Computation of values of p-adic Dirichlet L-functions using PARI/GP

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Let $f \geq 2$ and let $a, c \in \mathbb{N}$ such that $(a, f) = (c, f) = 1$ and $c > 1$. For $s \in \mathbb{C}$ with $\Re(s) > 1$, set

$$Z_f(a, s) := \sum_{n \equiv a(f)} n^{-s} \text{ and } Z_f(a, c, s) := c^{1-s} Z_f(ac^{-1}, s) - Z_f(a, s).$$

Both functions have continuations to $\mathbb{C}$, with a single pole at $s = 1$ for the first one, and no pole for the second one.

Let χ be a non-trivial Dirichlet character of conductor δ. Let f be a multiple of δ and let c be such that $\chi(c) \neq 1$. Then we have

$$\sum_{\substack{a=1 \\ (a,f)=1}}^{f-1} \chi(a) Z_f(a, c, s) = (c^{1-s}\chi(c) - 1) \prod_{\ell | f} (1 - \chi(\ell)\ell^{-s}) L(\chi, s).$$

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Lemma

For $r \in \mathbb{Q}$, *set* $\xi_r := \exp(2i\pi r)$. *Then*

$$Z_f(a,c,s) = \sum_{i=1}^{c-1} \sum_{n \equiv a(f)} \xi_{n/c}^{i}\, n^{-s}.$$

Let η be a root of unity such that $\eta^f \neq 1$. Define

$$Z_f(a,\eta,s) := \sum_{n \equiv a(f)} \eta^n n^{-s},$$

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$$Z_f(a,c,s) = \sum_{i=1}^{c-1} Z_f(a,\xi_{i/c},s).$$

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Define the following power series in T

$$F_{f,a,\eta}(T) := \frac{\eta^a (1+T)^a}{1 - \eta^f (1+T)^f}$$

and the operator Δ on power series by

$$\Delta := (1+T)\frac{d}{dT}.$$

Theorem

For any $k \geq 0$, we have

$$Z_f(a, \eta, -k) = \Delta^k F_{f,a,\eta}(T)_{|T=0}.$$

False proof. Expand the power series $F_{f,a,\eta}$ in terms of $(1+T)$ (which of course is not possible)

$$F_{f,a,\eta}(T) = \eta^a (1+T)^a \sum_{n \geq 0} \eta^{nf} (1+T)^{nf} = \sum_{n \equiv a (f)} \eta^n (1+T)^n.$$

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For any $k \geq 0$, we have

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A measure μ is a linear form on $\mathcal{C} := \mathcal{C}(\mathbb{Z}_p, \mathbb{C}_p)$ such that there exists $B > 0$ with

$$|\mu(f)| \leq B|f|,$$

where $|f| = \sup_{x \in \mathbb{Z}_p} |f(x)|$. We write $\int f \, d\mu := \mu(f)$.

Any function $f \in \mathcal{C}$ has a unique Mahler expansion

$$f(x) = \sum_{n \geq 0} a_n \binom{x}{n}$$

with $(a_n) \subset \mathbb{C}_p$. Furthermore $\lim a_n = 0$.

Theorem

A linear form μ on $\mathcal{C}$ is a measure iff there exits $B > 0$ such that for all $n \geq 0$

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Measures on $\mathcal{C}$ and power series in $\mathbb{C}_p[[T]]^{bd}$ are in 1-to-1 correspondance by the formula

$$F_\mu(T) := \int (1+T)^x \, d\mu(x) = \sum_{n \geq 0} \int \binom{x}{n} \, d\mu \cdot T^n.$$

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Embed $\overline{\mathbb{Q}} \subset \mathbb{C}_p$, then

$$F_{f,a,c}(T) := \sum_{i=1}^{c-1} F_{f,a,\xi_{i/c}}(T) \in \mathbb{C}_p[[T]]^{bd}.$$

Call $\mu_{f,a,c}$ the measure associated to $F_{f,a,c}$. Then

Theorem

For all $k \geq 0$, we have

$$\int x^k \, d\mu_{f,a,c}(x) = Z_f(a, c, -k).$$

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Recall that $\mathbb{Z}_p^\times := \mathbb{Z}_p \setminus p\mathbb{Z}_p$ splits as $\mathbb{Z}_p^\times = W_p \times \mathcal{U}_p$ with W_p the torsion part and $\mathcal{U}_p = 1 + q\mathbb{Z}_p$. For $x \in \mathbb{Z}_p^\times$, write $\omega(x)$ (resp. $\langle x \rangle$) the projection of x onto W_p (resp. $\mathcal{U}_p$).

