ON THE SEMI-SIMPLE CASE OF THE GALOIS BRUMER-STARK CONJECTURE FOR MONOMIAL GROUPS

X.-F. ROBLOT

Abstract. In a previous work, we stated a conjecture, called the Galois Brumer-Stark conjecture, that generalizes the (abelian) Brumer-Stark conjecture to Galois extensions. Other generalizations of the Brumer-Stark conjecture to non-abelian Galois extensions are due to Nickel. Nomura proved that the Brumer-Stark conjecture implies the weak non-abelian Brumer-Stark conjecture of Nickel when the group is monomial. In this paper, we use the methods of Nomura to prove that the Brumer-Stark conjecture implies the Galois Brumer-Stark conjecture for monomial groups in the semi-simple case.

1. Introduction

Let K/k be an abelian extension of number fields. The Brumer-Stark conjecture [14] predicts that a group ring element, called the Brumer-Stickelberger element, constructed from special values of L-functions associated to K/k, annihilates (after multiplication by a suitable factor) the ideal class group of K and specifies special properties for the generators obtained. In [5], we introduced a generalization of the conjecture to Galois extensions, called the Galois Brumer-Stark conjecture. Later, in [6], we introduced a refined version of the conjecture that focused on the contribution of the non-linear irreducible characters. Since the new version in [6] supersedes the version in [5], we will from now on refer to it as the Galois Brumer-Stark conjecture (and not call it anymore the refined Galois Brumer-Stark conjecture). Also, to avoid confusion, we call the original conjecture the abelian Brumer-Stark conjecture.

In [10], Nickel introduced another generalization of the abelian Brumer-Stark conjecture to Galois extensions and in [12], Nomura proved, among other things, that the abelian Brumer-Stark conjecture implies the (weak) non-abelian Brumer-Stark conjecture of Nickel when the Galois group of K/k is monomial. In this paper, we adapt the method used by Nomura to prove that the abelian Brumer-Stark conjecture implies the Galois Brumer-Stark conjecture in the semi-simple case when the Galois group of K/k is monomial (see Theorem 3.1). Furthermore, using the fact that the local abelian Brumer-Stark conjecture is known to hold in several cases, we prove unconditionally some cases of the local Galois Brumer-Stark conjecture (see Corollary 3.3).

2. The Galois Brumer-Stark conjecture

Before stating the Galois Brumer-Stark conjecture, we recall the statement of the abelian Brumer-Stark conjecture, see [15, IV.§6] or [14]. Let K/k be an abelian extension of number fields. Denote by G its Galois group. Let S be a finite set of places of k containing the infinite places and the finite places ramified in K. To simplify matters, we assume that the cardinality of S is at least 2. The interested reader can refer to [15, Sec. IV.§6] for the statement of the conjecture when $|S| = 1$. For $\chi \in \hat{G}$, where $\hat{G}$ denotes the group of irreducible characters of G, denote by $L_{K/k,S}(s, \chi)$ the Hecke L-function of the character χ with Euler factors associated to prime ideals in S deleted. The Brumer-Stickelberger element associated to the extension K/k and the set S is defined by

$$\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \chi)\, e_{\bar{\chi}}$$

where $e_\chi := \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$ is the idempotent of χ. It follows from [7] (see also [4]) that

$$\xi\, \theta_{K/k,S} \in \mathbb{Z}[G] \tag{2.1}$$

for any $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, the annihilator in $\mathbb{Z}[G]$ of the group μ_K of roots of unity in K. In particular, we have $w_K \theta_{K/k,S} \in \mathbb{Z}[G]$ where w_K denotes the cardinality of μ_K. We need one last notation before stating the abelian Brumer-Stark conjecture. We say that a non-zero element $\alpha \in K$ is an anti-unit if all its conjugates have absolute value equal to 1. The group of anti-units of K is denoted by K°.

Conjecture 2.1 (The abelian Brumer-Stark conjecture $\mathbf{BS}(K/k, S)$)**.** *For any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{w_K \theta_{K/k,S}}$ is principal and admits a generator $\alpha \in K^\circ$ such that $K(\alpha^{1/w_K})/k$ is abelian.*

We refer to [5, §2] for a review of the current state of the abelian Brumer-Stark conjecture. The following consequence of the abelian Brumer-Stark conjecture will be useful later on. (The conclusion of the proposition is known as the Brumer conjecture; thus, the proposition just states the well-known fact that the Brumer-Stark conjecture implies the Brumer conjecture.)

