NUMERICAL EVIDENCE TOWARD A 2-ADIC EQUIVARIANT “MAIN CONJECTURE”

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1. The conjecture

Let K be a totally real finite Galois extension of $\mathbb{Q}$ with Galois group G dihedral of order 8, and suppose that $\sqrt{2}$ is not in K. Fix a finite set S of primes of $\mathbb{Q}$ including $2, \infty$ and all primes that ramify in K. Let C be the cyclic subgroup of G of order 4 and F the fixed field of C acting on K. Fix a 2-adic unit $u \equiv 5 \mod 8\mathbb{Z}_2$.

Write $L_F(s,\chi)$ for the 2-adic L-functions, normalized as in [W], of the 2-adic characters χ of C or, equivalently by class field theory, of the corresponding 2-adic primitive ray class characters. We always work with their S-*truncated* forms

$$L_{F,S}(s,\chi) = L_F(s,\chi) \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})} \langle N(\mathfrak{p}) \rangle^{1-s}\right)$$

where $\mathfrak{p}$ runs through all primes of F above $S \backslash \{2,\infty\}$, and $\langle\,\rangle : \mathbb{Z}_2^\times \to 1+4\mathbb{Z}_2$ is the unique function with $\langle x \rangle x^{-1} \in \{-1, 1\}$ for all x. Now our interest is in the 2-adic function

$$f_1(s) = \frac{\rho_{F,S} \log(u)}{8(u^{1-s}-1)} + \frac{1}{8}\left(L_{F,S}(s,1) + L_{F,S}(s,\beta^2) - 2L_{F,S}(s,\beta)\right)$$

where β is a faithful irreducible 2-adic character of C and

$$\rho_{F,S} = \lim_{s \to 1} (s-1) L_{F,S}(s,1).$$

It follows from known results that $\frac{1}{2}\rho_{F,S} \in \mathbb{Z}_2$ and that $f_1(s)$ is an *Iwasawa analytic* function of $s \in \mathbb{Z}_2$, in the sense of [R]. This means that there is a unique power series $F_1(T) \in \mathbb{Z}_2[[T]]$ so that

$$F_1(u^n - 1) = f_1(1-n) \quad \text{for} \quad n = 1, 2, 3, \dots .$$

The conjecture we want to test is

Conjecture 1.

$$\frac{1}{2}\rho_{F,S} \in 4\mathbb{Z}_2 \quad \text{and} \quad F_1(T) \in 4\mathbb{Z}_2[[T]].$$

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Testing the conjecture amounts to calculating $\frac{1}{2}\rho_{F,S}$ and (many of) the power series coefficients of

$$F_1(T) = \sum_{j=1}^{\infty} x_j T^{j-1}$$

modulo $4\mathbb{Z}_2$. Were the conjecture false we would expect to find a counterexample in this way.

The idea of the calculation is, roughly, to express the coefficients of the power series $F_1(T)$ as integrals over suitable 2-adic continuous functions with respect to the measures used to construct the 2-adic L-functions.

The conjecture has been tested for 60 fields K determined by the size of their discriminant and the splitting of 2 in the field F. For this purpose, it is convenient to replace the datum K by F together with the ray class characters of F which determine K (cf §5). A description of the results is in §6: they are affirmative.

Where does $f_1(s)$ come from? It is an example which arises by attempting to refine the Main Conjecture of Iwasawa theory. This connection will be discussed next in order to prove that $F_1(T)$ is in $\mathbb{Z}_2[[T]]$.

2. The motivation

The Main Conjecture of classical Iwasawa theory was proved by Wiles [W] for odd prime numbers ℓ. More recently [RW2], an equivariant "main conjecture" has been proposed, which would both generalize and refine the classical one for the same ℓ. When a certain μ-invariant vanishes, as is expected for odd ℓ (by a conjecture of Iwasawa), this equivariant "main conjecture", up to its uniqueness assertion, depends only on properties of ℓ-adic L-functions, by Theorem A of [RW3].

The point is that it is possible to numerically test this Theorem A property of ℓ-adic L-functions, at least in simple special cases when it may be expressed in terms of congruences and the special values of these L-functions can be computed. The conjecture of §1 is perhaps the simplest non-abelian example when this happens, but with the price of taking $\ell = 2$. Although there are some uncertainties about the formulation of the "main conjecture" for $\ell = 2$, partly because [W] applies only in the cyclotomic case, it seems clearer what the 2-adic analogue of the Theorem A properties of L-functions should be, in view of their "extra" 2-power divisibilities [DR].

More precisely, let $L_{k,S} \in \mathrm{Hom}^*\left(R_\ell(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times\right)$ be the "power series" valued function of ℓ-adic characters χ of $G_\infty = \mathrm{Gal}(K_\infty/k)$ defined in §4 of [RW2]. This is made from the values of ℓ-adic L-functions by viewing them as a quotient of Iwasawa analytic functions, by the proof of Proposition 11 in [RW2]. When $\ell \neq 2$, the vanishing of the μ-invariant mentioned above means precisely that the coefficients of these power series have no nontrivial common divisor; and the Theorem A property of L-functions is that then $L_{k,S}$ is in $\mathrm{Det}\left(K_1(\Lambda(G_\infty)_\bullet)\right)$ (see next section for precise definitions).

