THE GALOIS BRUMER-STARK CONJECTURE FOR $\mathrm{SL}_2(\mathbb{F}_3)$-EXTENSIONS

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Abstract. In a previous work, we stated a conjecture, called the Galois Brumer-Stark conjecture, that generalizes the (abelian) Brumer-Stark conjecture to Galois extensions. We also proved that, in several cases, the Galois Brumer-Stark conjecture holds or reduces to the abelian Brumer-Stark conjecture. The first open case is the case of extensions with Galois group isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. This is the case studied in this paper. These extensions split naturally into two different types. For the first type, we prove the conjecture outside of 2. We also prove the conjecture for 59 $\mathrm{SL}_2(\mathbb{F}_3)$-extensions of $\mathbb{Q}$ using computations. The version of the conjecture that we study is a stronger version, called the Refined Galois Brumer-Stark conjecture, that we introduce in the first part of the paper.

1. Introduction

Let K/k be an abelian extension of number fields. The Brumer-Stark conjecture [14] predicts that a group ring element called the Brumer-Stickelberger element and constructed from special values of L-functions associated to K/k, annihilates (after multiplication by a suitable factor) the ideal class group of K and specifies special properties for the generators obtained. In [2], we introduced a generalization of the conjecture to Galois extensions, called the Galois Brumer-Stark conjecture. To avoid confusion, we call the original conjecture the abelian Brumer-Stark conjecture. In the paper mentioned above, we prove also that the Galois Brumer-Stark conjecture holds or reduces to the abelian Brumer-Stark conjecture in many cases. The first case not being covered by our results is the case of Galois extensions with Galois group isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. In the present paper, we study this case for a stronger version of the conjecture, called the Refined Galois Brumer-Stark conjecture, that we introduce in the first part of the paper.

The plan of the paper is the following. In the second section, we state the abelian Brumer-Stark conjecture and the Galois Brumer-Stark conjecture. We recall some of the properties of the Galois Brumer-Stark conjecture and what is known about it. The third section is devoted to the statement of the Refined Galois Brumer-Stark conjecture and our reasons to introduce this stronger version. The next section deals with the properties of the group $\mathrm{SL}_2(\mathbb{F}_3)$ and of $\mathrm{SL}_2(\mathbb{F}_3)$-extensions, and states an explicit expression for the Brumer-Stickelberger element. We also explain how $\mathrm{SL}_2(\mathbb{F}_3)$-extensions split naturally into two types depending on the expression of the Brumer-Stickelberger element. In Section 5, we prove that the Refined Galois Brumer-Stark conjecture holds outside of 2 for $\mathrm{SL}_2(\mathbb{F}_3)$-extensions of the first type. Finally, in the last section, we explain how we proved, using computations, the Refined Galois Brumer-Stark conjecture for 59 $\mathrm{SL}_2(\mathbb{F}_3)$-extensions of $\mathbb{Q}$.

□

Key words and phrases. Galois extensions, Brumer-Stark conjecture, Artin L-functions, Ideal class group annihilators.

2. The Galois Brumer-Stark conjecture

Before stating the Galois Brumer-Stark conjecture, we recall the statement of the abelian Brumer-Stark conjecture, see [15, IV.§6] for a more complete reference or [2, §2]. Let K/k be an abelian extension of number fields. Denote by G its Galois group. Let S be a finite set of places of k containing the infinite places and the finite places ramified in K. To simplify matters, we assume that the cardinality of S is at least 2. The interested reader can refer to [15, IV.§6] for the statement of the conjecture when $|S| = 1$. For $\chi \in \hat{G}$, the group of characters of G, denote by $L_{K/k,S}(s, \chi)$ the Hecke L-function associated to χ with Euler factors associated to prime ideals in S deleted. The Brumer-Stickelberger element associated to the extension K/k and the set S is defined by

$$\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \chi)\, e_{\bar{\chi}}$$

where e_χ is the idempotent associated to χ. It follows from the works of Deligne and Ribet [3] (see also [1]) that $\xi\, \theta_{K/k,S} \in \mathbb{Z}[G]$ for any $\xi \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, the annihilator in $\mathbb{Z}[G]$ of the group μ_K of roots of unity in K. In particular, denoting by w_K the cardinality of μ_K, we have $w_K \theta_{K/k,S} \in \mathbb{Z}[G]$. We need one last notation before stating the abelian Brumer-Stark conjecture. We say that a non-zero element $\alpha \in K$ is an anti-unit if all its conjugates have absolute value equal to 1. The group of anti-units of K is denoted by K°.

Conjecture (The abelian Brumer-Stark conjecture $\mathbf{BS}(K/k, S)$)**.**
For any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{w_K \theta_{K/k,S}}$ is principal and admits a generator $\alpha \in K^\circ$ such that $K(\alpha^{1/w_K})/k$ is abelian.

We refer to [2, §2] for a review of the current state of the abelian Brumer-Stark conjecture. We now introduce the Galois Brumer-Stark conjecture. Note that generalizations by Nickel to the non-abelian case of the Brumer-Stark conjecture and also of the Brumer conjecture are stated in [12].

Let K/k be a Galois extension of number fields. Denote by G its Galois group. Let S be a finite set of places of k containing the infinite places and the finite places ramified in K. Assume that the cardinality of S is at least 2. Following Hayes [8], we define the Brumer-Stickelberger element associated to K/k and S by

$$\theta_{K/k,S} := \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \chi)\, e_{\bar{\chi}}$$

where the sum is over the set $\hat{G}$ of irreducible characters of G and, for $\chi \in \hat{G}$, $L_{K/k,S}(s, \chi)$ denotes the Artin L-function of χ with Euler factors associated to prime ideals in S deleted and

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$$

denotes the central idempotent associated to χ.

Remark. One can prove that, if k is not totally real or K is not totally complex, then $\theta_{K/k,S} = 0$ and thus all the variant of the Brumer-Stark conjecture considered in this paper are trivially true.

It follows from the principal rank zero Stark conjecture, proved by Tate [15], that the Brumer-Stickelberger element lies in $\mathbb{Q}[G]$. The first conjecture introduced in [2] gives a denominator for this element. Recall that the commutator subgroup $[\Gamma, \Gamma]$ is the subgroup of Γ generated by the commutators $[\gamma_1, \gamma_2] := \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ with $\gamma_1, \gamma_2 \in \Gamma$. We give here a slightly different but equivalent formulation.

Conjecture (The Integrality Conjecture).
Let m_G be the lcm of the cardinalities of the conjugacy classes of G and let s_G be the order of the commutator subgroup $[G, G]$ of G. Let d_G be the lcm of m_G and s_G. Then, for any $\sigma \in G$ and any $n_\sigma \in \mathbb{Z}$ such that $\sigma - n_\sigma \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, we have

$$d_G(\sigma - n_\sigma)\,\theta_{K/k,S} \in \mathbb{Z}[G].$$

Before stating the Galois Brumer-Stark conjecture, we need to define the notion of strong central extensions. Let E/F be a Galois extension of number fields. Let N be a finite extension of E. We say that N is a strong central extension of E/F if the extension N/F is Galois and $A \cap [\Gamma, \Gamma] = \{e\}$ where $A := \mathrm{Gal}(N/E)$ and $\Gamma := \mathrm{Gal}(N/F)$. Denote by N^{ab} the maximal sub-extension of N/F that is abelian over F. Then, N is a strong central extension of E/F if and only if $N = EN^{\mathrm{ab}}$.

Conjecture (The Galois Brumer-Stark conjecture $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$).
Let K/k be a Galois extension of number fields and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S| \geq 2$. The Integrality Conjecture holds for the extension K/k and the set of places S, and, for any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{d_G w_K \theta_{K/k,S}}$ is principal and admits a generator $\alpha \in K^\circ$ such that $K(\alpha^{1/w_K})$ is a strong central extension of K/k.

Remark. Francesc Bars has kindly informed us of the formulation of the abelian Brumer-Stark conjecture for function fields contained in [13, Chap. 15] and suggested the following formulation for the Galois Brumer-Stark conjecture in positive characteristic. *Assume that K/k is a Galois geometric extension of function fields. The statement of the Integrality Conjecture stays unchanged (replacing the objects involved by their positive characteristic counterparts) and, for $\mathcal{D}$ a divisor of degree zero of K, $d_G w_K\,\theta_{K/k,S}\mathcal{D}$ is a principal divisor with a generator $\alpha \in K$ such that α has absolute value 1 at all archimedean places of K, if any, and $K(\alpha^{1/w_K})/k$ is a strong central extension of K/k.* We will not investigate the different aspects of the conjecture in positive characteristic in this paper.

The Galois Brumer-Stark conjecture implies the abelian Brumer-Stark conjecture. The following theorem sums up the different cases where the Galois Brumer-Stark conjecture is proved or is implied by the abelian Brumer-Stark conjecture.

Theorem 2.1 ([2, Th. 5.2]). *The Galois Brumer-Stark conjecture is satisfied in the following cases*

(1) $\mathrm{Gal}(K/k)$ *is a non-abelian simple group,*
(2) $\mathrm{Gal}(K/k) \simeq D_{2n}$ *where D_{2n} is the dihedral group of order $2n$ with n odd,*
(3) $\mathrm{Gal}(K/k) \simeq S_n$ *where S_n is the symmetric group on n letters with $n \geq 1$,*
(4) $\mathrm{Gal}(K/k)$ *is non-abelian of order* 8.

Assume that the abelian Brumer-Stark conjecture holds. Then, the Galois Brumer-Stark conjecture is satisfied in the following cases

(5) $\mathrm{Gal}(K/k)$ *is abelian,*
(6) $\mathrm{Gal}(K/k)$ *contains a normal abelian subgroup of prime index,*
(7) $\mathrm{Gal}(K/k)$ *is of order* < 32 *and not isomorphic to* $\mathrm{SL}_2(\mathbb{F}_3)$.

