associated to χ.

The following results are extracted from [13].

Theorem 3.1 (Hayes)**.** *Denote by $\mathscr{C}_G$ the set of conjugacy classes of G. The Brumer-Stickelberger element lies in the center $Z(\mathbb{C}[G])$ of $\mathbb{C}[G]$ and is the only element of $Z(\mathbb{C}[G])$ such that*

$$\phi_\chi(\theta_{K/k,S}) = L_{K/k,S}(0,\bar{\chi}) \tag{7}$$

for all $\chi \in \hat{G}$, where ϕ_χ is the ring homomorphism from $Z(\mathbb{C}[G])$ to $\mathbb{C}$ defined by

$$\phi_\chi(C) := \frac{\chi(C)}{\chi(1)}$$

for all $C \in \mathscr{C}_G$.

Let B be a normal subgroup of G. Then we have

$$\theta_{K^B/k,S} = \pi(\theta_{K/k,S})$$

where $\pi : \mathrm{Gal}(K/k) \to \mathrm{Gal}(K^B/k)$ is the canonical surjection induced by the restriction to K^B.

Let H be a subgroup of G. Denote by S_H the set of places of K^H above the places in S. Let $\mathrm{INorm}_{G\to H} : Z(\mathbb{C}[G]) \longrightarrow Z(\mathbb{C}[H])$ be the inhomogeneous norm defined by

$$\mathrm{INorm}_{G\to H}(a) := \sum_{\phi\in\hat{H}} \Big(\prod_{\chi\in\hat{G}} a(\chi)^{\langle \chi, \mathrm{Ind}_H^G \phi\rangle_G} \Big) e_\phi$$

for $a := \sum_{\chi\in\hat{G}} a(\chi) e_\chi \in Z(\mathbb{C}[G])$, where $\langle\cdot,\cdot\rangle_G$ is the inner product on the characters of G and e_ϕ is the central idempotent of $\mathbb{C}[H]$ associated to ϕ. Then we have

$$\theta_{K/K^H,S_H} = \mathrm{INorm}_{G\to H}(\theta_{K/k,S}).$$

We are now interested in generalizing properties (4) and (5). We start with (4).

Proposition 3.2. *For v a place of k, define*

$$N_v := \sum_{\sigma\in D_w} \frac{1}{|C_\sigma|} C_\sigma \in \mathbb{Q}[G]$$

where w is a fixed place of K above v, D_w is the decomposition group of w in G and $C_\sigma \in \mathscr{C}_G$ is the conjugacy class of σ in G. Then, for any place v in S, we have

$$N_v\, \theta_{K/k,S} = 0.$$

Proof. Since N_v is in $Z(\mathbb{C}[G])$, it is enough, with the notations of Theorem 3.1, to prove that $\phi_\chi(N_v\,\theta_{K/k,S}) = \phi_\chi(N_v)\phi_\chi(\theta_{K/k,S}) = 0$ for all $\chi \in \hat{G}$. Let $\chi \in \hat{G}$ be such that $\phi_\chi(N_v) \neq 0$. By (7), we need to prove that the order $r(\bar{\chi}) = r(\chi)$ of vanishing at $s = 0$ of $L_{K/k,S}(s,\chi)$ is at least 1. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation with character χ. By [22, Prop. I.3.4], we have

$$r(\chi) = \sum_{v'\in S} \dim V^{D_{w'}} - \dim V^G \tag{8}$$

where w' is a fixed place of K above v' and $D_{w'}$ denotes the decomposition group of w' in G. Assume first that χ is the trivial character. Then the above formula yields $r(\chi) = |S| - 1$ and the result follows from our hypothesis that S contains at least two places. Assume now that χ is non-trivial. We compute

$$\phi_\chi(N_v) = \sum_{\sigma\in D_w} \frac{1}{|C_\sigma|}\phi_\chi(C_\sigma) = \frac{1}{\chi(1)} \sum_{\sigma\in D_w} \chi(\sigma) = \frac{|D_w|}{\chi(1)} \big\langle \mathbf{1}_{D_w}, \chi_{|D_w} \big\rangle_{D_w}$$

where $\mathbf{1}_{D_w}$ is the trivial character of D_w. By the above hypothesis, $\phi_\chi(N_v) \neq 0$ and thus the trivial character $\mathbf{1_{D_w}}$ appears in the decomposition of $\chi_{|D_w}$. Therefore the space V^{D_w} has dimension at least 1. On the other hand, $V^G = \{0\}$ since χ is irreducible. It follows that $r(\chi) \geq 1$ and the result is proved. □

Assume that there exists $v \in S$ that is totally split in K/k. Then $N_v = 1$ and the Brumer-Stickelberger element is trivial in this case. Therefore, as in the abelian case, we assume that both k is totally real and K is totally complex, otherwise the Brumer-Stickelberger element is trivial. In fact, we can say more than that. Recall that a number field E is CM if it is a totally complex quadratic extension of a totally real field. If furthermore E is Galois over some totally real subfield F, then $\mathrm{Gal}(E/F)$ has a unique complex conjugation and we say that a character χ of $\mathrm{Gal}(E/F)$ is totally odd if the eigenvalues of an associated representation evaluated at the complex conjugation are all equal to -1. The following result is due to Tate, see [22, p. 71].

Proposition 3.3 (Tate)**.** *Let $\chi \in \hat{G}$ be a character such that $L_{K/k,S}(0,\chi) \neq 0$. Then χ is the inflation of a totally odd character of a Galois CM sub-extension F/k of K/k.*

Corollary 3.4. *If K/k does not contain a Galois CM sub-extension then $\theta_{K/k,S} = 0$.*

Proof. Assume that $\theta_{K/k,S} \neq 0$. Then, by Theorem 3.1 and the fact that $(\phi_\chi)_{\chi \in \hat{G}}$ is a basis of the dual of $Z(\mathbb{C}[G])$, we get that there exists an irreducible character $\chi \in \hat{G}$ such that $\phi_\chi(\theta_{K/k,S}) = L_{K/k,S}(0,\chi) \neq 0$. This character comes from a Galois CM sub-extension by the proposition. □

Corollary 3.5. *Let τ be a complex conjugation of G. Then $(\tau + 1) \cdot \theta_{K/k,S} = 0$.*

Proof. By the proposition, it is enough to prove that $(\tau + 1) \cdot e_\chi = 0$ for any character $\chi \in \hat{G}$ that is the inflation of a totally odd character $\tilde{\chi}$ of a Galois CM sub-extension. Since $\tilde{\chi}$ is totally odd, we have $\chi(g\tau) = -\chi(g)$ for all $g \in G$. Let R be a set of representatives of $G/\{1,\tau\}$. We now compute

$$\begin{aligned}(\tau+1)\cdot e_\chi &= (\tau+1)\cdot\frac{\chi(1)}{|G|}\sum_{\rho\in R}\Big(\chi(\rho)\rho^{-1} + \chi(\rho\tau)(\rho\tau)^{-1}\Big)\\ &= (\tau+1)\cdot\frac{\chi(1)}{|G|}\sum_{\rho\in R}\Big(\chi(\rho)\rho^{-1} - \chi(\rho)\tau\rho^{-1}\Big)\\ &= (\tau+1)(1-\tau)\cdot\frac{\chi(1)}{|G|}\sum_{\rho\in R}\chi(\rho)\rho^{-1} = 0.\end{aligned}$$ □

The following result generalizes (5) to the Galois case.

Proposition 3.6. *Let $\mathfrak{p}_0$ be a prime ideal of k not in S. Then*

$$\theta_{K/k,S\cup\{\mathfrak{p}_0\}} = \theta_{K/k,S}\sum_{\chi\in\hat{G}}\det(1-\rho_\chi(\sigma_{\mathfrak{P}_0}))e_{\bar{\chi}}$$

where $\mathfrak{P}_0$ is a fixed prime ideal of K above $\mathfrak{p}_0$, $\sigma_{\mathfrak{P}_0}$ is the Frobenius automorphism of $\mathfrak{P}_0$ in G, and, for $\chi \in \hat{G}$, ρ_χ denotes a fixed irreducible representation of G with character χ.

Proof. With the notations of Theorem 3.1, it is enough to prove, for all $\psi \in \hat{G}$, that

$$\begin{aligned}\phi_\psi(\theta_{K/k,S\cup\{\mathfrak{p}_0\}}) &= \phi_\psi(\theta_{K/k,S})\,\phi_\psi\Big(\sum_{\chi\in\hat{G}}\det(1-\rho_\chi(\sigma_{\mathfrak{P}_0}))e_{\bar{\chi}}\Big)\\ &= L_{K/k,S}(0,\bar{\psi})\sum_{\chi\in\hat{G}}\det(1-\rho_\chi(\sigma_{\mathfrak{P}_0}))\phi_\psi(e_{\bar{\chi}}).\end{aligned}$$

On the other hand, from the definition of Artin L-functions, we see that

$$\phi_\psi(\theta_{K/k,S\cup\{\mathfrak{p}_0\}}) = L_{K/k,S\cup\{\mathfrak{p}_0\}}(0,\bar{\psi}) = L_{K/k,S}(0,\bar{\psi})\,\det(1-\rho_{\bar{\psi}}(\sigma_{\mathfrak{P}_0})).$$

The result follows from the fact that $\phi_\psi(e_{\bar\chi}) = 1$ if $\psi = \bar\chi$ and zero otherwise. $\square$

We now turn to the question of the rationality of the Brumer-Stickelberger element $\theta_{K/k,S}$ when G is non-abelian. As noted on page 2584 of [14], it is a consequence of the principal rank zero Stark conjecture that was proved by Tate [22]. We recall the argument of the proof. For any character χ of G, the principal rank zero Stark conjecture states that

$$L_{K/k,S}(0,\chi^\alpha) = L_{K/k,S}(0,\chi)^\alpha \text{ for all } \alpha \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{C}) \tag{9}$$

where $\chi^\alpha := \alpha \circ \chi$. We write

$$\theta_{K/k,S} = \sum_{\chi\in\hat G} L_{K/k,S}(0,\chi)\frac{\bar\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma = \sum_{\sigma\in G} x_\sigma\, \sigma$$

where

$$x_\sigma := \frac{1}{|G|}\sum_{\chi\in\hat G}\bar\chi(1)\chi(\sigma)L_{K/k,S}(0,\chi).$$

Let α be an automorphism of $\mathbb{C}$. We compute

$$\begin{aligned}\alpha(x_\sigma) &= \frac{1}{|G|}\sum_{\chi\in\hat G}\bar\chi^\alpha(1)\chi^\alpha(\sigma)L_{K/k,S}(0,\chi)^\alpha \\ &= \frac{1}{|G|}\sum_{\chi\in\hat G}\bar\chi^\alpha(1)\chi^\alpha(\sigma)L_{K/k,S}(0,\chi^\alpha) = x_\sigma\end{aligned}$$

since the map $\chi \mapsto \chi^\alpha$ is a bijection on the set $\hat G$. It follows that $x_\sigma \in \mathbb{Q}$ for all $\sigma \in G$, and thus the Brumer-Stickelberger element $\theta_{K/k,S}$ lies in $\mathbb{Q}[G]$.

An interesting problem is to find a suitable denominator for the Brumer-Stickelberger element in the non-abelian case. In the abelian case, as noted above, $w_K\theta_{K/k,S}$ is always integral. In the Galois case, however, one can see from examples that it is not true anymore. Let $[G,G]$ be the commutator subgroup of G, that is the subgroup generated by the commutators $[g_1,g_2] := g_1g_2g_1^{-1}g_2^{-1}$ with $g_1,g_2 \in G$. We make the following conjecture.

Conjecture (The Integrality Conjecture)**.**
Define m_G to be the lcm of the cardinalities of the conjugacy classes of G and let s_G be the order of the commutator subgroup $[G,G]$ of G. Let d_G be the lcm of m_G and s_G. Then, for almost all prime ideals $\mathfrak{P}$ of K, we have

$$d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S} \in \mathbb{Z}[G] \tag{10}$$

where $\mathfrak{p}$ is the prime ideal of k below $\mathfrak{P}$ and $\sigma_{\mathfrak{P}}$ is the Frobenius automorphism of $\mathfrak{P}$ in G.