When $\ell = 2$, we can still form $L_{k,S}$, but now its values at characters χ of degree 1 have numerators divisible by $2^{[k:\mathbb{Q}]}$, because of (4.8), (4.9) of [R]. Define

$$\widetilde{L}_{k,S}(\chi) = 2^{-[k:\mathbb{Q}]\chi(1)} L_{k,S}(\chi)$$

for all 2-adic characters χ of G_∞, so that the deflation and restriction properties of Proposition 12 of [RW2] are maintained. Then the analogous coprimality condition on coefficients of numerator, denominator of the values $\widetilde{L}_{k,S}(\chi)$ will be referred to as vanishing of the $\widetilde{\mu}$-invariant of K_∞/k: the Theorem A property we want to test is therefore

Conjecture 2.

$$\widetilde{L}_{k,S} \quad \text{is in} \quad \mathrm{Det}\left(K_1(\Lambda(G_\infty)_\bullet)\right).$$

Remark 2.1. a) When Conjecture 2 holds, then $\widetilde{L}_{k,S}(\chi)$ is in $\Lambda^c(\Gamma_k)_\bullet^\times$ for all $\chi \in R_2(G_\infty)$, implying the vanishing of the $\widetilde{\mu}$-invariant of K_∞/k.

b) For $\ell \neq 2$, some cases of the equivariant "main conjecture" have recently been proved ([RW]).

3. Interpreting Conjecture 2 as a congruence

We now specialize to the situation of §1, so use the notation of its first paragraph, in order to exhibit a congruence equivalent to Conjecture 2 (see Figure 1).

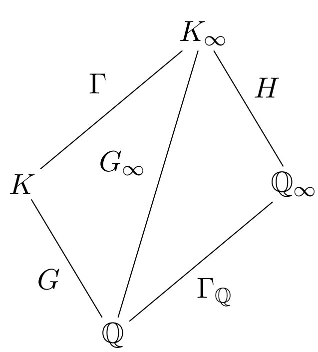

Figure 1

Let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$, i.e. the maximal totally real subfield of the field obtained from $\mathbb{Q}$ by adjoining all 2-power roots of unity, and set $\Gamma_{\mathbb{Q}} = \mathrm{Gal}\,(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_2$. Let $K_\infty = K\mathbb{Q}_\infty$, noting that $K \cap \mathbb{Q}_\infty = \mathbb{Q}$ follows from $\sqrt{2} \notin K$, and set $G_\infty = \mathrm{Gal}(K_\infty/\mathbb{Q})$. Defining $\Gamma = \ker(G_\infty \to G)$, $H = \ker(G_\infty \to \Gamma_{\mathbb{Q}})$, we now have $H \hookrightarrow G_\infty \twoheadrightarrow \Gamma_{\mathbb{Q}}$ in the notation of [RW2].

Since $G_\infty = \Gamma \times H$ with $\Gamma \simeq \Gamma_{\mathbb{Q}}$ and $H \simeq G$ dihedral of order 8 we can understand the structure of

$$\Lambda(G_\infty)_\bullet = \Lambda(\Gamma)_\bullet \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[H] = \Lambda(\Gamma)_\bullet[H]$$

where $\bullet$ means "invert all elements of $\Lambda(\Gamma)\backslash 2\Lambda(\Gamma)$."

Namely, choose σ, τ in G so that $C = \langle\tau\rangle$ with $\sigma^2 = 1$, $\sigma\tau\sigma^{-1} = \tau^{-1}$ and extend them to K_∞, with trivial action on $\mathbb{Q}_\infty$, to get s, t respectively. Then the abelianization of H is $H^{ab} = H/\langle t^2\rangle$ and we get a pullback diagram

$$\begin{array}{ccc} \Lambda(G_\infty)_\bullet = \Lambda(\Gamma)_\bullet[H] & \longrightarrow & (\Lambda(\Gamma)_\bullet(\zeta_4)) * \langle s\rangle \\ \downarrow & & \downarrow \\ \Gamma(G_\infty^{ab})_\bullet = \Lambda(\Gamma)_\bullet[H^{ab}] & \longrightarrow & \Lambda(\Gamma)_\bullet[H^{ab}]/2\Lambda(\Gamma)_\bullet[H^{ab}] \end{array}$$

where the upper right term is the crossed product order with $\Lambda(\Gamma)_\bullet$-basis $1, \zeta_4, \widetilde{s}, \zeta_4\widetilde{s}$ with $\zeta_4^2 = -1$, $\widetilde{s}^{\,2} = 1$, $\widetilde{s}\zeta_4 = \zeta_4^{-1}\widetilde{s} = -\zeta_4\widetilde{s}$ and the top map takes t, s to $\zeta_4, \widetilde{s}$ respectively, the right map takes $\zeta_4, \widetilde{s}$ to t^{ab}, s^{ab}. This diagram originates in the pullback diagram for the cyclic group $\langle t\rangle$ of order 4, then going to the dihedral group ring $\mathbb{Z}_2[H]$ by incorporating the action of s, and finally applying $\Lambda(\Gamma)_\bullet \otimes_{\mathbb{Z}_2} -$.

We now turn to getting the first version of our congruence in terms of the pullback diagram above. This is possible since $R^\times \to K_1(R)$ is surjective for all the rings considered there. We also simplify notation a little by setting $\mathfrak{A} = (\Lambda(\Gamma)_\bullet(\zeta_4)) * \langle s\rangle$ and writing $\widetilde{L}_{k,S}$ as $\widetilde{L}_{K_\infty/k}$, because we will now have to vary the fields and S is fixed anyway. The dihedral group G has 4 degree 1 irreducible characters $1, \eta, \nu, \eta\nu$ with $\eta(\tau) = 1$, $\nu(\sigma) = 1$ and a unique degree 2 irreducible α, which we view as characters of G_∞ by inflation.