Proposition 2.2. *Assume that the abelian Brumer-Stark conjecture $\mathbf{BS}(K/k, S)$ holds. Let $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$. Then, $\xi\, \theta_{K/k,S}$ annihilates the class group Cl_K of K.*

Proof. Under the assumption that $\mathbf{BS}(K/k, S)$ holds, there exists by [14, Proposition §2] a family $(a_i)_{i \in I}$ of elements of $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, generating $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, and such that $a_i\, \theta_{K/k,S}$ annihilates Cl_K for all $i \in I$. The result follows directly. □

We now introduce the Galois Brumer-Stark conjecture (more precisely, as noted in the introduction, the refined version stated in [6]). Assume now that K/k is a Galois extension of number fields. Denote by G its Galois group. Let S be a finite set of places of k containing the infinite places and the finite places ramified in K. Assume that the cardinality of S is at least 2. Denote by $\hat{G}^{(>1)}$ the set of non-linear irreducible characters of G and define the non-linear Brumer-Stickelberger element by

$$\theta^{(>1)}_{K/k,S} := \sum_{\chi \in \hat{G}^{(>1)}} L_{K/k,S}(0, \chi) e_{\bar{\chi}} \tag{2.2}$$

where, for $\chi \in \hat{G}^{(>1)}$, $L_{K/k,S}(s, \chi)$ denotes the Artin L-function of χ with Euler factors associated to prime ideals in S deleted, and

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$$

is the central idempotent of χ. It follows from the principal rank zero Stark conjecture, proved by Tate [15], that the non-linear Brumer-Stickelberger element lies in $\mathbb{Q}[G]$. Denote by $[G,G]$ the commutator subgroup of G, i.e., the subgroup of G generated by the commutators $[g_1,g_2] := g_1g_2g_1^{-1}g_2^{-1}$ with $g_1,g_2 \in G$. Let $G^{\mathrm{ab}} := G/[G,G]$ be the maximal abelian quotient of G and let $K^{\mathrm{ab}} := K^{[G,G]}$ be the maximal sub-extension of K/k that is abelian over k; we have $\mathrm{Gal}(K^{\mathrm{ab}}/k) = G^{\mathrm{ab}}$. Let s_G denote the order of $[G,G]$, let m_G be the lcm of the cardinalities of the conjugacy classes of G, and let d_G be the lcm of m_G and s_G.

Conjecture 2.3 (The Galois Brumer-Stark conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$)**.** *Let K/k be a Galois extension of number fields and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S| \geq 2$. Then, $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ holds, we have $d_G\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$ and, for any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{d_G\,\theta^{(>1)}_{K/k,S}}$ is principal and admits a generator in K°.*

For p a prime, denote by $\mathrm{Cl}_K\{p\}$ the p-part of Cl_K, that is the subgroup of Cl_K of classes of p-power order.

Conjecture 2.4 (The local Galois Brumer-Stark conjecture $\mathbf{BS}^{(p)}_{\mathrm{Gal}}(K/k,S)$)**.** *Let K/k be a Galois extension of number fields and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S| \geq 2$. Then, the local abelian Brumer-Stark conjecture at p for the extension K^{ab}/k and the set of places S holds, we have $d_G\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}_p[G]$ and, for any fractional ideal $\mathfrak{A}$ of K whose class lies in $\mathrm{Cl}_K\{p\}$, the ideal $\mathfrak{A}^{d_G\,\theta^{(>1)}_{K/k,S}}$ is principal and admits a generator in K°.*

The statement of the local abelian Brumer-Stark conjecture at p is that, for all ideals $\mathfrak{A}$ whose class lies in $\mathrm{Cl}_K\{p\}$, the ideal $\mathfrak{A}^{w_K\theta_{K/k,S}}$ is principal and admits a generator $\alpha \in K^\circ$ such that $K(\alpha^{1/w_{K,p}})/k$ is abelian where $w_{K,p}$ is the order of the p-part of the group of roots of unity in K, see [8]. One checks readily that the Galois Brumer-Stark conjecture is equivalent to the local Galois Brumer-Stark conjecture at p for all primes p. Some evidence for these conjectures is given in [6]. Relations between the Galois Brumer-Stark conjecture and the weak non-abelian Brumer-Stark conjecture of Nickel are discussed in the appendix of [5]. To conclude this section, we prove that the local versions of the weak non-abelian Brumer-Stark conjecture and of the Galois Brumer-Stark conjecture are equivalent for primes p not dividing $w_K|G|$. First, we recall briefly the statement of the local weak non-abelian Brumer-Stark conjecture, see [10] for more details.