Fix $s \in \mathbb{Z}_p$ and $m \in \mathbb{Z}$, then the function

$$\begin{aligned} \varphi_s^{(m)} : \mathbb{Z}_p^\times &\to \mathbb{Z}_p \\ x &\mapsto \omega(x)^m \langle x \rangle^{-s} \end{aligned}$$

is continuous. The function is extended to $\mathbb{Z}_p$ by setting $\varphi_s^{(m)}(x) := 0$ for $x \in p\mathbb{Z}_p$. The function $\varphi_s^{(m)}$ will serve as the "p-adic analogue" of x^{-s}.

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The p-adic twisted partial zeta function (with character ω^m) is defined for $s \in \mathbb{Z}_p$ by

$$Z_{f,p}^{(m)}(a, c, s) := \int \varphi_s^{(m)}(x)\, d\mu_{f,a,c}(x).$$

This is a continous function on $\mathbb{Z}_p$ that depends only upon the class of m modulo $\phi(q)$.

The function

$$Z_{f,p}(a, c, s) := Z_{f,p}^{(-1)}(a, c, s)$$

is *the* p-adic twisted partial zeta function.

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Assume p divides f. Then for all $k \geq 0$ with $k \equiv -1 \pmod{\phi(q)}$

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Let $s \in \mathbb{Z}_p$. Write

$$\varphi_s^{(-1)}(x) = \sum_{n \geq 0} \phi_n \binom{x}{n}$$

and

$$F_{f,a,c}(T) = \sum_{n \geq 0} f_n T^n.$$

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Let χ be a character of conductor δ, and let $s \in \mathbb{Z}_p$. We want to compute $L_p(s, \chi)$ up to a given precision.

Types: p-adic numbers will be represented by p-adic numbers and power series by power series

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Code: initialization of the roots of unity

```
p1 = x^(p-1) - 1;
p2 = vector(p - 1, j, x - j);
p1 = polhensellift(p1, p2, p, precp);
return(vector(p - 1, j, -polcoeff(p1[j], 0)
                       + O(p^precp)));
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Call vp the vector returned by the previous function.

Code: computation of $\omega(x)$

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return (vp[x%p]);
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Call `teichm(x)` the function computing $\omega(x)$ assuming $x \in \mathbb{Z}_p^\times$. Recall that

$$\varphi_k^{(m)}(x) := \begin{cases} 0 & \text{if } x \in p\mathbb{Z}_p, \\ \omega(x)^m \langle x \rangle^{-k} & \text{otherwise.} \end{cases}$$

Code: computation of $\varphi_k^{(m)}(x)$

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if (x%p == 0, return(0));
p1 = teichm(x)^(k + m);
p2 = x^(-k);
return (p1*p2);
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Now, if $A \in \mathbb{N}$, then we have $\binom{A}{n} = 0$ for all $n > A$.

Code: computation of $\phi_0, \ldots, \phi_{N-1}$

```
v = vector(N);
for (A = 1, N,
  val = phi_k(A, k, m);
  v[A] = val - sum(j = 1, A - 1,
                        binomial(A, j)*v[j])
);
return (v);
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Let e be a root of unity of order c, recall that

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Code: Compute $F_{f,a,\eta}$

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mon = (1 + O(p^precp))*(1 + X + O(X^precX));
den = 1 - e^f*mon^f;
num = e^a*mon^a;
return (num/den);
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where ξ is any primitive c-th root of unity.

Code: "formal" c-th root of unity

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e = Mod(y, (y^c - 1)/(y - 1));
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Code: compute $F_{f,a,c}$

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F = computeF_eta(N, f, e, a);
return (trace(F));
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The corresponding p-adic L-function is defined by

$$L_p(\chi, s) := (c\,\varphi_s^{(-1)}(c)\chi(c) - 1)^{-1} \sum_{\substack{a=1 \\ (a,f)=1}}^{f-1} \chi(a) Z_{f,p}(a, c, s).$$

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