Numerical Verification of the Stark-Chinburg Conjecture for Some Icosahedral Representations

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October 6, 2003

Abstract

In this paper, we give fourteen examples of icosahedral representations for which we have numerically verified the Stark-Chinburg conjecture.

1 Introduction

Let $\mathcal{K}/k$ be a Galois extension of number fields, with Galois group $G = \mathrm{Gal}(\mathcal{K}/k)$, and suppose $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is a non-trivial irreducible representation of G. Stark's conjectures [Tate 1984] aim to unravel the arithmetic information encoded in the leading coefficient of the Taylor series for the Artin L-function $L(s, \rho)$ of ρ at $s = 0$. When G is abelian and one modifies the Artin L-function by removing the factors in the Euler product at primes in a finite set S which contains all of the infinite primes, Stark formulated an especially precise conjecture for the case of a first-order zero at 0 [Stark 1980]. It states that the exact value of this coefficient may be obtained from an "L-function evaluator" element in $\mathcal{K}$ which is an S-unit in the typical case. Rubin [Rubin 1996], Popescu [Popescu 2003], Burns [Burns 2001], Sands [Sands 1987] and others have formulated similarly precise conjectures for abelian L-functions with any order of zero at $s = 0$.

In the general non-abelian case with $L(s, \rho)$ possessing a zero at $s = 0$ of order $r = r(\rho)$, the conjecture states that the L-function coefficient equals an algebraic factor multiplied by the determinant of a regulator matrix defined in terms of a set of r special units and the representation ρ. But this algebraic factor is not fully specified and in particular may be multiplied by any nonzero rational factor without affecting the truth of the conjecture. Hence the conjecture in this generality is considered to be a conjecture "over $\mathbb{Q}$", as opposed to the more precise conjectures "over $\mathbb{Z}$" mentioned above in the abelian case.

Chinburg [Chinburg 1983] has formulated a conjecture "over $\mathbb{Z}$," in the non-abelian case when the order of the zero at 0 is $r(\rho) = 1$, the base field is $k = \mathbb{Q}$, and the dimension of the irreducible representation ρ is $n = 2$. We will show that this conjecture is closely related to a Question of Stark appearing in [Stark 1981], and hence we will use the term "Stark-Chinburg conjecture." Here the regulator matrix is 1 by 1, and involves a single special unit, so we have the possibility of actually constructing this special unit from the first derivatives at 0 of certain Artin L-functions. This method of constructing S-units while simultaneously gaining numerical confirmation of the conjecture at hand appears in [Dummit et al. 1997] and [Roblot 2000] for the abelian case. A difference in this paper is that the extension field $\mathcal{K}$ is no longer a class field which can be explicitly constructed from abelian L-functions by means of the conjecture. We will choose our non-abelian extension field $\mathcal{K}$ beforehand in order to define the L-functions.

Irreducible two-dimensional representations are classified according to the isomorphism type of their images in $\mathrm{PGL}_2(\mathbb{C})$, the four possible types being dihedral, tetrahedral (A_4), octahedral (S_4), and icosahedral (A_5). Stark [Stark 1981] has provided illuminating examples in the dihedral cases; Chinburg [Chinburg 1983] has confirmed the conjecture numerically for five tetrahedral representations; and Fogel [Fogel 1998] has confirmed it numerically for eight octahedral representations with $\mathcal{K}$ of degree 48. Our aim in this paper is to provide the first numerical confirmation of the Stark-Chinburg conjecture for some icosahedral representations. As we will see, the minimal type of field $\mathcal{K}$ providing such an example is a complex field of degree 240 over $\mathbb{Q}$, while the Stark unit ε lies in a subfield $\mathcal{K}^+$ of degree 120 admitting a real embedding. This subfield $\mathcal{K}^+$ is Galois over a field M of degree 30. We identify ε by obtaining its minimal polynomial over M.

The outline of the article is the following: in section 2 we state the Stark-Chinburg conjecture, but also a question of Stark related to the same situation. In section 3 we look at $\hat{A}_5$-extensions which provide the simplest cases for testing the conjecture on icosahedral representations, where $\hat{A}_5$ is a central extension of A_5 by a cyclic group of order 4 (see Section 3 for details). We briefly explain how to construct those extensions and how to compute the value of the derivative of the corresponding L-functions at $s = 0$. Finally, in the last section, we describe the computations performed, give some remarks on the results obtained and conclude with an example.

2 The Stark-Chinburg conjecture

2.1 Odd Representations

A standard formula [Tate 1984, p. 24] for the order $r(\rho)$ of the zero of $L(s, \rho)$ at $s = 0$ calls for the choice of a single prime w of K above each infinite prime v of k. One then defines τ_v to be the generator of the decomposition group of the prime w over v, which is thus either the identity or a complex conjugation. Assuming that ρ is a non-trivial irreducible representation, $r(\rho)$ may then be obtained by taking the dimension of the eigenspace of $\rho(\tau_v)$ corresponding to the eigenvalue 1, and summing over v. Now suppose that our representation ρ is as in the Stark-Chinburg conjecture. Since $k = \mathbb{Q}$, there is a single infinite prime $v = \infty$. Since $\tau = \tau_v$ has order 1 or 2, all eigenvalues of $\rho(\tau)$ must be ± 1.

Since ρ is 2-dimensional and $r(\rho) = 1$, τ must be a complex conjugation of order 2 and $\rho(\tau)$ must have eigenvalues 1 and -1. Thus $\det \rho(\tau) = -1$, a condition which is described by saying that ρ is "odd." The associated character ψ defined as the trace of ρ then clearly takes the value $\psi(\tau) = 0$. Conversely, it is easy to see that if $\psi(\tau) = 0$, for the character ψ of degree 2, then the corresponding representation ρ is odd.

2.2 Regulators

For simplicity, in this section, we continue to assume (with Stark and Chinburg) that $k = \mathbb{Q}$. We also fix a subfield $\mathcal{K}$ of $\mathbb{C}$ which is Galois over $\mathbb{Q}$ with group G, and assume ρ is an irreducible non-trivial n-dimensional representation of $G = \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$ with character ψ. Let τ denote the restriction of complex conjugation to $\mathcal{K}$, which may be trivial, and $\| \; \|$ denote the normalized absolute value on $\mathcal{K}$ corresponding to the embedding of $\mathcal{K}$ in $\mathbb{C}$ (this is the square of the usual one if $\mathcal{K}$ is complex). Also let w be the infinite prime of $\mathcal{K}$ corresponding to this absolute value, so the decomposition group of this prime is $G_w = \langle \tau \rangle$. Following Stark, we may assume by conjugating the representation that $\rho(\tau)$ is diagonal and the diagonal elements consist of a certain number a of 1's followed by a certain number $b = n - a$ of -1's. In [Stark 1975, p. 62], Stark introduces a regulator which we will denote (with Tate) as $R(\psi, \varepsilon)$; this also calls for a choice of element $\varepsilon \in \mathcal{K}$. Then

$$R(\psi, \varepsilon) = \det \left(\sum_{\sigma \in G} \rho_a(\sigma) \log \|\varepsilon^\sigma\| \right),$$

where $\rho_a(\sigma)$ denotes the $a \times a$ matrix in the top left corner of $\rho(\sigma)$.

Like Tate and unlike Stark, our convention will be that G acts on $\mathcal{K}$ on the left, so that $\varepsilon^{\sigma\tau} = \sigma(\tau(\varepsilon))$.

In this regulator, Stark uses a choice of a unit $\varepsilon \in \mathcal{K} \cap \mathbb{R}$ for which the only relation among the conjugates ε^σ is $\prod_{\sigma \in G/G_w} \varepsilon^\sigma = \pm 1$. Such a unit is called a "Minkowski unit," since its existence is guaranteed by [Minkowski 1900].