Let K/k be a Galois CM-extension with group G. Let S be a finite set of places of k such that S contains the infinite places of k and the finite places of k that ramify in K/k. Let $\mathrm{Hyp}(S)$ denote the set of finite sets T of places of k such that: S and T are disjoint, and the group $E_K(S,T)$ is torsion-free. Here, $E_K(S,T)$ denotes the group of (S,T)-units of K, that is the group of elements $u \in K^\times$ such that $v_{\mathfrak{P}}(u) = 0$ for all prime ideals $\mathfrak{P}$ of K such that $(\mathfrak{P} \cap k) \notin S$ and $u \equiv 1 \pmod{\mathfrak{Q}}$ for all prime ideals $\mathfrak{Q}$ of K such that $(\mathfrak{Q} \cap k) \in T$. For $T \in \mathrm{Hyp}(S)$, define

$$\delta_T := \mathrm{nr}\Big(\prod_{\mathfrak{p}\in T} 1 - \sigma_{\mathfrak{P}}^{-1}\mathcal{N}(\mathfrak{p})\Big)$$

where $\mathfrak{P}$ is a fixed prime ideal of K above $\mathfrak{p}$, $\sigma_{\mathfrak{P}}$ is the Frobenius element of $\mathfrak{P}$ in G, and $\mathrm{nr} : \mathbb{Q}[G] \to Z(\mathbb{Q}[G])$ is the reduced norm (see [13, §9]). Let Λ' denote a fixed maximal order of $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$ and denote by $\mathfrak{F}(G) := \{x \in Z(\Lambda') : x\Lambda' \subset \mathbb{Z}[G]\}$ the central conductor of Λ' over $\mathbb{Z}[G]$.

Conjecture 2.5 (The local weak non-abelian Brumer-Stark conjecture [10]). *Let $\mathfrak{w}_K := \mathrm{nr}(w_K)$. Then $\mathfrak{w}_K\theta_{K/k,S} \in Z(\Lambda')\otimes\mathbb{Z}_p$. Furthermore, for any fractional ideal $\mathfrak{A}$ of K whose class lies in $\mathrm{Cl}_K\{p\}$ and for each $x \in \mathfrak{F}(G)$, there exists an anti-unit $\alpha_x \in K^\circ$ such that*

$$\mathfrak{A}^{x\mathfrak{w}_K\theta_{K/k,S}} = \alpha_x\mathcal{O}_K$$

and, for any set of places $T \in \mathrm{Hyp}(S\cup S_{\alpha_x})$, there exists $\alpha_{x,T} \in E_K(S_{\alpha_x}, T)$ such that, for all $z \in \mathfrak{F}(G)$

$$\alpha_x^{z\delta_T} = \alpha_{x,T}^{z\mathfrak{w}_K}$$

where S_{α_x} is the set of prime ideals $\mathfrak{p}$ of k such that $v_\mathfrak{p}(N_{K/k}(\alpha_x)) \neq 0$.

Observe that the conjecture stated above is slightly different from the original conjecture given by Nickel in [10]. Indeed, Nickel does not state explicitly the local conjecture in this paper but writes instead that one should restrict to ideals whose class lies in $\mathrm{Cl}_K\{p\}$ in the global conjecture to get the local conjecture at p. In particular, the local conjecture does not have an specific statement on where $\mathfrak{w}_K\theta_{K/k,S}$ should lie. However, it seems reasonable to only asks for $\mathfrak{w}_K\theta_{K/k,S}$ to be in $Z(\Lambda')\otimes\mathbb{Z}_p$ in this case.

Theorem 2.6. *Let K/k be a Galois CM-extension of number fields with Galois group G and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S| \geq 2$. Let p be a prime number not dividing $w_K|G|$. Then, the local Galois Brumer-Stark conjecture $\mathbf{BS}^{(p)}_{\mathrm{Gal}}(K/k, S)$ is equivalent to the local weak non-abelian Brumer-Stark conjecture at p for the extension K/k and the set of prime ideals S.*

Proof. We will use the following fact several times whose proof is direct and left to the reader: let t be an integer not divisible by p and let H be a group of fractional ideals containing the principal ideals and all the ideals $\mathfrak{B}^t$ where $\mathfrak{B}$ runs through the fractional ideals of K whose class lies in $\mathrm{Cl}_K\{p\}$. Then, H is the group of fractional ideals whose class lies in $\mathrm{Cl}_K\{p\}$.

Assume that $\mathbf{BS}^{(p)}_{\mathrm{Gal}}(K/k, S)$ holds. Then, $\theta_{K/k,S} \in Z(\mathbb{Z}_p[G])$ and therefore we have $\mathfrak{w}_K\theta_{K/k,S} \in Z(\Lambda')\otimes\mathbb{Z}_p = Z(\mathbb{Z}_p[G])$. Let $\mathfrak{B}$ be a fractional ideal of K whose class lies in $\mathrm{Cl}_K\{p\}$. By [5, Prop. A.1], for any $x \in \mathfrak{F}(G)$, there exists $\beta_x \in K^\circ$ such that

$$\mathfrak{B}^{d_G x\mathfrak{w}_K\theta_{K/k,S}} = \beta_x\mathcal{O}_K$$

and, for any set of places $T \in \mathrm{Hyp}(S\cup S_{\beta_x})$, there exists $\beta_{x,T} \in K^\times$ with $\beta_{x,T}^{w_K} \in E_K(S_{\beta_x}, T)$ such that, for all $z \in \mathfrak{F}(G)$