One could weaken the Integrality Conjecture by just asking that there exists an integer d_G, depending only on the isomorphism class of G, such that (10) holds without specify its value. However, heuristic arguments lead us to predict this specific value of d_G. First, observe that $m_G = 1$ if and only if $s_G = 1$ if and only if G is abelian. Therefore, when the extension K/k is abelian, the Integrality Conjecture is equivalent to the statement before (3) using Lemma 3.7 below. We now explain why we conjecture that the factor s_G is necessary. Let $G^{\mathrm{ab}} := G/[G,G]$ be the maximal abelian quotient of G and $K^{\mathrm{ab}} := K^{[G,G]}$ be the maximal sub-extension of K/k that is abelian over k; we have $\mathrm{Gal}(K^{\mathrm{ab}}/k) = G^{\mathrm{ab}}$. Denote by $\pi^{\mathrm{ab}} : G \to G^{\mathrm{ab}}$ the canonical surjection induced by the restriction to K^{ab}. Let ν^{ab} be the map from $\mathbb{C}[G^{\mathrm{ab}}]$ to $\mathbb{C}[G]$ defined for $\tilde g \in G^{\mathrm{ab}}$ by

$$\nu^{\mathrm{ab}}(\tilde g) := \frac{1}{s_G}\sum_{\pi^{\mathrm{ab}}(g)=\tilde g} g \tag{11}$$

where the sum is over elements $g \in G$ whose image by $\pi^{\rm ab}$ is equal to $\tilde{g}$. The map $\nu^{\rm ab}$ is extended to $\mathbb{C}[G^{\rm ab}]$ by linearity.[4] Let $\kappa \in \mathbb{C}[G^{\rm ab}]$, we have $(\pi^{\rm ab} \circ \nu^{\rm ab})(\kappa) = \kappa$ and, if $\xi \in \mathbb{C}[G]$, then $\xi \nu^{\rm ab}(\kappa) = \nu^{\rm ab}(\pi^{\rm ab}(\xi)\kappa)$. The 1-dimensional characters of G are exactly the ones that are inflations of characters of $G^{\rm ab}$. For such a character χ, denote by $\tilde{\chi}$ the character of $G^{\rm ab}$ such that $\chi = \tilde{\chi} \circ \pi^{\rm ab}$. One checks readily that $e_\chi = \nu^{\rm ab}(e_{\tilde{\chi}})$ where $e_{\tilde{\chi}}$ is the idempotent of $\mathbb{C}[G^{\rm ab}]$ associated to $\tilde{\chi}$. By the properties of Artin L-functions, we have

$$\begin{aligned}\sum_{\substack{\chi \in \hat{G} \\ \chi(1)=1}} L_{K/k,S}(0,\chi) e_{\bar{\chi}} &= \sum_{\tilde{\chi} \in \hat{G}^{\rm ab}} L_{K^{\rm ab}/k,S}(0,\tilde{\chi}) \nu^{\rm ab}(e_{\bar{\tilde{\chi}}}) \\ &= \nu^{\rm ab}\Big(\sum_{\tilde{\chi} \in \hat{G}^{\rm ab}} L_{K^{\rm ab}/k,S}(0,\tilde{\chi}) e_{\bar{\tilde{\chi}}} \Big) = \nu^{\rm ab}(\theta_{K^{\rm ab}/k,S}).\end{aligned}$$

We define

$$\theta^{(>1)}_{K/k,S} := \sum_{\substack{\chi \in \hat{G} \\ \chi(1)>1}} L_{K/k,S}(0,\chi) e_{\bar{\chi}}.$$

By the above computation, we find that

$$\theta_{K/k,S} = \nu^{\rm ab}(\theta_{K^{\rm ab}/k,S}) + \theta^{(>1)}_{K/k,S}. \tag{12}$$

For all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, we have $s_G\, \xi\, \nu^{\rm ab}(\theta_{K^{\rm ab}/k,S}) = s_G \nu^{\rm ab}(\tilde{\xi}\theta_{K^{\rm ab}/k,S}) \in \mathbb{Z}[G]$ by the remark before (3) since $\tilde{\xi} := \pi^{\rm ab}(\xi) \in \mathrm{Ann}_{\mathbb{Z}[G^{\rm ab}]}(\mu_{K^{\rm ab}})$. Therefore the factor s_G is there to ensure that the part of the Brumer-Stickelberger element coming from the 1-dimensional characters is integral.

The first open case for the Integrality Conjecture is when $G \simeq \mathrm{SL}_2(\mathbb{F}_3)$, see Theorem 5.2. In fact, for relative Galois extensions K/k of degree ≤ 31 and with Galois group not isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$, one can prove that s_G is a suitable denominator for $\theta_{K/k,S}$. However, numerical experiments in the case $G \simeq \mathrm{SL}_2(\mathbb{F}_3)$ show that s_G is not a suitable denominator in general and, in fact, it is necessary to use $3s_G$ in some cases, see [7, Chap. 5]. It is therefore necessary to add an extra factor. After Hayes, define, for $s \in \mathbb{C}$, the meromorphic function

$$\Theta_{K/k,S}(s) := \sum_{\chi \in \hat{G}} L_{K/k,S}(s,\chi)\, e_{\bar{\chi}}.$$

Note that $\Theta_{K/k,S}(0) = \theta_{K/k,S}$. Using this function, Hayes defines in [13, §5] the partial zeta function $\zeta_{K/k,S}(s, C)$ of a class $C \in \mathscr{C}_G$ by the formula

$$\Theta_{K/k,S}(s) = \sum_{C \in \mathscr{C}_G} \zeta_{K/k,S}(s,C) \frac{1}{|C|}\, C^{-1}. \tag{13}$$

Note that this definition makes sense because the values of $\Theta_{K/k,S}$ are in $Z(\mathbb{C}[G])$. Applying ϕ_χ on both sides, for $\chi \in \hat{G}$, he gets

$$L_{K/k,S}(s,\chi) = \frac{1}{\chi(1)} \sum_{C \in \mathscr{C}_G} \zeta_{K/k,S}(s,C)\, \chi(\sigma_C) \tag{14}$$

where σ_C denotes a fixed element in C. Equations (13) and (14) should be thought as generalizations to the non-abelian case of equations (2) and (1) respectively. Assuming that the partial zeta functions satisfy similar properties in the non-abelian case as in the abelian case and comparing (2) and (13) evaluated at $s = 0$, it is therefore natural to assume that the factor m_G, the lcm of the cardinalities of the conjugacy classes of G, is needed to make the Galois

[4]Note that the image of $\nu^{\rm ab}$ is in fact contained in $Z(\mathbb{C}[G])$.

Brumer-Stickelberger element integral. This explains the value of d_G given in the Integrality Conjecture.[5]

The next result is proved in [22, Lemme IV.1.1] for abelian extensions. It is straightforward to extend the proof to Galois extensions, also see [14, Lemma 2.2].

Lemma 3.7. *Let $\mathcal{T}$ be a set of prime ideals containing all the unramified prime ideals of K that do not divide w_K except, possibly, a finite number. Then $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ is generated as a $\mathbb{Z}$-module by the elements $\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})$ where $\mathfrak{P}$ runs through the prime ideals in $\mathcal{T}$ and $\mathfrak{p}$ denotes the prime ideal of k below $\mathfrak{P}$. Furthermore, we have*

$$w_K = \gcd_{\substack{\mathfrak{P}\in\mathcal{T}\\ \sigma_{\mathfrak{P}}=1}} (1 - \mathcal{N}(\mathfrak{p})). \qquad \square$$

From this, we deduce equivalent formulations of the Integrality Conjecture.

Proposition 3.8. *The following assertions are equivalent*

(1). *For almost all prime ideals $\mathfrak{P}$ of K, $d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S} \in \mathbb{Z}[G]$.*

(2). *For all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, $d_G\, \xi\, \theta_{K/k,S} \in \mathbb{Z}[G]$.*

(3). *For almost all prime ideals $\mathfrak{P}$ of K, $d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$.*

(4). *For all $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, $d_G\, \xi\, \theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$.*

Proof. The equivalences (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) are consequences of (12) and the discussion that follows. The direction (2) $\Rightarrow$ (1) is trivial. The other direction comes from Lemma 3.7. $\square$

4. Strong central extensions

Before we generalize the abelian Brumer-Stark conjecture to Galois extensions, we introduce the notion of strong central extensions that will play a crucial role. For that, we stop assuming for a moment that G is the Galois group of the extension K/k and just consider G as a finite group. Let Γ and Δ be two other finite groups with Δ a normal subgroup of Γ such that the following sequence is exact

$$1 \longrightarrow \Delta \longrightarrow \Gamma \xrightarrow{\ \pi\ } G \longrightarrow 1, \qquad (15)$$

that is, Γ is a group extension of G by Δ. Recall that the extension is said to be central if Δ is a subgroup of the center of Γ. This implies, in particular, that Δ is an abelian group. If, furthermore, the extension is split, that is there exists an homomorphism $s : G \to \Gamma$ such that $s \circ \pi$ is the identity, then the extension is trivial, that is $\Gamma \simeq \Delta \times G$.

We say that Γ is a strong central extension of G by Δ if $\Delta \cap [\Gamma, \Gamma] = 1$ where $[\Gamma, \Gamma]$ is the commutator subgroup of Γ. The choice of terminology is explained by the following lemma.

Lemma 4.1. *Let Γ be a strong central extension of G by Δ. Then Γ is a central extension of G by Δ.*

Proof. Let $\gamma \in \Gamma$ and $\delta \in \Delta$. We see that

$$[\gamma, \delta] = (\gamma\delta\gamma^{-1})\delta^{-1} \in \Delta$$

since Δ is normal in Γ. Thus, $[\gamma, \delta] = 1$ and γ and δ commute. Therefore Δ is in the center of Γ and the extension is central. $\square$

The trivial extension $\Delta \times G$ is always a strong central extension. As noted above, a strong central extension is trivial if and only if it is split. By the Schur-Zassenhaus theorem, this is the case when the orders of Δ and G are relatively prime. For strong central extensions, the extension is also trivial in an additional case. First, we have the following characterization of strong central extensions.

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[5]Note that, for $G \simeq \mathrm{SL}_2(\mathbb{F}_3)$, we have $s_G = 8$ and $m_G = 12$.

Lemma 4.2. *Consider the group extension* (15)*. This extension is strong central if and only if the map π restricts to an isomorphism between $[\Gamma,\Gamma]$ and $[G,G]$.*

Proof. It is straightforward to see that π restricts to a surjective map from $[\Gamma,\Gamma]$ to $[G,G]$. This map is injective if and only if $[\Gamma,\Gamma]\cap\mathrm{Ker}(\pi)=1$. The result follows since $\mathrm{Ker}(\pi)=\Delta$. □

Lemma 4.3. *Let Γ be a strong central extension of G by Δ. Assume that $G=[G,G]$. Then $\Gamma\simeq\Delta\times G$.*

Proof. Indeed, by Lemma 4.2, the sequence is split. □

It is not true however that all strong central extensions are split and give rise to a direct product as we show in the following example.

Example. Let Γ be the dicyclic group of order 12. It is the group generated by the two elements a and b with the following relations: $a^3=b^4=1$ and $bab^{-1}=a^{-1}$. Let $\Delta:=\langle b^2\rangle\simeq\mathbb{Z}/2\mathbb{Z}$; it is the center of Γ and one can verify that $\Gamma/\Delta\simeq S_3$, the symmetric group on 3 letters. We compute $[\Gamma,\Gamma]=\langle a\rangle$, thus $\Delta\cap[\Gamma,\Gamma]=\{1\}$ and we have the strong central extension

$$1\longrightarrow\mathbb{Z}/2\mathbb{Z}\longrightarrow\Gamma\longrightarrow S_3\longrightarrow 1.$$

However, the group Γ is not isomorphic to $\mathbb{Z}/2\mathbb{Z}\times S_3$ since the latter group does not have any element of order 4.

The following lemma provides us with yet another characterization of strong central extensions.

Lemma 4.4. *Consider the group extension* (15)*. This extension is strong central if and only if, for every abelian subgroup H of G, the subgroup $\pi^{-1}(H)$ of Γ is abelian.*

Proof. Assume that the extension is strong central. Let H be an abelian subgroup of G. Let $\gamma_1,\gamma_2\in\pi^{-1}(H)$, say $\pi(\gamma_1)=h_1$, $\pi(\gamma_2)=h_2$ with $h_1,h_2\in H$. We compute

$$\pi([\gamma_1,\gamma_2])=[h_1,h_2]=1.$$

By hypothesis, this implies that $[\gamma_1,\gamma_2]=1$ and therefore $\pi^{-1}(H)$ is abelian.

Reciprocally, we assume that, for any abelian subgroup H of G, the group $\pi^{-1}(H)$ is abelian. Let $\gamma_1,\gamma_2\in\Gamma$ be such that $[\gamma_1,\gamma_2]\in\Delta$. Then $\pi([\gamma_1,\gamma_2])=1$ and $\pi(\gamma_1)$ and $\pi(\gamma_2)$ commute. The subgroup of G that they generate is abelian and, by hypothesis, it follows that γ_1 and γ_2 commute, that is $[\gamma_1,\gamma_2]=1$. Therefore the extension Γ of G by Δ is strong central. □

We note another property of strong central extensions that will be useful later on. For a finite group A, recall that m_A denote the lcm of the cardinalities of the conjugacy classes of A, s_A is the order of the commutator subgroup $[A,A]$ of A and d_A is the lcm of m_A and s_A.

Lemma 4.5. *Consider the group extension* (15)*. Assume that the extension is strong central. Then we have $d_\Gamma=d_G$.*

Proof. It is enough to show that $m_\Gamma=m_G$ and $s_\Gamma=s_G$. The fact that $s_\Gamma=s_G$ is a direct consequence of Lemma 4.2. We now show that $m_\Gamma=m_G$. Let $\gamma\in\Gamma$. Denote by C and Z respectively the conjugacy class of γ in Γ and the centralizer of γ in Γ. We have

$$\begin{aligned}|C|&=(\Gamma:Z)=(\pi(\Gamma):\pi(Z))(\mathrm{Ker}(\pi):\mathrm{Ker}(\pi)\cap Z)=(G:\pi(Z))(\Delta:\Delta\cap Z)\\&=(G:Z_0)(Z_0:\pi(Z))(\Delta:\Delta\cap Z)=|C_0|(Z_0:\pi(Z))(\Delta:\Delta\cap Z)\end{aligned}$$

where C_0 is the conjugacy class of $\pi(\gamma)$ in G and Z_0 is the centralizer of $\pi(\gamma)$ in G. Since Δ is in the center of Γ by Lemma 4.1, we have $\Delta\subset Z$ and $(\Delta:\Delta\cap Z)=1$. Now, let $\rho_0\in Z_0$ and let $\rho\in\pi^{-1}(\rho_0)$. We have $\pi([\rho,\gamma])=[\rho_0,\pi(\gamma)]=1$ since ρ_0 commutes with $\pi(\gamma)$. Therefore $[\rho,\gamma]\in[\Gamma,\Gamma]\cap\Delta=\{1\}$ and $\rho\in Z$. Thus, $\pi(Z)=Z_0$ and we have finally $|C|=|C_0|$. As any conjugacy class of G is the image by π of a conjugacy class of Γ, we see that $m_\Gamma=m_G$ and the result is proved. □

We now come back to our previous setting and assume that G is the Galois group of an extension K/k. Let L be a finite extension of K. We say that L is a strong central extension of K/k if L/k is Galois and the group extension

$$1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

is strong central where $\Delta := \mathrm{Gal}(L/K)$ and $\Gamma := \mathrm{Gal}(L/k)$. The following result is a direct consequence of Lemma 4.2 (see also Figure 1).