Proposition 3.1. *Let K_∞^{ab} be the fixed field of $\langle t^2\rangle$, hence* $\mathrm{Gal}(K_\infty^{ab}/\mathbb{Q}) = G_\infty^{ab}$. *Then*

a) $\widetilde{L}_{K_\infty^{ab}/\mathbb{Q}} = \mathrm{Det}(\widetilde{\Theta}^{ab})$ *for some* $\widetilde{\Theta}^{ab} \in \Lambda(G_\infty^{ab})_\bullet^\times$

b) $\widetilde{L}_{K_\infty/\mathbb{Q}} \in \mathrm{Det}(K_1(\Lambda(G_\infty)_\bullet))$ *if, and only if, any* $y \in \mathfrak{A}$ *mapping to* $\widetilde{\Theta}^{ab} \mod 2$ *in* $\Lambda(G_\infty^{ab})_\bullet/2\Lambda(G_\infty^{ab})_\bullet$ *has*

$$nr(y) \equiv \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha) \mod 4\Lambda(\Gamma_\mathbb{Q})_\bullet$$

where nr is the reduced norm of (the total ring of fractions of) $\mathfrak{A}$ to its centre $\Lambda(\Gamma)_\bullet$ and we identify $\Lambda(\Gamma)_\bullet$ with $\Lambda(\Gamma_\mathbb{Q})_\bullet$ via $\Gamma \xrightarrow{\simeq} \Gamma_\mathbb{Q}$.

Proof. a) The vanishing of $\widetilde{\mu}$ for $K_\infty/\mathbb{Q}$, in the sense of §2, is known by [FW], i.e. $\widetilde{L}_{K_\infty^{ab}/\mathbb{Q}}(\chi)$ is a unit in $\Lambda(\Gamma_\mathbb{Q})_\bullet$ for all 2-adic characters χ of G_∞^{ab}. By the proof of Theorem 9 in [RW3] we have $L_{K_\infty^{ab}/\mathbb{Q}} = \mathrm{Det}(\lambda)$ with $\lambda \in \Lambda(G_\infty^{ab})_\bullet$ the pseudomeasure of Serre. The point is then that $\lambda = 2\widetilde{\Theta}^{ab}$ with $\widetilde{\Theta}^{ab} \in \Lambda(G_\infty^{ab})_\bullet$, which follows from Theorem 3.1b) of [R], because of Theorem 4.1 (loc.cit.) and the relation between λ and μ_c discussed just after it. Then $\widetilde{L}_{K_\infty^{ab}/\mathbb{Q}} = \mathrm{Det}(\widetilde{\Theta}^{ab})$ and now the proof of the Corollary to Theorem 9 in [RW3] shows that $\widetilde{\Theta}^{ab}$ is a unit of $\Lambda(G_\infty^{ab})$.

b) *Claim:* $nr(1 + 2\mathfrak{A}) = 1 + 4\Lambda(\Gamma)_\bullet$.

Proof of the claim. If $x = a1 + b\zeta_4 + c\widetilde{s} + d\zeta_4\widetilde{s}$ with $a,b,c,d \in \Lambda(\Gamma)_\bullet$, one computes $nr(x) = (a^2+b^2) - (c^2+d^2)$ from which $nr(1+2\mathfrak{A}) \subseteq 1 + 4\Lambda(\Gamma)_\bullet$; equality follows from $nr\big((1+2a)+2a\widetilde{s}\big) = (1+2a)^2 - (2a)^2 = 1+4a$ for $a \in \Lambda(\Gamma)_\bullet$. □

Suppose first that the congruence for $\widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$ holds. Start with $\widetilde{\Theta}^{ab}$ from a) in the lower left corner of the pullback square and map it to $\widetilde{\Theta}^{ab}$ mod 2 in the lower right corner. Choosing any $y_0 \in \mathfrak{A}$ mapping to $\widetilde{\Theta}^{ab}$ mod 2, we note that $y_0 \in \mathfrak{A}^\times$ because the maps in the pullback diagram are ring homomorphisms and the kernel $2\mathfrak{A}$ of the right one is contained in the radical of $\mathfrak{A}$. Thus $nr(y_0) \in \Lambda(\Gamma)_\bullet^\times$ has $nr(y_0)^{-1}\widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha) \in 1 + 4\Lambda(\Gamma_\mathbb{Q})_\bullet$ by the congruence, hence, by the Claim, $nr(y_0)^{-1}\widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha) = nr(z)$, $z \in 1+2\mathfrak{A}$. So $y_1 = y_0 z$ is another lift of $\widetilde{\Theta}^{ab}$ mod 2 and $nr(y_1) = \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$. By the pullback diagram we get $Y \in \Lambda(G_\infty)_\bullet^\times$ which maps to $\widetilde{\Theta}^{ab}$ and y_1, where $nr(y_1) = \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$.

It follows that $\operatorname{Det} Y = \widetilde{L}_{K_\infty/\mathbb{Q}}$. To see this we check that their values agree at every irreducible character χ of G_∞; it even suffices to check it on the characters $1, \eta, \nu, \eta\nu, \alpha$ of G by Theorem 8 and Proposition 11 of [RW2], because every irreducible character of G_∞ is obtained from these by multiplying by a character of type W. It works for the characters $1, \eta, \nu, \eta\nu$ of G_∞^{ab} by Proposition 12, 1b) (loc.cit.) since the deflation of Y equals $\widetilde{\Theta}^{ab}$ and Det $\widetilde{\Theta}^{ab} = \widetilde{L}_{K_\infty^{ab}/\mathbb{Q}}$ by a). Finally, (Det $Y)(\alpha) = j_\alpha\big(nr(Y)\big) = nr(y_1) = \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$ by the commutative triangle before Theorem 8 (loc.cit.), the definition of j_α, and $G_\infty = \Gamma \times H$.