On the other hand, Tate's regulator $R(\psi, F)$ ([Tate 1984]) is attached to a choice of a $\mathbb{Q}[G]$-isomorphism F. Let $U = U_K$ denote the unit group of K, $\mathbb{Q}U = \mathbb{Q} \otimes_{\mathbb{Z}} U$, $\mathbb{Q}Y$ be the $\mathbb{Q}$-vector space with basis consisting of the infinite primes of K, and $\mathbb{Q}X$ be the subspace of elements whose coordinates in this basis sum up to 0. There exists a $\mathbb{Q}[G]$-isomorphism $F : \mathbb{Q}X \to \mathbb{Q}U$ (by a theorem of Herbrand [Herbrand 1930, Herbrand 1931], in general), and this is used to define $R(\psi, F)$. We will not repeat the construction of this regulator found in [Tate 1984] but wish to note the important connection with Stark's regulator, described on page 41 there. The unique $\mathbb{Q}[G]$-homomorphism from $\mathbb{Q}Y$ to $\mathbb{Q}U$ which sends the fixed infinite prime w of K to ε induces an isomorphism $F_\varepsilon : \mathbb{Q}X \to \mathbb{Q}U$, and

$$R(\psi, F_\varepsilon) = |G_w|^a R(\psi, \varepsilon).$$

Now we make the connection between a regulator of the form $R(\psi,\pi)$, for π in $\mathcal{K}$ and the conjecture considered in this paper. So assume that ρ is a 2-dimensional irreducible representation which is odd. As seen above, we may take $\rho(\tau) = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. So we now have $a=1$, and $\rho_1(\sigma)$ is the single entry in the top left corner of $\rho(\sigma)$. Using the fact that the absolute value $\|\ \|$ is fixed by τ, it follows that

$$
\begin{aligned}
R(\psi,\pi) &= \sum_{\sigma\in G}\rho_1(\sigma)\log\|\pi^\sigma\| = \frac{1}{2}\sum_{\sigma\in G}\mathrm{Tr}\left(\left(\begin{smallmatrix}2 & 0\\ 0 & 0\end{smallmatrix}\right)\rho(\sigma)\right)\log\|\pi^\sigma\| \\
&= \frac{1}{2}\sum_{\sigma\in G}\mathrm{Tr}\left(\left\{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)+\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\right\}\rho(\sigma)\right)\log\|\pi^\sigma\| \\
&= \frac{1}{2}\sum_{\sigma\in G}\mathrm{Tr}\left(\{\rho(1)+\rho(\tau)\}\rho(\sigma)\right)\log\|\pi^\sigma\| \\
&= \frac{1}{2}\sum_{\sigma\in G}\mathrm{Tr}\left(\rho(\sigma)+\rho(\tau\sigma)\right)\log\|\pi^\sigma\| \\
&= \frac{1}{2}\sum_{\sigma\in G}\left(\mathrm{Tr}(\rho(\sigma))+\mathrm{Tr}(\rho(\tau\sigma))\right)\log\|\pi^\sigma\| \\
&= \frac{1}{2}\sum_{\sigma\in G}\left(\psi(\sigma)+\psi(\tau\sigma)\right)\log\|\pi^\sigma\| \\
&= \frac{1}{2}\sum_{\sigma\in G}\left(\psi(\sigma)\log\|\pi^\sigma\|+\psi(\tau\sigma)\log\|\pi^{\tau\sigma}\|\right) \\
&= \sum_{\sigma\in G}\psi(\sigma)\log\|\pi^\sigma\|
\end{aligned}
$$

This computation reconciles the difference in appearance between equations in [Stark 1975] and [Stark 1981] (see the comment in the middle of page 263 of [Stark 1981]).

2.3 Stark's Non-Abelian Question

The preceding computation relates specifically to the following question of Stark from [Stark 1981].

Question 2.1 (Stark). *Suppose that $\mathcal{K}$ is a complex Galois extension of $\mathbb{Q}$ with group G, and W is the number of roots of unity in $\mathcal{K}$. Fix a rational integer f divisible by the conductor of every character ψ of G which corresponds to an odd irreducible representation ρ of dimension 2, and let $L(s,\psi,f)$ be the Artin L-function of ρ with the Euler factors at primes dividing f removed. Is there an algebraic integer π in $\mathcal{K}$ such that*

1. *π^σ/π is a unit for each $\sigma\in G$, and some power of π is real,*

2. π^σ/π^p *is a* W*th power in* $\mathcal{K}$ *whenever* p *is a prime not dividing* Wf *times the discriminant of* $\mathcal{K}$ *and whose associated Frobenius automorphisms are conjugate to* σ *in* G*, and*

3. *For every* ψ *corresponding to an odd irreducible representation of dimension 2, we have*

$$L'(0, \psi, f) = \frac{-1}{2W} \sum_{\sigma \in G} \psi(\sigma) \log ||\pi^\sigma||? \tag{2.1}$$

Remark 1. *From our preceding discussion, one can see that condition (2.1) of Question 2.1 does indeed refine the conjecture of [Stark 1975], specifying that*

$$L'(0, \psi, f) = \frac{-1}{2W} R(\psi, \pi) = \frac{-1}{4W} R(\psi, F_\pi).$$

It is a question "over $\mathbb{Z}$*."*

Remark 2. *Since* $\psi(\sigma\tau) = \psi(\tau\sigma)$ *and* τ *fixes the absolute value* $\|\ \|$*, it is clear that an affirmative answer to the question would imply that*

$$L'(0, \psi, f) = \frac{-1}{4W} \sum_{\sigma \in G} (\psi(\sigma) + \psi(\sigma\tau)) \log \|\pi^\sigma\|.$$

In several cases where $\mathcal{K}$ is a class field of a real quadratic field, Stark has confirmed numerically that his Question has an affirmative answer [Stark 1976, Stark 1980]. But the Question itself does not suggest an effective means of constructing the distinguished element π because it does not provide enough information about the conjugates of π. However, the Stark-Chinburg conjecture does provide such information about a special element of $\mathcal{K}$.

Stark's contribution to this conjecture is as follows. Fix a character ψ as in Stark's Question. The field $E = \mathbb{Q}(\psi)$, obtained by adjoining to $\mathbb{Q}$ all the values of the character ψ, is contained in a cyclotomic extension of $\mathbb{Q}$. We let Γ denote the abelian Galois group of E over $\mathbb{Q}$. Following the formulation of his Question in [Stark 1981], Stark gave an argument which implies the following.

Proposition 2.2. *Suppose the Question has an affirmative answer with* π *real. If* $d \in E$ *has the property that* $\sum_{\gamma \in \Gamma} d^\gamma \psi(\sigma)^\gamma \in \mathbb{Z}$ *for all* $\sigma \in G$*, then there exists a positive real unit* $\varepsilon_f(d) \in \mathcal{K}$ *such that*

$$\sum_{\gamma \in \Gamma} d^\gamma L'(0, \psi^\gamma, f) = -\log(\varepsilon_f(d)) = -\frac{1}{2} \log \|\varepsilon_f(d)\|$$

Indeed $\varepsilon_f(d)$ *can be defined by*

$$\varepsilon_f(d)^W = \prod_{\sigma \in G} \pi^{\sum_{\gamma \in \Gamma} d^\gamma \psi(\sigma)^\gamma \sigma}.$$

Remark 3. *When π is real, and therefore fixed by τ, it is clear that we can also write*

$$\varepsilon_f(d)^{2W} = \prod_{\sigma\in G} \pi^{\sum_{\gamma\in\Gamma} d^\gamma(\psi(\sigma)+\psi(\sigma\tau))^\gamma \sigma}.$$

We now derive a strengthening of Proposition 2.2 which incorporates the conjugates of $\varepsilon_f(d)$ and thus leads to the conjecture formulated by Chinburg. First we record a preliminary step.

Lemma 2.3. *In the setting of Stark's non-abelian question, let σ_0 and σ be elements of G, and let ψ be an odd character of degree 2 corresponding to a representation ρ of G. Then*

$$(\psi(\sigma_0)+\psi(\sigma_0\tau))\,(\psi(\sigma)+\psi(\sigma\tau)) = \psi(\sigma_0\sigma)+\psi(\sigma_0\sigma\tau)+\psi(\sigma_0\tau\sigma)+\psi(\sigma_0\tau\sigma\tau).$$

Proof. We may again assume that $\rho(\tau) = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. Then

$$\begin{aligned}
&(\psi(\sigma_0)+\psi(\sigma_0\tau))\,(\psi(\sigma)+\psi(\sigma\tau))\\
&\quad= \operatorname{Tr}\left(\rho(\sigma_0)(\rho(1)+\rho(\tau))\right)\operatorname{Tr}\left(\rho(\sigma)(\rho(1)+\rho(\tau))\right)\\
&\quad= \operatorname{Tr}\left(\rho(\sigma_0)\left(\begin{smallmatrix}2 & 0\\ 0 & 0\end{smallmatrix}\right)\right)\operatorname{Tr}\left(\rho(\sigma)\left(\begin{smallmatrix}2 & 0\\ 0 & 0\end{smallmatrix}\right)\right)\\
&\quad= (2\rho_1(\sigma_0))(2\rho_1(\sigma)) = \operatorname{Tr}\left(\begin{smallmatrix}2\rho_1(\sigma_0)2\rho_1(\sigma) & 0\\ * & 0\end{smallmatrix}\right)\\
&\quad= \operatorname{Tr}\left(\left(\begin{smallmatrix}2\rho_1(\sigma_0) & 0\\ * & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}2\rho_1(\sigma) & 0\\ * & 0\end{smallmatrix}\right)\right) = \operatorname{Tr}\left(\rho(\sigma_0)\left(\begin{smallmatrix}2 & 0\\ 0 & 0\end{smallmatrix}\right)\rho(\sigma)\left(\begin{smallmatrix}2 & 0\\ 0 & 0\end{smallmatrix}\right)\right)\\
&\quad= \operatorname{Tr}\left(\rho(\sigma_0)\,(\rho(1)+\rho(\tau))\,\rho(\sigma)\,(\rho(1)+\rho(\tau))\right)\\
&\quad= \psi(\sigma_0\sigma)+\psi(\sigma_0\sigma\tau)+\psi(\sigma_0\tau\sigma)+\psi(\sigma_0\tau\sigma\tau)
\end{aligned}$$