$$\beta_x^{z\delta_T} = \beta_{x,T}^{z\mathfrak{w}_K}.$$

Observe that the proof of [5, Prop. A.1] uses the original formulation of the (global) Galois Brumer-Stark conjecture but that is not a concern since the refined version that we use now implies the original conjecture; furthermore, one can check readily that the local version of the conjecture is enough for the proof of the result in this case. Let $\mathfrak{A} := \mathfrak{B}^{d_G w_K}$. We set $\alpha_x := \beta_x^{w_K}$ and $\alpha_{x,T} := \beta_{x,T}^{w_K} \in E_K(S_{\beta_x}, T) = E_K(S_{\alpha_x}, T)$ for all $T \in \mathrm{Hyp}(S\cup S_{\beta_x}) = \mathrm{Hyp}(S\cup S_{\alpha_x})$. Then, it is direct to check that these elements satisfy the required properties for the statement of the local weak non-abelian conjecture to be satisfied for the ideal $\mathfrak{A}$. Since it is proved in [10] that the set of ideals satisfying the properties of the weak non-abelian Brumer-Stark conjecture is a group containing the principal ideals, it follows by the above remark that the local weak non-abelian Brumer-Stark conjecture holds at p for the extension K/k and the set S.

Reciprocally, assume that the local weak non-abelian Brumer-Stark conjecture holds at p for the extension K/k and the set S. We first prove that this implies that the local abelian Brumer-Stark conjecture holds at p for the extension $K^{\rm ab}/k$ and the set S. Let $\mathfrak{b}$ be a fractional ideal of $K^{\rm ab}$ whose class lies in $\mathrm{Cl}_{K^{\rm ab}}\{p\}$, thus the class of $\mathfrak{b}\mathcal{O}_K$ is in $\mathrm{Cl}_K\{p\}$. Thanks to [13, Th. 41.1], we can take $x = |G| \in \mathcal{F}(G)$ in the local weak non-abelian Brumer-Stark conjecture, and thus there exists $\beta_0 \in K^\circ$ such that $(\mathfrak{b}\mathcal{O}_K)^{|G|\mathfrak{w}_K\theta_{K/k,S}} = \beta_0\mathcal{O}_K$. Taking norms down to $K^{\rm ab}$ and using the properties of the Brumer-Stickelberger element, we deduce that

$$\mathfrak{b}^{s_G|G|w_K\theta_{K^{\rm ab}/k,S}} = \mathfrak{a}^{w_K\theta_{K^{\rm ab}/k,S}} = \alpha_0\mathcal{O}_{K^{\rm ab}}$$

where $\mathfrak{a} := \mathfrak{b}^{s_G|G|}$ and $\alpha_0 := N_{K/K^{\rm ab}}(\beta_0) \in (K^{\rm ab})^\circ$. Now, since p does not divide w_K, we have $w_{K,p} = 1$ and thus $K^{\rm ab}(\alpha_0^{1/w_{K,p}}) = K^{\rm ab}$ is abelian over k. The set of ideals that satisfy the local abelian Brumer-Stark conjecture is a group containing the principal ideals therefore, by the remark at the start of the proof, the local abelian Brumer-Stark conjecture holds at p for the extension $K^{\rm ab}/k$ and the set S. To prove the part of the statement of the local Galois Brumer-Stark conjecture concerning the non-linear Brumer-Stickelberger element, we proceed in a similar way. Observe that it follows from [5, Eq. (12)] that one can write $\theta_{K/k,S} = \theta_0 + \theta^{(>1)}_{K/k,S}$ with $s_G\, w_K\theta_0 \in \mathbb{Z}[G]$. In particular, we have $\theta_0 \in \mathbb{Z}_p[G]$ and thus $\theta^{(>1)}_{K/k,S} \in \mathbb{Z}_p[G]$. Now, let $\mathfrak{B}$ be a fractional ideal of K whose class is in $\mathrm{Cl}_K\{p\}$. Let ℓ be the maximum of the $\chi(1)$'s for $\chi \in \hat{G}$. Let

$$x = |G|^2 \sum_{\chi\in\hat{G}^{(>1)}} w_K^{\ell-\chi(1)} e_\chi \in |G|\mathbb{Z}[G].$$

As noted above $|G| \in \mathcal{F}(G)$, therefore $x \in \mathcal{F}(G)$ and there exists $\alpha \in K^\circ$ such that $\mathfrak{B}^{x\mathfrak{w}_K\theta_{K/k,S}} = \alpha\mathcal{O}_K$. Observe that

$$x\mathfrak{w}_K\theta_{K/k,S} = |G|^2 w_K^\ell \theta^{(>1)}_{K/k,S}.$$

Let $\mathfrak{A} := \mathfrak{B}^{|G|^2 w_K^\ell/d_G}$. Then, we have $\mathfrak{A}^{d_G\theta^{(>1)}_{K/k,S}} = \alpha\mathcal{O}_K$. Since the set of ideals that satisfy the statement of the local Galois Brumer-Stark conjecture is group containing the principal ideals, it follows that the local Galois Brumer-Stark conjecture holds at p for the extension K/k and the set of places S. This concludes the proof. □

3. The semi-simple case for monomial group

In this section, we prove the main result of this paper.