The converse depends on related ingredients. More precisely, $\widetilde{L}_{K_\infty/\mathbb{Q}} \in \operatorname{Det} K_1\big((\Lambda G_\infty)_\bullet\big)$ implies $\widetilde{L}_{K_\infty/\mathbb{Q}} = $ Det Y with $Y \in (\Lambda G_\infty)_\bullet^\times$ by surjectivity of $(\Lambda G_\infty)_\bullet^\times \to K_1\big((\Lambda G_\infty)_\bullet\big)$. Since $(\Lambda G_\infty^{ab})_\bullet^\times \to K_1\big((\Lambda G_\infty^{ab})_\bullet\big)$ is an isomorphism, we get that the deflation of Y equals $\widetilde{\Theta}^{ab}$ in $\Lambda(G_\infty^{ab})^\times$. Letting $y_1 \in \mathfrak{A}^\times$ be the image of Y in the pullback diagram, it follows that $nr(y_1) = \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$ and that y_1 maps to $\widetilde{\Theta}^{ab}$ mod 2 in $\Lambda(G_\infty^{ab})_\bullet/2\Lambda(G_\infty^{ab})_\bullet$. Given any y as in b), then $y_1^{-1}y$ maps to 1 hence is in $1+2\mathfrak{A}$ and our congruence follows from the Claim on applying nr. □

4. Rewriting the congruence in testable form

Set $F_0 = \dfrac{\widetilde{L}_{K_\infty/F,S}(1) + \widetilde{L}_{K_\infty/F,S}(\beta^2)}{2} - \widetilde{L}_{K_\infty/F,S}(\beta)$.

Proposition 4.1. a) *F_0 is in $\Lambda(\Gamma_\mathbb{Q})_\bullet$*

b) $\widetilde{L}_{K_\infty/\mathbb{Q}} \in$ Det $K_1\big(\Lambda(G_\infty)_\bullet\big)$ *if, and only if, $F_0 \in 4\Lambda(\Gamma_\mathbb{Q})_\bullet$*

Proof. Note that $\operatorname{ind}_C^G 1_C = 1_G + \eta$, $\operatorname{ind}_C^G \beta^2 = \nu + \eta\nu$, $\operatorname{ind}_C^G \beta = \alpha$. When we inflate β to a character of $\mathrm{Gal}(K_\infty/F)$ then $\operatorname{ind}_{\mathrm{Gal}(K_\infty/F)}^{G_\infty} \beta = \alpha$ with α inflated to G_∞, etc.

By Proposition 3.1 of the previous section we can write $\widetilde{L}_{K_\infty^{ab}/\mathbb{Q}} = \operatorname{Det}(\widetilde{\Theta}^{ab})$ with

$$\widetilde{\Theta}^{ab} = a + bt^{ab} + cs^{ab} + ds^{ab}t^{ab}$$

for some a, b, c, d in $\Lambda(\Gamma)_\bullet$. It follows that

$$\begin{aligned} \widetilde{L}_{K_\infty/\mathbb{Q}}(1) &= a+b+c+d \\ \widetilde{L}_{K_\infty/\mathbb{Q}}(\eta) &= a+b-c-d \\ \widetilde{L}_{K_\infty/\mathbb{Q}}(\nu) &= a-b+c-d \\ \widetilde{L}_{K_\infty/\mathbb{Q}}(\eta\nu) &= a-b-c+d. \end{aligned}$$

Form $y = a + b\zeta_4 + c\widetilde{s} + d\zeta_4\widetilde{s}$ in $\big(\Lambda(\Gamma)_\bullet(\zeta_4)\big) * \langle s\rangle$. By the computation in the Claim in the proof of Proposition 3.1, we have

$$\begin{aligned} nr(y) &= (a+c)(a-c) + (b+d)(b-d) \\ &= \frac{\widetilde{L}_{\mathbb{Q}}(1)+\widetilde{L}_{\mathbb{Q}}(\nu)}{2}\ \frac{\widetilde{L}_{\mathbb{Q}}(\eta)+\widetilde{L}_{\mathbb{Q}}(\eta\nu)}{2} + \frac{\widetilde{L}_{\mathbb{Q}}(1)-\widetilde{L}_{\mathbb{Q}}(\nu)}{2}\ \frac{\widetilde{L}_{\mathbb{Q}}(\eta)-\widetilde{L}_{\mathbb{Q}}(\eta\nu)}{2} \\ &= \frac{1}{4}\big(\widetilde{L}_{\mathbb{Q}}(1+\eta)+\widetilde{L}_{\mathbb{Q}}(1+\eta\nu)+\widetilde{L}_{\mathbb{Q}}(\nu+\eta)+\widetilde{L}_{\mathbb{Q}}(\nu+\eta\nu)\big) \\ &\quad + \frac{1}{4}\big(\widetilde{L}_{\mathbb{Q}}(1+\eta)-\widetilde{L}_{\mathbb{Q}}(1+\eta\nu)-\widetilde{L}_{\mathbb{Q}}(\nu+\eta)+\widetilde{L}_{\mathbb{Q}}(\nu+\eta\nu)\big) \\ &= \frac{\widetilde{L}_{\mathbb{Q}}(1+\eta)+\widetilde{L}_{\mathbb{Q}}(\nu+\eta\nu)}{2} = \frac{\widetilde{L}_F(1)+\widetilde{L}_F(\beta^2)}{2}, \end{aligned}$$

because

$$\widetilde{L}_{K_\infty/\mathbb{Q}}(\operatorname{ind}^{G_\infty}_{\operatorname{Gal}(K_\infty/F)}\chi) = \widetilde{L}_{K_\infty/F}(\chi)$$

for all characters χ of $\operatorname{Gal}(K_\infty/F)$. Thus also $\widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha) = \widetilde{L}_{K_\infty/F}(\beta)$, so we now have shown that

$$F_0 = nr(y) - \widetilde{L}_{K_\infty/\mathbb{Q}}(\alpha)$$

proving a), since $\widetilde{L}_{K_\infty/F}(\beta) \in (\Lambda\Gamma_F)_\bullet$ by §2, as β has degree 1.