□

Proposition 2.4. *Assume that Stark's question has an affirmative answer with π real. Fix $d\in E$ such that $\sum_{\gamma\in\Gamma} d^\gamma\psi(\sigma)^\gamma \in \mathbb{Z}$ for all $\sigma\in G$. Define $\varepsilon_f(d)$ as in Proposition 2.2. Then for each $\sigma_0\in G$, we have*

$$\sum_{\gamma\in\Gamma} \left(d(\psi(\sigma_0)+\psi(\sigma_0\tau))\right)^\gamma L'(0,\psi^\gamma,f) = -\log\|\varepsilon_f(d)^{\sigma_0^{-1}}\|$$

Proof.

$$
\begin{aligned}
&\sum_{\gamma\in\Gamma} \quad (d(\psi(\sigma_0)+\psi(\sigma_0\tau)))^\gamma\, L'(0,\psi^\gamma,f) \\
= \quad & \frac{-1}{4W}\sum_{\gamma\in\Gamma}\sum_{\sigma\in G} d^\gamma\,(\psi(\sigma_0)+\psi(\sigma_0\tau))^\gamma\,(\psi(\sigma)+\psi(\sigma\tau))^\gamma \log||\pi^\sigma|| \quad \text{(by Remark 2)} \\
= \quad & \frac{-1}{4W}\sum_{\gamma}\sum_{\sigma} d^\gamma\,(\psi(\sigma_0\sigma)+\psi(\sigma_0\sigma\tau))^\gamma \log\|\pi^\sigma\| \\
& -\frac{1}{4W}\sum_{\gamma}\sum_{\sigma} d^\gamma\,(\psi(\sigma_0\tau\sigma)+\psi(\sigma_0\tau\sigma\tau))^\gamma \log\|\pi^\sigma\| \quad \text{(by the lemma)} \\
= \quad & \frac{-1}{4W}\sum_{\gamma}\sum_{\sigma} d^\gamma\,(\psi(\sigma)+\psi(\sigma\tau))^\gamma \log\|\pi^{\sigma_0^{-1}\sigma}\| \quad \text{(on replacing } \sigma \text{ by } \sigma_0^{-1}\sigma) \\
& -\frac{1}{4W}\sum_{\gamma}\sum_{\sigma} d^\gamma\,(\psi(\sigma)+\psi(\sigma\tau))^\gamma \log\|\pi^{(\sigma_0\tau)^{-1}\sigma}\| \quad \text{(on replacing } \sigma \text{ by } (\sigma_0\tau)^{-1}\sigma) \\
= \quad & \frac{-2W}{4W}\log\|\varepsilon_f(d)^{\sigma_0^{-1}}\| - \frac{2W}{4W}\log\|\varepsilon_f(d)^{\tau\sigma_0^{-1}}\| \quad \text{by Remark 3} \\
= \quad & -\log\|\varepsilon_f(d)^{\sigma_0^{-1}}\| \quad \text{since the chosen absolute value is fixed by } \tau
\end{aligned}
$$

□

In the Stark-Chinburg conjecture, there is a further restriction on the choice of d which is formulated in terms of the Dirichlet series for $L(s,\psi)$. In return for this restriction, one gains in that the conjecture concerns the primitive Artin L-series without Euler factors removed.

2.4 Dirichlet Series

For this section, we need only assume that $\mathcal{K}/k$ is a finite Galois extension with group G, and that ρ is a representation of G with character ψ. Artin's expression for his L-series involves the choice of a representative Frobenius element $\sigma_\mathfrak{p}\in G$ for each prime ideal $\mathfrak{p}$ of k. That is, one picks a prime ideal $\mathfrak{P}$ of $\mathcal{K}$ above $\mathfrak{p}$ and selects $\sigma_\mathfrak{p}\in G$ which acts as the Frobenius in the corresponding residue field extension. Since ψ is a class function, $\psi(\sigma_\mathfrak{p}^n)$ is well-defined for $\mathfrak{p}$ unramified in K/k. For ramified $\mathfrak{p}$, define $\psi(\sigma_\mathfrak{p}^n)$ to be the average over the coset of $\sigma_\mathfrak{p}^n$ by the appropriate inertia group $I_\mathfrak{p}=I_{\mathfrak{P}/\mathfrak{p}}\subseteq G$. Then for $\Re(s)>1$,

$$
\begin{aligned}
L(s,\rho) &= L(s,\psi)=\exp\left(\sum_{\mathfrak{p}}\sum_{n=1}^{\infty}\frac{\psi(\sigma_\mathfrak{p}^n)}{n\,\mathrm{N}(\mathfrak{p})^{ns}}\right) \\
&= \sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{\mathfrak{p}}\sum_{n=1}^{\infty}\frac{\psi(\sigma_\mathfrak{p}^n)}{n\,\mathrm{N}(\mathfrak{p})^{ns}}\right)^m=\sum_{n=1}^{\infty}\frac{a_n}{n^s},
\end{aligned}
$$

which clearly shows that each a_n lies in $\mathbb{Q}(\psi)$, and that

$$a_p = \psi(\sigma_{\mathfrak{p}})$$

when p is a rational prime which is the norm of a first degree prime $\mathfrak{p}$ of k not ramifying in $\mathcal{K}$.

On the other hand, one can express $L(s,\rho)$ using the eigenvalues $\lambda_{\mathfrak{p},i}$ (listed with multiplicity) for $\rho(\sigma_p)$ acting on the subspace $V_{\mathfrak{p}}$ fixed by the inertia group I_p. These eigenvalues are necessarily roots of unity and hence algebraic integers. Again for $\Re(s) > 1$,

$$\begin{aligned} L(s,\rho) &= \prod_{\mathfrak{p}} \det\left(I - \mathrm{N}(\mathfrak{p})^{-s}\rho(\sigma_p)\mid_{V_{\mathfrak{p}}}\right)^{-1} \\ &= \prod_{\mathfrak{p}}\prod_{i}\left(1 - \lambda_{\mathfrak{p},i}\,\mathrm{N}(\mathfrak{p})^{-s}\right)^{-1} \\ &= \prod_{\mathfrak{p}}\prod_{i}\left(\sum_{j=0}^{\infty}\frac{\lambda_{\mathfrak{p},i}^{j}}{\mathrm{N}(\mathfrak{p})^{js}}\right) \\ &= \sum_{n=1}^{\infty}\frac{a_n}{n^s} \end{aligned}$$

This expression clearly shows that each a_n is an algebraic integer. Thus we have supplied a proof of an important fact.

Proposition 2.5. *For any finite Galois extension $\mathcal{K}/k$ with group G and any representation ρ of G with character ψ, the Artin L-function $L(s,\rho) = L(s,\psi)$ has a Dirichlet series $\sum \frac{a_n}{n^s}$ whose coefficients a_n are algebraic integers lying in $\mathbb{Q}(\psi)$, and $a_p = \psi(\sigma_{\mathfrak{p}})$ when p is a rational prime which is the norm of a first degree prime $\mathfrak{p}$ of k not ramifying in $\mathcal{K}$.*

2.5 The Stark-Chinburg Conjecture

Assume from now on that ρ is an irreducible 2-dimensional odd representation of the group $G = \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$, with associated character ψ. The statement of the conjecture involves the Dirichlet series expansion

$$L(s,\rho) = L(s,\psi) = \sum_{n}\frac{a_n}{n^s}$$

for $\Re(s) > 1$.

For $d \in E = \mathbb{Q}(\psi)$ and $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$, define the function

$$f_d(s) := \sum_{\gamma\in\Gamma} d^{\gamma} L(s,\psi^{\gamma}).$$

Thus $f_d'(0) = \sum_{\gamma\in\Gamma} d^{\gamma} L'(0,\psi^{\gamma})$, which acts as an analog for the primitive L-function of the quantity in Proposition 2.2 in which the Euler factors for the primes dividing f have been removed.

For $\Re(s) > 1$, we also have the expression

$$f_d(s) = \sum_{n\geq 1} A_n n^{-s}$$

with $A_n = \sum_{\gamma\in\Gamma} (da_n)^\gamma = \mathrm{Tr}_{E/\mathbb{Q}}(da_n) \in \mathbb{Q}$.