A main feature of the Galois Brumer-Stark conjecture is the fact that it decomposes into an abelian part and a non-abelian part. We now explain this decomposition, see

[2, §3] for details. Let $G^{\mathrm{ab}} := G/[G,G]$ be the maximal abelian quotient of G and let $K^{\mathrm{ab}} := K^{[G,G]}$ be the maximal sub-extension of K/k that is abelian over k; we have $\mathrm{Gal}(K^{\mathrm{ab}}/k) = G^{\mathrm{ab}}$. Denote by $\pi^{\mathrm{ab}} : G \to G^{\mathrm{ab}}$ the canonical surjection induced by the restriction to K^{ab}. Let ν^{ab} be the map from $\mathbb{C}[G^{\mathrm{ab}}]$ to $\mathbb{C}[G]$ defined for $\tilde{g} \in G^{\mathrm{ab}}$ by

$$\nu^{\mathrm{ab}}(\tilde{g}) := \frac{1}{s_G} \sum_{\pi^{\mathrm{ab}}(g)=\tilde{g}} g$$

where the sum is over all elements $g \in G$ whose image by π^{ab} is equal to $\tilde{g}$, and extended to $\mathbb{C}[G^{\mathrm{ab}}]$ by linearity. Note that the image of ν^{ab} is in the center of $\mathbb{C}[G]$ and that the map $\pi^{\mathrm{ab}} \circ \nu^{\mathrm{ab}}$ is the identity on $\mathbb{C}[G^{\mathrm{ab}}]$. We have

$$\theta_{K/k,S} = \nu^{\mathrm{ab}}(\theta_{K^{\mathrm{ab}}/k,S}) + \theta^{(>1)}_{K/k,S}$$

where

$$\theta^{(>1)}_{K/k,S} := \sum_{\substack{\chi \in \hat{G} \\ \chi(1)>1}} L_{K/k,S}(0,\chi) e_{\bar{\chi}}.$$

We call $\theta^{(>1)}_{K/k,S}$ the non-linear Brumer-Stickelberger element (associated to K/k and S). Using the properties of $\theta_{K^{\mathrm{ab}}/k,S}$, one can prove easily that the Integrality Conjecture is equivalent to the fact that $d_G(\sigma - n_\sigma)\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$ for all $\sigma \in G$ and $n_\sigma \in \mathbb{Z}$ such that $\sigma - n_\sigma \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, see [2, Prop. 3.8] for details; in particular, it implies that $d_G w_K\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$. Thanks to this decomposition of the Brumer-Stickelberger element, we have the following result that shows that the Galois Brumer-Stark conjecture splits into an abelian part, corresponding to the abelian Brumer-Stark conjecture, and a non-abelian part.

Theorem 2.2 ([2, Th. 6.3]). *Assume that* $\mathbf{BS}(K^{\mathrm{ab}}/k,S)$ *holds and that the Integrality Conjecture is satisfied for the extension* K/k *and the set of places* S. *Then,* $\mathbf{BS}_{\mathrm{Gal}}(K/k,S)$ *is satisfied if, for any fractional ideal* $\mathfrak{A}$ *of* K, *the ideal* $\mathfrak{A}^{d_G w_K\,\theta^{(>1)}_{K/k,S}}$ *is principal and admits a generator* $\beta \in K^\circ$ *such that* $K(\beta^{1/w_K})$ *is a strong central extension of* K/k.

3. The Refined Galois Brumer-Stark conjecture

In this section, we introduce a refined version of the Galois Brumer-Stark conjecture. For that, we start with Theorem 2.2 and the remark before it to deduce that the Galois Brumer-Stark conjecture is equivalent to the abelian Brumer-Stark conjecture and the following two assertions

(A1) $d_G(\sigma - n_\sigma)\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$ for all $\sigma \in G$ and $n_\sigma \in \mathbb{N}$ such that $\sigma - n_\sigma \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$,

(A2) for any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{d_G w_K\,\theta^{(>1)}_{K/k,S}}$ is principal and admits a generator $\beta \in K^\circ$ such that $K(\beta^{1/w_K})$ is a strong central extension of K/k.

An interesting consequence of (A1) is the following fact. Write

$$d_G\,\theta^{(>1)}_{K/k,S} = \sum_{g \in G} t_g\, g$$

with $t_g \in \mathbb{Q}$. Assume that there exists $h \in G$ such that $t_h = 0$. For $g \in G$, use (A1) with $\sigma := gh^{-1}$. It follows that $n_\sigma t_g \in \mathbb{Z}$. On the other hand, we have $w_K t_g \in \mathbb{Z}$ since $d_G w_K\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$. But, n_σ and w_K are relatively prime and thus $t_g \in \mathbb{Z}$. We have proved the following result.

Lemma 3.1. *Assume that* (A1) *holds and that there is at least one coefficient in the expression of* $\theta_{K/k,S}^{(>1)}$ *as an element of* $\mathbb{Q}[G]$ *that is equal to zero. Then, we have* $d_G\,\theta_{K/k,S}^{(>1)} \in \mathbb{Z}[G]$. □

Of course, a similar reasoning could be done for the abelian Brumer-Stickelberger element and, in fact, this kind of relation for the abelian Brumer-Stickelberger element was first noticed by Hayes [7]. The point here is that, contrary to the abelian case, there are many situations where the non-linear Brumer-Stickelberger element has a zero coefficient. Indeed, by definition, it is a linear combination of the idempotents of the non-linear irreducible characters of G, and, by a result of Burnside [9, Th. 3.14], if χ is a non-linear irreducible character of G, there exists $g \in G$ such that $\chi(g) = 0$. The following lemma gives an equivalent condition for all the non-linear irreducible characters to take the value zero on the same element of G and thus for Lemma 3.1 to apply.

Lemma 3.2. *Let* $\sigma \in G$. *Then,* $\chi(\sigma) = 0$ *for all non-linear irreducible characters of* G *if and only if* $|[\sigma]| = |[G,G]|$ *where* $[\sigma]$ *denote the conjugacy class of* σ *in* G.

Proof. Let ϕ be the characteristic function of $[\sigma]$. Then, ϕ is a class function and thus we have

$$\phi = \sum_{\chi \in \hat{G}} \lambda_\chi\, \chi.$$

where

$$\lambda_\chi = \frac{1}{|G|} \sum_{g \in G} \bar{\chi}(g)\phi(g) = \frac{|[\sigma]|}{|G|}\bar{\chi}(\sigma).$$

In particular, we have $\lambda_\chi = 0$ if and only if $\chi(\sigma) = 0$. Therefore, $\chi(\sigma) = 0$ for all non-linear irreducible characters of G if and only if ϕ is a linear combination of the linear irreducible characters of G. The linear irreducible characters of G are exactly the irreducible characters that factorize through the canonical surjection $\pi^{\mathrm{ab}} : G \to G^{\mathrm{ab}}$. Thus, ϕ is a linear combination of the linear irreducible characters of G if and only if there exists a function $\psi : G^{\mathrm{ab}} \to \{0,1\}$ such that $\phi = \psi \circ \pi^{\mathrm{ab}}$. It is immediate to see that the function ψ is the characteristic function of $\pi^{\mathrm{ab}}(\sigma)$. Hence, $\chi(\sigma) = 0$ for all non-linear irreducible characters of G if and only if $[\sigma] = [G,G]\sigma$. Let $g\sigma g^{-1} \in [\sigma]$, then $g\sigma g^{-1} = (g\sigma g^{-1}\sigma^{-1})\sigma \in [G,G]\sigma$, thus $[\sigma] \subset [G,G]\sigma$. It follows that $[\sigma] = [G,G]\sigma$ if and only if $|[\sigma]| = |[G,G]\sigma| = |[G,G]|$ and the proof is complete. □

Lemma 3.2 provides a quick way to check whether, for a given group G, there exists $\sigma \in G$ such that $\chi(\sigma) = 0$ for all non-linear irreducible characters of G. Using the GAP system [4], we find that it is the case for 81 678 classes out of the 91 356 isomorphism classes of non-abelian groups of even order ≤ 500, that is about 85%. So, for extensions K/k with Galois group isomorphic to one of these groups, the assertion (A1) implies that d_G is a suitable denominator for the non-linear Brumer-Stickelberger element. Observe furthermore that this is also the case for other groups such as $\mathrm{SL}_2(\mathbb{F}_3)$, see next section, and for groups with a normal abelian subgroup of prime index, see [2, §7]. Indeed, by a result of Tate, see [15, p. 71], if the character χ is not the inflation of a totally odd character of a Galois CM sub-extension, we have $L_{K/k,S}(0,\chi) = 0$ and thus it does not contribute to $\theta_{K/k,S}^{(>1)}$. Hence, the condition tested by Lemma 3.2 is stronger than what is actually needed for Lemma 3.1 to apply. With so many occurrences where (A1) is equivalent to the stronger assertion that $d_G\,\theta_{K/k,S}^{(>1)}$ is integral, it seems reasonable to conjecture that this stronger assertion is in fact true in general. With that in mind, it seems logical to further conjecture that the factor w_K may not be necessary in assertion

(A2) and that $\mathfrak{A}^{d_G\,\theta^{(>1)}_{K/k,S}}$ is principal and admits a generator $\gamma \in K^\circ$. Then, up to a root of unity, $\gamma = \beta^{1/w_K}$ and the property that $K(\beta^{1/w_K})$ is a strong central extension of K/k is automatically satisfied. This leads us to the following stronger conjecture.

Conjecture (The Refined Galois Brumer-Stark conjecture $\mathbf{RBS}_{\mathrm{Gal}}(K/k, S)$)**.**
Let K/k be a Galois extension of number fields and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S| \geq 2$. Then $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ holds, we have $d_G\,\theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$ and, for any fractional ideal $\mathfrak{A}$ of K, the ideal $\mathfrak{A}^{d_G\,\theta^{(>1)}_{K/k,S}}$ is principal and admits a generator in K°.

We now discuss some evidence for this conjecture. First, using Theorem 2.2, it is immediate to prove that the Refined Galois Brumer-Stark conjecture implies the Galois Brumer-Stark conjecture. It is also obvious that it implies the abelian Brumer-Stark conjecture.

Proposition 3.3. $\mathbf{RBS}_{\mathrm{Gal}}(K/k, S)$ *implies* $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$. □

Furthermore, in all cases listed in Theorem 2.1 where the Galois Brumer-Stark conjecture is satisfied or implied by the abelian Brumer-Stark conjecture, the same is true for the Refined Galois Brumer-Stark conjecture. That is, we have the following result.

