Numerical verification of the Stark-Chinburg conjecture for some icosahedral representations

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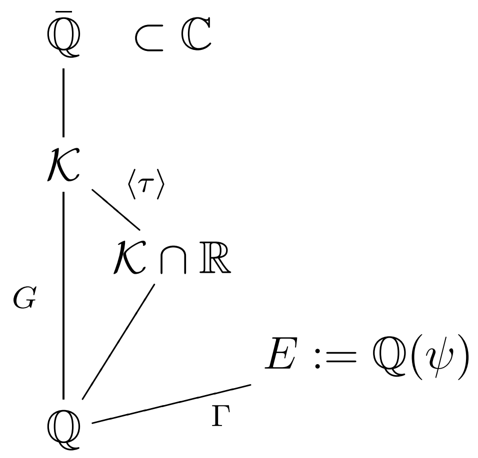
The Conjecture



$\rho : G \to \mathrm{GL}_2(\mathbb{C})$ odd and irreducible

$\psi : G \to \mathbb{C}$ corresponding character

$L(s, \rho)$ corresponding Artin L-function

for $d \in E$, define

$$f_d(s, \rho) := \sum_{\gamma \in \Gamma} d^\gamma L(s, \rho^\gamma)$$

so $f_d(s, \rho) = \sum_{n \geq 1} A_n \, n^{-s}$ with $A_n \in \mathbb{Q}$ for all $n \geq 1$.

Conjecture (Stark-Chinburg) *Let $d \in E$ be such that $A_n \in \mathbb{Z}$ for all $n \geq 1$, then there exists a unit $\varepsilon(d) \in \mathcal{K} \cap \mathbb{R}$ such that, for all $\sigma \in G$:*

$$\log ||\varepsilon(d)^{\sigma^{-1}}|| = f'_{d(\psi(\sigma)+\psi(\sigma\tau))}(0).$$

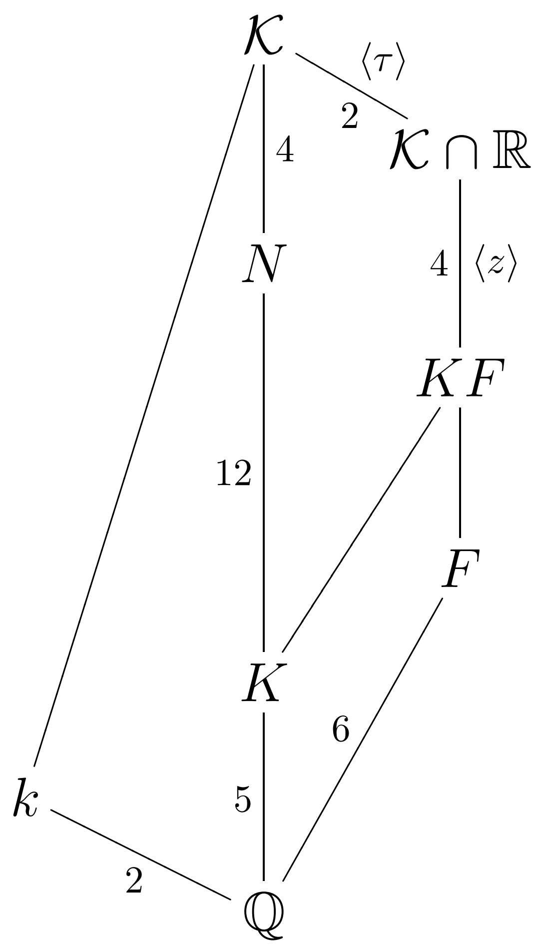
Present Status of the Conjecture

$$\bar{\rho} : G \xrightarrow{\rho} \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$$ projective representation

$\mathrm{Im}\bar{\rho} \simeq D_n$ (dihedral)	Stark (1980)	proved under some conditions
$\mathrm{Im}\bar{\rho} \simeq A_4$ (tetrahedral)	Chinburg (1980)	5 numerical examples
$\mathrm{Im}\bar{\rho} \simeq S_4$ (octahedral)	Fogel (1998)	8 numerical examples

$\mathrm{Im}\bar{\rho} \simeq A_5$ (icosahedral)
14 numerical examples

Constructing examples



$1 \longrightarrow C_4 \longrightarrow \hat{A}_5 \longrightarrow A_5 \longrightarrow 1$ (non-split)

$\mathrm{Im}\bar{\rho} \simeq A_5 \simeq \mathrm{PSL}_2(\mathbb{F}_5)$

$\mathrm{Im}\rho \simeq \hat{A}_5 \simeq \mathrm{ESL}_2(\mathbb{F}_5)$ (determinant ± 1)