Moreover, the image of y under the right arrow of the pullback diagram of §3 equals $\widetilde{\Theta}^{ab} \mod 2$, by construction, hence b) follows directly from Proposition 3.1b). □

Remark 4.2. Considering F_0 in $\Lambda(\Gamma_{\mathbb{Q}})_\bullet$, instead of its natural home $\Lambda(\Gamma_F)_\bullet$, is done to be consistent with the identification in b) of Proposition 3.1, via the natural isomorphisms $\Gamma \to \Gamma_F \to \Gamma_{\mathbb{Q}}$: this is the sense in which $L_{K_\infty/\mathbb{Q}}(\alpha) = L_{K_\infty/F}(\beta)$.

The congruence $F_0 \equiv 0 \mod 4\Lambda(\Gamma_{\mathbb{Q}})_\bullet$ can now be put in the more testable form of Conjecture 1. Let $\gamma_{\mathbb{Q}}$ be the generator of $\Gamma_{\mathbb{Q}}$ which, when extended to $\mathbb{Q}(\sqrt{-1})$ as the identity, acts on all 2-power roots of unity in $\mathbb{Q}_\infty(\sqrt{-1})$ by raising them to the u^{th} power, where $u \equiv 5 \mod 8\mathbb{Z}_2$ as fixed before. Then the Iwasawa isomorphism $\Lambda(\Gamma_{\mathbb{Q}}) \simeq \mathbb{Z}_2[[T]]$, under which $\gamma_{\mathbb{Q}} - 1$

corresponds to T, makes $F_0 \in \Lambda(\Gamma_{\mathbb{Q}})_\bullet$ correspond to some $F_0(T) \in \mathbb{Z}_1\big[[T]\big]_\bullet$ and the congruence of Proposition 4.1b) to

$$F_0(T) \equiv 0 \mod 4\mathbb{Z}_2\big[[T]\big]_\bullet$$

Since β is an abelian character, we know that $\widetilde{L}_{F,S}(\beta^2)$, $\widetilde{L}_{F,S}(\beta)$ correspond to elements of $\mathbb{Z}_2\big[[T]\big]$, not just $\mathbb{Z}_2\big[[T]\big]_\bullet$ (cf. §4 of [RW2]), and $\widetilde{L}_F(1)$ to one of $T^{-1}\mathbb{Z}_2\big[[T]\big]$. We thus have

$$F_0(T) = \frac{x_0}{T} + \sum_{j=1}^{\infty} x_j T^{j-1}$$

with $x_j \in \mathbb{Z}_2$ for all $j \geq 0$.

By the interpolation definition of $\big(\widetilde{L}_{F,S}(\beta^i)\big)(T)$ (cf §4 of [R]), it follows that

$$F_0(u^s - 1) = \frac{1}{2}\Big(\frac{L_{F,S}(1-s,1)}{4} + \frac{L_{F,S}(1-s,\beta^2)}{4} - 2\,\frac{L_{F,S}(1-s,\beta)}{4}\Big).$$

We abbreviate the right hand side of the equality as $f_0(1-s)$. This implies

$$x_0 = -\,\frac{\rho_{F,S}\log(u)}{8}\ ,$$

because the left side is

$$\lim_{T\to 0} TF_0(T) = \lim_{s\to 1}\frac{u^{1-s}-1}{s-1}\ (s-1)f_0(s) = -\log(u)\lim_{s\to 1}(s-1)\ \frac{L_{F,S}(s,1)}{8}$$

as required. Note that $u \equiv 5 \mod 8$ implies that $\frac{\log(u)}{4}$ is a 2-adic unit, hence $\frac{1}{2}\,\rho_{F,S} \in \mathbb{Z}_2$ is in $4\mathbb{Z}_2$ if, and only if, $x_0 \in 4\mathbb{Z}_2$.

Define $F_1(T) = F_0(T) - x_0T^{-1} = \sum\limits_{j=1}^{\infty} x_jT^{j-1} \in \mathbb{Z}_2\big[[T]\big]$. It follows that

$$F_1(u^s-1) = -\,\frac{x_0}{u^s-1} + F_0(u^s-1) = \frac{\rho_{F,S}\log(u)}{8(u^s-1)} + f_0(1-s)$$

which is $f_1(1-s)$, with f_1 as in §1, hence our present $F_1(T)$ is also the same as in §1. Thus Conjecture 1 of §1 is equivalent to Conjecture 2 of §2 for the special case $K_\infty/\mathbb{Q}$ of §1.

5. Testing Conjecture 1

Let χ be a 2-adic character of the Galois group C of K/F and let $\mathfrak{f}$ be the conductor of K/F. By class field theory, we view χ as a map on the group of ideals relatively prime to $\mathfrak{f}$. Fix a prime ideal $\mathfrak{c}$ not dividing $\mathfrak{f}$. For $\mathfrak{a}$, a fractional ideal relatively prime to $\mathfrak{c}$ and $\mathfrak{f}$, let $\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a},\mathfrak{c};s)$ denote the associated 2-adic twisted partial zeta function [CN]. Thus, we have

$$L_{F,S}(s,\chi) = \frac{1}{\chi(\mathfrak{c})\langle N\mathfrak{c}\rangle^{1-s}-1}\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{N\mathfrak{p}}\langle N\mathfrak{p}\rangle^{1-s}\right)\sum_{\sigma\in G}\chi(\sigma)^{-1}\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a}_\sigma^{-1},\mathfrak{c};s)$$

where $\mathfrak{p}$ runs through the prime ideals of F in S not dividing $2\mathfrak{f}$, $\mathfrak{a}_\sigma$ is a (fixed) integral ideal coprime with $2\mathfrak{f}\mathfrak{c}$ whose Artin symbol is σ