Theorem 3.4. *The Refined Galois Brumer-Stark conjecture is satisfied in the following cases*

(1) $\mathrm{Gal}(K/k)$ *is a non-abelian simple group,*
(2) $\mathrm{Gal}(K/k) \simeq D_{2n}$ *where* D_{2n} *is the dihedral group of order* $2n$ *with* n *odd,*
(3) $\mathrm{Gal}(K/k) \simeq S_n$ *where* S_n *is the symmetric group on* n *letters with* $n \geq 1$*,*
(4) $\mathrm{Gal}(K/k)$ *is non-abelian of order* 8.

Assume that the abelian Brumer-Stark conjecture holds. Then, the Refined Galois Brumer-Stark conjecture is satisfied in the following cases

(5) $\mathrm{Gal}(K/k)$ *is abelian,*
(6) $\mathrm{Gal}(K/k)$ *contains a normal abelian subgroup of prime index,*
(7) $\mathrm{Gal}(K/k)$ *is of order* < 32 *and not isomorphic to* $\mathrm{SL}_2(\mathbb{F}_3)$.

Proof. First, note that in all cases for which Proposition 6.5 of [2] applies and reduces $\mathbf{BS}_{\mathrm{Gal}}(K/k, S)$ to $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$, we also have that $\mathbf{RBS}_{\mathrm{Gal}}(K/k, S)$ reduces to $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ since then the non-linear Brumer-Stickelberger element is zero. Cases (1), (2) and (3) follow from this observation. Case (5) is trivial. For case (6), we see that $d_G\,\theta^{(>1)}_{K/k,S}$ is integral using the expression given by Theorem 7.1 of [2] and the results of Proposition 7.3, *ibid.* A close look at the proof of Theorem 7.4, *ibid.*, shows that $d_G\,\theta^{(>1)}_{K/k,S}$ annihilates the class group of K and that one can find a generator in K°. For case (4), using case (6), we reduce its validity to that of $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$, as it is done in Proposition 7.7, *ibid.*, and use similar reasoning to prove that $\mathbf{BS}(K^{\mathrm{ab}}/k, S)$ holds in that case. Finally, for case (7), we proceed as in the remark after Theorem 5.2, *ibid.*, and check that for all the extensions considered one can reduce to one of the other cases. □

Our last piece of evidence for the Refined Galois Brumer-Stark conjecture is in the case of $\mathrm{SL}_2(\mathbb{F}_3)$-extensions, that is extensions with Galois group isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$, studied in the rest of this paper. These extensions splits naturally into two different types, see Section 4.2, and we prove in Section 5 that the local version of the Refined Galois Brumer-Stark conjecture, stated below, holds for $\mathrm{SL}_2(\mathbb{F}_3)$ extensions of type I at p for p odd. Finally, in Section 6, we prove using computations that the Refined Galois Brumer-Stark conjecture is true for 59 extensions $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$.

Conjecture (The local Refined Galois Brumer-Stark conjecture $\mathbf{RBS}^{(p)}_{\mathrm{Gal}}(K/k,S)$)**.** *Let K/k be a Galois extension of number fields and let S be a finite set of places of k that contains the infinite places and the finite places that ramify in K with $|S|\geq 2$. Then, the local abelian Brumer-Stark conjecture at p for the extension K^{ab}/k and the set of places S holds, we have $d_G\,\theta^{(>1)}_{K/k,S}\in\mathbb{Z}_p[G]$ and, for any fractional ideal $\mathfrak{A}$ of K whose class in Cl_K has p-power order, the ideal $\mathfrak{A}^{d_G\,\theta^{(>1)}_{K/k,S}}$ is principal and admits a generator in K°.*

For the definition of the local abelian Brumer-Stark conjecture, see [6]. One checks readily that the Refined Galois Brumer-Stark conjecture is equivalent to the local Refined Galois Brumer-Stark conjecture at p for all primes p.

4. Extensions with Galois group isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$

In the rest of the paper, we focus on the case of extensions K/k with Galois group isomorphic to $\mathrm{SL}(\mathbb{F}_3)$. We start with some considerations on the group $\mathrm{SL}_2(\mathbb{F}_3)$ itself.

4.1. **The group** $\mathrm{SL}_2(\mathbb{F}_3)$**.** The group $\mathrm{SL}_2(\mathbb{F}_3)$ is the group of 2×2 matrices with coefficients in $\mathbb{F}_3$ and of determinant 1. It is of order 24 and is generated by the two matrices

$$a := \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix} \quad\text{and}\quad b := \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

of order 6 and 4 respectively. Let $c = a^3 = b^2$. It is the only element of order 2 in $\mathrm{SL}_2(\mathbb{F}_3)$ and, in fact, it generates the center of $\mathrm{SL}_2(\mathbb{F}_3)$ and we have $\mathrm{SL}_2(\mathbb{F}_3)/\langle c\rangle \simeq A_4$, the alternating group on 4 letters. The commutator subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$ is isomorphic to the quaternion group, and is generated by b and the commutator $[a,b]$. We have $\mathrm{SL}(\mathbb{F}_3)^{\mathrm{ab}} \simeq \mathbb{Z}/3\mathbb{Z}$ with the quotient being generated by the image of a. For $x\in\mathrm{SL}_2(\mathbb{F}_3)$, denote by $[x]$ its conjugacy class. Depending on context, we will also view $[x]$ as the group ring element obtained by summing all the conjugates of x. The group $\mathrm{SL}_2(\mathbb{F}_3)$ has seven conjugacy classes: $[1]$ and $[c]$ of cardinality 1, $[a]$, $[a^2]$, $[a^4]$ and $[a^5]$ of cardinality 4, and $[b]$ of cardinality 6. Note in particular that, by Lemma 3.2, there a no $\sigma\in\mathrm{SL}_2(\mathbb{F}_3)$ such that $\chi(\sigma)=0$ for all non-linear irreducible characters χ of $\mathrm{SL}_2(\mathbb{F}_3)$.

The elements of order 6 will play a crucial role in our study. The relations they satisfy are described by the following result which can be proved by lengthy hand computations using matrices or, more conveniently, using the GAP system [4].

Proposition 4.1. *Define the following elements of* $\mathrm{SL}_2(\mathbb{F}_3)$

$$a_0 := a,\ a_1 := ba_0b^{-1},\ a_2 := a_0a_1a_0^{-1},\ a_3 := a_0a_2a_0^{-1}.$$

Then, the elements a_i, $i=0,1,2,3$, are of order 6 and form a conjugacy class in $\mathrm{SL}_2(\mathbb{F}_3)$. The other elements of order 6 in $\mathrm{SL}_2(\mathbb{F}_3)$ are a_i^5, $i=0,1,2,3$. Furthermore, for all $i\in\{0,1,2,3\}$, there exists a 3-cycle π_i such that, for all $j\in\{0,1,2,3\}$ with $j\neq i$, we have

$$a_ia_ja_{\pi_i(j)} = e.$$

The 3-cycles are $\pi_0=(1,3,2)$, $\pi_1=(0,2,3)$, $\pi_2=(0,3,1)$, and $\pi_3=(0,1,2)$. □

These properties characterize the group $\mathrm{SL}_2(\mathbb{F}_3)$. Indeed, we have the following presentation

$$\mathrm{SL}_2(\mathbb{F}_3) \simeq \langle a_0,a_1,a_2,a_3 \mid a_0^6=e,\ a_j^3=a_0^3,\ a_0a_ja_{\pi_0(j)}=e,\ j=1,2,3\rangle.$$

Thanks to Proposition 4.1, we can deduce further relations between the a_i's.

Lemma 4.2. *For $i, j \in \{0, 1, 2, 3\}$ with $i \neq j$, we have*

$$a_i^2 a_j^2 = a_{\pi_j(i)} \text{ and } a_i a_j^4 = a_i^4 a_j = a_{\pi_i(j)}^2.$$

Proof. For the first assertion, we have $a_i^2 a_j^2 = a_i^2 cca_j^2 = a_i^5 a_j^5 = (a_j a_i)^{-1} = (a_{\pi_j(i)}^5)^{-1} = a_{\pi_j(i)}$. The other assertion is proved in a similar way. □

The two next lemmas give relations in the group ring $\mathbb{Q}[\mathrm{SL}_2(\mathbb{F}_3)]$ that will be used in the next section.

Lemma 4.3. *We have* $(2 + [a] - [a^2])(1 - c)(1 + a_0^2 + a_0^4) = 0$.

Proof. We use repeatedly Lemma 4.2. First, we compute

$$\begin{aligned}\left(\sum_{i=1}^3 a_i - \sum_{i=1}^3 a_i^2\right)(1 + a_0^2 + a_0^4) &= \sum_{i=1}^3 a_i + \sum_{i=1}^3 a_i a_0^2 + \sum_{i=1}^3 a_i a_0^4 - \sum_{i=1}^3 a_i^2 - \sum_{i=1}^3 a_i^2 a_0^2 - \sum_{i=1}^3 a_i^2 a_0^4 \\ &= \sum_{i=1}^3 a_i + \sum_{i=1}^3 a_i a_0^2 + \sum_{i=1}^3 a_i^2 - \sum_{i=1}^3 a_i^2 - \sum_{i=1}^3 a_i - \sum_{i=1}^3 a_i a_0^2 = 0.\end{aligned}$$

Now, we have

$$\begin{aligned}&(2+[a] - [a^2])(1 - c)(1 + a_0^2 + a_0^4) \\ &\quad= \left(2 + \sum_{i=0}^3 a_i - \sum_{i=0}^3 a_i^2\right)(1 - c)(1 + a_0^2 + a_0^4) \\ &\quad= (1 - c)\left(2 + 2a_0^2 + 2a_0^4 + \sum_{i=0}^3 a_i(1 + a_0^2 + a_0^4) - \sum_{i=0}^3 a_i^2(1 + a_0^2 + a_0^4)\right) \\ &\quad= (1 - c)\left(1 + a_0 + a_0^2 + a_0^3 + a_0^4 + a_0^5 + \left(\sum_{i=1}^3 a_i - \sum_{i=1}^3 a_i^2\right)(1 + a_0^2 + a_0^4)\right) \\ &\quad= (1 - c)(1 + a_0 + a_0^2 + a_0^3 + a_0^4 + a_0^5) = (1 - c)(1 + c)(1 + a_0 + a_0^2) = 0.\end{aligned}$$ □

The next lemma is proved by similar methods. We leave the proof to the reader.