Conjecture 2.6 (Stark-Chinburg). *Assume that $d \in E$ is such that all the coefficients A_n are in fact rational integers. Then there exists a unit $\varepsilon(d)$ in $\mathcal{K}^+ = \mathcal{K} \cap \mathbb{R}$, the so-called Stark unit, such that, for all $\sigma \in G$*

$$\log \|\varepsilon(d)^{\sigma^{-1}}\| = f'_{d(\psi(\sigma)+\psi(\sigma\tau))}(0).$$

Furthermore, the real conjugates of $\varepsilon(d)$ are positive.

Remark 4. *Note that the unit $\varepsilon(d)$, if it exists, is unique and it is given by the formula*

$$\varepsilon(d) = \exp\left\{ f'_{d(\psi(1)+\psi(\tau))}(0) \right\}.$$

Remark 5. *The condition on d in the conjecture can be restated as the requirement that d lie in the product of the co-different and the inverse of the ideal generated by the coefficients a_n.*

This conjecture is to be compared with Proposition 2.4. It is stronger in as much as the L-functions are primitive. However, in general the conjecture places a slightly greater restriction on the choice of $d \in E$. For the Čebatorev density theorem shows that, given any $\sigma \in G$, there are infinitely many primes p such that $\sigma_p = \sigma$, and Proposition 2.5 then gives $a_p = \psi(\sigma)$ for such a p. But in many cases, including all examples arising as ours do from $G \cong \hat{A}_5$, the set of possible d is the same in the Stark-Chinburg conjecture as it is in Stark's Question. Indeed this happens whenever the character values generate the ring of integers $\mathcal{O}_E$ of E as a $\mathbb{Z}$-module. For Proposition 2.5 also shows that all a_n are in the set $\mathcal{O}_E$, and we can easily deduce the following result.

Proposition 2.7. *Suppose $d \in E$, and let $\mathcal{D}(E)^{-1}$ be the co-different of E, defined by $\mathcal{D}(E)^{-1} = \{d \in E \mid \mathrm{Tr}_{E/\mathbb{Q}}(da) \in \mathbb{Z}\ \forall a \in \mathcal{O}_E\}$. Then*

$$d \in \mathcal{D}(E)^{-1} \Rightarrow A_n = \mathrm{Tr}_{E/\mathbb{Q}}(da_n) \in \mathbb{Z}\ \forall n \Rightarrow \mathrm{Tr}_{E/\mathbb{Q}}(d\psi(\sigma)) \in \mathbb{Z}\ \forall \sigma \in G.$$

Also, if the values of $\psi(\sigma)$ as σ ranges over G generate $\mathcal{O}_E$ as a $\mathbb{Z}$-module, then the third condition implies the first, so all three are equivalent.

To summarize, the Stark-Chinburg conjecture is a precise conjecture "over $\mathbb{Z}$" designed to be a close analog for primitive L-functions of a consequence of an affirmative answer to Stark's non-abelian question for imprimitive L-functions. It should be noted that we have not stated the most general form of the conjecture formulated by Chinburg in [Chinburg 1983]. The conjecture there applies to a finite linear combination of L-functions for odd irreducible 2-dimensional characters such that the resulting Dirichlet series has integral coefficients. For the specific group G used in the computations in this paper, all four characters of the appropriate type are conjugate, and the two conjectures are equivalent in this case.

3 Icosahedral representations

3.1 Minimal Icosahedral Representations

In this section, we briefly determine the minimal degree of an extension $\mathcal{K}/\mathbb{Q}$ supporting an icosahedral representation of the type appearing in Chinburg's conjecture, namely one for which the associated Artin L-function $L(s,\rho)$ has a first order zero at $s=1$.

So suppose that ρ is an odd icosahedral representation of $G = \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$. This means that the image of $\rho(G)$ in $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to the alternating group $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$, which has order 60 and can also be identified with the group of symmetries of the icosahedron. By minimality, we may assume that ρ is faithful, i.e. has trivial kernel, so $G \cong \rho(G)$. We have an exact sequence

$$1 \to A \to \rho(G) \to A_5 \to 1,$$

where the kernel A may be described as $\rho(G) \cap \mathbb{C}^*$, and is therefore a finite cyclic group lying in the center of $\rho(G)$. We seek the minimal possible order for A.

The kernel A cannot have order 1, because A_5 has no irreducible representation of degree 2.

Let C_n denote the cyclic group of order $|C_n| = n$ and V_4 denote a 2-Sylow subgroup of A_5, isomorphic to the Klein 4-group. If A has order 2, then $\rho(G)$ represents an element of $H^2(A_5, C_2)$, which we will show is isomorphic to C_2. The group $H^2(A_5, C_2)$ has exponent 2 since C_2 does. Then by the restriction map, $H^2(A_5, C_2) = H^2(A_5, C_2)_2 \subset H^2(V_4, C_2)$. This last group classifies central extensions of V_4 by C_2, of which there are 8: one trivial class represented by C_2^3, three classes represented by $C_2 \times C_4$, three classes represented by D_8, and one class represented by Q_8. Of these, the classes fixed by the action of A_5 actually constitute $H^2(A_5, C_2)$, by [Brown 1994, III.10.3]. There are two fixed classes: those of C_2^3 and Q_8. This gives two possibilities for $G \cong \rho(G)$ when $|A| = 2$, namely $G \cong A_5 \times C_2$ and $G \cong \mathrm{SL}_2(\mathbb{F}_5)$. Each irreducible character of $A_5 \times C_2$ is obtained as the product of an irreducible character of A_5 and an irreducible character of C_2, and thus cannot be of degree 2.

On the other hand, $\mathrm{SL}_2(\mathbb{F}_5)$ admits two conjugate characters of degree 2, but these have value -2 on each element of order 2 (see [Buhler 1978, p. 135] for the character table). Hence $r(\rho) = 0$, in violation of the assumption $r(\rho) = 1$. So $|A| = 2$ is impossible.

If $|A| = 3$ then $\rho(G)$ represents an element of $H^2(A_5, C_3) = 0$. This time, the group $H^2(A_5, C_3)$ has exponent 3 and is isomorphic to a subgroup of $H^2(C_3, C_3)$. The latter group classifies central extensions of C_3 by C_3 of which there are: one trivial class represented by $C_3 \times C_3$ and two classes represented by C_9. Thus $H^2(C_3, C_3)$ has order 3, while the only class fixed by the action of A_5 is the trivial one. So we must have $G \cong A_5 \times C_3$. Again the irreducible characters of this group are products and so none are of degree 2.

Thus the minimal order for A is at least 4, and A must be cyclic. The case of $|A| = 4$ is indeed realized by $G \cong \mathrm{ESL}_2(\mathbb{F}_5) = \{M \in \mathrm{GL}_2(\mathbb{F}_5) : \det(M) = \pm 1\}$, for which an odd irreducible representation ρ exists. One can show that there is only one equivalence class of extensions of A_5 by C_4; a representative of this equivalence class is usually denoted by $\hat{A}_5$. (Just check that $Q_8 \times C_2$ represents the only non-trivial class in $H^2(V_4, C_4)$ which is fixed by the action of A_5.) We have chosen a convenient realization $\mathrm{ESL}_2(\mathbb{F}_5) \cong \hat{A}_5$.

3.2 $\hat{A}_5$ extensions

From now on, we assume that $\mathcal{K}/\mathbb{Q}$ is a Galois extension of group G isomorphic to $\hat{A}_5$. As mentioned in the last section, the group $\hat{A}_5$, hence also G, can be identified with the group $\mathrm{ESL}_2(\mathbb{F}_5)$ of 2×2 matrix with coefficients in $\mathbb{F}_5$ and determinant ± 1. This group is generated by the two matrices: $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right)$. We choose such an identification of G so that $\tau = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (recall that τ is the complex conjugation in G). The center of G is cyclic of order 4 and generated by $z = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$.

The group G has 18 conjugacy classes, listed in the following table. For each conjugacy class $\mathcal{C}$, we list its cardinality $|\mathcal{C}|$, an representative element $\varkappa \in \mathcal{C}$, the order of $\varkappa$ and the values $\psi(\varkappa)$ and $\chi(\varkappa)$ where ψ is the character of ρ and χ is the abelian character obtained by composing ρ with the determinant map.