Denote the ring of integers of F by $\mathcal{O}_F$ and let $\gamma \in \mathcal{O}_F$ be such that $\mathcal{O}_F = \mathbb{Z}+\gamma\mathbb{Z}$. In [Rob] (see also [BBJR] for a slightly different presentation), it is shown that the function $\mathcal{Z}_\mathfrak{f}(\mathfrak{a}, \mathfrak{c}; s)$ is defined by the following integral

$$\mathcal{Z}_\mathfrak{f}(\mathfrak{a}, \mathfrak{c}; s) = \int \frac{\langle N\mathfrak{a}\, N(x_1 + x_2\gamma)\rangle^{1-s}}{N\mathfrak{a}\, N(x_1 + x_2\gamma)}\, d\mu_\mathfrak{a}(x_1, x_2)$$

where the integration domain is $\mathbb{Z}_2^2$, $\langle\,\rangle$ is extended to $\mathbb{Z}_2$ by $\langle x\rangle = 0$ if $x \in 2\mathbb{Z}_2$, and the measure $\mu_\mathfrak{a}$ is a measure of norm 1 (depending also on γ, $\mathfrak{f}$ and $\mathfrak{c}$).

Assume now, as we can do without loss of generality, that the ideal $\mathfrak{c}$ is such that $\langle N\mathfrak{c}\rangle \equiv 5 \pmod{8\mathbb{Z}_2}$ and take $u = \langle N\mathfrak{c}\rangle$. For $s \in \mathbb{Z}_2$, we let $t = t(s) = u^s - 1 \in 4\mathbb{Z}_2$, so that $s = \log(1 + t)/\log(u)$. For $x \in \mathbb{Z}_2^\times$, one can check readily that

$$\langle x\rangle^s = \left(u^{\mathcal{L}(x)}\right)^s = (1 + u^s - 1)^{\mathcal{L}(x)} = \sum_{n\geq 0} \binom{\mathcal{L}(x)}{n} t^n$$

where $\mathcal{L}(x) = \log\langle x\rangle/\log u \in \mathbb{Z}_2$. For $x \in \mathbb{Z}_2^\times$, we set

$$L(x; T) = \sum_{n\geq 0} \binom{\mathcal{L}(x)}{n} T^n \in \mathbb{Z}_2[[T]]$$

and $L(x; T) = 0$ if $x \in 2\mathbb{Z}_2$. Now, we define

$$R(\mathfrak{a}, \mathfrak{c}; T) = \int \frac{L\big(N\mathfrak{a}\, N(x_1 + x_2\gamma); T\big)}{N\mathfrak{a}\, N(x_1 + x_2\gamma)}\, d\mu_\mathfrak{a}(x_1, x_2) \in \mathbb{Z}_2[[T]]$$

$$B(\chi; T) = \chi(\mathfrak{c})(T + 1) - 1 \in \mathbb{Z}_2[\chi][T],$$

$$A(\chi; T) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}} L(N\mathfrak{p}; T)\right) \sum_{\sigma\in G} \chi(\sigma)^{-1} R(\mathfrak{a}_\sigma^{-1}, \mathfrak{c}; T) \in \mathbb{Z}_2[\chi][[T]]$$

where $\mathfrak{p}$ runs through the prime ideals of F in S not dividing $2\mathfrak{f}$.

Proposition 5.1. *We have, for all $s \in \mathbb{Z}_2$*

$$L_{F,S}(1 - s, \chi) = \frac{A(\chi; u^s - 1)}{B(\chi; u^s - 1)}.$$

We now specialize to our situation. For that, we need to make the additional assumption that $\beta^2(\mathfrak{c}) = -1$, so $\beta(\mathfrak{c})$ is a fourth root of unity in $\mathbb{Q}_2^c$ that we will denote by i. Thus, we have

$$B(1; T) = T,\ B(\beta; T) = i(T + 1) - 1,$$
$$B(\beta^2; T) = -T - 2,\ B(\beta^3; T) = -i(T + 1) - 1.$$

Let $x \mapsto \bar{x}$ be the $\mathbb{Q}_2$-automorphism of $\mathbb{Q}_2(i)$ sending i to $-i$. Then we have $\overline{L_{F,S}(1 - s, \beta)} = L_{F,S}(1 - s, \beta^3)$ by the expression of $L_{F,S}(s, \chi)$ given at the beginning of the section since the twisted partial zeta functions have values in $\mathbb{Q}_2$ and $\bar{\beta} = \beta^3$. And furthermore,

$$L_{F,S}(s,\beta^3) = L_{\mathbb{Q},S}(s,\mathrm{Ind}_C^G(\beta^3)) = L_{\mathbb{Q},S}(s,\mathrm{Ind}_C^G(\beta)) = L_{F,S}(s,\beta).$$

Therefore, by Prop. 5.1, we deduce that

$$\begin{aligned} A(\beta;u^s-1) + \bar{A}(\beta;u^s-1) &= \big(B(\beta;T) + B(\beta^3;T)\big) L_{F,S}(1-s,\beta) \\ &= -2L_{F,S}(1-s,\beta). \end{aligned}$$

Since

$$f_1(s) = \frac{\rho_{F,S}\log u}{8(u^{1-s}-1)} + \frac{1}{8}\left(L_{F,S}(s,1) + L_{F,S}(s,\beta^2) - 2L_{F,S}(s,\beta)\right)$$

we find that

$$F_1(T) = \frac{\rho_{F,S}\log u}{8T} + \frac{1}{8}\left(\frac{A(1;T)}{T} - \frac{A(\beta^2;T)}{T+2} + A(\beta;T) + \bar{A}(\beta,T)\right)$$

is such that $F_1(u^n-1) = f_1(1-n)$ for $n = 1,2,3,\ldots$.