Theorem 3.1. *Let K/k be a Galois extension of number fields with Galois group G and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S| \geq 2$. Assume that the group G is monomial and that the abelian Brumer-Stark conjecture* $\mathbf{BS}(E/F, S_F)$ *holds for any abelian extension E/F contained in K/k where S_F denotes the set of places of F above* *the places in S. Let p be a prime number such that $p \nmid |G|$. Then, the local Galois Brumer-Stark conjecture $\mathbf{BS}^{(p)}_{\mathrm{Gal}}(K/k,S)$ holds.*

Proof. In order to prove Theorem 3.1, it is enough to prove the following two facts:

(1) $\theta^{(>1)}_{K/k,S} \in \mathbb{Z}_p[G]$,
(2) $\theta^{(>1)}_{K/k,S}$ annihilates $\mathrm{Cl}_K\{p\}$.

Indeed, d_G is a divisor of G and thus it is invertible in $\mathbb{Z}_p[G]$, therefore $d_G\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}_p[G]$ if and only if $\theta^{(>1)}_{K/k,S} \in \mathbb{Z}_p[G]$. Now, assume that $\theta^{(>1)}_{K/k,S}$ annihilates $\mathrm{Cl}_K\{p\}$ then, by the previous remark, that means that $d_G\,\theta^{(>1)}_{K/k,S}$ also annihilates $\mathrm{Cl}_K\{p\}$. The only thing remaining to prove is that, for any ideal $\mathfrak{A}$ of K whose class lies in $\mathrm{Cl}_K\{p\}$, one can find a generator of $\mathfrak{A}^{d_G\,\theta^{(>1)}_{K/k,S}}$ that is an anti-unit. But, since p is odd (otherwise the conjecture is trivially true, see Remark 2.2 of [6]), this is always possible using the trick explained on page 299 of [8].

We now prove the two assertions. As we mentioned in the introduction, the method we use is a direct adaptation of the method used by Nomura in [12]. Since the result is trivial if G is abelian, we assume from now on that G is non-abelian. Let ν be a character defined on some subgroup H_ν of G. (From now on, we will always use the notation H_ν to denote the subgroup of G on which the character ν is defined.) For $g \in G$, define the character $\nu[g]$ of $g^{-1}H_\nu g$ by $\nu[g](x) := \nu(gxg^{-1})$ for all $x \in g^{-1}H_\nu g$. Note that $\chi[g] = \chi[g']$ if g and g' are in the same right coset of G modulo H_ν. Observe that $\nu^G = (\nu[g])^G$ for all $g \in G$, where we denote by ν^G the induced character of ν on G (see [9, Chapter 5]). In particular, for $\chi \in \hat{G}^{(>1)}$, the group G acts on the set of linear characters ν defined on some subgroup H_ν of G and such that $\nu^G = \chi$. (This is a non-empty set by hypothesis.) We denote by $\Omega(\chi)$ a fixed orbit of this set under the action of G. Then, we have

$$\chi = \sum_{\nu \in \Omega(\chi)} \dot{\nu} \tag{3.3}$$

where $\dot{\nu}$ denotes the function of G obtained by setting $\dot{\nu}(x) := \nu(x)$ if $x \in H_\nu$ and $\dot{\nu}(x) := 0$ otherwise. In particular, it follows that

$$e_\chi = \sum_{\nu \in \Omega(\chi)} e_\nu. \tag{3.4}$$

For $\nu \in \Omega(\chi)$, we denote by $\pi_\nu : H_\nu \to H_\nu/\operatorname{Ker}(\nu)$ the canonical surjection and by $\hat{\nu}$ the unique linear character of $H_\nu/\operatorname{Ker}(\nu)$ such that $\nu = \hat{\nu} \circ \pi_\nu$. We also associate to ν two extensions: $E_\nu := K^{\operatorname{Ker}(\nu)}$ and $F_\nu := K^{H_\nu}$. Thus, E_ν/F_ν is a cyclic extension with Galois group isomorphic to $H_\nu/\operatorname{Ker}(\nu)$. Finally, we define S_ν to be the set of places of F_ν above the places in S. From the properties of Artin L-functions, we see that

$$\chi(\theta^{(>1)}_{K/k,S}) = L_{K/k,S}(\bar{\chi},0) = L_{K/F_\nu,S_\nu}(\bar{\nu},0) = L_{E_\nu/F_\nu,S_\nu}(\bar{\hat{\nu}},0) = \hat{\nu}(\theta_{E_\nu/F_\nu,S_\nu}).$$

Let $\Omega := \bigcup_{\chi \in \hat{G}^{(>1)}} \Omega(\chi)$ (note that it is a disjoint union). Combining the previous equalities with (2.2) and (3.4), we obtain the following identity (which is the non-linear equivalent of [12, Lemma 4.4])

$$\theta^{(>1)}_{K/k,S} = \sum_{\nu \in \Omega} \hat{\nu}(\theta_{E_\nu/F_\nu,S_\nu})\, e_\nu. \tag{3.5}$$

The following result plays a crucial role in the proof.