#	1	2	3	4	5	6	7	8	9
$\lvert\mathcal{C}\rvert$	1	30	12	20	12	20	30	20	12
$\varkappa$	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	τ	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 4 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 3 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 4 & 0 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 4 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 4 & 2 \end{smallmatrix}\right)$
order	1	2	20	12	20	12	4	6	5
$\psi(\varkappa)$	2	0	$-i\frac{1+\sqrt{5}}{2}$	$-i$	$i\frac{1+\sqrt{5}}{2}$	i	0	1	$\frac{\sqrt{5}-1}{2}$
$\chi(\varkappa)$	1	-1	-1	-1	-1	-1	1	1	1

#	10	11	12	13	14	15	16	17	18
$\lvert\mathcal{C}\rvert$	20	12	12	12	12	12	1	1	1
$\varkappa$	$\left(\begin{smallmatrix} 0 & 1 \\ 4 & 4 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 1 \\ 4 & 3 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 2 \\ 2 & 2 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 2 \\ 2 & 3 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 2 \\ 3 & 1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & 2 \\ 3 & 4 \end{smallmatrix}\right)$	z	z^2	z^3
order	3	10	5	10	20	20	4	2	4
$\psi(\varkappa)$	-1	$\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$i\frac{\sqrt{5}-1}{2}$	$i\frac{1-\sqrt{5}}{2}$	$-2i$	-2	$2i$
$\chi(\varkappa)$	1	1	1	1	-1	-1	-1	1	-1

The field E generated over $\mathbb{Q}$ by the values of ψ is $\mathbb{Q}(i, \sqrt{5})$, an integral basis of E is given by $\left\{1, i, \frac{1+\sqrt{5}}{2}, i\frac{1+\sqrt{5}}{2}\right\}$, and its Galois group Γ is generated by γ_1 and γ_2 with

$$\begin{aligned} \gamma_1(i) &= -i, \quad \gamma_1(\sqrt{5}) = \sqrt{5} \\ \gamma_2(i) &= i, \qquad \gamma_2(\sqrt{5}) = -\sqrt{5} \end{aligned}$$

A $\mathbb{Z}$-basis of $\mathcal{D}(E)^{-1}$ is $\left\{\frac{1}{2}, \frac{i}{2}, \frac{5+\sqrt{5}}{20}, i\frac{5+\sqrt{5}}{20}\right\}$. If d_1 and d_2 are two elements of $\mathcal{D}(E)^{-1}$ for which Conjecture 2.6 is true, so the units $\varepsilon(d_1)$ and $\varepsilon(d_2)$ exist, then the Conjecture is also true of md_1 ($m \in \mathbb{Z}$) and $d_1 + d_2$ simply by taking

$$\varepsilon(md_1) = \varepsilon(d_1)^m \text{ and } \varepsilon(d_1 + d_2) = \varepsilon(d_1)\varepsilon(d_2)$$

Hence, Conjecture 2.6 is true for all $d \in \mathcal{D}(E)^{-1}$ if and only if it is true for $d = \frac{1}{2}$, $\frac{i}{2}$, $\frac{5+\sqrt{5}}{20}$, and $i\frac{5+\sqrt{5}}{20}$.

Moreover, one can readily prove that $i\psi(\sigma) = \psi(z^3\sigma)$ for all $\sigma \in G$. Thus

$$f_{id(\psi(\sigma)+\psi(\sigma\tau))}(s) = f_{d(\psi(z^3\sigma)+\psi(z^3\sigma\tau))}(s),$$

and the truth of Conjecture 2.6 for some $d \in E$ implies the truth of the conjecture for di simply by taking

$$\varepsilon(di) = \varepsilon(d)^z.$$

We have proved the following

Lemma 3.1. *Conjecture 2.6 is true if and only if it is true for*

$$d = \frac{1}{2} \text{ and } d = \frac{5+\sqrt{5}}{20}.$$

3.3 Computations of $L'(0,\rho)$

In order to test Conjecture 2.6, we have to compute the value of $L'(0,\rho^\gamma)$ to a high accuracy. For this, we follow the method used by Stark [Stark 1977]. Recall that

$$L(s,\rho) = \sum_{n\geq 1} a_n n^{-s}$$

is the expansion as a Dirichlet series of the L-function for $\Re(s) > 1$. Let C be the conductor of ρ, then the function

$$\xi(s,\rho) = \left(\frac{C}{4\pi^2}\right)^{s/2} \Gamma(s) L(s,\rho)$$

satisfies the functional equation

$$\xi(s,\rho) = w\xi(1-s,\rho^\tau), \tag{3.2}$$

where w, the so-called Artin Root Number, is a complex number of modulus 1. Let

$$f(t,\rho) = \sum_{n\geq 1} a_n \exp\left(-\frac{2\pi n}{\sqrt{C}} t\right).$$

Then

$$\xi(s,\rho) = \int_0^\infty t^{s-1} f(t,\rho) dt \tag{3.3}$$

for $\Re(s) > 1$, that is ξ is the Mellin transform of f. By the inverse Mellin transform formula, we get

$$f(t,\rho) = \frac{1}{2i\pi} \int_{\Re(s)=\sigma} \xi(s,\rho) t^{-s} ds,$$

where σ is a real number > 1. Now, we have

$$\begin{aligned}
f(t^{-1},\rho) &= \frac{1}{2i\pi}\int_{\Re(s)=\sigma}\xi(s,\rho)t^s ds \\
&= \frac{1}{2i\pi}\int_{\Re(s)=1-\sigma}\xi(1-s,\rho)t^{1-s}ds \\
&= \frac{w^{-1}t}{2i\pi}\int_{\Re(s)=1-\sigma}\xi(s,\rho^\tau)t^{-s}ds \quad \text{(by the functional equation).}
\end{aligned}$$

We now assume that $\xi(s,\rho^\tau)$ is holomorphic over $\mathbb{C}$ (actually, we only need that there exists $\epsilon > 0$ such that $\xi(s,\rho^\tau)$ is holomorphic on the half-plane $\Re(s) > -\epsilon$, but, by the functional equation (3.2), this is equivalent to the holomorphy of $\xi(s,\rho^\tau)$ over $\mathbb{C}$). Then, we have for two numbers $\sigma_1 < \sigma_2$, see for example [Cohen 2000], Lemma 10.3.5:

$$\lim_{|T|\to\infty}\int_{\sigma_1+iT}^{\sigma_2+iT}\xi(s,\rho^\tau)t^{-s}ds = 0,$$

and thus we obtain

$$\frac{1}{2i\pi}\int_{\Re(s)=1-\sigma}\xi(s,\rho^\tau)t^{-s}ds = \frac{1}{2i\pi}\int_{\Re(s)=\sigma}\xi(s,\rho^\tau)t^{-s}ds = f(t,\rho^\tau)$$

and so

$$f(t^{-1},\rho) = w^{-1}tf(t,\rho^\tau). \tag{3.4}$$

For any fixed $u > 0$, we break the integral in (3.3) into two pieces: the first one from 0 to u, the second one from u to ∞. In the first integral, we replace t by t^{-1} and use (3.4) to get:

$$\xi(s,\rho) = w^{-1}\int_{u^{-1}}^{\infty}t^{-s}f(t,\rho^\tau)dt + \int_u^\infty t^{s-1}f(t,\rho)dt.$$

So finally

$$\begin{aligned}
L'(0,\rho) &= \xi(0,\rho) \\
&= w^{-1}\int_{u^{-1}}^{\infty}f(t,\rho^\tau)dt + \int_u^\infty f(t,\rho)\frac{dt}{t} \\
&= \frac{w^{-1}\sqrt{C}}{2\pi}\sum_{n\geq 1}\frac{\overline{a_n}}{n}\exp\left(-\frac{2\pi n}{u\sqrt{C}}\right) + \sum_{n\geq 1}a_n\int_u^\infty \exp\left(-\frac{2\pi n}{\sqrt{C}}t\right)\frac{dt}{t}.
\end{aligned}$$

We have proved the following result:

Proposition 3.2. *Assume the L-function $L(s,\rho)$ is holomorphic over $\mathbb{C}$. Then there exists a complex number w of modulus 1, such that, for any $u > 0$*

$$L'(0,\rho) = \frac{w^{-1}\sqrt{C}}{2\pi}\sum_{n\geq 1}\frac{\overline{a_n}}{n}\exp\left(-\frac{2\pi n}{u\sqrt{C}}\right) + \sum_{n\geq 1}a_n\mathrm{Ei}\left(\frac{2\pi un}{\sqrt{C}}\right)$$

where $\mathrm{Ei}(x) = \int_x^{+\infty} e^{-t} dt/t$ *is the exponential integral function.*

Remark 6. *The proposition requires the hypothesis that $L(s,\rho)$ is holomorphic over $\mathbb{C}$, that is, that ρ satisfies the Artin conjecture. This conjecture is not proved in general, but it has been proved in all examples we work with in this paper using either results of Kiming and Wang [Kiming and Wang 1994], or the work of Buzzard, Dickinson, Shepherd-Baron and Taylor which proves the Artin conjecture for infinitely many icosahedral odd representations [Buzzard et al. 2001], or results of Jehanne and Müller [Jehanne and Müller 2000, Jehanne and Müller 2001].*

The formula given by Proposition 3.2 can be used to compute approximations of $L'(0,\rho)$, or of $L'(0,\rho^\gamma)$, if one knows how to compute the coefficients a_n and the value of Artin Root Number w. The former can be computed using the method of [Jehanne 2001] (see also the next section), the latter can be found (following Stark) by computing the two sums in the formula for two different values of u and solving the system. This also provides a neat check of the computations since the complex number found must be of modulus 1.