Lemma 4.4. *We have*

$$\begin{aligned}(1 - c)(1 + a_2^2 + a_2^4) = \frac{1}{2}(1 - c)\big(&(1 + a_0^2 + a_0^4)(1 - a_2 + a_2^2 + a_3^2) \\ &+ (1 + a_1^2 + a_1^4)(1 - a_2 + a_2^2 - a_3)\big).\end{aligned}$$ □

The group $\mathrm{SL}_2(\mathbb{F}_3)$ has 3 irreducible linear representations, 3 irreducible representations of dimension 2 and 1 irreducible representation of dimension 3. The group of linear representations is generated by λ with

$$\lambda(a) := j \quad \text{and} \quad \lambda(b) := 1$$

where $j := e^{2i\pi/3}$ is a primitive third root of unity. The 3-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{F}_3)$ is trivial on c, and thus will play no role in this paper as we will see in the next subsection. Let ρ be the 2-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{F}_3)$ defined by

$$\rho(a) := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \rho(b) := \begin{pmatrix} j^2 & j^2 \\ 1 & -j^2 \end{pmatrix}.$$

The other two irreducible 2-dimensional representations are $\lambda\rho$ and $\lambda^2\rho$. The eigenvalues of the representations ρ, $\lambda\rho$ and $\lambda^2\rho$ are given (with multiplicity) in the following table.

	e	c	a	a^2	a^4	a^5	b
ρ	$1,1$	$-1,-1$	$-j,-j^2$	j,j^2	j,j^2	$-j,-j^2$	$-i,i$
$\lambda\rho$	$1,1$	$-1,-1$	$-1,-j^2$	$1,j$	$1,j^2$	$-1,-j$	$-i,i$
$\lambda^2\rho$	$1,1$	$-1,-1$	$-1,-j$	$1,j^2$	$1,j$	$-1,-j^2$	$-i,i$

(4.1)

Denote by χ the character of ρ. It is a rational character whereas the characters $\lambda\chi$ and $\lambda^2\chi$ of $\lambda\rho$ and $\lambda^2\rho$ respectively satisfy $\overline{\lambda\chi} = \lambda^2\chi$.

4.2. $\mathrm{SL}_2(\mathbb{F}_3)$**-extensions.** Let K/k be a Galois extension of number fields with Galois group G isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. We identify from now on G with $\mathrm{SL}_2(\mathbb{F}_3)$. We take S to be minimal, that is S is the set of the infinite places of k and of the finite places that ramify in K/k. Assume that K is totally complex and k is totally real, otherwise $\theta_{K/k,S} = 0$ and the conjecture is trivially true. The element c is the only element of order 2 in G, thus c is the unique complex conjugation and K is a CM-field. Denote by $K^+ := K^{\langle c\rangle}$ the maximal totally real subfield of K, and by F and E the subfields of K fixed respectively by a^2 and b (see the field diagram below). Note that E is totally real and that F is a CM-field. Denote by $F^+ := F^{\langle c\rangle}$ the maximal totally real subfield of F. We set $A := \langle a\rangle = \mathrm{Gal}(K/F^+)$ and $B := \langle b\rangle = \mathrm{Gal}(K/E)$.

$$\begin{tikzcd}
K \arrow[d, dash, "2"'] \arrow[ddr, dash, "3"] & \\
K^+ \arrow[d, dash, "2"'] \arrow[ddr, dash, "3"] & \\
E \arrow[d, dash, "2"'] & F \arrow[d, dash, "2"] \\
K^{\mathrm{ab}} \arrow[d, dash, "3"'] & F^+ \arrow[dl, dash, "4"] \\
k &
\end{tikzcd}$$

We start by finding an expression for the Brumer-Stickelberger element. First, observe that K^{ab}/k is of degree 3, thus K^{ab} is totally real and $\theta_{K^{\mathrm{ab}}/k,S} = 0$. Now, let ψ be a non-trivial irreducible character of G. By [15, Prop. I.3.4], the rank of vanishing of $L_{K/k}(\psi, s)$ at $s = 0$ is equal to $[k : \mathbb{Q}] \dim V_\psi^{\langle c\rangle}$ where $\rho_\psi : G \to \mathrm{GL}(V_\psi)$ is an irreducible representation of character ψ. In particular, if ψ is the 3-dimensional irreducible character of G, then, as noted in the previous subsection, ψ is trivial on c, and thus $L_{K/k,S}(0, \psi) = 0$. It follows that

$$\begin{aligned}
\theta^{(>1)}_{K/k,S} &= L_{K/k,S}(0,\chi)\, e_{\bar\chi} + L_{K/k,S}(0,\lambda\chi)\, e_{\overline{\lambda\chi}} + L_{K/k,S}(0,\lambda^2\chi)\, e_{\overline{\lambda^2\chi}} \\
&= L_{K/k,S}(0,\chi)\, e_{\chi} + L_{K/k,S}(0,\lambda\chi)\, e_{\overline{\lambda\chi}} + \overline{L_{K/k,S}(0,\lambda\chi)\, e_{\overline{\lambda\chi}}} \\
&= L_{K/k,S}(0,\chi)\, e_{\chi} + 2\,\mathrm{Re}(L_{K/k,S}(0,\lambda\chi)\, e_{\overline{\lambda\chi}}).
\end{aligned}$$

We now express the element in terms of the values of the primitive L-functions $L_{K/k}(0,\chi)$ and $L_{K/k}(0,\lambda\chi)$. Note that, by the above formula for the rank of vanishing at $s = 0$, these values are non-zero. Recall that, for ψ a character of G, we have

$$L_{K/k,S}(s,\psi) = L_{K/k}(s,\psi) \prod_{\mathfrak{p}\in S_0} E_p(s,\psi)$$

where S_0 is the set of prime ideals contained in S and, for $\mathfrak{p}$ a prime ideal, we have set

$$E_{\mathfrak{p}}(s,\psi) = 1 - \mathcal{N}\mathfrak{p}^{-s} \det\left(\sigma_{\mathfrak{P}} | V_\psi^{I_{\mathfrak{P}}}\right)$$

with $\mathfrak{P}$ a fixed prime ideal of K above $\mathfrak{p}$, $\sigma_{\mathfrak{P}}$ the Frobenius automorphism of $\mathfrak{P}$ in G and $I_{\mathfrak{P}}$ the inertia group of $\mathfrak{P}$. By our choice of S, $\mathfrak{p}$ is ramified in K and thus $I_{\mathfrak{P}}$ is non-trivial. Looking at table (4.1), we find that in all cases $V_\chi^{I_{\mathfrak{P}}} = \{0\}$, thus $E_{\mathfrak{p}}(s,\chi) = 1$ for all $\mathfrak{p} \in S_0$ and $L_{K/k,S}(s,\chi) = L_{K/k}(s,\chi)$. In the same way, we find that $V_{\lambda\chi}^{I_{\mathfrak{P}}} = \{0\}$, and thus $E_{\mathfrak{p}}(s,\lambda\chi) = 1$, unless $I_{\mathfrak{P}}$ is conjugate to $\langle a^2\rangle$. Observe that $I_{\mathfrak{P}}$ is conjugate to $\langle a^2\rangle$ if and only if the ramification index of $\mathfrak{P}$ in K/k is 3 since the subgroups of G of order 3 are exactly the conjugate groups of $\langle a^2\rangle$. Assume that it is the case. Replacing $\mathfrak{P}$ by one of its conjugates, we can assume, without loss of generality, that $I_{\mathfrak{P}} = \langle a^2\rangle$. The quotient $D_{\mathfrak{P}}/I_{\mathfrak{P}}$, where $D_{\mathfrak{P}}$ is the decomposition group of $\mathfrak{P}$, is a cyclic group. Thus $D_{\mathfrak{P}}$ is either $\langle a^2\rangle$ or $\langle a\rangle$. If $D_{\mathfrak{P}} = \langle a^2\rangle$, then the Frobenius $\sigma_{\mathfrak{P}}$ acts trivially on $V_\chi^{I_{\mathfrak{P}}}$, thus $E_{\mathfrak{p}}(s,\lambda\chi) = 1 - \mathcal{N}p^{-s}$ and it vanishes at $s = 0$. If $D_{\mathfrak{P}} = \langle a\rangle$, then the Frobenius is either a or a^5. It acts as an element of order 2 on the 1-dimensional vector space $V_\chi^{I_{\mathfrak{P}}}$ and thus it acts as the multiplication by -1. It follows that $E_{\mathfrak{p}}(s,\lambda\chi) = 1 + \mathcal{N}p^{-s}$ and $E_{\mathfrak{p}}(0,\lambda\chi) = 2$.

With these considerations in mind, for $\mathfrak{p}$ a prime ideal in S_0, we define a quantity $\delta_{\mathfrak{p}}$ in the following way. Let $e_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$ be the ramification index and inertia degree of $\mathfrak{p}$ in K/k, we set

$$\delta_{\mathfrak{p}} := \begin{cases} 1 & \text{if } e_{\mathfrak{p}} \neq 3, \\ 0 & \text{if } e_{\mathfrak{p}} = 3 \text{ and } f_{\mathfrak{p}} = 1, \\ 2 & \text{if } e_{\mathfrak{p}} = 3 \text{ and } f_{\mathfrak{p}} = 2. \end{cases}$$

We set $\delta_{K/k} := \prod_{\mathfrak{p}\in S_0} \delta_{\mathfrak{p}}$. We have proved the following proposition.

Proposition 4.5. *We have*

$$\theta_{K^{\mathrm{ab}}/k,S} = 0 \quad \textit{and} \quad \theta^{(>1)}_{K/k,S} = L_{K/k}(0,\chi)\, e_\chi + 2\delta_{K/k} \operatorname{Re}(L_{K/k}(0,\lambda\chi)\, e_{\overline{\lambda\chi}}). \qquad \square$$

Remark. Note that one consequence of this expression is that the coefficient of b in $\theta^{(>1)}_{K/k,S}$ is always zero and thus, by Lemma 3.1, the Galois Brumer-Stark conjecture, or more precisely the Integrality Conjecture, would imply that $d_G\, \theta^{(>1)}_{K/k,S} \in \mathbb{Z}[G]$.