Lemma 3.2. *Let $\nu \in \Omega$. Define $T_\nu := \pi_\nu(H_\nu \cap [G,G])$ and*

$$\mathcal{A}_\nu := \sum_{c \in T_\nu} (1 - c).$$

Then, the element $\mathcal{A}_\nu$ annihilates the group of roots of unity of E_ν and we have $\hat{\nu}(\mathcal{A}_\nu) = t_\nu$ where $t_\nu := |T_\nu|$.

Proof of the lemma. The first assertion follows from the fact that elements of T_ν are image of elements of $[G,G]$, thus they act trivially on roots of unity. For the second assertion, fix $g \in G \setminus H_\nu$ (g exists since χ is non-linear). Since ν^G is irreducible, it follows from Mackey's irreducibility criterion that the restriction to $H_\nu \cap g^{-1} H_\nu g$ of ν and $\nu[g]$ do not have a common irreducible constituent. Since the characters ν and $\nu[g]$ are linear characters, this implies that there exists $h \in H_\nu \cap g^{-1} H_\nu g$ such that $\nu[g](h) \neq \nu(h)$, i.e., $\nu(ghg^{-1}h^{-1}) \neq 1$. Therefore, $H_\nu \cap [G,G]$ is not contained in the kernel of ν and T_ν is non-trivial. It follows that

$$\hat{\nu}(\mathcal{A}_\nu) = t_\nu - \sum_{c \in T_\nu} \hat{\nu}(c) = t_\nu$$

and the lemma is proved. □

We prove the first assertion, that is $\theta^{(>1)}_{K/k,S} \in \mathbb{Z}_p[G]$. Let $\nu \in \Omega$. First, note that $t_\nu \hat{\nu}(\theta_{E_\nu/F_\nu,S_\nu}) = \hat{\nu}(\mathcal{A}_\nu\, \theta_{E_\nu/F_\nu,S_\nu})$ is an algebraic integer. Indeed, since the extension E_ν/F_ν is abelian, it follows from (2.1) and Lemma 3.2 that $\mathcal{A}_\nu\, \theta_{E_\nu/F_\nu,S_\nu}$ lies in $\mathbb{Z}[H_\nu/\operatorname{Ker}(\nu)]$. But, the integer t_ν divides $|G|$ and therefore $|G|\, \hat{\nu}(\theta_{E_\nu/F_\nu,S_\nu})$ is an algebraic integer for all $\nu \in \Omega$. Since $|G| e_\nu$ is also an algebraic integer for all $\nu \in \Omega$, we deduce using (3.5) that the coefficients of $|G|^2 \theta^{(>1)}_{K/k,S}$ are all algebraic integers and, since it is rational, it lies in $\mathbb{Z}[G]$. Finally, since p does not divide $|G|$, we get that $\theta^{(>1)}_{K/k,S} \in \mathbb{Z}_p[G]$.

We prove the second assertion, i.e., $\theta^{(>1)}_{K/k,S}$ annihilates $\mathrm{Cl}_K\{p\}$. Let ν be a character of a subgroup H_ν of G and let $\sigma \in \Gamma := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We denote by ν^σ the character of H_ν defined by $\nu^\sigma(x) := \sigma(\nu(x))$ for all $x \in H_\nu$. The group Γ acts on ν via its quotient $\Gamma(\nu) := \Gamma/\operatorname{Stab}_\Gamma(\nu)$ which is also the Galois group of $\mathbb{Q}(\nu)/\mathbb{Q}$ where $\mathbb{Q}(\nu)$ is the extension of $\mathbb{Q}$ generated by the values of ν. Assume now that $\nu \in \Omega(\chi)$ with $\chi \in \hat{G}^{(>1)}$. We see that $\Gamma(\nu) = \Gamma(\hat{\nu})$, but $\Gamma(\chi)$ is a quotient of $\Gamma(\nu)$ where $\chi := \nu^G$ (although we will not use this fact). Observe that, for $\sigma \in \Gamma(\nu)$, we have $H_{\nu^\sigma} = H_\nu$, $\operatorname{Ker}(\nu^\sigma) = \operatorname{Ker}(\nu)$, $\pi_{\nu^\sigma} = \pi_\nu$, $T_{\nu^\sigma} = T_\nu$, $\mathcal{A}_{\nu^\sigma} = \mathcal{A}_\nu$, $E_{\nu^\sigma} = E_\nu$, $F_{\nu^\sigma} = F_\nu$, and $S_{\nu^\sigma} = S_\nu$. Let Ω_0 be a set of representatives of Ω under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Then, we can rewrite equation (3.5) as

$$\theta^{(>1)}_{K/k,S} = \sum_{\nu \in \Omega_0} \sum_{\sigma \in \Gamma(\nu)} \hat{\nu}^\sigma(\theta_{E_\nu/F_\nu,S_\nu})\, e_{\nu^\sigma}. \tag{3.6}$$