Lemma 4.6. *Denote by L^{ab} the maximal sub-extension of L/k that is abelian over k. Then L is a strong central extension of K/k if and only if $L = KL^{\mathrm{ab}}$. Furthermore, in that case, restriction to L^{ab} yields an isomorphism between $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L^{\mathrm{ab}}/K^{\mathrm{ab}})$ where K^{ab} is the maximal sub-extension of K/k that is abelian over k.* □

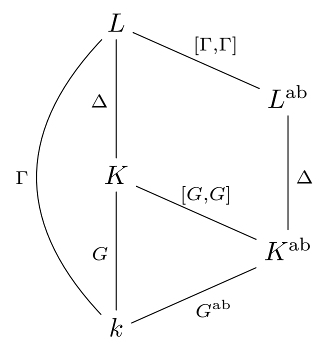

FIGURE 1. Some subfields of the strong central extension L/k of K/k

We conclude this section with a lemma that shows strong central extensions behave somewhat nicely.

Lemma 4.7. *Let L be a strong central extension of K/k.*

(1) Let L_0/K be a sub-extension of L/K. Then L_0 is a strong central extension of K/k.
(2) Let M be another strong central extension of K/k. Then LM is a strong central extension of K/k.

Proof. We use repeatedly the characterization of strong central extensions given by Lemma 4.6. We prove the first assertion. The group $\mathrm{Gal}(L/L_0)$ is a subgroup of $\mathrm{Gal}(L/K)$ and thus it is normal in $\mathrm{Gal}(L/k)$. Therefore, L_0/k is a Galois extension. Let $L_0^{\mathrm{ab}} = L^{\mathrm{ab}} \cap L_0$ be the maximal abelian sub-extension of L_0/k, then $[L_0^{\mathrm{ab}} : K^{\mathrm{ab}}] = [L_0 : K]$ since $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L^{\mathrm{ab}}/K^{\mathrm{ab}})$. Furthermore, since $L_0^{\mathrm{ab}} \cap K = K^{\mathrm{ab}}$, we find that

$$[KL_0^{\mathrm{ab}} : k] = \frac{[L_0^{\mathrm{ab}} : k][K : k]}{[K^{\mathrm{ab}} : k]} = [L_0^{\mathrm{ab}} : K^{\mathrm{ab}}][K : k] = [L_0 : k],$$

thus $KL_0^{\mathrm{ab}} = L_0$ and L_0 is a strong central extension of K/k.

We now prove the second assertion. The extension LM/k is Galois as the compositum of two Galois extensions of k. Let $F = L \cap M$. It is an extension of K. Then, a direct computation shows that $[LM : K] = [L^{\mathrm{ab}}M^{\mathrm{ab}} : K^{\mathrm{ab}}]$. We find that

$$[KL^{\mathrm{ab}}M^{\mathrm{ab}} : k] = \frac{[L^{\mathrm{ab}}M^{\mathrm{ab}} : k][K : k]}{[K^{\mathrm{ab}} : k]} = [L^{\mathrm{ab}}M^{\mathrm{ab}} : K^{\mathrm{ab}}][K : k] = [LM : k].$$

Thus, $KL^{\mathrm{ab}}M^{\mathrm{ab}} = LM$. Since the maximal abelian sub-extension $(LM)^{\mathrm{ab}}$ of LM/k that is abelian over k contains $L^{\mathrm{ab}}M^{\mathrm{ab}}$, it follows that $K(LM)^{\mathrm{ab}} = LM$ and LM is a strong central extension of K/k. □

5. The Galois Brumer-Stark conjecture

We are now ready to state our generalization of the abelian Brumer-Stark conjecture to Galois extensions.

Conjecture (The Galois Brumer-Stark conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$)**.**
Let K/k be a Galois extension of number fields and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S| \geq 2$. The Integrality Conjecture holds for the extension K/k and the set of places S, and, for any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{d_G w_K \theta_{K/k,S}}$ is principal and admits a generator $\alpha \in K^\circ$ such that $K(\alpha^{1/w_K})$ is a strong central extension of K/k.

Remark. As in the abelian case, the last assertion that $K(\alpha^{1/w_K})$ is a strong central extension of K/k does not depend on the choice of the w_K-th root of α since all of these generate the same extension of K.

Before studying conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$, we discuss briefly our evidence for it. Observe first that it is in some ways a natural generalization of the abelian Brumer-Stark conjecture. Indeed, we have the following result.

Proposition 5.1. *Assume that K/k is abelian. Then the Galois Brumer-Stark conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ is equivalent to the abelian Brumer-Stark conjecture $\mathbf{BS}(K/k, S)$.*

Proof. This is clear since $d_G = 1$ in that case and, by Lemma 4.4, we see that $K(\alpha^{1/w_K})/k$ is abelian if and if only if $K(\alpha^{1/w_K})$ is a strong central extension of K/k. □

Another piece of evidence is provided by the following result that sums up the cases where the conjecture is proved or reduces to the abelian Brumer-Stark conjecture. Examples where the conjecture is numerically proved are also given in [7, Chap. 5].

Theorem 5.2. *The Galois Brumer-Stark conjecture is satisfied in the following cases*

(1) $\mathrm{Gal}(K/k)$ *is a non-abelian simple group,*
(2) $\mathrm{Gal}(K/k) \simeq D_{2n}$ *where D_{2n} is the dihedral group of order $2n$ with n odd,*
(3) $\mathrm{Gal}(K/k) \simeq S_n$ *where S_n is the symmetric group on n letters with $n \geq 1$,*
(4) $\mathrm{Gal}(K/k)$ *is non-abelian of order* 8.

Assume that the abelian Brumer-Stark conjecture holds. Then the Galois Brumer-Stark conjecture is satisfied in the following cases

(5) $\mathrm{Gal}(K/k)$ *is abelian,*
(6) $\mathrm{Gal}(K/k)$ *contains a normal abelian subgroup of prime index,*
(7) $\mathrm{Gal}(K/k)$ *is of order < 32 and not isomorphic to* $\mathrm{SL}_2(\mathbb{F}_3)$.

Proof. Cases *1*, *2*, *3*, *4* and *5* follow respectively from Propositions 6.6, 6.7, 6.8, 7.7, and 5.1. The results of Section 7, and in particular Theorem 7.4, imply case *6*. Finally, case *7* follows from a direct inspection using the GAP system [9] and verifying that, in each case, one can reduce to the abelian case, one of the other listed cases or an application of Proposition 6.5 below. □

Remark. Using the GAP system [9], one can verify also by similar techniques that the Galois Brumer-Stark conjecture holds or reduces to the abelian Brumer-Stark conjecture for 730 out of the 1048 possible isomorphism types of Galois groups when $[K : k] \leq 100$.

Remark. The Integrality Conjecture actually holds in all the cases listed in Theorem 5.2 without having to assume the abelian Brumer-Stark conjecture for cases *5, 6, 7*. It also holds for the 730 isomorphism types of Galois groups mentioned in the previous remark.

The following result is the generalization to the non-abelian case of Theorem 2.1. Recall that, for a prime ideal $\mathfrak{P}$ of K, we denote by $\mathfrak{p}$ the prime ideal of k below $\mathfrak{P}$ and by $\sigma_{\mathfrak{P}}$ the Frobenius automorphism of $\mathfrak{P}$ in G.

Theorem 5.3. *Assume that the Integrality Conjecture holds for the extension K/k and the set of places S. Let $\mathfrak{A}$ be a fractional ideal of K. The following assertions are equivalent.*

(*i*). *There exists an anti-unit $\alpha \in K^\circ$ such that $\mathfrak{A}^{d_G w_K \theta_{K/k,S}} = \alpha\mathcal{O}_K$ and $K(\alpha^{1/w_K})$ is a strong central extension of K/k.*

(*ii*). *There exists a strong central extension L of K/k and an anti-unit $\gamma \in L^\circ$ such that $(\mathfrak{A}\mathcal{O}_L)^{d_G\theta_{K/k,S}} = \gamma\mathcal{O}_L$*

(*iii*). *For almost all prime ideals $\mathfrak{P}$ of K, there exists an anti-unit $\alpha_\mathfrak{P} \in K^\circ$ such that $\mathfrak{A}^{d_G(\sigma_\mathfrak{P}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S}} = \alpha_\mathfrak{P}\mathcal{O}_K$ and $\alpha_\mathfrak{P} \equiv 1 \pmod{^* \mathfrak{Q}}$ for all prime ideals $\mathfrak{Q}$ of K above $\mathfrak{p}$ such that $\sigma_\mathfrak{Q} = \sigma_\mathfrak{P}$.*

(*iv*). *For any abelian subgroup H of G, there exists a family $(a_i)_{i\in I}$ of elements of $\mathbb{Z}[H]$ generating $\mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$ as a $\mathbb{Z}$-module and a family of anti-units $(\alpha_i)_{i\in I}$ of K such that $\mathfrak{A}^{d_G a_i \theta_{K/k,S}} = \alpha_i\mathcal{O}_K$ and ${\alpha_j}^{a_i} = {\alpha_i}^{a_j}$ for all $i, j \in I$.*

Remark. In part (*ii*), $(\mathfrak{A}\mathcal{O}_L)^{d_G\theta_{K/k,S}}$ is defined by the formula $\big((\mathfrak{A}\mathcal{O}_L)^{nd_G\theta_{K/k,S}}\big)^{1/n}$ where $n \geq 1$ is any integer such that $nd_G\theta_{K/k,S} \in \mathbb{Z}[G]$. This is well-defined since the group of ideals of a number field is torsion-free.

Proof. We use repeatedly the fact that $\theta_{K/k,S}$ lies in the center of $\mathbb{C}[G]$.

(*i*) ⇒ (*ii*). Let $\gamma := \alpha^{1/w_K}$ and $L := K(\gamma)$. Then, L is a strong central extension of K/k and γ is an anti-unit in L. Furthermore, we have

$$\big(\gamma\mathcal{O}_L\big)^{w_K} = \alpha\mathcal{O}_L = (\mathfrak{A}\mathcal{O}_L)^{d_G w_K \theta_{K/k,S}}$$

and the result follows since the group of ideals of a number field is torsion-free.

(*ii*) ⇒ (*iii*). Denote by Γ the Galois group of L/k and by Δ the Galois group of L/K. Let $\mathcal{T}$ be the set of prime ideals of K, unramified in L/K and $K/\mathbb{Q}$, relatively prime with w_K and with $\mathfrak{A}$ and all its conjugates over k. Note that $\mathcal{T}$ contains all but finitely many prime ideals of K. Let $\mathfrak{P} \in \mathcal{T}$ and let $\tilde{\mathfrak{P}}$ be a prime ideal of L above $\mathfrak{P}$. Denote by $\sigma_{\tilde{\mathfrak{P}}}$ the Frobenius automorphism of $\tilde{\mathfrak{P}}$ in Γ. We set $\alpha_{\tilde{\mathfrak{P}}} := \gamma^{\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p})}$. Let $\tilde{\mathfrak{Q}}$ be another prime ideal of L above $\mathfrak{p}$ such that $\pi(\sigma_{\tilde{\mathfrak{P}}}) = \pi(\sigma_{\tilde{\mathfrak{Q}}})$ where $\pi : \Gamma \to G$ is the canonical surjection induced by the restriction to K and $\sigma_{\tilde{\mathfrak{Q}}}$ is the Frobenius automorphism of $\tilde{\mathfrak{Q}}$ in Γ. There exists $\rho \in \Gamma$ such that $\tilde{\mathfrak{Q}} = \rho(\tilde{\mathfrak{P}})$, and we have $\sigma_{\tilde{\mathfrak{Q}}} = \rho\sigma_{\tilde{\mathfrak{P}}}\rho^{-1}$. Since $\pi([\rho, \sigma_{\tilde{\mathfrak{P}}}]) = \pi(\sigma_{\tilde{\mathfrak{Q}}})\pi(\sigma_{\tilde{\mathfrak{P}}})^{-1} = 1$, this commutator lies in Δ and is therefore trivial. Thus $\sigma_{\tilde{\mathfrak{Q}}} = \sigma_{\tilde{\mathfrak{P}}}$ and $\alpha_{\tilde{\mathfrak{Q}}} = \alpha_{\tilde{\mathfrak{P}}}$. In particular, $\alpha_{\tilde{\mathfrak{P}}}$ does not depend on the choice of the prime ideal $\tilde{\mathfrak{P}}$ of L above $\mathfrak{P}$, and we can just denote it by $\alpha_\mathfrak{P}$. Furthermore, $\alpha_\mathfrak{P} = \gamma^{\sigma_{\tilde{\mathfrak{Q}}}-\mathcal{N}(\mathfrak{p})} \equiv 1 \pmod{^* \tilde{\mathfrak{Q}}}$ for all prime ideals $\tilde{\mathfrak{Q}}$ of L above $\mathfrak{p}$ such that $\sigma_\mathfrak{Q} = \sigma_\mathfrak{P}$ where $\mathfrak{Q}$ is the prime ideal of K below $\tilde{\mathfrak{Q}}$. We now prove that $\alpha_\mathfrak{P}$ lies in K. Let $\delta \in \Delta$. We have

$$\big(\alpha_\mathfrak{P}^{\delta-1}\big)^{w_K} = \Big(\big(\gamma^{w_K}\big)^{\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p})}\Big)^{\delta-1} = \Big(\alpha^{\sigma_\mathfrak{P}-\mathcal{N}(\mathfrak{p})}\Big)^{\delta-1} = 1$$

since α lies in K. Thus, there exists a root of unity $\xi \in \mu_K$ such that $\alpha_\mathfrak{P}^{\delta-1} = \xi$. We have $\alpha_\mathfrak{P} \equiv \alpha_\mathfrak{P}^\delta \equiv 1 \pmod{^* \tilde{\mathfrak{P}}}$ by the above remark, hence $\xi \equiv 1 \pmod{^* \tilde{\mathfrak{P}}}$ and thus $\xi = 1$ by the choice of $\mathfrak{P}$. Therefore, $\alpha_\mathfrak{P} \in K$ as desired. Furthermore, it is clear from its construction that it is an anti-unit and that we have $\alpha_\mathfrak{P} \equiv 1 \pmod{^* \mathfrak{Q}}$ for all prime ideals $\mathfrak{Q}$ above $\mathfrak{p}$ such that $\sigma_\mathfrak{Q} = \sigma_\mathfrak{P}$ by the above. Finally, we compute

$$\alpha_\mathfrak{P}\mathcal{O}_L = (\gamma\mathcal{O}_L)^{\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p})} = \big((\mathfrak{A}\mathcal{O}_L)^{d_G\theta_{K/k,S}}\big)^{\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p})} = (\mathfrak{A}\mathcal{O}_L)^{d_G(\sigma_{\tilde{\mathfrak{P}}}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S}},$$

and, since $\mathfrak{A}$ is an ideal of K and $d_G(\sigma_\mathfrak{P} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S} \in \mathbb{Z}[G]$ by the Integrality Conjecture, we get

$$\alpha_\mathfrak{P}\mathcal{O}_K = \mathfrak{A}^{d_G(\sigma_\mathfrak{P}-\mathcal{N}(\mathfrak{p}))\theta_{K/k,S}}.$$

The implication is proved.