3.4 Construction of $\hat{A}_5$ extensions

In this section, we explain how one can construct $\hat{A}_5$ extensions with an odd irreducible degree 2 representation with a *quadratic determinant*, that is representation whose determinant is a quadratic character. Any such extension $\mathcal{K}$ contains a quintic field K of A_5-type. Furthermore, since the representation is odd, K is a complex field. A table of quintic complex fields of A_5-type of discriminant less than 4027^2 has been computed by J. Basmaji [Basmaji 2002] using the methods of [Basmaji and Kiming 1994].

Such a A_5-type complex quintic field K yields two projective representations corresponding to the two embeddings of A_5 in $\mathrm{PGL}_2(\mathbb{C})$. Let ρ_{proj} be one of these two representations; methods to decide whether or not ρ_{proj} has a lifting ρ of given conductor and determinant are described in [Crespo 1992] or [Jehanne 2001]. Assume from now on such a lifting exists and call it ρ. Then the set of all liftings of ρ_{proj} is

$$\mathcal{E}(\rho) = \{\rho \otimes \nu \text{ with } \nu \text{ a Dirichlet character of } G_{\mathbb{Q}}\}.$$

Some of these $\rho\otimes\nu$ may have the same conductor as ρ. Also, $\det(\rho\otimes\nu) = \det\rho$ if and only if $\nu^2 = 1$. In particular, if $\det\rho$ is quadratic, then $\rho\otimes\det\rho$ has the same conductor and the same determinant as ρ.

By looking at a table of irreducible characters of $\hat{A}_5$ (which can be easily constructed from [Buhler 1978] p. 135), we see that there are four characters of degree 2, and that they are conjugate under the action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5}, i)/\mathbb{Q})$. The characters of ρ and $\rho\otimes\det\rho$ are conjugate by the complex conjugation, and the two other representations correspond to the other embedding of A_5 in $\mathrm{PGL}_2(\mathbb{C})$.

To find all the representations with odd quadratic determinant and conductor up to 3676, we use the table of [Basmaji 2002] and the following result,

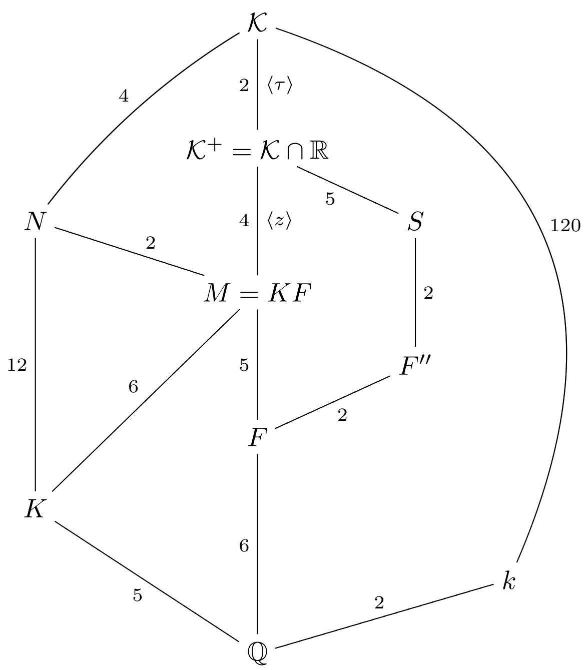

Figure 1: Some subfields of $\mathcal{K}$

obtained by local computations (for the computations of conductors, we refer to [Kiming 1994]).

Proposition 3.3. *Let N be an A_5-extension, let Δ be the discriminant of a quintic field contained in K and let C be the conductor of a representation ρ with quadratic determinant corresponding to N. Then $C \geq \sqrt{\Delta}$.*

Table 1 lists all icosahedral representations with odd quadratic determinant and conductor up to 3676^2. For each representation ρ, we read on this table: the conductor C, a polynomial for the corresponding quintic field, and the square-free integer δ such that $\det(\rho)$ fixes the field $k = \mathbb{Q}(\sqrt{\delta})$.

As mentioned above, we used the method described in [Jehanne 2001] to compute the coefficients of the L-series of the representations . We briefly explain this method (see also Figure 1). Let $f(X)$ be one of the polynomials in Table 1, and let K be the field defined by an arbitrary (fixed) complex root x_1 of f. Thus K is a complex quintic field of A_5-type. Let N be the Galois closure of $K/\mathbb{Q}$, so $\mathrm{Gal}(N/\mathbb{Q}) \simeq A_5$, and let $x_2, \ldots, x_5 \in N$ be the other roots of f. Define θ by

$$\theta = (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_4)^2(x_4 - x_5)^2(x_5 - x_1)^2$$

and let $F = \mathbb{Q}(\theta)$. The field F is a degree 6 field with Galois closure N. The degree 30 field $M = KF$ will be play an important part in the computations (see

C	polynomial defining K	δ
1948	$x^5 - 7x^3 - 17x^2 + 18x + 73$	-487
2083	$x^5 + 8x^3 + 7x^2 + 172x + 53$	-2083
2336	$x^5 + 2x^3 - 4x^2 - 2x + 4$	-73
2336	$x^5 + 2x^3 - 4x^2 - 2x + 4$	-146
2707	$x^5 - x^4 + 9x^3 - 6x^2 - 32x + 93$	-2707
2863	$x^5 + 12x^3 + 21x^2 + 22x + 7$	-409
2863	$x^5 + 12x^3 + 21x^2 + 22x + 7$	-2863
3004	$x^5 - 8x^3 + 10x^2 + 160x + 128$	-751
3203	$x^5 + 8x^3 + 5x^2 - 4x + 1$	-3203
3547	$x^5 - 8x^3 - 2x^2 + 31x + 74$	-3547
3548	$x^5 + 10x^3 + 10x^2 + 44x + 56$	-887
3587	$x^5 + 3x^3 + 24x^2 - 20x + 131$	-311
3587	$x^5 + 3x^3 + 24x^2 - 20x + 131$	-3587
3676	$x^5 - 8x^3 + 28x^2 - 40x + 48$	-919

Table 1: The first icosahedral representations with odd quadratic determinant

next section). Then we define $F'' = \mathbb{Q}(\sqrt{\theta\delta})$. In [Jehanne 2001], it is proved that there exists a quadratic extension S of F'' such that its Galois closure is the field $\mathcal{K}$ we are looking for. Since we know the conductor, the determinant and the image in $\mathrm{PGL}_2(\mathbb{C})$ of ρ, we know the ramification of S/F'' and thus can construct a finite set B of elements of F'' such that $S = F''(\sqrt{\beta})$ for some $\beta \in B$. We then use an explicit criterion to decide which element is the right one. Once the field S has been found, we can use the explicit decomposition of prime ideals in S (and possibly also some other subfields of $\mathcal{K}$) to compute the coefficients a_n of the L-function of ρ.

4 Computations

4.1 Numerical determination of the Stark unit

Let d be a fixed element of $\mathcal{D}(E)^{-1}$. In this section we explain how to find the conjectural Stark unit $\varepsilon(d)$, assuming from now on that it exists, using numerical approximations. (Actually, according to lemma 3.1, we have only performed these computations for $d = \frac{1}{2}$ and $d = \frac{5+\sqrt{5}}{20}$.)

As mentioned in the introduction, we find $\varepsilon(d)$ by constructing its minimal polynomial over the field M. This field has degree 30 and signature $(2, 14)$. Also, it is a subfield of $\mathcal{K}^+$. More precisely it is the subfield of $\mathcal{K}^+$ fixed by z, that is to say by the center of G. Since $\varepsilon(d)$ is real, its conjugates over M, *i.e.* $z^l(\varepsilon(d))$ with $0 \le l \le 3$, are real too and positive by the second part of the

Conjecture. Thus they are given by the formula

$$z^l(\varepsilon(d)) = \exp\left\{f'_{d(\psi(z^l)+\psi(z^l\tau))}(0)\right\}$$

for $0 \leq l \leq 3$.

Hence, we can compute approximations of $L'(0,\rho)$ and get from them approximations of the conjugates of $\varepsilon(d)$ over M, and then form the monic polynomial $\tilde{P}$ whose roots are these approximations. This polynomial is thus an approximation of the minimal polynomial $P(X)$ of $\varepsilon(d)$ over M. Now, since $\psi(z^3\sigma) = i\psi(\sigma)$, it follows that $\psi(z^2\sigma) = -\psi(\sigma)$, and thus $z^2(\varepsilon(d)) = \varepsilon(d)^{-1}$, $z^3(\varepsilon(d)) = z(\varepsilon(d))^{-1}$, and one can write

$$P(X) = X^4 + aX^3 + bX^2 + aX + 1$$

with $a, b \in \mathcal{O}_M$. We now need to be able to recover the coefficients a and b from the corresponding coefficients $\tilde{a}$ and $\tilde{b}$ of $\tilde{P}$.