Now, for $\nu \in \Omega_0$ and $\sigma \in \Gamma(\nu)$, one checks readily that $e_{\nu^\sigma} = \iota_{\nu^\sigma} \mathcal{N}_{\operatorname{Ker}(\nu)}$ where $\iota_\nu \in \bar{\mathbb{Q}}[H_\nu]$ is such that $\pi_\nu(\iota_{\nu^\sigma}) = e_{\hat{\nu}^\sigma}$ and $\mathcal{N}_{\operatorname{Ker}(\nu)} := \sum_{x \in \operatorname{Ker}(\nu)} x$. Thus, we have

$$\sum_{\sigma \in \Gamma(\nu)} \hat{\nu}^\sigma(\theta_{E_\nu/F_\nu,S_\nu})\, e_{\nu^\sigma} = \Bigg(\sum_{\sigma \in \Gamma(\nu)} \hat{\nu}^\sigma(\theta_{E_\nu/F_\nu,S_\nu})\, \iota_{\nu^\sigma}\Bigg) \mathcal{N}_{\operatorname{Ker}(\nu)}.$$

Let $\mathcal{C}$ be a class in Cl_K of p-power order. We compute

$$\begin{aligned} t_\nu \sum_{\sigma\in\Gamma(\nu)} \hat\nu^\sigma(\theta_{E_\nu/F_\nu,S_\nu})\, e_{\nu^\sigma}\,\mathcal{C} &= \Big(\sum_{\sigma\in\Gamma(\nu)} t_\nu\, \hat\nu^\sigma(\theta_{E_\nu/F_\nu,S_\nu})\, \iota_{\nu^\sigma}\Big)\mathcal{N}_{\mathrm{Ker}(\nu)}\,\mathcal{C}\\ &= \Big(\sum_{\sigma\in\Gamma(\nu)} \hat\nu^\sigma(\mathcal{A}_\nu\,\theta_{E_\nu/F_\nu,S_\nu})\, \iota_{\nu^\sigma}\Big)N_{K/E_\nu}(\mathcal{C})\\ &= \pi_\nu\Big(\sum_{\sigma\in\Gamma(\nu)} \hat\nu^\sigma(\mathcal{A}_\nu\,\theta_{E_\nu/F_\nu,S_\nu})\, \iota_{\nu^\sigma}\Big)N_{K/E_\nu}(\mathcal{C})\\ &= \Big(\sum_{\sigma\in\Gamma(\nu)} \hat\nu^\sigma(\mathcal{A}_\nu\,\theta_{E_\nu/F_\nu,S_\nu})\, e_{\hat\nu^\sigma}\Big)N_{K/E_\nu}(\mathcal{C})\\ &= \Big(\sum_{\sigma\in\Gamma(\nu)} \mathcal{A}_\nu\,\theta_{E_\nu/F_\nu,S_\nu}\, e_{\hat\nu^\sigma}\Big)N_{K/E_\nu}(\mathcal{C})\\ &= \mathcal{A}_\nu\,\theta_{E_\nu/F_\nu,S_\nu}\Big(\sum_{\sigma\in\Gamma(\nu)} e_{\hat\nu^\sigma}\Big)N_{K/E_\nu}(\mathcal{C}). \end{aligned}$$

The element $\sum_{\sigma\in\Gamma(\nu)} e_{\hat\nu^\sigma}$ is p-integral and thus $\big(\sum_{\sigma\in\Gamma(\nu)} e_{\hat\nu^\sigma}\big)N_{K/E_\nu}(\mathcal{C})$ is well-defined and is a class in $\mathrm{Cl}_{E_\nu}\{p\}$. But, by Lemma 3.2 and Proposition 2.2, $\mathcal{A}_\nu\,\theta_{E_\nu/F_\nu,S_\nu}$ annihilates Cl_{E_ν}. Since t_ν is prime to p, we deduce that the element $\sum_{\sigma\in\Gamma(\nu)} \hat\nu^\sigma(\theta_{E_\nu/F_\nu,S_\nu})\, e_{\hat\nu^\sigma}$ kills $\mathrm{Cl}_K\{p\}$. This is true for all $\nu\in\Omega_0$, hence we get by (3.6) that $\theta^{(>1)}_{K/k,S}$ annihilates $\mathrm{Cl}_K\{p\}$. This concludes the proof of Theorem 3.1. □

The local abelian Brumer-Stark conjecture is known to hold unconditionally in many cases. Combining Theorem 3.1 and several results in [1] by Burns and Flach, and in [2] and [3] by Burns, Kurihara, and Sano, we can thus deduce cases where the local Galois Brumer-Stark conjecture is satisfied.