(*iii*) ⇒ (*iv*). Let H be an abelian subgroup of G. Denote by $\mathcal{T}_H$ the set of prime ideals of K for which (*iii*) applies and that are unramified in L/K and K/k, relatively prime with w_K and with $\mathfrak{A}$ and all its conjugates over k, and whose Frobenius automorphism in G lies in H. Let I be a set indexing $\mathcal{T}_H$, so that $\mathcal{T}_H = \{\mathfrak{P}_i : i \in I\}$. For $i \in I$, we set $a_i := \sigma_{\mathfrak{P}_i} - \mathcal{N}(\mathfrak{p}_i) \in \mathbb{Z}[H]$ and $\alpha_i := \alpha_{\mathfrak{P}_i} \in K^\circ$. It follows from Lemma 3.7 that the family $(a_i)_{i\in I}$ generates $\mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$. By construction, we have also $\mathfrak{A}^{d_G a_i \theta_{K/k,S}} = \alpha_i \mathcal{O}_K$. It remains to prove that, for $i, j \in I$, we have $\alpha_j^{a_i} = \alpha_i^{a_j}$, that is, for two prime ideals $\mathfrak{P}$ and $\mathfrak{Q}$ in $\mathcal{T}_H$, the two elements $\alpha_{\mathfrak{P}}^{\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{q})}$ and $\alpha_{\mathfrak{Q}}^{\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})}$ are equal. We have

$$\begin{aligned}(\alpha_{\mathfrak{P}}\mathcal{O}_K)^{\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{p})} &= \left(\mathfrak{A}^{d_G(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S}}\right)^{\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{p})} \\ &= \left(\mathfrak{A}^{d_G(\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{p}))\theta_{K/k,S}}\right)^{\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})} = (\alpha_{\mathfrak{Q}}\mathcal{O}_K)^{\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})}\end{aligned}$$

where we used the fact that $\sigma_{\mathfrak{P}}$ and $\sigma_{\mathfrak{Q}}$ commute since they both belong to H. Since $\alpha_{\mathfrak{P}}$ and $\alpha_{\mathfrak{Q}}$ are both anti-units, there exists a root of unity $\xi \in \mu_K$ such that $\alpha_{\mathfrak{P}}^{\sigma_{\mathfrak{Q}} - \mathcal{N}(\mathfrak{q})} = \xi \alpha_{\mathfrak{Q}}^{\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p})}$. Reasoning as above, we see that $\xi \equiv 1 \pmod{^* \mathfrak{P}}$, thus $\xi = 1$ and the equality is proved.

$(iv) \Rightarrow (i)$. Let H be an abelian subgroup of G. Let $(a_i)_{i\in I}$ and $(\alpha_i)_{i\in I}$ be the corresponding families. There exists a family $(\lambda_i)_{i\in I}$ of integers, with only finitely many non-zero terms, such that

$$w_K = \sum_{i \in I} \lambda_i a_i.$$

We set $\alpha_H := \prod_{i\in I} \alpha_i^{\lambda_i}$. It is clear that α_H is an anti-unit of K and we have

$$\alpha_H \mathcal{O}_K = \mathfrak{A}^{d_G(\sum_i \lambda_i a_i)\theta} = \mathfrak{A}^{d_G w_K \theta_{K/k,S}}.$$

In particular, up to a root of unity in K, α_H does not depend upon the choices made, and we will therefore denote it simply by α. For any $h \in H$, there exists an integer $n_h \in \mathbb{N}$ such that $h - n_h$ annihilates μ_K. Therefore, there exists a family $(\lambda_{h,i})_{i\in I}$ of integers, with only finitely many non-zero terms, such that

$$h - n_h = \sum_{i \in I} \lambda_{h,i} a_i.$$

Furthermore, we have

$$\alpha^{h-n_h} = \prod_{i\in I} \big(\prod_{j\in I} \alpha_i^{a_j \lambda_{h,j}}\big)^{\lambda_i} = \prod_{i\in I} \big(\prod_{j\in I} \alpha_j^{\lambda_{h,j}}\big)^{\lambda_i a_i} = \alpha_h^{\sum_{i\in I} \lambda_i a_i} = \alpha_h^{w_K}$$

where $\alpha_h := \prod_{i\in I} \alpha_i^{\lambda_{h,i}}$. For g, another element of H, one can prove in the same way that $\alpha_h^{g-n_g} = \alpha_g^{h-n_h}$. Let $\gamma := \alpha^{1/w_K}$ and $L := K(\gamma)$. We now prove that L/K^H is an abelian extension. First, we prove that L/K^H is a Galois extension. For $h \in H$, let $\tilde{h}$ be any lift of h to L. We compute

$$(\gamma^{\tilde{h}-n_h})^{w_K} = (\gamma^{w_K})^{\tilde{h}-n_h} = \alpha^{h-n_h} = \alpha_h^{w_K}.$$

Thus, there exists $\xi_h \in \mu_K$ such that $\gamma^{\tilde{h}-n_h} = \xi_h \alpha_h$. Therefore, we have

$$\gamma^{\tilde{h}} = \xi_h \alpha_h \gamma^{n_h} \in L$$

and L/K^H is a Galois extension. Observe, in passing, that since we can take $H = \langle g \rangle$, where $g \in G$ is arbitrary, this implies that L/k is Galois. We now prove that $\mathrm{Gal}(L/K^H)$ is abelian. Let $\tilde{h}, \tilde{g}$ be two elements of $\mathrm{Gal}(L/K^H)$; denote by h and g their restriction to K. We have

$$\gamma^{(\tilde{g}-n_g)(\tilde{h}-n_h)} = (\xi_h \alpha_h)^{g-n_g} = \alpha_h^{g-n_g} = \alpha_g^{h-n_h} = (\xi_g \alpha_g)^{h-n_h} = \gamma^{(\tilde{h}-n_h)(\tilde{g}-n_g)}$$

and therefore $\gamma^{\tilde{g}\tilde{h}} = \gamma^{\tilde{h}\tilde{g}}$. Thus $\mathrm{Gal}(L/K^H)$ is abelian as desired. Since this is true for any abelian subgroup H of G, we get by Lemma 4.4 that L is a strong central extension of K/k. This concludes the proof. ∎

For a fractional ideal $\mathfrak{A}$ of K, we say that $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$ is satisfied if the Integrality Conjecture holds for the extension K/k and the set of places S, and the ideal $\mathfrak{A}$ verifies the equivalent properties of Theorem 5.3. Conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ is thus equivalent to the collection of conjectures $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$ where $\mathfrak{A}$ ranges through the fractional ideals of K.

6. Some properties of the Galois Brumer-Stark conjecture

In this section, we look at some properties satisfied by the Galois Brumer-Stark conjecture and, in particular, the generalizations of the properties of the abelian Brumer-Stark conjecture stated in Section 2.

Proposition 6.1. *The set of fractional ideals $\mathfrak{A}$ of K that satisfy $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$ is a subgroup of the group of ideals of K, stable under the action of G and that contains the principal ideals of K.*

Proof. We first prove that this set is a group. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two fractional ideals of K such that $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$ and $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{B})$ hold. Let α and β be anti-units satisfying part (i) of Theorem 5.3 for the ideals $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then $\alpha\beta$ is an anti-unit such that $\alpha\beta\mathcal{O}_K = (\mathfrak{A}\mathfrak{B})^{d_G w_K \theta_{K/k,S}}$. Furthermore, since $K((\alpha\beta)^{1/w_K}) \subset K(\alpha^{1/w_K},\beta^{1/w_K})$, it is a strong central extension of K/k by Lemma 4.7 and therefore $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A}\mathfrak{B})$ is satisfied. Thus the set of ideals $\mathfrak{A}$ such that $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$ holds is a subgroup of the group of fractional ideals of K.

Let σ be an element of G. We now prove that $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A}^\sigma)$ is satisfied assuming $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$ holds. Since $\theta_{K/k,S}$ is in the center of $\mathbb{C}[G]$, α^σ is a generator of

$$\left(\mathfrak{A}^{d_G w_K \theta_{K/k,S}}\right)^\sigma = \left(\mathfrak{A}^\sigma\right)^{d_G w_K \theta_{K/k,S}}.$$

Furthermore, α^σ is clearly an anti-unit. Let $\gamma := \alpha^{1/w_K}$ and $\delta := (\alpha^\sigma)^{1/w_K}$. Denote by $\tilde{\sigma}$ a lift of σ to $L := K(\gamma)$. Then there exists $\xi \in \mu_K$ such that $\delta = \xi\gamma^{\tilde{\sigma}}$. Since L/k is Galois, we get that $L' := K(\delta) \subset L$. This proves that L' is a strong central extension of K/k by Lemma 4.7 and thus concludes the proof that $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A}^\sigma)$ is satisfied.

Finally, we prove that $\mathbf{BS}_{\mathrm{Gal}}(K/k,S;\mathfrak{A})$ is satisfied if $\mathfrak{A}$ is a principal ideal, say $\mathfrak{A} = \eta\mathcal{O}_K$. For that, we use the equivalent formulation (iv) of Theorem 5.3. Let H be an abelian subgroup of G. For $h \in H$, let $n_h \in \mathbb{N}$ be such that $\xi^h = \xi^{n_h}$ for all $\xi \in \mu_K$ with the convention that $n_1 = w_K + 1$. Then the family $a_h := h - n_h$, for $h \in H$, generates $\mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$. For $h \in H$, we define $\alpha_h := \eta^{d_G a_h \theta_{K/k,S}}$. Note that $d_G a_h \theta_{K/k,S} \in \mathbb{Z}[G]$ by the Integrality Conjecture. For all $h \in H$, we have $(\eta\mathcal{O}_K)^{d_G a_h \theta_{K/k,S}} = \alpha_h\mathcal{O}_K$ by construction. Furthermore, let w be an infinite (complex) place of K. Denote by $\tau_w \in G$ the complex conjugation at w. By Corollary 3.5, we have that $(1+\tau_w)\theta_{K/k,S} = 0$ and thus $\alpha_h^{1+\tau_w} = 1$ for all complex places w of K. Therefore α_h is an anti-unit for all $h \in H$. It remains to prove that $\alpha_h^{a_g} = \alpha_g^{a_h}$ for all $g,h \in H$. But this is a direct consequence of the fact that $(h-n_h)(g-n_g) = (g-n_g)(h-n_h)$ since H is abelian. This concludes the proof. □

Corollary 6.2. *Assume that K is principal. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ is satisfied.* □

Using the decomposition of the Brumer-Stickelberger element given by (12), we can prove the following result that relates $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ and $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$.

Theorem 6.3. *Assume that the Integrality Conjecture is satisfied for the extension K/k and the set of places S and that $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ holds. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ is satisfied if, for any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{d_G w_K \theta^{(>1)}_{K/k,S}}$ is principal, and admits a generator $\beta \in K^\circ$ such that $K(\beta^{1/w_K})$ is a strong central extension of K/k.*

Proof. Let $\mathfrak{A}$ be a fractional ideal of K. Set $\mathfrak{a} := N_{K/K^{\mathrm{ab}}}(\mathfrak{A})$. An direct computation shows that

$$\mathfrak{A}^{d_G w_K \nu^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/k,S})} = \mathfrak{a}^{(d_G/s_G) w_K \theta_{K^{\mathrm{ab}}/k,S}} \mathcal{O}_K.$$

By hypothesis, there exists α_0, an anti-unit in K^{ab}, such that

$$\mathfrak{a}^{(d_G/s_G) w_K \theta_{K^{\mathrm{ab}}/k,S}} = \alpha_0 \mathcal{O}_{K^{\mathrm{ab}}}$$

and $K^{\mathrm{ab}}(\alpha_0^{1/w_K})/k$ is abelian. Let $\alpha := \alpha_0\beta$. Then α is an anti-unit of K and by (12), we have

$$\alpha\mathcal{O}_K = \mathfrak{A}^{d_G w_K \theta_{K/k,S}}.$$

It remains to prove that $K(\alpha^{1/w_K})$ is a strong central extension of K/k. It is a sub-extension of $K(\alpha_0^{1/w_K}, \beta^{1/w_K})/K$. But $K(\beta^{1/w_K})$ is a strong central extension of K/k by hypothesis and $K(\alpha_0^{1/w_K})$ is a strong central extension of K/k by Lemma 4.6. Thus, $K(\alpha^{1/w_K})$ is a strong central extension of K/k by Lemma 4.7 and the result is proved. ☐

For $\chi \in \hat{G}$, recall that K^χ denote the subfield of K fixed by the kernel of χ.