The conjecture that we wish to check states that

$$\frac{1}{2}\rho_{F,S} \in 4\mathbb{Z}_2 \quad \text{and} \quad F_1(T) \in 4\mathbb{Z}_2[[T]].$$

Now define $D(T) = 8T(T+2)F_1(T)$, so that

$$\begin{aligned} D(T) = (T+2)&\left(\rho_{F,S}\log u + A(1;T)\right) \\ &- TA(\beta^2;T) + T(T+2)\left(A(\beta;T) + \bar{A}(\beta,T)\right). \end{aligned}$$

We can now give a final reformulation of the conjecture which is the one that we actually tested.

Conjecture 3.

$$\rho_{F,S} \in 8\mathbb{Z}_2 \quad \textit{and} \quad D(T) \in 32\mathbb{Z}_2[[T]]$$

The computation of $\rho_{F,S}$ is done using the following formula [Col]

$$\rho_{F,S} = 2\,h_F\,R_F d_F^{-1/2} \prod_{\mathfrak{p}} (1 - 1/N(\mathfrak{p}))$$

where h_F, R_F, d_F are respectively the class number, 2-adic regulator and discriminant of F and $\mathfrak{p}$ runs through all primes of F above 2. Note that although R_F and $d_F^{-1/2}$ are only defined up to sign, the quantity $R_F d_F^{-1/2}$ is uniquely determined in the following way: Let ι be the embedding of F into $\mathbb{R}$ for which $\sqrt{d_F}$ is positive and let ε be the fundamental unit of F such that $\iota(\varepsilon) > 1$. Then for any embedding g of F into $\mathbb{Q}_2^c$, we have

$$R_F d_F^{-1/2} = \frac{\log_2 g(\varepsilon)}{g(\sqrt{d})}.$$

Now, for the computation of $D(T)$, the only difficult part is the computations of the $R(\mathfrak{a},\mathfrak{c};T)$. The measures $\mu_{\mathfrak{a}}$ are computed explicitly using the methods of [Rob] (see also [BBJR]), that is we construct a power series $M_{\mathfrak{a}}(X_1, X_2)$ in $\mathbb{Q}_2[X_1, X_2]$ with integral coefficients, such that

$$\int (1+t_1)^{x_1}(1+t_2)^{x_2}\, d\mu_{\mathfrak{A}}(x_1,x_2) = M_{\mathfrak{A}}(t_1,t_2) \quad \text{for all } t_1, t_2 \in 2\mathbb{Z}_2.$$

In particular, if f is a continuous function on $\mathbb{Z}_2^2$ with values in $\mathbb{C}_2$ and Mahler expansion

$$f(x_1,x_2) = \sum_{n_1,n_2\geq 0} f_{n_1,n_2}\binom{x_1}{n_1}\binom{x_2}{n_2}$$

then we have

$$\int f(x_1,x_2)\, d\mu_{\mathfrak{A}}(x_1,x_2) = \sum_{n_1,n_2\geq 0} f_{n_1,n_2} m_{n_1,n2}$$

where $M_{\mathfrak{A}}(X_1,X_2) = \sum_{n_1,n_2\geq 0} m_{n_1,n_2} X_1^{n_1} X_2^{n_2}$.

We compute this way the first few coefficients of the power series $A(\chi;T)$, for $\chi = \beta^j$, $j = 0,1,2,3$, and then deduce the first coefficients of $D(T)$ to see if they do indeed belong to $32\mathbb{Z}_2[[T]]$. We found that this was indeed always the case; see next section for more details.

To conclude this section, we remark that, in fact, we do not need the above formula to compute $\rho_{F,S}$ since the constant coefficient of $A(1;T)$ is $-\rho_{F,S}\log u$. (This can be seen directly from the expression of x_0 given at the end of Section 4 or using the fact that $D(T)$ has zero constant coefficient since $F_1(T) \in \mathbb{Z}_2[[T]]$.) However, we did compute it using this formula since it then provides a neat way to check that (at least one coefficient of) $A(1;T)$ is correct.

6. The numerical verifications

We have tested the conjecture in 60 examples. The examples are separated in three subcases of 20 examples according to the way 2 decomposes in the quadratic subfield F: ramified, split or inert. In each subcase, the examples are actually the first 20 extensions $K/\mathbb{Q}$ of the suitable form of the smallest discriminant. These are given in the following three tables of Figure 2 where the entries are: the discriminant d_F of F, the conductor $\mathfrak{f}$ of K/F (which is always a rational integer) and the discriminant d_K of K. In each example, we have computed $\rho_{F,S}$ and the first 30 coefficients of $D(T)$ to a precision of at least 2^8 and checked that they satisfy the conjecture.

We now give an example, namely the smallest example for the discriminant of K. We have $F = \mathbb{Q}(\sqrt{145})$ and K is the Hilbert class field of F. The prime 2 is split in $F/\mathbb{Q}$ and the primes above 2 in F are inert in K/F. We compute $\rho_{F,S}$ and find that