The problem can be stated in a more general setting as follows: given a real number $\tilde{x}$, and two positive real numbers C_1 and C_2, find, if it exists, an algebraic integer x in $\mathcal{O}_M$ such that

$$|\tilde{x} - x| < C_1 \quad \text{and} \quad |x'| < C_2 \text{ where } x' \text{ is any conjugate of } x\ (\neq x). \tag{4.5}$$

Note that here and it what follows, when we talk about the conjugates of x, we mean the conjugates of x *different* from x. It is not difficult to show that if we choose C_1 small enough, then we can make sure that there is at most one algebraic integer in M satisfying these conditions.

The method we used is a generalization of a method due to H. Cohen (see [Cohen 2000, section 6.2.4]). Let r_1, r_2 be the two real embeddings of M, with r_1 being the identity, and let $c_1, \ldots, c_{14}$ be a complete fixed set of non-conjugate complex embeddings of M. For $y \in M$ and $1 \leq l \leq 30$, we define

$$y^{(l)} = \begin{cases} r_l(y) & \text{if } l = 1 \text{ or } 2 \\ \Re(c_{l-2}(y)) + \Im(c_{l-2}(y)) & \text{if } 3 \leq l \leq 16 \\ \Re(c_{l-16}(y)) - \Im(c_{l-16}(y)) & \text{if } 17 \leq l \leq 30 \end{cases}$$

Let $v(y)$ be the 30-dimensional vector whose l-component is $y^{(l)}$. Then the map from M to $\mathbb{R}^{30}$ which sends y to $v(y)$ sends the ring of integers $\mathcal{O}_M$ to a full lattice in $\mathbb{R}^{30}$ of determinant the absolute value of the discriminant of M.

We will need the following lemma whose proof is direct.

Lemma 4.1. *Let $x \in M$ and assume that all the conjugates of x have absolute value less than C_2. Then*

$$\left|x^{(2)}\right| < C_2 \quad \text{and} \quad \left|x^{(l)}\right| < \sqrt{2}C_2 \text{ for } 3 \leq l \leq 30.$$

In the reverse direction, if

$$\left|x^{(l)}\right| < C_2 \text{ for } 2 \leq l \leq 30$$

then all the conjugates of x have absolute value less than C_2.

Let $\{\omega_1, \ldots, \omega_{30}\}$ be an integral basis of $\mathcal{O}_M$. For a fixed real number $\tilde{x}$, consider the following quadratic form on $\mathbb{Z}^{31}$:

$$Q(v_0, v_1, \ldots, v_{30}) = C_2^2 v_0^2 + (C_2/C_1)^2 \left(\sum_{j=1}^{30} v_j \omega_j^{(1)} - v_0 \tilde{x} \right)^2 + \sum_{l=2}^{30} \left(\sum_{j=1}^{30} v_j \omega_j^{(l)} \right)^2$$

If $x = x_1\omega_1 + \cdots + x_{30}\omega_{30} \in \mathcal{O}_M$ is a solution of (4.5) then

$$\begin{aligned} Q(1, x_1, \ldots, x_{30}) &< C_2^2 + (C_2/C_1)^2 (x - \tilde{x})^2 + \left(x^{(2)}\right)^2 + \sum_{l=3}^{30} \left(x^{(l)}\right)^2 \\ &< C_2^2 + C_2^2 + C_2^2 + 2 \sum_{l=3}^{30} C_2^2 = 59 C_2^2 \end{aligned}$$

Conversely, let $(x_0, \ldots, x_{30}) \in \mathbb{Z}^{31}$ be such that

$$Q(x_0, \ldots, x_{30}) < 59 C_2^2.$$

Then $C_2^2 x_0^2 < 59 C_2^2$ so $|x_0| \leq 7$. If x_0 is actually equal to ± 1, then we can set

$$x^* = x_0 \left(\sum_{j=1}^{30} x_j \omega_j \right) \in \mathcal{O}_M.$$

and x^* satisfies $|x^* - \tilde{x}| < \sqrt{59} C_1$ and all its conjugates are of absolute value less than $\sqrt{59} C_2$. Therefore, solutions to

$$Q(x_0, \ldots, x_{30}) < 59 C_2^2 \tag{4.6}$$

with $x_0 = \pm 1$ are *not too far* from being solutions to our original problem, and there are only a small number of solutions to (4.6) when C_1 is small enough.

In order to find solutions to (4.5), one can use the Fincke-Pohst algorithm [Cohen 1993, 2.7.3] to find solutions to (4.6), then discard those for which $x_0 \neq \pm 1$ (or even better, modify the algorithm in such a way that it only considers vectors with $x_0 = \pm 1$). Then for each solution found (with $x_0 = \pm 1$), compute the corresponding algebraic integer and check whether or not it satisfies the stronger conditions of (4.5).

In practice, this method works very well for small enough values of C_1 and gives only a few vectors satisfying (4.6), only one of those satisfying (4.5). For information, the size of C_1, that is the precision used, was between 10^{-100} and 10^{-200} for most examples, with a precision up to 400 decimal places in one case ($N = 3004$). However, the computations used to compute the value of the L-functions was higher, around 600 decimal places.

Once the (conjectural) unit $\varepsilon(d)$ has been found, we need to check that is satisfies the conjecture. Of course, one part of the check involves testing whether or not two real numbers, given by approximations, are equal, which is an impossible computational task. So we will not be able to prove the conjecture in these cases but only to give evidence pointing toward the truth of the conjecture. These checks are described in section 4.4 together with an example. We summarize these computations in the following result.

Theorem 4.2. *For the fourteen icosahedral representations with odd quadratic determinant listed in Table 1, Conjecture 2.6 has been numerically verified up to the precision of the computation.*

4.2 Square-root of the Stark unit

In all the examples, we have found that the unit $\varepsilon(d)$ was a square in $\mathcal{K}$. In fact, in almost all examples, it is actually a *fourth* power (see below). We have used this fact to simplify the computation. Indeed, in all examples, we have started by assuming that it was a fourth power, and instead of trying to recognize the coefficients of the minimal polynomial of $\varepsilon(d)$ over M, we searched for the coefficients of the minimal polynomial of $\varepsilon(d)^{1/4}$. In doing so, we always took the *positive* fourth root as conjugates of $\varepsilon(d)^{1/4}$. If we were not able to find those, then we searched for that of the minimal polynomial of $\varepsilon(d)^{1/2}$, assuming again that all the conjugates were positive. As stated above, in all cases, we were able to find these coefficients. Not only does this method prove directly that the unit $\varepsilon(d)$ is a fourth power (resp. a square), but it also greatly simplifies the computations since we had to deal with numbers having one fourth (resp. one half) as many digits! Of course, if we failed to recognize the coefficients of the minimal polynomial of $\varepsilon(d)^{1/4}$ that did not prove that it was not a fourth power, since we arbitrarily decided to consider only the *positive* fourth root. So, in those cases, we did check once the unit had been found that it was not a fourth power.

The following table sums up the information mentioned above. For each conductor N, an entry 2 (resp. 4) means that the unit $\varepsilon(d)$ was a square (resp. a fourth power) in $\mathcal{K}$. We do not specify in the table the value of d ($\frac{1}{2}$ or $\frac{5+\sqrt{5}}{20}$), or the representation (if there are more than one to test of the same conductor) since in all the examples we have found that this property does not depend on these.