Corollary 6.4. *Assume that* $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ *is satisfied and that, for all* $\chi \in \hat{G}$ *such that* $\chi(1) > 1$*,* K^χ *is not a CM extension. Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ *holds.*

Proof. Indeed, in that case, $\theta^{(>1)}_{K/k,S} = 0$ by Proposition 3.3. ☐

As an application of Corollary 6.4, we can prove that $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ implies $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ for some isomorphic types of group $\mathrm{Gal}(K/k)$.

Proposition 6.5. *Let* $\mathcal{G}$ *be a finite group such that, for all irreducible characters* χ *of* $\mathcal{G}$ *with* $\chi(1) > 1$*, the center of* $\mathcal{G}/\ker(\chi)$ *does not contain an element of order* 2*. Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ *holds for any Galois extension* K/k *of number fields with* $\mathrm{Gal}(K/k) \simeq \mathcal{G}$ *and such that* $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ *is satisfied.*

Proof. The result is trivial if k is not totally real or if K is not totally complex. Assume therefore that k is totally real and K is totally complex. Let χ be an irreducible character of $\mathrm{Gal}(K/k)$ with $\chi(1) > 1$. It is enough to prove that K^χ is not a CM extension. Assume it is a CM extension. Then the complex conjugation is an element of order 2 in its Galois group, which is isomorphic to $\mathcal{G}/\ker(\chi)$, and it commutes with all the elements of the group since it is the unique complex conjugation. This is a contradiction, thus K^χ is not CM and the result follows from Corollary 6.4. ☐

We give several applications of this result.

Proposition 6.6. *Assume that* $\mathrm{Gal}(K/k)$ *is a non-abelian simple group. Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ *holds.*

Proof. The commutator subgroup $[G, G]$ is normal in G, thus it is equal to G and $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ trivially holds since $K^{\mathrm{ab}} = k$. Now, let χ be an irreducible character of G with $\chi(1) > 1$. Then χ is faithful because $\ker(\chi)$ is a normal subgroup of G. But the center of G is trivial and therefore $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ holds by Proposition 6.5. ☐

Proposition 6.7. *Assume that* $\mathrm{Gal}(K/k)$ *is isomorphic to the dihedral group* D_{2n} *of order* $2n$ *where* $n \geq 3$ *is odd. Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ *holds.*

Proof. The group D_{2n} is the group generated by two elements a and b with the following relations: $a^2 = b^n = 1$ and $aba = b^{-1}$. When n is odd, its maximal abelian quotient is the cyclic group of order 2, thus K^{ab}/k is quadratic and $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ holds. Furthermore, by [20, §I.5.3], its non-linear irreducible representations are the representations ρ_h, for $1 \leq h \leq (n-1)/2$, defined by

$$\rho_h(b^k) = \begin{pmatrix} \omega^{kh} & 0 \\ 0 & \omega^{-kh} \end{pmatrix} \quad \text{and} \quad \rho_h(ab^k) = \begin{pmatrix} 0 & \omega^{-hk} \\ \omega^{hk} & 0 \end{pmatrix}$$

for $k \in \mathbb{Z}$, where ω is a fixed primitive n-th root of unity. In particular, the kernel of ρ_h is a subgroup of $\langle b \rangle$, distinct from $\langle b \rangle$. It follows that $D_{2n}/\ker(\rho_h)$ is isomorphic to D_{2m} for some integer $m \geq 3$ dividing n. But the center of D_{2m}, for $m \geq 3$ odd, is trivial. The result follows from Proposition 6.5. □

Proposition 6.8. *Assume that* $\mathrm{Gal}(K/k)$ *is isomorphic to the symmetric group* S_m *on* m *letters with* $m \geq 2$. *Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ *holds.*

Proof. The result is clear if $m = 2$. Assume $m \geq 3$. We use Proposition 6.5 again. The commutator subgroup of S_m is the alternating group A_m. Therefore K^{ab} is a quadratic extension of k and $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ holds. Assume first that $m \geq 5$. Then A_m is the only non-trivial normal subgroup of G and therefore the non-trivial irreducible representations of S_m are either faithful or have A_m as kernel. In particular, the non-linear irreducible representations of S_m must be faithful and the result follows since the center of S_m is trivial. For $m = 3$ and $m = 4$, the result follows from direct inspection. Indeed, for $m = 3$, the unique non-linear irreducible representation is faithful and the center of S_3 is trivial. For $m = 4$, there is only one non-linear irreducible representation ρ that is not faithful. Its kernel is isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the quotient is $S_4/\ker(\rho) \simeq S_3$ and thus has again trivial center. □

Remark. Using Proposition 6.5 and similar techniques, one can prove that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ follows from $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ for some other families of groups, eg. the group of affine bijective maps of a finite field $\mathbb{F}_q$ which is isomorphic to $\mathbb{F}_q \rtimes \mathbb{F}_q^\times$.

We now turn to the question of the change of extension for the Galois Brumer-Stark conjecture. We will prove that it is satisfied in many cases up to a factor.

Proposition 6.9. *Let* K'/k *be a Galois sub-extension of* K/k *with* $G' := \mathrm{Gal}(K'/k)$. *Denote by* $\widetilde{\mathbf{BS}}_{\mathrm{Gal}}(K'/k, S)$ *the Galois Brumer-Stark conjecture for the extension* K'/k *and the set of places* S *with the factor* $d_{G'}$ *replaced by* d_G *including in the statement of the Integrality Conjecture. Assume that* w_K *is relatively prime with the degree of the extension* $K/K'K^{\mathrm{ab}}$. *Then* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ *implies* $\widetilde{\mathbf{BS}}_{\mathrm{Gal}}(K'/k, S)$.

Remark. If G is abelian then $K^{\mathrm{ab}} = K$, thus $K = K'K^{\mathrm{ab}}$ and the condition of the proposition is always satisfied. Furthermore, we have $d_G = d_{G'} = 1$ and we recover the fact that $\mathbf{BS}(K/k, S)$ implies $\mathbf{BS}(K'/k, S)$.

Remark. We prove actually a slighter stronger statement: if $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ holds then, for all fractional ideal $\mathfrak{A}'$ of K', there exists an anti-unit $\alpha \in K'$ such that

$$\mathfrak{A}'^{\, d_G w_{K'} \theta_{K'/k,S}} = (\alpha).$$

The extra hypothesis that w_K is relatively prime with the degree of $K/K'K^{\mathrm{ab}}$ is only used to prove the fact that $K'(\alpha^{1/w_{K'}})$ is a strong central extension of K'/k.

In order to see that the statement of Proposition 6.9 makes sense, we have the following lemma.

Lemma 6.10. *Let* A *be a finite group and let* B *be a quotient group of* A. *Then* d_B *divides* d_A.

Proof. It is enough to prove that s_B divides s_A and m_B divides m_A. Let $\pi : A \to B$ be the canonical surjection and denote by D its kernel. It is clear that s_B divides s_A since $\pi([A, A]) = [B, B]$. We now prove that m_B divides m_A. Let $b \in B$ and let $a \in A$ be such that $\pi(a) = b$. Denote by Z the centralizer of a in A and by Z_0 the centralizer of b in B. Note that $\mathcal{Z} := \pi^{-1}(Z_0)$ is a subgroup of A containing Z and that

$$|Z_0| = \frac{|\mathcal{Z}|}{|D|} = \frac{(\mathcal{Z} : Z)\, |Z|}{|D|}.$$

Denote by C and C_0 the conjugacy classes of a and b in A and B respectively. We find that

$$|C| = \frac{|A|}{|Z|} = \frac{|A|(\mathcal{Z} : Z)}{|D|\,|Z_0|} = (\mathcal{Z} : Z)\frac{|B|}{|Z_0|} = (\mathcal{Z} : Z)\,|C_0|.$$

Thus $|C_0|$ divides $|C|$ and therefore m_B divides m_A. □

Proof of Proposition 6.9. To start, observe that, thanks to Theorem 3.1, the Integrality Conjecture for the extension K/k and the set of places S implies the Integrality Conjecture for the extension K'/k and the set of places S with $d_{G'}$ replaced by d_G. We first prove the result when $K = K'K^{\mathrm{ab}}$. In this situation, we shall actually prove that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ implies $\mathbf{BS}_{\mathrm{Gal}}(K'/k, S)$. Indeed, we have $d_G = d_{G'}$ by Lemma 4.5 since one can see, thanks to Lemma 4.6, that K is a strong central extension of K'/k. Let $\mathfrak{A}'$ be a fractional ideal of K'. By our assumption that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ holds, taking $\mathfrak{A} := \mathfrak{A}'\mathcal{O}_K$, we see that there exists an anti-unit α in K such that

$$\alpha\mathcal{O}_K = (\mathfrak{A}\mathcal{O}_K)^{d_G w_K \theta_{K/k,S}} = \mathfrak{A}'^{\,d_G w_K \theta_{K/k,S}}\mathcal{O}_K = \mathfrak{A}'^{\,d_{G'} w_K \theta_{K'/k,S}}\mathcal{O}_K. \tag{16}$$

Furthermore, $L := K(\gamma)$ is a strong central extension of K/k where $\gamma := \alpha^{1/w_K}$. Clearly, we have

$$\gamma\mathcal{O}_L = (\mathfrak{A}'\mathcal{O}_L)^{\,d_{G'}\theta_{K'/k,S}}.$$

We now use Theorem 5.3(ii) with the extension L/K' and the element γ. The only assertion that needs to be checked is the fact that L is a strong central extension of K'/k. By Lemma 4.6, this is equivalent to the fact that $L = K'L^{\mathrm{ab}}$ where L^{ab} is the maximal sub-extension of L/k that is abelian over k. Clearly, $K^{\mathrm{ab}} \subset L^{\mathrm{ab}}$ thus we have $K'K^{\mathrm{ab}} = K \subset K'L^{\mathrm{ab}}$. Since $KL^{\mathrm{ab}} = L$, it follows that $L \subset K'L^{\mathrm{ab}}$, thus $K'L^{\mathrm{ab}} = L$ and L is a strong central extension of K'/k. Therefore $\mathbf{BS}_{\mathrm{Gal}}(K'/k, S; \mathfrak{A}')$ holds for all fractional ideals $\mathfrak{A}'$ of K' and $\mathbf{BS}_{\mathrm{Gal}}(K'/k, S)$ is satisfied.

We now prove the general case. By the first part, replacing K' by $K'K^{\mathrm{ab}}$ if necessary, we can assume that K' contains K^{ab} and therefore, by hypothesis, w_K is relatively prime with the degree of K/K'. Let $\mathfrak{A}'$ be a fractional ideal of K'. Reasoning as above, we see that there exists $\alpha \in K^\circ$ such that

$$\alpha\mathcal{O}_K = \mathfrak{A}'^{\,d_G w_K \theta_{K'/k,S}}\mathcal{O}_K$$

and L is a strong central extension of K/k where $L := K(\gamma)$ and $\gamma := \alpha^{1/w_K}$. Denote by Γ the Galois group of L/k. For $\sigma \in \Gamma$, $L^\sigma = L$ is a Kummer extension of $K^\sigma = K$ generated by γ^σ. Thus there exist an integer n_σ relatively prime to w_K with $1 \le n_\sigma \le d := [L : K]$, and an element $\kappa_\sigma \in K^\times$ such that $\gamma^\sigma = \kappa_\sigma\gamma^{n_\sigma}$. Observe that, for $\delta \in \Delta := \mathrm{Gal}(L/K)$, we have $n_\delta = 1$ and κ_δ is a root of unity in K. Furthermore, using the fact that σ and δ commute, we get

$$\gamma^{\delta\sigma} = (\kappa_\sigma\gamma^{n_\sigma})^\delta = \kappa_\sigma\kappa_\delta^{n_\sigma}\gamma^{n_\sigma} = \gamma^{\sigma\delta} = (\kappa_\delta\gamma)^\sigma = \kappa_\delta^\sigma\kappa_\sigma\gamma^{n_\sigma}$$

and thus $\kappa_\delta^\sigma = \kappa_\delta^{n_\sigma}$. As δ runs through the elements of Δ, κ_δ runs through the roots of unity of order d, thus $\sigma - n_\sigma$ annihilates the group μ_d of d-th roots of unity. Assume now that σ lies in $A := \mathrm{Gal}(L/K')$. Therefore, σ fixes the group of roots of unity $\mu_{K'} = \mu_K$ and $n_\sigma = 1$. Using the fact that $\theta_{K'/k,S}$ is in the center of $\mathbb{C}[G]$, we get

$$\alpha^\sigma\mathcal{O}_K = (\mathfrak{A}'^\sigma)^{\,d_G w_K \theta_{K'/k,S}}\mathcal{O}_K = \mathfrak{A}'^{\,d_G w_K \theta_{K'/k,S}}\mathcal{O}_K = \alpha\mathcal{O}_K.$$

Since α is an anti-unit, there exists a root of unity ξ_σ in $K^\times$ such that $\alpha^\sigma = \xi_\sigma\alpha$. Combining with the above expression for γ^σ, we find that $\kappa_\sigma^{w_K} = \xi_\sigma$. Thus κ_σ is a root of unity in K and $\xi_\sigma = 1$. It follows that $\alpha \in K'$. Again we use Theorem 5.3(ii) to prove that $\widetilde{\mathbf{BS}}_{\mathrm{Gal}}(K/k, S)$ holds for $\mathfrak{A}'$. It remains to prove that there is a strong central extension of K'/k containing γ. Let $L' := K'L^{\mathrm{ab}}$ where L^{ab} is the maximal sub-extension of L/k that is abelian over k. The Galois group of the extension L/L' is $[\Gamma, \Gamma] \cap A$. Hence, by Lemma 4.6, L' is the maximal sub-extension of L/k that is strong central for K'/k. We now prove that $\gamma \in L'$. Denote by $\pi : \Gamma \to G$ the canonical surjection induced by the restriction to K. Its kernel is Δ, thus it restricts to an isomorphism between $[\Gamma, \Gamma]$ and $[G, G]$ (see also Lemma 4.2). We have $\gamma \in L'$ if and only if

$\pi(\mathrm{Gal}(L/L')) \subset \pi(\mathrm{Gal}(L/K'(\gamma)))$, that is $\pi([\Gamma,\Gamma]\cap A) \subset \mathrm{Gal}(K/N)$ where $N = K \cap K'(\gamma)$. But N/K' is a sub-extension of K/K' of degree dividing w_K and therefore $N = K'$ and the above condition is always satisfied. Hence $\widetilde{\mathbf{BS}}_{\mathrm{Gal}}(K'/k, S)$ holds and this concludes the proof. □

We conclude this section with a proof of when the validity of the conjecture is preserved when one enlarges the set S. For $\chi \in \hat{G}$, denote by ρ_χ a fixed irreducible representation of G of character χ.