$K/\mathbb{Q}$ a non-real quintic field of type A_5

$N/\mathbb{Q}$ its Galois closure, so $\mathrm{Gal}(N/\mathbb{Q}) \simeq A_5$

$\mathcal{K}/\mathbb{Q}$ obtained by solving $\hat{A}_5 \to A_5$

$\psi : G \xrightarrow{\rho} \mathrm{GL}_2(\mathbb{C}) \xrightarrow{\mathrm{Tr}} \mathbb{C}$

$\chi : G \xrightarrow{\rho} \mathrm{GL}_2(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times$ $(\mathrm{Gal}(\mathcal{K}/k) = \mathrm{Ker}(\chi))$

$E := \mathbb{Q}(\psi) = \mathbb{Q}(\sqrt{5}, i)$

$F/\mathbb{Q}$ sextic resolvent of $K/\mathbb{Q}$

$[KF : \mathbb{Q}] = 30$ and $\mathrm{Gal}(\mathcal{K}\cap\mathbb{R}/KF) \simeq C_4$

Some reductions

$E := \mathbb{Q}(\psi) = \mathbb{Q}(\sqrt{5}, i)$, so $\mathrm{Gal}(E/\mathbb{Q}) = \langle \gamma_1, \gamma_2 \rangle$ with

$$\gamma_1(i) = -i, \quad \gamma_1(\sqrt{5}) = \sqrt{5},$$
$$\gamma_2(i) = i, \quad \gamma_2(\sqrt{5}) = -\sqrt{5}.$$

Let $d \in E$,

$$f_d(s) := \sum_{\gamma \in \Gamma} d^\gamma L(s, \rho^\gamma) = 2\Re \left(dL(s, \rho) + d^{\gamma_2} L(s, \rho^{\gamma_2}) \right) = \sum_{n \geq 1} A_n \, n^{-s}$$

with $A_n = \mathrm{Tr}_{E/\mathbb{Q}}(da_n)$, so $A_n \in \mathbb{Z}$ for all $n \geq 1$ iff

$$d \in \mathcal{D}(E)^{-1} = \frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z} + \frac{5+\sqrt{5}}{20}\mathbb{Z} + i\frac{5+\sqrt{5}}{20}\mathbb{Z}.$$

Note: $\varepsilon(m\, d_1) = \varepsilon(d_1)^m$ and $\varepsilon(d_1 + d_2) = \varepsilon(d_1) \times \varepsilon(d_2)$ for $d_1, d_2 \in \mathcal{D}(E)^{-1}$, but also $\varepsilon(id_1) = \varepsilon(d_1)^z$, so:

Conjecture true for all $d \in \mathcal{D}(E)^{-1}$ $\iff$ Conjecture true for $d = \frac{1}{2}$ and $d = \frac{5+\sqrt{5}}{20}$.

Computing L-functions

There exists $W \in \mathbb{C}$ with $|W| = 1$ such that, for any $u > 0$:

$$L'(0,\rho) = \frac{W\sqrt{N}}{2\pi} \sum_{n\geq 1} \frac{\overline{a_n}}{n} \exp\left(-\frac{2\pi n}{u\sqrt{N}}\right) + \sum_{n\geq 1} a_n \mathrm{E_i}\left(\frac{2\pi u n}{\sqrt{N}}\right)$$

where $N :=$ conductor of ρ and $\mathrm{E_i}(x) = \int_x^{+\infty} e^{-t} dt/t$ exponential integral function.

Method: compute the two sums for two different values of u, deduce W (check $|W| = 1$) and then get $L'(0,\rho)$.

Finding the unit

Conjecture gives $||\varepsilon^\sigma||$ for all $\sigma \in G = \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$, but ε, ε^z, $\varepsilon^{z^2} = \varepsilon^{-1}$ and $\varepsilon^{z^3} = \varepsilon^{-z}$ are all *real*. Also, since $\mathrm{Gal}(\mathcal{K} \cap \mathbb{R}/KF) = \langle z \rangle$:

$$\begin{aligned} P(X) &= (X-\varepsilon)(X-\varepsilon^z)(X-\varepsilon^{z^2})(X-\varepsilon^{z^3}) \\ &= X^4 + aX^3 + bX^2 + aX + 1 \qquad \in \mathcal{O}_{KF}[X]. \end{aligned}$$

Need to reconstruct a and b as elements of $\mathcal{O}_{KF}$ with a and b known to a large precision (as real numbers) and with an upper bound for the absolute value of their conjugates over $\mathbb{Q}$.

Method: Use a special quadratic form, also use $\varepsilon^{1/4}$ or $\varepsilon^{1/2}$ whenever it is possible.