In view of the expression for $\theta^{(>1)}_{K/k,S}$ given by Proposition 4.5, we distinguish between two types of extensions K/k. Type I extensions are the extensions such that there is a prime ideal $\mathfrak{P}$ of K that is totally split in F/k and ramified in K/F, for these we have $\delta_{K/k} = 0$; the other extensions are of type II and satisfy $\delta_{K/k} \neq 0$. We will prove in the next section that the local Refined Galois Brumer-Stark conjecture is true at p for all odd primes p for type I extensions.

Let ν be the linear character of A defined by $\nu(a) = -j^2$ and let ϕ be the linear character of B defined by $\phi(b) = i$. The characters ν and ϕ generate respectively the group of characters of A and of B. We compute that

$$\chi = \operatorname{Ind}_B^G(\phi) - \operatorname{Ind}_A^G(\nu^3) \quad \text{and} \quad \lambda\chi = \operatorname{Ind}_B^G(\phi) - \operatorname{Ind}_A^G(\nu).$$

By the properties of Artin L-functions, this implies that

$$L_{K/k}(s,\chi) = \frac{L_{K/E}(s,\phi)}{L_{K/F^+}(s,\nu^3)} \quad \text{and} \quad L_{K/k}(s,\lambda\chi) = \frac{L_{K/E}(s,\phi)}{L_{K/F^+}(s,\nu)}.$$

Observe that

$$\zeta_K(s) = \zeta_{K^+}(s)\cdot L_{K/E}(s,\phi)\cdot L_{K/E}(s,\phi^3) = \zeta_{K^+}(s)\cdot L_{K/E}(s,\phi)^2.$$

Indeed, $\mathrm{Ind}_B^G(\phi) = \mathrm{Ind}_B^G(\phi^3)$ thus the Artin L-functions associated to ϕ and ϕ^3 are equal. By the formula for the leading term at $s=0$ of Dedekind zeta functions, see for example [15, Cor. I.1.2], we obtain that

$$L_{K,E}(0,\phi)^2 = \lim_{s\to 0}\frac{\zeta_K(s)}{\zeta_{K^+}(s)} = \frac{h_K R_K}{h_{K^+}R_{K^+}}.$$

Here, we used the fact that $w_K = 2$ since K^{ab} is totally real. Now, the kernel of ν^3 is the subgroup $\langle a^2\rangle$ of A and therefore

$$L_{K/F^+}(s,\nu^3) = L_{F/F^+}(s,\eta)$$

where η is the non-trivial character of the quadratic extension F/F^+. It follows that

$$L_{F/F^+}(0,\eta) = \lim_{s\to 0}\frac{\zeta_F(s)}{\zeta_{F^+}(s)} = \frac{h_F R_F}{h_{F^+}R_{F^+}}.$$

Putting everything together, we have proved the following result.

Theorem 4.6. *We have*

$$\theta_{K/k,S}^{(>1)} = \pm\Big(\frac{h_K R_K}{h_{K^+}R_{K^+}}\Big)^{1/2}\left(\frac{h_{F^+}R_{F^+}}{h_F R_F}\, e_\chi + 2\delta_{K/k}\,\mathrm{Re}\big(L_{K/F^+}(0,\nu)^{-1}\, e_{\overline{\lambda\chi}}\big)\right). \qquad \square$$

Remark. Using similar methods, one can prove that

$$|L_{K/F+}(0,\nu)|^2 = \frac{h_K R_K}{h_{K^+}R_{K^+}}\,\frac{h_{F^+}R_{F^+}}{h_F R_F}.$$

However, there does not appear to be a way to deduce an explicit expression for $L_{K/F^+}(0,\nu)$.

One last simplification to the formula appearing in Theorem 4.6 is to compute the quotients of regulators.

Lemma 4.7. *If all the units of K are totally real, then we have*

$$\frac{R_K}{R_{K^+}} = 2^{11} \quad \textit{and} \quad \frac{R_{F^+}}{R_F} = 2^3.$$

Otherwise, the units of K that are not totally real are totally imaginary and we have

$$\frac{R_K}{R_{K^+}} = 2^{10} \quad \textit{and} \quad \frac{R_{F^+}}{R_F} = 2^2.$$

Proof. It follows from [17, Prop. 4.16] that

$$\frac{R_K}{R_{K^+}} = \frac{1}{Q_K}2^{11} \quad \text{and} \quad \frac{R_F}{R_{F^+}} = \frac{1}{Q_F}2^3$$

where $Q_K := (U_K : U_{K^+})$ and $Q_F := (U_F : U_{F^+})$. It is also known, see *ibid.*, that $Q_K, Q_F \in \{1,2\}$ with $Q_K = 2$, resp. $Q_F = 2$, if and only if the map $\varepsilon \mapsto \varepsilon/\bar\varepsilon$ from U_K to $\{\pm 1\}$, resp. U_F to $\{\pm 1\}$, is surjective. Assume that $Q_K = 2$ and let $\varepsilon \in U_K$ be such that ε is not totally real. Then $\varepsilon/\bar\varepsilon = -1$ and ε is totally imaginary. Furthermore, $N_{K/F}(\varepsilon)$ is also totally imaginary and thus $Q_F = 2$. The result follows in that case. Assume now that $Q_K = 1$. Then $U_K = U_{K^+}$ and the units of K are totally real. Since $U_F \subset U_K$, it follows that $U_F = U_K \cap F = U_{K^+} \cap F = U_{F+}$ and $Q_F = 1$. This proves the result in the second case and concludes the proof. $\square$

Remark. Since the character χ is rational, the value $L_{K/k}(0, \chi)$ is also rational, that is we have
$$\Big(\frac{h_K R_K}{h_{K^+} R_{K^+}}\Big)^{1/2} \frac{h_{F^+} R_{F^+}}{h_F R_F} \in \mathbb{Q}.$$
Therefore, using Lemma 4.7, we find that the p-adic valuation of h_K/h_{K^+} is even for p odd, and the 2-adic valuation of h_K/h_{K^+} is odd if all the units of K are totally real and even otherwise. We will give in passing an algebraic proof of the statement for odd primes p in the next section.

To conclude this section, we compute the expressions of the needed idempotents, We have
$$e_\chi = \frac{2}{24}\big(2 - 2c + [a] - [a^2] - [a^4] + [a^5]\big) = \frac{1-c}{12}\big(2 + [a] - [a^2]\big) \tag{4.2}$$
since $[a^4] = [ca] = c[a]$ and $[a^5] = [ca^2] = c[a^2]$. In the same way, we find that
$$e_{\overline{\lambda\chi}} = \frac{1-c}{12}\big(2 + j^2[a] - j[a^2]\big).$$

5. The Conjecture for $\mathrm{SL}_2(\mathbb{F}_3)$-extensions of type I

In this section, we prove the following result.

Theorem 5.1. *Let K/k be a Galois extension of number fields with Galois group isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$ and let S be the set of infinite places of k and of the finite places that ramify in K/k. Assume that K/k is of type I. Then, the local Refined Galois Brumer-Stark conjecture $\mathbf{RBS}^{(p)}_{\mathrm{Gal}}(K/k, S)$ holds for all odd primes p.*

Recall that type I extensions are the ones for which where there exists a prime ideal $\mathfrak{P}$ of K that is totally split in F/k and totally ramified in K/F. For these extensions, we have $\delta_{K/k} = 0$ and only the character χ contributes in the definition of the non-linear Brumer-Stickelberger element. The order of $[G, G]$ is 8, and the conjugacy classes of G are of size 1, 4 or 6, thus $d_G = 24$. From this and the results of the previous section, we get the following expression
$$d_G\, \theta^{(>1)}_{K/k,S} = \pm 2^{\varepsilon} \left(\frac{h_K}{h_{K^+}}\right)^{1/2} \frac{h_{F^+}}{h_F}(1-c)\big(2 + [a] - [a^2]\big) \tag{5.3}$$
for some $\varepsilon \in \frac{1}{2}\mathbb{N}$.

Let p be an odd prime number. For M a finite abelian group, we denote by $M\{p\} := M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ its p-part. In order to prove Theorem 5.1, we study the structure of $\mathrm{Cl}_K\{p\}$. From now on, we use the additive notation to denote the action of group rings on class groups. For L a subfield of K, denote by $\iota_{K/L} : I_L \to I_K$ the map that sends an ideal $\mathfrak{a}$ of L to the ideal $\mathfrak{a}\mathcal{O}_K$ of K and by $N_{K/L} : I_K \to I_L$ the norm map. By abuse, we use the same notation for the induced maps on class groups. Let $\mathcal{C}_0 \in \mathrm{Cl}_L$, then we have $(N_{K/L} \circ \iota_{K/L})(\mathcal{C}_0) = [K : L]\,\mathcal{C}_0$. It follows that $\iota_{K/L}(\mathrm{Cl}_L\{p\}) \simeq \mathrm{Cl}_L\{p\}$ if p does not divide the degree $[K : L]$. If $\mathcal{C} \in \mathrm{Cl}_K$, then we have $(\iota_{K/L} \circ N_{K/L})(\mathcal{C}) = \sum_{h \in H} h\,\mathcal{C}$ where $H := \mathrm{Gal}(K/L)$.

Define Cl_K^- and Cl_F^- to be respectively the kernels of the norm maps $N_{K/K^+} : \mathrm{Cl}_K \to \mathrm{Cl}_{K^+}$ and $N_{F/F^+} : \mathrm{Cl}_F \to \mathrm{Cl}_{F^+}$. Let $\mathcal{C} \in \mathrm{Cl}_K^-$. Observe that $(1+c)\,\mathcal{C} = \iota_{K/K^+}(N_{K/K^+}(\mathcal{C})) = 0$ and therefore $c\,\mathcal{C} = -\mathcal{C}$. Similarly, c acts as the multiplication by -1 on Cl_F^-. The extensions K/K^+ and F/F^+ are ramified at the infinite places, thus the norm maps on class groups are surjective by Class Field theory and we have the following result.

Lemma 5.2. *We have* $|\,\mathrm{Cl}_K^-\,| = h_K/h_{K^+}$ *and* $|\,\mathrm{Cl}_F^-\,| = h_F/h_{F^+}$. □

It is easy to see then $(1-c)\,\mathrm{Cl}_K\{p\}$ is included in $\mathrm{Cl}_K^-\{p\}$ and similarly $(1-c)\,\mathrm{Cl}_F\{p\} \subset \mathrm{Cl}_F^-\{p\}$. For the inclusions in the other direction, we have the following lemma.