N	power	N	power	N	power	N	power
1948	4	2083	4	2336	2	2707	4
2863	4	3004	2	3203	4	3547	4
3548	2	3587	4	3676	2		

4.3 The abelian condition

The so-called abelian rank one Stark conjecture for an abelian extension K_1/K_2 of number fields (see [Tate 1984, Chap. IV]) predicts the existence of a unit $\epsilon \in K_1$ satisfying conditions similar to that of Conjecture 2.6 and such that $\epsilon^{1/e}$ defines an abelian extension of K_2, where e is the number of roots of unity in K_1. A similar condition for the Stark-Chinburg conjecture has not yet been stated. However, following a suggestion made by Stark, we have checked in all fourteen of our examples that the fourth root of the Stark unit always generates an abelian extension of M (of course, this is trivially satisfied when the Stark unit is already of fourth power). We ask the following question:

Question 4.3. *In this setting, is it true that the extension $\mathcal{K}(\varepsilon(d)^{1/4})/M$ is always an abelian extension?*

4.4 An example

$$
\begin{gathered}
X^{120} - 14X^{119} + 10X^{118} + 528X^{117} - 721X^{116} - 12186X^{115} - 1022X^{114} + 272500X^{113} + 511431X^{112} \\
- 5891390X^{111} - 13723194X^{110} + 105936396X^{109} + 206807275X^{108} - 1534364982X^{107} \\
- 2148339186X^{106} + 17880418336X^{105} + 15711872585X^{104} - 168995037912X^{103} \\
- 67266744140X^{102} + 1311505408046X^{101} - 89291738681X^{100} - 8494623508910X^{99} \\
+ 4235000319554X^{98} + 45100336757892X^{97} - 44699345526371X^{96} - 198789238356076X^{95} \\
+ 318871476120728X^{94} + 744008040343916X^{93} - 1682328456079649X^{92} - 2135167290906516X^{91} \\
+ 7510163330929880X^{90} + 4496703792844188X^{89} - 28166525613664065X^{88} - 3766815912764636X^{87} \\
+ 90442638583264834X^{86} - 22239156146839804X^{85} - 255223687887713211X^{84} + 144746005476226278X^{83} \\
+ 637495087229463472X^{82} - 538617388539670046X^{81} - 1405725781756458029X^{80} + 1586294443053165584X^{79} \\
+ 2827558653955188558X^{78} - 3855005474440939128X^{77} - 5224727512198411575X^{76} + 8024368738725683150X^{75} \\
+ 9089153383531781324X^{74} - 14516564995099084468X^{73} - 15279740471791884367X^{72} \\
+ 22886584508143805334X^{71} + 24920013640589593880X^{70} - 31520286306610904124X^{69} \\
- 38951665554067587963X^{68} + 37417793964722184658X^{67} + 56660611016658159176X^{66} \\
- 37074726381332624176X^{65} - 74983502912940337331X^{64} + 27894404791194859718X^{63} \\
+ 89065895937691036822X^{62} - 10418769031801600904X^{61} - 94322409774714428665X^{60} \\
- 10418769031801600904X^{59} + 89065895937691036822X^{58} + 27894404791194859718X^{57} \\
- 74983502912940337331X^{56} - 37074726381332624176X^{55} + 56660611016658159176X^{54} \\
+ 37417793964722184658X^{53} - 38951665554067587963X^{52} - 31520286306610904124X^{51} \\
+ 24920013640589593880X^{50} + 22886584508143805334X^{49} - 15279740471791884367X^{48} \\
- 14516564995099084468X^{47} + 9089153383531781324X^{46} + 8024368738725683150X^{45} - 5224727512198411575X^{44} \\
- 3855005474440939128X^{43} + 2827558653955188558X^{42} + 1586294443053165584X^{41} - 1405725781756458029X^{40} \\
- 538617388539670046X^{39} + 637495087229463472X^{38} + 144746005476226278X^{37} - 255223687887713211X^{36} \\
- 22239156146839804X^{35} + 90442638583264834X^{34} - 3766815912764636X^{33} - 28166525613664065X^{32} \\
+ 4496703792844188X^{31} + 7510163330929880X^{30} - 2135167290906516X^{29} - 1682328456079649X^{28} \\
+ 744008040343916X^{27} + 318871476120728X^{26} - 198789238356076X^{25} - 44699345526371X^{24} \\
+ 45100336757892X^{23} + 4235000319554X^{22} - 8494623508910X^{21} - 89291738681X^{20} \\
+ 1311505408046X^{19} - 67266744140X^{18} - 168995037912X^{17} + 15711872585X^{16} + 17880418336X^{15} \\
- 2148339186X^{14} - 1534364982X^{13} + 206807275X^{12} + 105936396X^{11} - 13723194X^{10} - 5891390X^{9} \\
+ 511431X^{8} + 272500X^{7} - 1022X^{6} - 12186X^{5} - 721X^{4} + 528X^{3} + 10X^{2} - 14X + 1
\end{gathered}
$$

Figure 2: The irreducible polynomial over $\mathbb{Q}$ of $\varepsilon(\frac{5+\sqrt{5}}{20})^{1/4}$

We conclude with an example of a computation. We will look at the representation of conductor $N = 2863$ and with determinant the quadratic character of the field $\mathbb{Q}(\sqrt{-409})$, and the value $d = \frac{5+\sqrt{5}}{20}$. (This example is the one for which the irreducible polynomial over $\mathbb{Q}$ of the Stark unit has the smallest coefficients.) In what follows we will write ε instead of $\varepsilon(\frac{5+\sqrt{5}}{20})$ to denote the Stark unit.

First, we compute the field F using the explicit formula for θ and then find that the field M is generated over $\mathbb{Q}$ by a (fixed) real root of the polynomial:

$$\begin{aligned}X^{30} - 11X^{29} + 60X^{28} - 184X^{27} + 282X^{26} - 93X^{25} + 1155X^{24} - 15102X^{23}\\ + 81876X^{22} - 295153X^{21} + 824690X^{20} - 1918902X^{19} + 3838834X^{18}\\ - 6617268X^{17} + 9651756X^{16} - 11548871X^{15} + 10886632X^{14}\\ - 7709825X^{13} + 3980211X^{12} - 1749801X^{11} + 1033046X^{10}\\ - 526435X^{9} - 55897X^{8} + 112042X^{7} + 213353X^{6} - 221284X^{5}\\ - 31311X^{4} + 78204X^{3} + 4802X^{2} - 12005X - 2401\end{aligned}$$

We compute the values of $f'_{d(\psi(g)+\psi(g\tau))}(0)$ for all $g \in G$ with a precision of 620 decimal places for $g \in \langle z \rangle$ and of 100 decimal places for $g \notin \langle z \rangle$. Using the (positive) fourth root of these values, we find that, if these choices are correct, then $\varepsilon^{1/4}$ must be a root of the following polynomial which must have coefficients in $\mathcal{O}_M$ if the fourth root belongs to $\mathcal{K}^+$: †

$$\begin{aligned}X^4 - 11.073358292740063818493289707579 6398...X^3\\ + 26.453851797607365861412492238042 8030...X^2\\ - 11.073358292740063818493289707579 6398...X + 1.\end{aligned}$$

Also, we find that the other conjugates of the coefficient a of X^3 (which is also that of X) are bounded in absolute value by 14, and those of the coefficient b of X^2 are bounded by 44. These bounds are also found by using Conjecture 2.6. Using the method explained above, we recognize the two coefficients a and b using $C_1 = 10^{-120}$. We will not list those since each one would require several pages just to write it down! However, once these coefficients have been found, we construct the corresponding polynomial and compute its roots to a precision of 600 decimal digits. We then check than these values agree with the one computed *via* the conjecture. This is already a first good check since only a precision of 120 decimal digits was used to recover the coefficients. (Actually, the first good check is that there are indeed elements a and b in $\mathcal{O}_M$ satisfying the conditions we imposed.)

We then compute the irreducible polynomial of $\varepsilon^{1/4}$ over $\mathbb{Q}$; it is a degree 120 polynomial with quite big coefficients even though we are only looking at the fourth root of the Stark unit (see Figure 2). We compute its roots $(e_i)_{1\leq i\leq 120}$ to a precision of 100 decimal digits. We then look for a one-to-one correspondence between the absolute values of the e_i's and the values of $\exp\{f'_{d(\psi(g)+\psi(g\tau))}(0)\}$, for g in a set of representatives of $G/\langle\tau\rangle$, such that corresponding values agree up to the precision. Such a correspondence must exist if the conjecture is true and, indeed, we find that it does. This is also quite a good check since we used the values of the other conjugates of ε which are not conjugates over M *only*

† In this example, we will, of course, give all the numerical results with a much smaller precision than the one used in the computations.

through the upper bound that they provide on the conjugates of the coefficients a and b. Actually, if the conjecture is true, this correspondence should give us *explicitly* the Galois action of G on the conjugates of the Stark unit. However, it is not practical to recover this (conjectural) correspondence by trying to match the values of the absolute values of the conjugates with the values predicted by the conjecture since there are too many conjugates of the Stark unit with the same absolute value and thus too many possible correspondences (in this example, the number of possible correspondences is around 10^{26}).

Finally, we check that the Stark unit generates the field $\mathcal{K}^+$ over $\mathbb{Q}$ in the following way. Recall that the method of [Jehanne 2001] gives us an explicit construction of S. Now, looking at Figure 1, it is clear that $\mathcal{K}^+ = SK = SM$. We find a primitive element α of S over F, so $\mathcal{K}^+ = M(\alpha)$. Next, we compute the compositum field over M of $M(\varepsilon)$ and $M(\alpha)$ using the method of [Cohen 2000, 2.1.3] and find that it has degree 4. Thus $M(\varepsilon) = M(\alpha) = \mathcal{K}^+$.

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