Reconstructing an algebraic integer

Let $M = KF$: $M = r_1(M), r_2(M) \subset \mathbb{R}$ and $c_1(M), \ldots, c_{14}(M) \subset \mathbb{C}$.

Problem: Find $a \in \mathcal{O}_M$ such that $|a - \alpha| < \epsilon$ and $|r_2(a)|, |c_j(a)| < C$ for $j = 1, \ldots, 14$.

Define $x^{(1)} := x$, $x^{(2)} := r_2(x)$, $x^{(j)} := \Re(c_{l-2}(x)) + \Im(c_{l-2}(x))$ $(3 \leq j \leq 16)$ and $x^{(j)} := \Re(c_{l-16}(x)) - \Im(c_{l-16}(x))$ $(17 \leq j \leq 30)$.

Let $\mathcal{O}_M = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \cdots + \mathbb{Z}\omega_{30}$ and for set

$$Q(v_0, v_1, \ldots, v_{30}) = C^2 v_0^2 + (C/\epsilon)^2 \left(v_1 \omega_1^{(1)} + \cdots + v_{30}\omega_{30}^{(1)} - v_0 \alpha \right)^2 + \sum_{j=2}^{30} \left(v_1 \omega_1^{(j)} + \cdots + v_{30}\omega_{30}^{(j)} \right)^2$$

- If $a = a_1\omega_1 + \cdots + a_{30}\omega_{30}$ is a solution, then $Q(1, a_1, \ldots, a_{30}) < 31C^2$.
- If $Q(v_0, v_1, \ldots, v_3) < 31C^2$ then $|v_0| \leq 5$, and if furthermore $v_0 = \pm 1$, then $a = v_0(v_1\omega_1 + \cdots + v_{30}\omega_{30})$ satisfies:

$$|a - \alpha| < \sqrt{30}\epsilon, \quad |r_2(a)| < \sqrt{30}C \quad \text{and} \quad |c_j(a)| \leq \sqrt{15}C \ \ (1 \leq j \leq 14).$$

Some further verifications

Let $P(X) = \mathrm{Irr}(\tilde{\varepsilon}, KF)$ and $Q(X) = \mathrm{Irr}(\tilde{\varepsilon}, \mathbb{Q})$.

- Check the roots of $P(X)$ agree with $\exp\left(f'_{d(\psi(\sigma)+\psi(\sigma\tau))}(0)\right)$ for $\sigma \in \langle z \rangle$.
- Check there exists an ordering $\tilde{\varepsilon}_\sigma$ ($\sigma \in G$) on the roots of Q such that

$$\log ||\tilde{\varepsilon}_\sigma|| \simeq f'_{d(\psi(\sigma^{-1})+\psi(\sigma^{-1}\tau))}(0).$$

- Given $J/\mathbb{Q}$ a degree 24 number field such that $KJ = \mathcal{K} \cap \mathbb{R}$, check that $KJ = \mathbb{Q}(\tilde{\varepsilon})$.
- Galois action?

The examples

N	d_k	K	power	+twist?
$1948 = 2^2 \cdot 487$	-487	$X^5 - 7X^3 - 17X^2 + 18X + 73$	$1/4$	No
2083	-2083	$X^5 + 8X^3 + 7X^2 + 172X + 53$	$1/4$	No
$2336 = 2^5 \cdot 73$	$-2^2 \cdot 73$	$X^5 + 2X^3 - 4X^2 - 2X + 4$	$1/2$	χ_{-4}
2707	-2707	$X^5 - 2X^4 + 2X^3 - 8X^2 + 21X - 62$	$1/4$	No
$2863 = 7 \cdot 409$	$-7 \cdot 409$	$X^5 - 2X^4 + 6X^3 + X^2 + 14X + 49$	$1/4$	χ_{-7}
$3004 = 2^2 \cdot 751$	-751	$X^5 - 8X^3 + 10X^2 + 160X + 128$	$1/2$	No
3203	-3203	$X^5 - X^4 - X^3 - 9X^2 + 20X - 11$	$1/4$	No
3547	-3547	$X^5 - 8X^3 - 2X^2 + 31X + 74$	$1/4$	No
$3548 = 2^2 \cdot 887$	-887	$X^5 + 10X^3 + 10X^2 + 44X + 56$	$1/2$	No
$3587 = 17 \cdot 211$	$-17 \cdot 211$	$X^5 - 2X^4 - 3X^3 - 13X^2 + 51X - 17$	$1/4$	χ_{17}
$3676 = 2^2 \cdot 919$	-919	$X^5 - 8X^3 + 28X^2 - 40X + 48$	$1/2$	No