Lemma 5.3. *We have* $\mathrm{Cl}_K^-\{p\} = (1-c)\,\mathrm{Cl}_K\{p\} = ((1-c)\,\mathrm{Cl}_K)\{p\}$. *The same result holds with* K *replaced by* F.

Proof. Let $\mathcal{C} \in \mathrm{Cl}_K^-\{p\}$. Then, as noted above, we have $(1+c)\,\mathcal{C} = 0$. But, $H^1(\langle c\rangle, \mathrm{Cl}_K\{p\})$ is trivial since the orders of the groups $\langle c\rangle$ and $\mathrm{Cl}_K\{p\}$ are relatively prime. Therefore, $\mathcal{C} \in (1-c)\,\mathrm{Cl}_K\{p\}$ and the equality $\mathrm{Cl}_K^-\{p\} = (1-c)\,\mathrm{Cl}_K\{p\}$ follows. It is immediate to see that $(1-c)\,\mathrm{Cl}_K\{p\} \subset ((1-c)\,\mathrm{Cl}_K)\{p\}$. Now, let $\mathcal{C} \in ((1-c)\,\mathrm{Cl}_K)\{p\}$, say $\mathcal{C} = (1-c)\,\mathcal{D}$ with $\mathcal{D} \in \mathrm{Cl}_K$. Let $\mathcal{C}_0 \in \mathrm{Cl}_K\{p\}$ be such that $2\mathcal{C}_0 = \mathcal{C}$. Observe that $(1-c)\,\mathcal{C} = (1-c)^2\,\mathcal{D} = 2(1-c)\,\mathcal{D} = 2\mathcal{C}$, thus $\mathcal{C} = (1-c)\,\mathcal{C}_0 \in (1-c)\,\mathrm{Cl}_K\{p\}$ and $(1-c)\,\mathrm{Cl}_K\{p\} = ((1-c)\,\mathrm{Cl}_K)\{p\}$. The assertions for F are proved in a similar way. □

Recall that F is the subfield of K fixed by $a^2 = a_0^2$.

Lemma 5.4. *We have* $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\} = \iota_{K/F}(\mathrm{Cl}_F^-\{p\}) \simeq \mathrm{Cl}_F^-\{p\}$.

Proof. By hypothesis, the extension K/F is ramified and thus $N_{K/L}(\mathrm{Cl}_K\{p\}) = \mathrm{Cl}_F\{p\}$ by Class Field theory. (Note that, for $p \neq 3$, this can be proved directly by the above remarks without the assumption that K/F is ramified.) We have $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K\{p\} = (\iota_{K/F} \circ N_{K/F})(\mathrm{Cl}_K\{p\}) = \iota_{K/F}(\mathrm{Cl}_F\{p\})$. Multiplying both sides by $(1-c)$ and using Lemma 5.3 gives the first equality.

It remains to prove that the map $\iota_{K/F} : \mathrm{Cl}_F^-\{p\} \to \mathrm{Cl}_K^-\{p\}$ is injective. For $p \neq 3$, the result is direct by the remarks at the beginning of the section. Assume now that $p = 3$. The kernel of this map is studied in general in [5]. We adapt the methods of this article to our case. Let $H := \mathrm{Gal}(K/F) = \langle a_0^2\rangle$. To simplify notations, let Cl_K° and Cl_F° denote respectively $(1-c)\,\mathrm{Cl}_K$ and $(1-c)\,\mathrm{Cl}_F$. Define $(\mathrm{Cl}_K^\circ)^H_{\mathrm{str}}$ to be the subgroup of strong ambiguous classes of Cl_K°, that is the group of the classes in Cl_K° generated by ideals fixed by H. Observe that $\iota_{K/F}(\mathrm{Cl}_F^\circ)$ is a subgroup of $(\mathrm{Cl}_K^\circ)^H_{\mathrm{str}}$. Furthermore, we have the commutative diagram with exact rows and the vertical maps are the natural maps

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_F \cap I_F^{1-c} & \longrightarrow & I_F^{1-c} & \longrightarrow & \mathrm{Cl}_F^\circ & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota_{K/F}} & & \\ 1 & \longrightarrow & (P_K \cap I_K^{1-c})^H & \longrightarrow & (I_K^{1-c})^H & \longrightarrow & (\mathrm{Cl}_K^\circ)^H_{\mathrm{str}} & \longrightarrow & 1 \end{array} \tag{5.4}$$

where P_L denote the groups of principal ideals of a number field L. The map $I_F^{1-c} \to (I_K^{1-c})^H$ is injective and thus we get from the Snake Lemma that $\mathrm{Ker}(\iota_{K/F} : \mathrm{Cl}_F^\circ \to (\mathrm{Cl}_K^\circ)^H_{\mathrm{str}}) = \mathrm{Ker}(\iota_{K/F} : \mathrm{Cl}_F^\circ \to \mathrm{Cl}_K^\circ)$ is isomorphic to a subgroup of $\mathrm{Coker}(P_F \cap I_F^{1-c} \to (P_K \cap I_K^{1-c})^H)$. Now, consider the following variation of the classical exact sequence

$$1 \longrightarrow U_K \cap (K^\times)^{1-c} \longrightarrow (K^\times)^{1-c} \longrightarrow P_K^{1-c} \longrightarrow 1.$$

Note that $(K^\times)^{1-c}$ is also the kernel of the map $N_{K/K^+} : K^\times \to (K^+)^\times$ by Hilbert's Theorem 90. Hence $((K^\times)^{1-c})^H = (F^\times)^{1-c}$ and, taking invariants for the action of H, we get the short exact sequence

$$(F^\times)^{1-c} \longrightarrow (P_K^{1-c})^H \longrightarrow H^1(H, U_K \cap (K^\times)^{1-c}).$$

Let $u \in U_K \cap (K^\times)^{1-c}$. Then, all conjugates of u have absolute value 1 and thus u is a root of unity. But the only roots of unity in K are ± 1 and thus $U_K \cap (K^\times)^{1-c} \subset \{\pm 1\}$. It follows that $H^1(H, U_K \cap (K^\times)^{1-c})$ is trivial and the natural map $P_F^{1-c} \to (P_K^{1-c})^H$ is surjective (it is in fact an isomorphism). Now, we have the commutative diagram with exact rows and the vertical maps are the natural maps

$$\begin{tikzcd} 1 \arrow[r] & P_F^{1-c} \arrow[r] \arrow[d] & P_F \cap I_F^{1-c} \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & (P_K^{1-c})^H \arrow[r] & (P_K \cap I_K^{1-c})^H \arrow[r] & Q_1 \arrow[r] & 1. \end{tikzcd} \tag{5.5}$$

As noted above the map $P_F^{1-c} \to (P_K^{1-c})^H$ is surjective and thus $\operatorname{Coker}(P_F \cap I_F^{1-c} \to (P_K \cap I_K^{1-c})^H)$ is isomorphic to a subgroup of $\operatorname{Coker}(Q_0 \to Q_1)$. Let $\mathfrak{A} \in (P_K \cap I_K^{1-c})^H$, say $\mathfrak{A} = (\alpha) = \mathfrak{B}^{1-c}$ for some $\alpha \in K^\times$ and $\mathfrak{B} \in I_K$. Note that $\mathfrak{A}^{1-c} = (\alpha^{1-c}) = \mathfrak{B}^{(1-c)^2} = (\mathfrak{B}^{1-c})^2 = \mathfrak{A}^2$, hence $\mathfrak{A}^2 \in (P_K^{1-c})^H$. Therefore, $Q_1 = \operatorname{Coker}((P_K^{1-c})^H \to (P_K \cap I_K^{1-c})^H)$ is killed by 2. Thus, the same is true for $\operatorname{Coker}(P_F \cap I_F^{1-c} \to (P_K \cap I_K^{1-c})^H)$ and for $\operatorname{Ker}(\iota_{K/F} : \mathrm{Cl}_F^\circ \to \mathrm{Cl}_K^\circ)$. It follows that $\operatorname{Ker}(\iota_{K/F} : \mathrm{Cl}_F^\circ\{3\} \to \mathrm{Cl}_K^\circ\{3\})$ is injective. By Lemma 5.3, this implies that the map $\iota_{K/F} : \mathrm{Cl}_F^-\{3\} \to \mathrm{Cl}_K^-\{3\}$ is injective and the proof is complete. (In fact, since the non-trivial classes in $\operatorname{Ker}(\iota_{K/F} : \mathrm{Cl}_F^- \to \mathrm{Cl}_K^-)$ have order 3, we have proved that this kernel is trivial.) $\square$

Lemma 5.5. *Let $\mathcal{M}_p$ denote the subgroup of $\mathrm{Cl}_K^-\{p\}$ generated by $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\}$ under the action of $\mathbb{Z}_p[G]$. Then $\mathcal{M}_p \simeq \mathrm{Cl}_F^-\{p\}^2$.*

Proof. First, note that $b(1 + a_0^2 + a_0^4)b^{-1} = 1 + a_1^2 + a_1^4$, thus $b(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\} = (1 + a_1^2 + a_1^4)b\,\mathrm{Cl}_K^-\{p\} = (1 + a_1^2 + a_1^4)\,\mathrm{Cl}_K^-\{p\}$ is also contained in $\mathcal{M}_p$. We prove that $\mathcal{M}_p = (1+a_0^2+a_0^4)\,\mathrm{Cl}_K^-\{p\} \oplus (1+a_1^2+a_1^4)\,\mathrm{Cl}_K^-\{p\}$. Since it is clear that $(1+a_0^2+a_0^4)\,\mathrm{Cl}_K^-\{p\}$ and $(1 + a_1^2 + a_1^4)\,\mathrm{Cl}_K^-\{p\}$ are isomorphic, the result will follow using Lemma 5.4.