Lemma 6.11. *Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_t$ be prime ideals of K. We have*

$$\prod_{i=1}^{t} \sum_{\chi\in\hat{G}} \det(1-\rho_\chi(\sigma_{\mathfrak{P}_i}))e_{\bar{\chi}} \in \frac{1}{|G|} Z(\mathbb{Z}[G]).$$

Proof. Let $\alpha \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. One can see that the above expression is invariant under the action of α using the fact that the map $\chi \mapsto \chi^\alpha$ is a bijection on $\hat{G}$. Therefore, it lies in $\mathbb{Q}[G] \cap Z(\mathbb{C}[G]) = Z(\mathbb{Q}[G])$. Now, by the orthogonality of characters, we have

$$\prod_{i=1}^{t} \sum_{\chi\in\hat{G}} \det(1-\rho_\chi(\sigma_{\mathfrak{P}_i}))e_{\bar{\chi}} = \sum_{\chi\in\hat{G}} \prod_{i=1}^{t} \det(1-\rho_\chi(\sigma_{\mathfrak{P}_i}))e_{\bar{\chi}}.$$

Finally, for all $\chi \in \hat{G}$, $|G|\, e_\chi$ and $\det(1-\rho_\chi(\sigma_{\mathfrak{P}_i}))$, for $i = 1, \ldots, t$, are algebraic integers and thus the result follows. □

Proposition 6.12. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be distinct prime ideals of k not belonging to S. Define*

$$\omega := \prod_{i=1}^{t} \sum_{\chi\in\hat{G}} \det(1-\rho_\chi(\sigma_{\mathfrak{P}_i}))e_{\bar{\chi}} \in \frac{1}{|G|} Z(\mathbb{Z}[G])$$

where $\mathfrak{P}_i$ is a prime ideal of K above $\mathfrak{p}_i$ for $i = 1, \ldots, t$. Let $d \geq 1$ be the smallest integer such that $d\omega \in \mathbb{Z}[G]$. Assume that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ holds and let $S' := S \cup \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k, S'; \mathfrak{A})$ is satisfied for any fractional ideal $\mathfrak{A}$ of K whose class in Cl_K has order relatively prime to d.

Proof. Assume that $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ holds. Let $\mathfrak{A}$ be an ideal of K whose class in Cl_K has order relatively prime to d. Thus there exists an ideal $\mathfrak{A}_0$ of K and $\eta \in K^\times$ such that $\mathfrak{A} = \eta\mathfrak{A}_0^d$. Let α_0 be an anti-unit of K such that

$$\alpha_0\mathcal{O}_K = \mathfrak{A}_0^{d_G w_K \theta_{K/k,S}}$$

and the extension $K(\alpha_0^{1/w_K})$ is a strong central extension of K/k. Define

$$\alpha := \alpha_0^{d\omega}\eta^{d_G w_K \theta_{K/k,S'}}.$$

One checks directly that

$$\alpha\mathcal{O}_K = \mathfrak{A}^{d_G w_K \theta_{K/k,S'}}.$$

From the proof of Proposition 6.1, we see that $\delta := \eta^{d_G w_K \theta_{K/k,S'}}$ is an anti-unit and that the extension $K(\delta^{1/w_K})$ is a strong central extension of K/k. Therefore, α is an anti-unit and the extension $K(\alpha^{1/w_K}) \subset K(\alpha_0^{1/w_K}, \delta^{1/w_K})$ is a strong central extension of K/k by Lemma 4.7. Thus $\mathbf{BS}_{\mathrm{Gal}}(K/k, S'; \mathfrak{A})$ holds. □

7. Groups with a normal abelian subgroup of prime index

In this section, we consider the case where the Galois group G contains an abelian normal subgroup H of prime index. We prove in this setting that the Integrality Conjecture is satisfied and that the Galois Brumer-Stark conjecture follows from the abelian Brumer-Stark conjecture for suitable abelian sub-extensions. The methods used in this section are similar in spirit to the ones used by Nomura [17] to prove that the weak non-abelian Brumer-Stark conjecture of Nickel for monomial groups follows from the abelian Brumer-Stark conjecture (and similar results for the weak Brumer conjecture of Nickel). However, a big difference is that, in Nomura's paper, he can work (rational) character by character whereas this does not seem to possible with the Galois Brumer-Stark conjecture.

We assume from now on that the group G is not abelian and contains a normal abelian subgroup H of index ℓ, a prime number. Let m denote the order of H, thus $|G| = m\ell$. We have $[G,G] \subset H$ since G/H is cyclic of order ℓ and therefore K^H is a subfield of K^{ab}. Let S_H denote the set of places of K^H that are above the places in S. The set S_H contains the infinite places of K^H and the finite places that ramify in K/K^H. The first result of this section gives a decomposition of the Brumer-Stickelberger element in this situation.

Theorem 7.1. *With the notations and setting as above, we have*

$$\theta^{(>1)}_{K/k,S} = \Big(1 - \frac{1}{s_G} N_{[G,G]}\Big)\theta_{K/K^H,S_H}$$

where $N_{[G,G]} := \sum_{c\in[G,G]} c \in \mathbb{Z}[G]$.

Proof. Since the group G contains an abelian normal subgroup of index ℓ, the dimensions of the irreducible characters of G divide ℓ. Hence any character in $\hat{G}$ with $\chi(1) > 1$ is of dimension ℓ. Denote by $\hat{G}_\ell$ the set of irreducible characters of G of dimension ℓ.

Lemma 7.2. *Let $\hat{H}_\ell$ be the set of irreducible characters of H whose kernel does not contain $[G,G]$. For $\chi \in \hat{G}_\ell$, define $\hat{H}_\ell(\chi)$ to be the subset of those characters in $\hat{H}_\ell$ whose induction to G is χ. Then, we have*

$$\hat{H}_\ell = \bigcup_{\chi\in\hat{G}_\ell} \hat{H}_\ell(\chi) \quad (\textit{disjoint union})$$

and each $\hat{H}_\ell(\chi)$ has exactly ℓ elements. Furthermore, for all $\chi \in \hat{G}_\ell$ and $g \in G$, we have

$$\chi(g) = \begin{cases} 0 & \text{if } g \notin H, \\ \sum_{\varphi\in\hat{H}_\ell(\chi)} \varphi(g) & \text{if } g \in H. \end{cases}$$

Proof of the lemma. Let φ be a character in $\hat{H}_\ell$ and let $\chi := \mathrm{Ind}_H^G(\varphi)$. Then χ is of dimension ℓ. Assume χ is not irreducible. Then it is a sum of 1-dimensional characters and all these characters are trivial on $[G,G]$. By Frobenius reciprocity, the restriction of any of these characters to H is equal to φ. Thus φ is trivial on $[G,G]$, a contradiction. Therefore χ is irreducible and lies in $\hat{G}_\ell$. The restriction of χ to H is the sum of ℓ characters of H, and using once again Frobenius reciprocity, we see that these characters are exactly the characters of H whose induction to G is χ and that they are all distinct. Therefore, we have proved that, if $\chi \in \hat{G}_\ell$ is the induction of some character in $\hat{H}_\ell$, then the set $\hat{H}_\ell(\chi)$ contains ℓ distinct characters, say $\varphi_1, \ldots, \varphi_\ell$, such that $\chi_{|H} = \varphi_1 + \cdots + \varphi_\ell$. Furthermore, if χ' is a character of $\hat{G}_\ell$ induced from a character in $\hat{H}_\ell$ with $\chi \neq \chi'$, the sets $\hat{H}_\ell(\chi)$ and $\hat{H}_\ell(\chi')$ are clearly disjoint. This implies that $\hat{H}_\ell$ is the disjoint union of the $\hat{H}_\ell(\chi)$'s for $\chi \in \hat{G}_\ell$. We now prove that $\hat{H}_\ell(\chi)$ is non-empty for all $\chi \in \hat{G}_\ell$. This amounts to proving that any character in $\hat{G}_\ell$ is the induction of some character in $\hat{H}_\ell$. Characters of H whose kernel contains $[G,G]$ are in bijection with characters of $H/[G,G]$. Denote by t the index of $[G,G]$ in H. The number of characters in $\hat{H}_\ell$ is therefore $m-t$ and, by the above discussion, the inductions of these characters yield $(m-t)/\ell$ characters in $\hat{G}_\ell$. On the other hand, we have the formula $m\ell = t\ell + a\ell^2$, where a is the number of characters in $\hat{G}_\ell$, since the sum of the square of the dimensions of the irreducible characters of G is equal to $|G|$ and using the fact that $(G:[G,G]) = t\ell$. Therefore, we have $a = (m-t)/\ell$ and all the characters of $\hat{G}_\ell$ are inductions of characters in $\hat{H}_\ell$. To conclude, it remains to prove the expression for $\chi \in \hat{G}_\ell$. Let $\varphi \in \hat{H}_\ell(\chi)$. For all $g \in G$, we have

$$\chi(g) = \frac{1}{m} \sum_{\substack{r \in G \\ rgr^{-1} \in H}} \varphi(rgr^{-1}).$$

Since the group H is normal in G, $rgr^{-1} \in H$ if and only $g \in H$. Thus $\chi(g) = 0$ if $g \notin H$. If $g \in H$, the expression follows from the fact that $\chi_{|H} = \sum_{\varphi \in \hat{H}_\ell(\chi)} \varphi$. □

As a consequence of Lemma 7.2, we have, for $\chi \in \hat{G}_\ell$, that

$$e_\chi = \sum_{\varphi \in \hat{H}_\ell(\chi)} e_\varphi$$

where e_φ is the idempotent of $\mathbb{C}[H]$ associated to the character φ. We now compute

$$\begin{aligned}
\theta^{(>1)}_{K/k,s} &= \sum_{\chi \in \hat{G}_\ell} L_{K/k,S}(0,\chi)\, e_{\bar\chi} = \sum_{\chi \in \hat{G}_\ell} L_{K/k,S}(0,\chi) \sum_{\varphi \in \hat{H}_\ell(\chi)} e_{\bar\varphi} \\
&= \sum_{\chi \in \hat{G}_\ell} \sum_{\varphi \in \hat{H}_\ell(\chi)} L_{K/k,S}(0, \mathrm{Ind}_H^G \varphi)\, e_{\bar\varphi} = \sum_{\chi \in \hat{G}_\ell} \sum_{\varphi \in \hat{H}_\ell(\chi)} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} \\
&= \sum_{\varphi \in \hat{H}_\ell} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} = \sum_{\varphi \in \hat{H}} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} - \sum_{\varphi \in \hat{H} \setminus \hat{H}_\ell} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} \\
&= \theta_{K/K^H,S_H} - \sum_{\substack{\varphi \in \hat{H} \\ [G,G] \subset \mathrm{Ker}\,\varphi}} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi}.
\end{aligned}$$

Let φ be a character of H whose kernel contains $[G,G]$. Let $\tilde\varphi$ be the only character of $J := H/[G,G]$ such that the inflation of $\tilde\varphi$ to H is equal to φ. From the properties of Artin L-functions, we have $L_{K/K^H,S_H}(0,\varphi) = L_{K^{\mathrm{ab}}/K^H,S_H}(0,\tilde\varphi)$ and a direct calculation shows that $e_\varphi = \nu_H^{\mathrm{ab}}(e_{\tilde\varphi})$ where $e_{\tilde\varphi}$ is the idempotent of $\mathbb{C}[G^{\mathrm{ab}}]$ associated to $\tilde\varphi$, $\nu_H^{\mathrm{ab}} : \mathbb{C}[J] \to \mathbb{C}[H]$ is the map defined for $\tilde g \in J$ by

$$\nu_H^{\mathrm{ab}}(\tilde g) := \frac{1}{s_G} \sum_{\pi_H^{\mathrm{ab}}(g) = \tilde g} g,$$

and extended by linearity to $\mathbb{C}[J]$, and $\pi_H^{\mathrm{ab}} : H \to J$ is the canonical surjection. Therefore, we have

$$\begin{aligned}
\sum_{\substack{\varphi \in \hat{H} \\ [G,G] \subset \mathrm{Ker}\,\varphi}} L_{K/K^H,S_H}(0,\varphi)\, e_{\bar\varphi} &= \sum_{\tilde\varphi \in \hat{J}} L_{K^{\mathrm{ab}}/K^H,S_H}(0,\tilde\varphi) \nu_H^{\mathrm{ab}}(e_{\bar{\tilde\varphi}}) \\
&= \nu_H^{\mathrm{ab}}\Big(\sum_{\tilde\varphi \in \hat{J}} L_{K^{\mathrm{ab}}/K^H,S_H}(0,\tilde\varphi)\, e_{\bar{\tilde\varphi}} \Big) \\
&= \nu_H^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/K^H,S_H}).
\end{aligned}$$