An example: $N = 7 \cdot 409$, $d = (5 + \sqrt{5})/20$

$$
\begin{gathered}
X^{120} + 26X^{119} - 94X^{118} - 13600X^{117} - 281311X^{116} - 3321996X^{115} - 26695600X^{114} - 154147794X^{113} - 633454733X^{112} - 1715520846X^{111} \\
- 3470848290X^{110} - 30730083146X^{109} - 442488173187X^{108} - 4241975499068X^{107} - 29683203312874X^{106} - 160018034060140X^{105} \\
- 663689188701923X^{104} - 2021804070374122X^{103} - 4006528158838154X^{102} - 4884414082098948X^{101} - 3658168442792465 7X^{100} \\
- 487528951945104522X^{99} - 3783005384356929126X^{98} - 20403327017395397024X^{97} - 84955151957338226945X^{96} - 294293366619039740386X^{95} \\
- 913545334089156134170X^{94} - 2697514791586967527902X^{93} - 7663680434531394529531X^{92} - 20456291234678745439466X^{91} \\
- 51289870206760479995338X^{90} - 127082471589904301709030X^{89} - 325889924154119365692755X^{88} - 836465690752396946216978X^{87} \\
- 1996086627778243950531172X^{86} - 4292965781360113755325812X^{85} - 8612234037145139046881703X^{84} - 17205248221768863457571082X^{83} \\
- 34980994234137834004988972X^{82} - 69290016038721990395949600X^{81} - 128003082848154913592158939X^{80} - 220610572593979107867316098X^{79} \\
- 364225668247380528514085586X^{78} - 578852895069490862382330272X^{77} - 850511718430215025448509179X^{76} - 1081353097405617020513141512X^{75} \\
- 1096185494282559030925178618X^{74} - 742769471405376488763892122X^{73} - 150339344175747387360456 91X^{72} + 914406880475444830539894282X^{71} \\
+ 1775424928237598246145749032X^{70} + 2320289738861916855748839350X^{69} + 2425873616614254748075556939X^{68} \\
+ 2187960822311042822094511360X^{67} + 1983824193814413999886771962X^{66} + 2372728685456958572223960652X^{65} \\
+ 3760223467003604255883124473X^{64} + 6039722171959017044736565850X^{63} + 8564829773982177534419666602X^{62} \\
+ 10499011099495497101938960668X^{61} + 11217034388314946629995466683X^{60} + 10499011099495497101938960668X^{59} \\
+ 8564829773982177534419666602X^{58} + 6039722171959017044736565850X^{57} + 3760223467003604255883124473X^{56} \\
+ 2372728685456958572223960652X^{55} + 1983824193814413999886771962X^{54} + 2187960822311042822094511360X^{53} \\
+ 2425873616614254748075556939X^{52} + 2320289738861916855748839350X^{51} + 1775424928237598246145749032X^{50} + 914406880475444830539894282X^{49} \\
- 150339344175747387360456 91X^{48} - 742769471405376488763892122X^{47} - 1096185494282559030925178618X^{46} - 1081353097405617020513141512X^{45} \\
- 850511718430215025448509179X^{44} - 578852895069490862382330272X^{43} - 364225668247380528514085586X^{42} - 220610572593979107867316098X^{41} \\
- 128003082848154913592158939X^{40} - 69290016038721990395949600X^{39} - 34980994234137834004988972X^{38} - 17205248221768863457571082X^{37} \\
- 8612234037145139046881703X^{36} - 4292965781360113755325812X^{35} - 1996086627778243950531172X^{34} - 836465690752396946216978X^{33} \\
- 325889924154119365692755X^{32} - 127082471589904301709030X^{31} - 51289870206760479995338X^{30} - 20456291234678745439466X^{29} \\
- 7663680434531394529531X^{28} - 2697514791586967527902X^{27} - 913545334089156134170X^{26} - 294293366619039740386X^{25} \\
- 84955151957338226945X^{24} - 20403327017395397024X^{23} - 3783005384356929126X^{22} - 487528951945104522X^{21} \\
- 3658168442792465 7X^{20} - 4884414082098948X^{19} - 4006528158838154X^{18} - 2021804070374122X^{17} - 663689188701923X^{16} \\
- 160018034060140X^{15} - 29683203312874X^{14} - 4241975499068X^{13} - 442488173187X^{12} - 30730083146X^{11} - 3470848290X^{10} \\
- 1715520846X^{9} - 633454733X^{8} - 154147794X^{7} - 26695600X^{6} - 3321996X^{5} - 281311X^{4} - 13600X^{3} - 94X^{2} + 26X + 1
\end{gathered}
$$