We first prove that the sum is direct. Let $\mathcal{C}$ be a class in $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\} \cap (1 + a_1^2 + a_1^4)\,\mathrm{Cl}_K^-\{p\}$. Then, it is fixed by a_0^2 and a_1^2 and thus by $c = (a_0^2 a_1^2)^3$. Thus, $2\mathcal{C} = (1 + c)\mathcal{C} = 0$. Hence $\mathcal{C}$ is trivial and the sum between $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\}$ and $(1 + a_1^2 + a_1^4)\,\mathrm{Cl}_K^-\{p\}$ is direct. To conclude the proof, it is enough to prove that $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\} \oplus (1 + a_1^2 + a_1^4)\,\mathrm{Cl}_K^-\{p\}$ is stable under the action of G. It is clear by the above that it is stable under the action of b. Now, observe that $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\}$ is stable under the action of a_0; in fact $a_0\,\mathcal{C} = -\mathcal{C}$ for all $\mathcal{C} \in (1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\}$. Let $\mathcal{C} \in (1 + a_1^2 + a_1^4)\,\mathrm{Cl}_K^-\{p\}$. Since $a_2 = a_0 a_1 a_0^{-1}$, we have $a_0\,\mathcal{C} \in (1 + a_2^2 + a_2^4)\,\mathrm{Cl}_K^-\{p\}$. But, by Lemma 4.4 and Lemma 5.3, we get that $(1 + a_2^2 + a_2^4)\,\mathrm{Cl}_K^-\{p\}$ is contained in $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\} \oplus (1 + a_1^2 + a_1^4)\,\mathrm{Cl}_K^-\{p\}$. Therefore, $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\} \oplus (1 + a_1^2 + a_1^4)\,\mathrm{Cl}_K^-\{p\}$ is also stable under the action of a, and thus it is stable under the action of $G = \langle a, b\rangle$. The proof is complete. $\square$

Lemma 5.6. *The subgroup $\mathcal{M}_p$ of $\mathrm{Cl}_K^-\{p\}$ is annihilated by $2 + [a] - [a^2]$.*

Proof. We have $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\} = (1 - c)(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K\{p\}$ by Lemma 5.3 and thus, by Lemma 4.3, $2 + [a] - [a^2]$ kills $(1 + a_0^2 + a_0^4)\,\mathrm{Cl}_K^-\{p\}$. The result follows by the definition of $\mathcal{M}_p$ and the fact that $2 + [a] - [a^2]$ is in the center of $\mathbb{Z}_p[G]$. $\square$

We prove Theorem 5.1 for $p \equiv 1 \pmod 4$. In this case, there exists $i \in \mathbb{Z}_p$ with $i^2 = -1$ and i acts on $\mathrm{Cl}_K\{p\}$. Define

$$e^\wedge := \frac{1 + ib}{2} \quad \text{and} \quad e^\vee := \frac{1 - ib}{2}.$$

Then $e^\wedge$ and $e^\vee$ are orthogonal idempotents acting on $\mathrm{Cl}_K^-\{p\}$, and thus we have the decomposition

$$\mathrm{Cl}_K^-\{p\} = \mathrm{Cl}_K^\wedge\{p\} \oplus \mathrm{Cl}_K^\vee\{p\} \tag{5.6}$$

where $\mathrm{Cl}_K^\wedge\{p\} := e^\wedge \mathrm{Cl}_K^-\{p\}$ and $\mathrm{Cl}_K^\vee\{p\} := e^\vee \mathrm{Cl}_K^-\{p\}$. We define in the same way $\mathcal{M}_p^\wedge := e^\wedge \mathcal{M}_p$ and $\mathcal{M}_p^\vee := e^\vee \mathcal{M}_p$. Clearly, $M_p^\wedge \subset \mathrm{Cl}_K^\wedge\{p\}$, $\mathcal{M}_p^\vee \subset \mathrm{Cl}_K^\vee\{p\}$, and we have the decomposition

$$\mathcal{M}_p = \mathcal{M}_p^\wedge \oplus \mathcal{M}_p^\vee. \tag{5.7}$$

Observe now that $(a_0a_1^2)b(a_0a_1^2)^{-1} = b^3$, thus the map $\mathcal{C} \mapsto a_0a_1^2\,\mathcal{C}$ yields an isomorphism between $\mathrm{Cl}_K^\wedge\{p\}$ and $\mathrm{Cl}_K^\vee\{p\}$ that restricts to an isomorphism between $\mathcal{M}_p^\wedge$ and $\mathcal{M}_p^\vee$. In particular, there exist integers $n \geq m \geq 0$ such that $|\mathrm{Cl}_K^\wedge\{p\}| = |\mathrm{Cl}_K^\vee\{p\}| = p^n$ and $|\mathcal{M}_p^\wedge| = |\mathcal{M}_p^\vee| = p^m$. Furthermore, $|\mathrm{Cl}_K^-\{p\}| = p^{2n}$ by (5.6) and, by (5.7) and Lemma 5.5, $|\mathrm{Cl}_F^-\{p\}| = p^m$.

From (5.3) and Lemma 5.2, there exists $u \in \mathbb{Z}_p^\times$ such that

$$d_G\,\theta_{K/k,S}^{(>1)} = up^{n-m}(1-c)\big(2 + [a] - [a^2]\big) \tag{5.8}$$

and therefore $d_G\,\theta_{K/\mathbb{Q},S}^{(>1)} \in \mathbb{Z}_p[G]$. Now, let $\mathcal{C} \in \mathrm{Cl}_K\{p\}$ be a class with p-power order. Write $\mathcal{C} = \mathcal{C}^\wedge + \mathcal{C}^\vee$ with $\mathcal{C}^\wedge := e^\wedge\,\mathcal{C}$ and $\mathcal{C}^\vee := e^\vee\,\mathcal{C}$. Then $(1-c)\,\mathcal{C}^\wedge \in \mathrm{Cl}_K^\wedge\{p\}$ and thus $p^{n-m}(1-c)\,\mathcal{C}^\wedge \in \mathcal{M}_p^\wedge$ since $(\mathrm{Cl}_K^\wedge\{p\} : \mathcal{M}_p^\wedge) = p^{n-m}$. In the same way, we have $p^{n-m}(1-c)\,\mathcal{C}^\vee \in \mathcal{M}_p^\vee$. Therefore, by (5.7), we get that $p^{n-m}(1-c)\,\mathcal{C} \in \mathcal{M}_p$. It follows from Lemma 5.6 that $d_G\,\theta_{K/k,S}^{(>1)}\mathcal{C} = 0$. Hence, for any $\mathfrak{A} \in \mathcal{C}$, the ideal $\mathfrak{A}^{d_G\theta_{K/k,S}^{(>1)}}$ is principal. To prove that it admits a generator that is also an anti-unit, we use the same method as in [6, p. 299]. Since p is odd, there exists an ideal $\mathfrak{B}$ of K and an element $\eta \in K^\times$ such that $\mathfrak{A} = (\eta)\mathfrak{B}^2$ and the class of $\mathfrak{B}$ lies in $\mathrm{Cl}_K\{p\}$. Thus there exists $\beta \in K^\times$ such that $\mathfrak{B}^{d_G\theta_{K/k,S}^{(>1)}} = (\beta)$. We have $(1+c)\,\theta_{K/k,S}^{(>1)} = 0$ by a straightforward modification of [2, Cor. 3.5], hence $(1-c)\,\theta_{K/k,S}^{(>1)} = 2\theta_{K/k,S}^{(>1)}$ and $\alpha := \eta^{d_G\theta_{K/k,S}^{(>1)}}\beta^{1-c}$ is a generator of $\mathfrak{A}$. Note that $\alpha^{1+c} = 1$ and therefore α is indeed an anti-unit and the theorem is proved in this case.

We now turn to the case $p \equiv 3 \pmod 4$. Let $i \in \bar{\mathbb{Q}}_p$ be such that $i^2 = -1$, thus $\mathbb{Z}_p[i] \simeq \mathbb{Z}_p[X]/(X^2+1)$. Observe that $b^2 = c$ acts as -1 on $\mathrm{Cl}_K^-\{p\}$ by the remark before Lemma 5.2. Therefore, one can see $\mathrm{Cl}_K^-\{p\}$ and $\mathcal{M}_p$ as $\mathbb{Z}_p[i]$-modules with i acting via b.

Lemma 5.7. *Let Γ be an finite abelian p-group. Assume that Γ is also a $\mathbb{Z}_p[i]$-module. Then there exists subgroups $\Gamma^\wedge$ and $\Gamma^\vee$ of Γ such that $\Gamma^\wedge \simeq \Gamma^\vee$ and $\Gamma = \Gamma^\wedge \oplus \Gamma^\vee$.*

Proof. Let $\mathcal{O} := \mathbb{Z}_p[i]$. Observe that $p\mathcal{O}$ is the only prime ideal of $\mathcal{O}$. Let $\gamma \in \Gamma$, then $\mathrm{Ann}_{\mathcal{O}}(\gamma) = p^w\mathcal{O}$ for some $w \geq 0$. We have

$$\mathcal{O}\,\gamma \simeq \mathcal{O}/p^w\mathcal{O} = \mathbb{Z}_p/p^w\mathbb{Z}_p \oplus i\mathbb{Z}_p/p^w\mathbb{Z}_p.$$

We now construct the required subgroups. The ring $\mathcal{O}$ is principal thus, by the elementary divisors theorem, there exists $\gamma_1, \ldots, \gamma_s \in \Gamma$ such that

$$\Gamma = \bigoplus_{k=1}^{s} \mathcal{O}\,\gamma_k.$$

We define $\Gamma^\wedge := \bigoplus_k \mathbb{Z}_p\,\gamma_k$ and $\Gamma^\vee := \bigoplus_k \mathbb{Z}_p\,(i\gamma_k)$. It is clear from the above discussion that these groups satisfy the required properties. $\square$

The proof now goes along the same lines as in the case $p \equiv 1 \pmod 4$. We apply Lemma 5.7 to the groups $\mathrm{Cl}_K^-\{p\}$ and $\mathcal{M}_p$ to obtain the subgroups $\mathrm{Cl}_K^\wedge\{p\}$, $\mathrm{Cl}_K^\vee\{p\}$, $\mathcal{M}_p^\wedge$ and $\mathcal{M}_p^\vee$. Let $n \geq m \geq 0$ be integers such that $|\,\mathrm{Cl}_K^\wedge\{p\}| = |\,\mathrm{Cl}_K^\vee\{p\}| = p^n$ and $|\mathcal{M}_p^\wedge| = |\mathcal{M}_p^\vee| = p^m$. The equations (5.6) and (5.7) are still verified and thus we have $|\,\mathrm{Cl}_K^-\{p\}| = p^{2n}$ and $|\,\mathrm{Cl}_F^-\{p\}| = p^m$.