Corollary 3.3. *Let K/k be a Galois CM-extension of number fields and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S|\geq 2$. Assume that $\mathrm{Gal}(K/k)$ is monomial. Then, for any odd prime p such that p does not divide $[K:k]$ and at least one of the two following condition is satisfied:* (1) *p is unramified in $K/\mathbb{Q}$, or* (2) *at most one prime ideal of k above p splits in K/K^+, the local Galois Brumer-Stark conjecture $\mathbf{BS}^{(p)}_{\mathrm{Gal}}(K/k,S)$ holds.*

Proof. Let E/F be an abelian CM-extension of number fields. It is known that the abelian Brumer-Stark conjecture for the extension E/F follows from the equivariant Tamagawa number conjecture [1] for the pair $(h^0(\mathrm{Spec}(E)),\mathbb{Z}[H])$ where $H:=\mathrm{Gal}(E/F)$. For example, using the results of [3], we get that this special case of the equivariant Tamagawa number conjecture is equivalent to Conjecture 3.1 of *ibid* by Remark 3.2 of *ibid*, which is in turn equivalent to the 'leading term conjecture' (Conjecture 3.6 of *ibid*) and 'the leading term conjecture' implies the abelian Brumer-Stark conjecture by Remark 1.11(i) of *ibid.* More precisely, the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}(E)),\mathbb{Z}_p[H]^-)$ implies the local abelian Brumer-Stark conjecture at p for E/F. (Here, $\mathbb{Z}_p[H]^- := \mathbb{Z}_p[H]/(1+\tau)$ where τ is the complex conjugation in H.) Therefore, cases where this special case of the equivariant Tamagawa number conjecture is proved together with Theorem 3.1

yield cases where the local Galois Brumer-Stark conjecture holds unconditionally. Case (1) follows from Theorem 4 of [11]. Indeed, for any CM-subextension E/F of K/k, the prime p is unramified in E and thus the conditions of the theorem are satisfied by the remark just before the theorem; since $p \nmid [E : F]$, the condition on the vanishing of the Iwasawa μ-invariant is not necessary (see Remark 6 of *ibid*). For case (2), we use [2, Cor. 1.2] since, in every abelian subextension E/F of K/k, there is at most one prime ideal of F above p that splits in E/E^+; for the same reasons as in case (1), the condition on the vanishing of the Iwasawa μ-invariant is not necessary. □

References

[1] D. Burns and M. Flach. Tamagawa numbers for motives with (non-commutative) coefficients. *Doc. Math.*, 6:501–570 (electronic), 2001.

[2] D. Burns, M. Kurihara, and T. Sano. Iwasawa theory and zeta elements for $\mathbb{G}_m$. *ArXiv e-prints*, ArXiv:1506.07935, 2015.

[3] D. Burns, M. Kurihara, and T. Sano. On zeta elements for $\mathbb{G}_m$. *Doc. Math.*, 21:555–626, 2016.

[4] Pi. Cassou-Noguès. Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques. *Invent. Math.*, 51(1):29–59, 1979.

[5] G. Dejou and X.-F. Roblot. A Brumer-Stark conjecture for non-abelian Galois extensions. *J. Number Theory*, 142:51–88, 2014.

[6] G. Dejou and X.-F. Roblot. The Galois Brumer-Stark conjecture for $\mathrm{SL}_2(\mathbb{F}_3)$-extensions. *Int. J. Number Theory*, 12(1):165–188, 2016.

[7] P. Deligne and K. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

[8] C. Greither, X.-F. Roblot, and B. Tangedal. The Brumer-Stark conjecture in some families of extensions of specified degree. *Math. Comp.*, 73(245):297–315, 2004.

[9] I. M. Isaacs. *Character theory of finite groups*. Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original.

[10] A. Nickel. On non-abelian Stark-type conjectures. *Ann. Inst. Fourier (Grenoble)*, 61(6):2577–2608, 2011.

[11] A. Nickel. On the equivariant Tamagawa number conjecture in tame CM-extensions. *Math. Z.*, 268(1-2):1–35, 2011.

[12] J. Nomura. On non-abelian Brumer and Brumer-Stark conjecture for monomial CM-extensions. *Int. J. Number Theory*, 10(4):817–848, 2014.

[13] I. Reiner. *Maximal orders*, volume 28 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, Oxford, 2003.

[14] J. Tate. Brumer-Stark-Stickelberger. In *Seminar on Number Theory, 1980–1981* (*Talence, 1980–1981*), pages 16, Exp. No. 24. Univ. Bordeaux I, Talence, 1981.

[15] J. Tate. *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, volume 47 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1984.