$$\rho_{F,S} \equiv 2^7 \pmod{2^8}$$

2 ramified in F			2 inert in F			2 split in F		
d_F	$\mathfrak{f}$	d_K	d_F	$\mathfrak{f}$	d_K	d_F	$\mathfrak{f}$	d_K
44	3	2 732 361 984	445	1	39 213 900 625	145	1	442 050 625
156	2	9 475 854 336	5	21	53 603 825 625	41	5	44 152 515 625
220	2	37 480 960 000	205	3	143 054 150 625	505	1	65 037 750 625
12	14	39 033 114 624	221	3	193 220 905 761	689	1	225 360 027 841
156	4	151 613 669 376	61	5	216 341 265 625	777	1	364 488 705 441
380	2	333 621 760 000	205	4	452 121 760 000	793	1	395 451 064 801
152	3	389 136 420 864	221	4	610 673 479 936	17	13	403 139 914 489
24	11	587 761 422 336	901	1	659 020 863 601	897	1	647 395 642 881
876	1	588 865 925 376	29	15	895 152 515 625	905	1	670 801 950 625
220	4	599 695 360 000	1 045	1	1 192 518 600 625	305	3	700 945 700 625
444	2	621 801 639 936	5	16	1 911 029 760 000	377	3	1 636 252 863 921
12	28	624 529 833 984	109	5	2 205 596 265 625	1 145	1	1 718 786 550 625
44	12	699 484 667 904	1 221	1	2 222 606 887 281	145	8	1 810 639 360 000
92	6	835 600 748 544	29	20	2 829 124 000 000	305	4	2 215 334 560 000
60	8	849 346 560 000	29	13	3 413 910 296 329	1 313	1	2 972 069 112 961
44	10	937 024 000 000	205	7	4 240 407 600 625	377	4	5 171 367 076 096
12	19	975 543 388 416	149	5	7 701 318 765 625	545	3	7 146 131 900 625
12	26	1 601 419 382 784	1 677	1	7 909 194 404 241	17	21	7 163 272 192 041
44	15	1 707 726 240 000	21	19	9 149 529 982 761	1 705	1	8 450 794 350 625
1 164	1	1 835 743 170 816	341	3	9 857 006 530 569	329	3	8 541 047 165 049

FIGURE 2

Using the method of the previous section, we compute the first 30 coefficients of the power series $A(\cdot\,;T)$ to a 2-adic precision of 2^8. We get

$$\begin{aligned}A(1;T) \equiv 2^2\big(&16T + 57T^3 + 44T^4 + 8T^5 + 40T^6 + 21T^7 + 40T^8 + 30T^9\\&+ 16T^{10} + 49T^{11} + 56T^{12} + 29T^{13} + 32T^{14} + 50T^{15}\\&+ 62T^{16} + 47T^{17} + 48T^{18} + 60T^{19} + 32T^{20} + 16T^{21}\\&+ 8T^{22} + 21T^{23} + 30T^{24} + 26T^{25} + 2T^{26} + 9T^{27}\\&+ 56T^{28} + 34T^{29}\big) + O(T^{30}) \pmod{2^8}\end{aligned}$$

$$\begin{aligned}A(\beta;T) \equiv 2^2\big(&(28 + 1124i) + (36 + 1728i)T + (47 + 45i)T^2 + (56 + 153i)T^3\\&+ (46 + 154i)T^4 + (56 + 282i)T^5 + (55 + 433i)T^6\\&+ (54 + 435i)T^7 + (40 + 386i)T^8 + (48 + 392i)T^9\\&+ (63 + 65i)T^{10} + (48 + 257i)T^{11} + (63 + 161i)T^{12}\\&+ (20 + 477i)T^{13} + (38 + 182i)T^{14} + (56 + 66i)T^{15}\\&+ (37 + 35i)T^{16} + (6 + 341i)T^{17} + (20 + 446i)T^{18}\\&+ (40 + 412i)T^{19} + 368iT^{20} + (56 + 336i)T^{21}\\&+ (61 + 291i)T^{22} + (40 + 427i)T^{23} + (34 + 38i)T^{24}\\&+ (48 + 94i)T^{25} + (9 + 47i)T^{26} + (6 + 497i)T^{27}\\&+ (40 + 42i)T^{28} + (44 + 52i)T^{29}\big) + O(T^{30}) \pmod{2^8}\end{aligned}$$

$$\begin{aligned}A(\beta^2;T) \equiv 2^2\big(&32 + 32T + 22T^2 + 39T^3 + 36T^4 + 20T^5 + 62T^6 + 27T^7 \\ &+ 16T^8 + 62T^9 + 46T^{10} + 23T^{11} + 30T^{12} + 51T^{13} \\ &+ 4T^{14} + 2T^{15} + 56T^{16} + 33T^{17} + 44T^{18} + 12T^{19} \\ &+ 40T^{20} + 8T^{21} + 54T^{22} + 11T^{23} + 34T^{24} + 42T^{25} \\ &+ 43T^{27} + 56T^{28} + 46T^{29}\big) + O(T^{30}) \pmod{2^8}\end{aligned}$$

Therefore

$$\begin{aligned}D(T) \equiv 2^5\big(&6T + 7T^2 + 4T^3 + 5T^4 + 4T^7 + 2T^8 + 4T^9 + 2T^{10} + 4T^{11} \\ &+ T^{12} + 6T^{13} + 7T^{14} + 3T^{16} + 5T^{17} + 2T^{18} + 3T^{19} \\ &+ 7T^{20} + 5T^{21} + 7T^{22} + 4T^{23} + 4T^{24} + T^{25} + 7T^{26} \\ &+ 3T^{27} + 7T^{28} + 6T^{29}\big) + O(T^{30}) \pmod{2^8}\end{aligned}$$

and the conjecture is satisfied by the first 30 coefficients of the series D associated to the extension.

Note, as a final remark, that we have tested the conjecture in the same way for 30 additional examples where F is real quadratic, K/F is cyclic of order 4 but K is not a dihedral extension of $\mathbb{Q}$ (either $K/\mathbb{Q}$ is not Galois or its Galois group is not the dihedral group of order 8). In all of these examples, we found that the conjecture was not satisfied, that is either $\rho_{F,S}$ did not belong to $8\mathbb{Z}_2$ or one of the first 30 coefficients of the associated power series D did not belong to $32\mathbb{Z}_2$.

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