Now, for $\alpha \in \mathbb{C}[H]$ and $\beta \in \mathbb{C}[J]$, one checks readily that $\alpha \nu_H^{\mathrm{ab}}(\beta) = \nu_H^{\mathrm{ab}}(\tilde\alpha\beta)$ where $\tilde\alpha := \pi_H^{\mathrm{ab}}(\alpha)$. Thus, we find that

$$\nu_H^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/K^H,S_H}) = \theta_{K/K^H,S_H}\, \nu_H^{\mathrm{ab}}(1) = \frac{1}{s_G} N_{[G,G]}\, \theta_{K/K^H,S_H}.$$

The result then follows by substituting in the above expression. ☐

The main advantage of the decomposition given by Theorem 7.1 is the fact that the extensions involved are all abelian. Therefore, in our study of $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ in that setting, we can reduce to the abelian case. In particular, it follows from (3) that

$$d_G w_K \theta_{K/k}^{(>1)} = \frac{d_G}{s_G}(s_G - N_{[G,G]}) w_K \theta_{K/K^H, S_H} \in \mathbb{Z}[G].$$

However, for $\mathfrak{p}$ a prime ideal of k, unramified in K/k, and $\mathfrak{P}$ a prime ideal of K above $\mathfrak{p}$, we do not have necessarily that $(\sigma_{\mathfrak{P}} - \mathcal{N}(\mathfrak{p}))\theta_{K/K^H,S_H}$ lies in $\mathbb{Z}[G]$, since $\sigma_{\mathfrak{P}}$ might not belong to H. However, we can still prove the Integrality Conjecture is satisfied.

Proposition 7.3. *We have*

$$(s_G - N_{[G,G]})\theta_{K/K^H,S_H} \in \mathbb{Z}[H].$$

In particular, the Integrality Conjecture holds for the extension K/k and the set S.

Proof. First note that, by Theorem 7.1 and the discussion after (12), the first assertion implies the Integrality Conjecture in this case. Now, we have

$$(s_G - N_{[G,G]})\theta_{K/K^H,S_H} = \sum_{c\in[G,G]} (1-c)\theta_{K/K^H,S_H}.$$

But $1-c \in \mathrm{Ann}_{\mathbb{Z}[H]}(\mu_K)$ for all $c \in [G,G]$ and thus, by the properties of the abelian Brumer-Stickelberger element, all the terms in the sum are in $\mathbb{Z}[H]$. The first assertion and the proof of the proposition follow. ☐

We now prove that, in this situation, the Galois Brumer-Stark conjecture is a consequence of the abelian Brumer-Stark conjecture.

Theorem 7.4. *Let K/k be a Galois extension of number fields whose Galois group G contains an abelian normal subgroup H of prime index. Assume that $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ and $\mathbf{BS}(K/K^H,S_H)$ hold where S_H denotes the set of places of K^H above the places in S. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ is satisfied.*

Proof. We will prove the result using Theorem 6.3. Let $\mathfrak{A}$ be a fractional ideal of K. By our hypothesis, there exists $\alpha_1 \in K^\circ$ such that

$$\mathfrak{A}^{w_K\theta_{K/K^H,S_H}} = \alpha_1\mathcal{O}_K$$

and the extension $K(\gamma_1)/K^H$ is abelian where $\gamma_1 := \alpha_1^{1/w_K}$. Define

$$\beta := \alpha_1^{(d_G/s_G)(s_G - N_{[G,G]})} = \Big(\prod_{c\in[G,G]} \alpha_1^{1-c}\Big)^{d_G/s_G}.$$

By construction, β is an anti-unit of K and satisfies

$$\beta\mathcal{O}_K = \mathfrak{A}^{d_G w_K \theta_{K/k,S}^{(>1)}}.$$

It remains to prove that $K(\beta^{1/w_K})$ is a strong central extension of K/k. We will actually prove that $K(\beta^{1/w_K}) = K$. Let L_1 be the Galois closure of $K(\gamma_1)/k$. Denote by Γ_1 the Galois group $\mathrm{Gal}(L_1/k)$. Let $c_1 \in [\Gamma_1,\Gamma_1]$. Note that $c_1 \in \mathrm{Gal}(L_1/K^H)$ since K^H/k is abelian. Thus, by Theorem 2.1, there exists a prime ideal $\mathfrak{P}_1$ of L_1, relatively prime to the order of μ_{L_1}, whose Frobenius automorphism in Γ_1 is equal to c_1, and an anti-unit $\alpha_{1,\mathfrak{p}_H} \in K^\circ$ such that $\alpha_{1,\mathfrak{p}_H} \equiv 1 \pmod{^*\ \mathfrak{p}_H\mathcal{O}_K}$ and

$$\alpha_{1,\mathfrak{p}_H}\mathcal{O}_K = \mathfrak{A}^{(\sigma_{\mathfrak{p}_H} - \mathcal{N}(\mathfrak{p}_H))\theta_{K/K^H,S_H}}$$

where $\mathfrak{p}_H$ is the prime ideal of K^H below $\mathfrak{P}_1$ and $\sigma_{\mathfrak{p}_H}$ is the Frobenius automorphism of $\mathfrak{p}_H$ in H. We have

$$\begin{aligned}\gamma_1^{c_1-1}\mathcal{O}_{L_1} &= \mathfrak{A}^{(c_1-1)\theta_{K/K^H,S_H}}\mathcal{O}_{L_1}\\ &= \mathfrak{A}^{(\sigma_{\mathfrak{p}_H}-\mathcal{N}(\mathfrak{p}_H))\theta_{K/K^H,S_H}}\mathfrak{A}^{(\mathcal{N}(\mathfrak{p}_H)-1)\theta_{K/K^H,S_H}}\mathcal{O}_{L_1}\\ &= \alpha_{1,\mathfrak{p}_H}\alpha_1^{(\mathcal{N}(\mathfrak{p}_H)-1)/w_K}\mathcal{O}_{L_1}.\end{aligned}$$

Observe that γ_1, $\alpha_{1,\mathfrak{p}_H}$ and α_1 are anti-units, thus there exists a root of unity $\xi \in \mu_{L_1}$ such that $\xi\gamma_1^{c_1-1} = \alpha_{1,\mathfrak{p}_H}\alpha_1^{(\mathcal{N}(\mathfrak{p}_H)-1)/w_K}$. Furthermore, since c_1 acts trivially on the group μ_K, w_K divides $\mathcal{N}(\mathfrak{p}_H)-1$ and $\alpha_{1,\mathfrak{p}_H}\alpha_1^{(\mathcal{N}(\mathfrak{p}_H)-1)/w_K}$ belongs to K°. Raising to the power w_K, we get

$$\xi^{w_K}\alpha_1^{\sigma_{\mathfrak{p}_H}-1} = \alpha_{1,\mathfrak{p}_H}^{w_K}\alpha_1^{\mathcal{N}(\mathfrak{p}_H)-1}$$

and therefore

$$\xi^{w_K} \equiv \alpha_1^{\mathcal{N}(\mathfrak{p}_H)-\sigma_{\mathfrak{p}_H}} \equiv 1 \ (\mathrm{mod}^*\ \mathfrak{p}_H\mathcal{O}_K).$$

Therefore we find that $\xi^{w_K}=1$, hence $\xi \in \mu_K$ and $\gamma_1^{c_1-1} \in K$.

Now, for all $c \in [G,G]$, fix an element c_1 in $[\Gamma_1,\Gamma_1]$ whose restriction to K is equal to c, and define

$$\delta := \Big(\prod_{c\in[G,G]}\gamma_1^{1-c_1}\Big)^{d_G/s_G}.$$

By the above computation, we see that $\delta \in K$ and, by construction, that $\delta^{w_K}=\beta$. Therefore $K(\beta^{1/w_K}) = K$ and the result follows. □

Corollary 7.5. *Assume that the order of H is odd. Then $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ implies $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$.*

Proof. Indeed, since the degree of K/K^H is odd, $\mathbf{BS}(K/K^H,S)$ is trivially true as we cannot have both K totally complex and K^H totally real. □

We proved already that $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ holds when G is isomorphic to the dihedral group of order $2m$ with m odd (see Proposition 6.7). We prove a similar statement when m is even.

Proposition 7.6. *Assume that G is isomorphic to the dihedral group D_{2m} of order $2m$, with m even, and that $\mathbf{BS}(K/K^H,S_H)$ holds where H is the unique cyclic subgroup of G of order m. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ is satisfied.*

Proof. The cyclic subgroup H is normal and of index 2. Therefore, by Theorem 7.4 and the hypothesis, it is enough to prove that $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ is satisfied. The maximal abelian quotient of D_{2m} is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, thus $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ holds by the results of [19]. □

We conclude this section with another application of Theorem 7.4.

Proposition 7.7. *Assume that G is non-abelian of order 8. Then $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ is satisfied.*

Proof. The non-abelian groups of order 8 are, up to isomorphism, the dihedral group D_8 of order 8 and the quaternion group Q_8. We start with $\mathrm{Gal}(K/k)\simeq D_8$. Thanks to Proposition 7.6, it remains to prove that $\mathbf{BS}(K/K^H,S)$ holds where H is the cylic subgroup of order 4 contained in $\mathrm{Gal}(K/k)$. But K/K^H is a degree 4 abelian extension contained in the degree 8 non-abelian Galois extension K/k, so $\mathbf{BS}(K/K^H,S)$ is true by results of Tate (see Theorem 2.2).

Assume now that $\mathrm{Gal}(K/k)\simeq Q_8$. A presentation for the group Q_8 is given by the three generators a,b,c with the relations $a^2=b^2=c^2=abc$. It is customary to denote the element abc by -1 since it is of order 2 and lies in the center of Q_8. (In fact, it generates the center of Q_8.) The subgroup H generated by a is cyclic of order 4. Thus, we can apply Theorem 7.4 and $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ follows from $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ and $\mathbf{BS}(K/K^H,S_H)$. Now, the commutator subgroup of Q_8 is the subgroup $\{\pm1\}$ and $\mathrm{Gal}(K^{\mathrm{ab}}/k)\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Thus, $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ is satisfied by the results of [19]. Finally, as above, the extension K/K^H is a degree 4 abelian extension contained in the degree 8 non-abelian Galois extension K/k so $\mathbf{BS}(K/K^H, S_H)$ follows again from Theorem 2.2. □

Appendix. The non-abelian Brumer-Stark conjecture of Nickel

In this appendix, we state the weak non-abelian Brumer-Stark conjecture of Nickel [14] and compare it with our conjecture. Note that Nickel states also a strong version of the non-abelian Brumer-Stark conjecture and similar generalizations of the Brumer conjecture. We change slightly the notations used by Nickel to match the notations used in the previous sections.

Let K/k be a Galois CM-extension with group G. Fix a finite set S of places of k such that S contains the infinite places of k and the finite places of k that ramify in K/k. Let $\mathrm{Hyp}(S)$ be the set of finite set T of places of k such that

- S and T are disjoint,
- the group $E_K(S,T)$ is torsion-free.

Here, $E_K(S,T)$ denotes the group of (S,T)-units of L, that is the group of elements $u \in K^\times$ such that $v_{\mathfrak{P}}(u) = 0$ for all prime ideals $\mathfrak{P}$ of K such that $(\mathfrak{P} \cap k) \notin S$ and $u \equiv 1 \pmod{^* \mathfrak{Q}}$ for all prime ideals $\mathfrak{Q}$ of K such that $(\mathfrak{Q} \cap k) \in T$. For $T \in \mathrm{Hyp}(S)$, define

$$\delta_T := \mathrm{nr}\Big(\prod_{\mathfrak{p}\in T} 1 - \sigma_{\mathfrak{P}}^{-1}\mathcal{N}(\mathfrak{p})\Big)$$

where $\mathfrak{P}$ is a fixed prime ideal of K above $\mathfrak{p}$ and $\mathrm{nr} : \mathbb{Q}[G] \to Z(\mathbb{Q}[G])$ is the reduced norm (see [18, §9]). Let Λ' denote a fixed maximal order of $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$. Denote by $\mathfrak{F}(G) := \{x \in Z(\Lambda') : x\Lambda' \subset \mathbb{Z}[G]\}$ the central conductor of Λ' over $\mathbb{Z}[G]$.

Conjecture (The weak non-abelian Brumer-Stark conjecture of Nickel)**.**
Let $\mathfrak{w}_K := \mathrm{nr}(w_K)$. Then $\mathfrak{w}_K\theta_{K/k,S} \in Z(\Lambda')$. Furthermore, for any fractional ideal $\mathfrak{A}$ of K and for each $x \in \mathfrak{F}(G)$, there exists an anti-unit $\alpha_x \in K^\circ$ such that

$$\mathfrak{A}^{x\mathfrak{w}_K\theta_{K/k,S}} = \alpha_x\mathcal{O}_K$$

and, for any set of places $T \in \mathrm{Hyp}(S \cup S_{\alpha_x})$, there exists $\alpha_{x,T} \in E_K(S_{\alpha_x}, T)$ such that, for all $z \in \mathfrak{F}(G)$

$$\alpha_x^{z\delta_T} = \alpha_{x,T}^{z\mathfrak{w}_K}$$

where S_{α_x} is the set of prime ideals $\mathfrak{p}$ of k such that $v_{\mathfrak{p}}(N_{K/k}(\alpha_x)) \neq 0$.

Remark. The strong version of the conjecture is similar with the modules $Z(\Lambda')$ and $\mathfrak{F}(G)$ replaced respectively by the modules $\mathcal{I}(G)$ and $\mathcal{H}(G)$ where $\mathcal{I}(G)$ is the module generated by the reduced norms of matrices with coefficients in $\mathbb{Z}[G]$; the definition of $\mathcal{H}(G)$ is more intricate, see [14, p. 2582]

The results of Greither-Popescu [10], mentioned at the end of Section 2, have been generalized by Nickel in [16] where he proves that the p-part of his non-abelian Brumer conjecture and non-abelian Brumer-Stark conjecture hold if S contains all the prime ideals above p and some appropriate μ-invariant vanishes. As mentioned in the previous section, Nomura [17] proves that the weak non-abelian Brumer-Stark conjecture of Nickel is implied by the abelian Brumer-Stark conjecture when the Galois group of K/k is monomial (and also obtain additional results on the strong version and local versions of the conjecture).

We now briefly compare our conjecture with the weak non-abelian Brumer-Stark conjecture of Nickel. Both conjectures have two parts: an integrality statement for the Brumer-Stickelberger element and an annihilation of the class group statement that also predicts special properties for the generators obtained. We first prove that the Galois Brumer-Stark conjecture implies the annihilation statement in the weak non-abelian Brumer-Stark conjecture of Nickel up to a factor d_G, including the existence of generators that satisfy almost all the required properties.

Proposition 7.8. *Assume that* $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ *holds. Let* $\mathfrak{A}$ *be a fractional ideal of* K*. Then, for all* $x \in \mathfrak{F}(G)$*, there exists an anti-unit* $\alpha_x \in K^\circ$ *such that*

$$\mathfrak{A}^{d_G x \mathfrak{w}_K \theta_{K/k,S}} = \alpha_x \mathcal{O}_K$$

and, for any set of places $T \in \mathrm{Hyp}(S \cup S_{\alpha_x})$*, there exists* $\alpha_{x,T} \in K^\times$ *with* $\alpha_{x,T}^{w_K} \in E_K(S_{\alpha_x}, T)$ *such that, for all* $z \in \mathfrak{F}(G)$

$$\alpha_x^{z\delta_T} = \alpha_{x,T}^{z\mathfrak{w}_K}.$$

Proof. Let $\alpha \in K^\circ$ be such that $\mathfrak{A}^{d_G w_K \theta_{K/k,S}} = \alpha \mathcal{O}_K$ and $K(\alpha^{1/w_K})$ is a strong central extension of K/k. For $x \in \mathfrak{F}(G)$, let $\alpha_x := \alpha^{x\mathfrak{m}_K}$ where $\mathfrak{m}_K = w_K^{-1}\mathfrak{w}_K$. Note that by (18) below, we have $\mathfrak{m}_K \in Z(\Lambda')$ and thus $x\mathfrak{m}_K \in \mathbb{Z}[G]$. We compute

$$\mathfrak{A}^{d_G x \mathfrak{w}_K \theta_{K/k,S}} = (\mathfrak{A}^{d_G w_K \theta_{K/k,S}})^{x\mathfrak{m}_K} = \alpha_x \mathcal{O}_K.$$

Now, let $T \in \mathrm{Hyp}(S \cup S_{\alpha_x})$. Let $\mathfrak{M}_T$ be the product of the prime ideals of K above the prime ideals in T. Then, for any $a \in \mathfrak{F}(G)$, the element $a\delta_T \in \mathbb{Z}[G]$ kills $(\mathcal{O}_K/\mathfrak{M}_T)^\times$. We give the proof of this result due to Nickel (personal communication). Let ℓ be a prime number. The module $(\mathcal{O}_K/\mathfrak{M}_T)^\times \otimes \mathbb{Z}_\ell$ admits a quadratic presentation induced by the following exact sequences

$$\mathbb{Z}_\ell[D_{\mathfrak{P}}] \longrightarrow \mathbb{Z}_\ell[D_{\mathfrak{P}}] \longrightarrow (\mathcal{O}_K/\mathfrak{P})^\times \otimes \mathbb{Z}_\ell \longrightarrow 1$$

where $\mathfrak{p}$ ranges through the prime ideals in T, $\mathfrak{P}$ is a fixed prime ideal of K above $\mathfrak{p}$, $D_{\mathfrak{P}}$ is the decomposition group of $\mathfrak{P}$ in G, the first map is the multiplication by $1 - \sigma_{\mathfrak{P}}^{-1}\mathcal{N}(\mathfrak{p})$ and the second map is induced by the action of $D_{\mathfrak{P}}$ on a fixed generator of $(\mathcal{O}_K/\mathfrak{P})^\times$. Thus, the Fitting invariant of $(\mathcal{O}_K/\mathfrak{M}_T)^\times \otimes \mathbb{Z}_\ell$ is generated by δ_T, see [14, p. 2580]. By Theorem 1.2, *ibid.*, we get that $a\delta_T$ annihilates $(\mathcal{O}_K/\mathfrak{M}_T)^\times \otimes \mathbb{Z}_\ell$. Since this is true for all primes ℓ and $(\mathcal{O}_K/\mathfrak{M}_T)^\times$ is a finite abelian group, the result follows. In particular, because $E_K(S,T)$ is torsion-free, we get that $x\delta_T$ lies in $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$. Using Lemma 3.7, we can write

$$x\delta_T = \sum_{i=1}^{t} \lambda_i(\sigma_{\mathfrak{P}_i} - \mathcal{N}(\mathfrak{p}_i)),$$

where λ_i's are rational integers and the $\mathfrak{P}_i$'s are prime ideals of K such that Theorem 5.3(*iii*) applies. Let $\gamma := \alpha^{1/w_K}$, $L := K(\gamma)$ and $\Gamma := \mathrm{Gal}(L/K)$. Define

$$\tilde{x}_T := \sum_{i=1}^{t} \lambda_i(\sigma_{\tilde{\mathfrak{P}}_i} - \mathcal{N}(\mathfrak{p}_i)) \in \mathbb{Z}[\Gamma]$$

where $\tilde{\mathfrak{P}}_i$ is a fixed prime ideal of L above $\mathfrak{P}_i$ and $\sigma_{\tilde{\mathfrak{P}}_i}$ is the Frobenius automorphism of $\tilde{\mathfrak{P}}_i$ in Γ. We set $\alpha_{x,T} := \gamma^{\tilde{x}_T}$. From the proof of Theorem 5.3(*iii*), we see that $\gamma^{\sigma_{\tilde{\mathfrak{P}}_i} - \mathcal{N}(\mathfrak{p}_i)}$ is an element of $K^\times$ (it is equal to $\alpha_{\mathfrak{P}_i}$ with the notations of Theorem 5.3(*iii*)) and thus $\alpha_{x,T} \in K^\times$. Furthermore, we have

$$\alpha_{x,T}\mathcal{O}_L = (\mathfrak{A}^{d_G \theta_{K/k,S}}\mathcal{O}_L)^{\tilde{x}_T} = \mathfrak{A}^{d_G x \delta_T \theta_{K/k,S}}\mathcal{O}_L$$

and, using the fact that $d_G x \delta_T \theta_{K/k,S} \in \mathbb{Z}[G]$ by the Integrality Conjecture, we get that $\alpha_{x,T}\mathcal{O}_K = \mathfrak{A}^{d_G x \delta_T \theta_{K/k,S}}$. Now, let N be a large enough integer such that $N\mathfrak{m}_K^{-1}\delta_T \in \mathbb{Z}[G]$. Then, we find (compare with proof of [14, Lemma 2.12])

$$\alpha_{x,T}^N \mathcal{O}_K = \left(\mathfrak{A}^{d_G x \mathfrak{m}_K \theta_{K/k,S}}\right)^{N\mathfrak{m}_K^{-1}\delta_T} = \alpha_x^{N\mathfrak{m}_K^{-1}\delta_T}\mathcal{O}_K$$

and $\alpha_{x,T}$ is supported only by prime ideals above S_{α_x}. Also, we have $\alpha_{x,T}^{w_K} = \alpha^{x\delta_T} \equiv 1 \ (\mathrm{mod}^* \ \mathfrak{M}_T)$ and therefore $\alpha_{x,T}^{w_K} \in E_K(S_{\alpha_x}, T)$. Finally, for $z \in \mathfrak{F}(G)$, we compute

$$\alpha_{x,T}^{z\mathfrak{w}_K} = \gamma^{\tilde{x}_T z \mathfrak{w}_K} = (\gamma^{w_K})^{\tilde{x}_T z \mathfrak{m}_K} = \alpha^{x\delta_T z \mathfrak{m}_K} = (\alpha^{x\mathfrak{m}_K})^{z\delta_T} = \alpha_x^{z\delta_T}.$$

□

It is an interesting question to ask if there is any result in the other direction: assuming the non-abelian Brumer-Stark conjecture of Nickel, can we deduce some results on our Galois Brumer-Stark conjecture? We were not able yet to obtain significant results on this question. Similarly, it appears that the connections between the Integrality Conjecture and the integrality statement in the conjectures of Nickel are quite thin and, indeed, the two statements appear to be of quite different nature. To illustrate this point, we look at the proof of the integrality statement of the weak non-abelian Brumer-Stark conjecture in the setting of the previous section. (Of course, this case also follows from the results of [17].)

We start with some general results and facts. Let $\mathfrak{X}$ be the set of irreducible $\mathbb{Q}$-characters of G or, equivalently, the set of orbits of $\hat{G}$ under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. For $X \in \mathfrak{X}$, define $e_X := \sum_{\phi \in X} e_\phi \in \mathbb{Q}[G]$. Then, as a $\mathbb{Q}[G]$-module, one has

$$\mathbb{Q}[G] = \bigoplus_{X \in \mathfrak{X}} e_X \mathbb{Q}[G].$$

For each $X \in \mathfrak{X}$, fix a character $\phi_X \in X$ and set $\mathbb{Q}_X := \mathbb{Q}(\phi_X)$ and $n_X := \phi_X(1)$. Indeed these do not depend on the choice of the character $\phi_X \in X$. Then $\bigoplus_{X \in \mathfrak{X}} \mathbb{Q}_X$ is isomorphic to $Z(\mathbb{Q}[G])$ by the map

$$(\alpha_X)_{X \in \mathfrak{X}} \mapsto \sum_{X \in \mathfrak{X}} \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_X/\mathbb{Q})} \alpha_X^\sigma e_{\phi_X^\sigma} \tag{17}$$

where $\phi_X^\sigma := \sigma \circ \phi_X$. This map restricts to an isomorphism between $\bigoplus_{X \in \mathfrak{X}} \mathcal{O}_X$ and $Z(\Lambda')$ where $\mathcal{O}_X$ denotes the ring of integers of $\mathbb{Q}_X$. A direct computation shows that

$$\mathfrak{w}_K = \sum_{X \in \mathfrak{X}} w_K^{n_X} e_X \tag{18}$$

and thus

$$\mathfrak{w}_K \theta_{K/k,S} = \sum_{X \in \mathfrak{X}} \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_X/\mathbb{Q})} w_K^{n_X} L_{K/k,S}(0, \bar{\phi}_X)^\sigma e_{\phi_X^\sigma}$$

using (9). Thus, the assertion that $\mathfrak{w}_K \theta_{K/k,S}$ lies in $Z(\Lambda')$ is equivalent to the fact that $w_K^{n_X} L_{K/k,S}(0, \phi_X)$ lies in $\mathcal{O}_X$ for all $X \in \mathfrak{X}$. Assume that $X \in \mathfrak{X}$ is such that $n_X = 1$. Let $\tilde{\phi}_X$ be the unique character of G^{ab} whose inflation to G is equal to ϕ_X. Then, we have

$$w_K^{n_X} L_{K/k,S}(0, \phi_X) = w_K L_{K^{\mathrm{ab}}/k,S}(0, \tilde{\phi}_X) \in \mathcal{O}_X$$

using (1) and (3). Thus $\mathfrak{w}_K \theta_{K/k,S} \in Z(\Lambda')$ if and only if $\mathfrak{w}_K \theta_{K/k,S}^{(>1)} \in Z(\Lambda')$.

We now specialize to the setting of the last section and assume that G contains a normal abelian subgroup H of prime index ℓ. Thanks to the above remark and Theorem 7.1, we have $\mathfrak{w}_K \theta_{K/k,S} \in Z(\Lambda')$ if and only if $\mathfrak{e} w_K^\ell \theta_{K/K^H,S_H} \in Z(\Lambda')$ where

$$\mathfrak{e} := 1 - \frac{1}{s_G} N_{[G,G]} = \sum_{\substack{X \in \mathfrak{X} \\ n_X > 1}} e_X \in Z(\Lambda'). \tag{19}$$

Since $w_K \theta_{K/K^H,S_H} \in \mathbb{Z}[H] \subset Z(\Lambda')$ by (3), we recover the fact that $\mathfrak{w}_K \theta_{K/k,S} \in Z(\Lambda')$ in that setting.

The different roles played by the idempotent $\mathfrak{e}$ in both proofs show, in our opinion, that the integrality statements of the two conjectures are, somewhat, of different nature (at least in this setting). Indeed, for the weak non-abelian Brumer-Stark conjecture of Nickel, this factor plays no role at all whereas it plays an essential part in the proof of the Integrality Conjecture, see the proof of Proposition 7.3. This leads us to believe that there is no direct easy connection between the two integrality conjectures.

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