As above, from (5.3) and Lemma 5.2, there exists $u \in \mathbb{Z}_p^\times$ such that

$$d_G\,\theta_{K/k,S}^{(>1)} = up^{n-m}(1-c)\big(2+[a]-[a^2]\big)$$

and therefore $d_G\,\theta_{K/k,S}^{(>1)} \in \mathbb{Z}_p[G]$. Now, consider the quotient group $\mathrm{Cl}_K^-\{p\}/\mathcal{M}_p$. Its order is $p^{2(n-m)}$ and, since it is also a $\mathbb{Z}_p[i]$-module, its exponent divides p^{n-m} by Lemma 5.7. Therefore, for $\mathcal{C} \in \mathrm{Cl}_K\{p\}$ a class with p-power order, we have $p^{n-m}(1-c)\,\mathcal{C} \in \mathcal{M}_p$. It follows from Lemma 5.6 that $d_G\,\theta_{K/k,S}^{(>1)}\,\mathcal{C} = 0$. Hence, for any ideal $\mathfrak{A} \in \mathcal{C}$, the ideal $\mathfrak{A}^{d_G\theta_{K/k,S}^{(>1)}}$ is principal. We prove that it admits a generator that is also an anti-unit as above. This concludes the proof of Theorem 5.1.

Remark. To conclude this section, we remark that most of the results of this section apply also in the general case, see the proposition below. For $p = 3$, we need the fact that there is prime ideal that is ramified in K/F in the proof of Lemma 5.4.

Proposition 5.8. *Let K/k be a Galois extension with Galois group G isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. Let $p \geq 5$ be a prime number. Then, $L_{K/k}(0,\chi)\,e_\chi \in \mathbb{Z}_p[G]$ and annihilates $\mathrm{Cl}_K\{p\}$.* □

6. Numerical verifications

In this section, we explain how we proved the Refined Galois Brumer-Stark conjecture for many extensions $K/\mathbb{Q}$ with Galois group isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$ using the PARI/GP system [16]. First, we say a few words on how we found these extensions. The database [10], created by Klüners and Malle, contains tables of number fields up to degree 19. We extracted from the database all degree 8 totally complex number fields whose normal closure has Galois group over $\mathbb{Q}$ isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. These fields correspond to the field F. The database contains 73 such fields. For each field, we compute its Galois closure, corresponding to the field K. Then, we compute the class group of K; we find that only one field K is principal and discard it since the Refined Galois Brumer-Stark conjecture is satisfied for principal fields. For the 72 remaining fields, the class numbers range from 8 to 87 684 589 640 025. Of these 72 fields, 43 fields correspond to extensions of type I and 29 to extensions of type II. Fix S to be the set containing the infinite place of $\mathbb{Q}$ and the prime numbers that ramify in K. Among the 43 extensions of type I, 13 have odd class number and thus the Refined Galois Brumer-Stark conjecture for these extensions and the set of places S follows from Theorem 5.1. For the remaining 59 fields, we prove that the Refined Galois Brumer-Stark conjecture holds using the method explained below on an example.

Theorem 6.1. *The Refined Galois Brumer-Stark conjecture $\mathbf{RBS}_{\mathrm{Gal}}(K/\mathbb{Q},S)$ holds for the 30 extensions K of type I with even class number listed in Table 1 and the 29 non-principal extensions of type II listed in Table 2.*

Tables 1 and 2 are given below and contains for each extension: an irreducible monic polynomial defining a totally complex degree 8 extension F of $\mathbb{Q}$ whose Galois closure is a field K with $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$; the discriminant of F; the structure of the class group Cl_K as a vector $(d_1^{a_1}, \dots, d_r^{a_r})$ with $d_{i+1} \mid d_i$ such that

$$\mathrm{Cl}_K \simeq \bigoplus_{i=1}^{r} (\mathbb{Z}/d_i\mathbb{Z})^{a_i}.$$

Polynomial defining F	d_F	Cl_K
$X^8 - 2X^7 + 15X^6 - 22X^5 + 68X^4 - 37X^3 + 188X^2 - 269X + 599$	163^2	$(18^2, 2^3)$
$X^8 - 3X^7 + 7X^6 - 5X^5 + 16X^4 - 31X^3 + 289X^2 - 180X + 69$	277^2	$(36, 18, 2)$
$X^8 + 13X^6 + 52X^4 + 65X^2 + 9$	277^2	$(42^2, 2^4)$
$X^8 + 18X^6 + 92X^4 + 112X^2 + 16$	163^2	$(84^2, 4, 2)$
$X^8 - X^7 + 8X^6 + 8X^5 + 32X^4 + 66X^3 + 128X^2 + 204X + 179$	$7^2 \cdot 73^2$	$(18^2, 2)$
$X^8 + 20X^6 + 42X^4 + 24X^2 + 4$	$2^6 \cdot 31^2$	$(18^2, 6, 3)$
$X^8 + 12X^6 + 42X^4 + 40X^2 + 4$	$2^6 \cdot 31^2$	$(78^2, 2)$
$X^8 - 4X^7 + 23X^6 - 34X^5 + 74X^4 + 102X^3 + 465X^2 + 686X + 296$	349^2	$(60^2, 12, 6)$
$X^8 - 3X^7 + 16X^6 - 35X^5 + 153X^4 - 325X^3 + 1148X^2 + 841X + 571$	277^2	$(420, 210, 2)$
$X^8 + 27X^6 + 207X^4 + 378X^2 + 81$	163^2	$(168, 84, 4, 2^4)$
$X^8 - 4X^7 + 4X^6 - 47X^5 + 194X^4 - 95X^3 + 169X^2 - 1582X + 1960$	$3^4 \cdot 127^2$	$(252, 126, 6)$
$X^8 - 2X^7 + 8X^6 + 6X^5 - 106X^4 - 6X^3 + 260X^2 + 325X + 502$	$19^2 \cdot 37^2$	$(252, 126, 6, 2^2)$
$X^8 - X^7 + 19X^6 + 11X^5 + 291X^4 - 37X^3 + 227X^2 + 1523X + 1557$	163^2	$(126^2, 6^2, 2)$
$X^8 - 3X^7 + 19X^6 - 84X^5 + 216X^4 - 699X^3 + 1930X^2 - 36X + 751$	$3^2 \cdot 61^2$	$(120, 60, 4^2, 2^2)$
$X^8 + 15X^6 + 47X^4 + 38X^2 + 9$	607^2	$(390^2, 2^2)$
$X^8 + 30X^6 + 252X^4 + 432X^2 + 144$	$3^2 \cdot 61^2$	$(210^2, 2^5)$
$X^8 + 12X^6 + 44X^4 + 52X^2 + 9$	$2^8 \cdot 43^2$	$(126^2, 2^2)$
$X^8 - X^7 + 28X^6 + 12X^5 + 467X^4 - 13X^3 + 753X^2 + 2929X + 2996$	163^2	$(126^2, 6^3, 3)$
$X^8 + 17X^6 + 60X^4 + 29X^2 + 1$	$3^4 \cdot 79^2$	$(468, 234, 6, 2^5)$
$X^8 + 57X^6 + 1017X^4 + 5508X^2 + 1296$	$31^2 \cdot 43^2$	$(60^2, 12^2, 3)$
$X^8 + 28X^6 + 204X^4 + 148X^2 + 25$	$2^8 \cdot 3^4 \cdot 7^2$	$(210^2, 6, 2)$
$X^8 - X^7 + 13X^6 + 41X^5 + 233X^4 - 423X^3 + 3197X^2 - 1768X + 2836$	853^2	$(210^2, 6^2, 2)$
$X^8 + 63X^6 + 1127X^4 + 4802X^2 + 2401$	163^2	$(2184, 1092, 4, 2^4)$
$X^8 + 17X^6 + 78X^4 + 53X^2 + 4$	$3^4 \cdot 127^2$	$(4662^2, 6, 2)$
$X^8 + 21X^6 + 108X^4 + 81X^2 + 9$	$3^2 \cdot 13^2 \cdot 19^2$	$(210^2, 6, 2^6)$
$X^8 - 2X^7 + 2X^6 - 66X^5 + 157X^4 - 554X^3 + 1827X^2 - 1831X + 5575$	1777^2	$(84^2, 28, 14, 2)$
$X^8 + 30X^6 + 240X^4 + 280X^2 + 16$	$7^2 \cdot 97^2$	$(774^2, 6^3)$
$X^8 + 44X^6 + 188X^4 + 184X^2 + 16$	$19^2 \cdot 37^2$	$(1116, 558, 6^3)$
$X^8 + 40X^6 + 268X^4 + 560X^2 + 324$	$2^6 \cdot 7^2 \cdot 19^2$	$(1110^2, 6)$
$X^8 + 57X^6 + 882X^4 + 4229X^2 + 1296$	$3^4 \cdot 601^2$	$(58776^2, 12, 2^3)$

TABLE 1. Extensions of type I

We illustrate our method of verification using an example of type II. The field F is the extension of $\mathbb{Q}$ generated by a root of the irreducible degree 8 polynomial

$$X^8 + 45X^6 + 707X^4 + 4430X^2 + 8649.$$

The field K is the Galois closure of F and it is generated over $\mathbb{Q}$ by a root of the irreducible polynomial

$$X^{24} - 9X^{23} + 118X^{22} - 566X^{21} + 3283X^{20} - 13171X^{19} + 50621X^{18} - 543733X^{17}$$
$$- 1072192X^{16} + 2804188X^{15} + 60060688X^{14} + 135957502X^{13} - 452543574X^{12}$$
$$- 2828250562X^{11} - 2262669513X^{10} + 7810609167X^9 + 8314237510X^8 + 15300973158X^7$$
$$+ 292802218851X^6 + 872090144149X^5 + 378022263593X^4 - 1912472285563X^3$$

Polynomial defining F	d_F	Cl_K
$X^8+X^6-3X^5-3X^4-6X^3+4X^2+16$	$3^2\cdot 61^2$	(2^3)
$X^8-X^7-10X^6+32X^5+43X^4-245X^3+393X^2-192X+68$	853^2	(11^2)
$X^8-3X^7+6X^5-6X^4+8X^3+162X^2+284X+173$	937^2	(19^2)
$X^8+19X^6+116X^4+255X^2+169$	$13^2\cdot 73^2$	$(33,11)$
$X^8+15X^6+63X^4+54X^2+9$	$3^2\cdot 61^2$	$(52,26,2^4)$
$X^8+19X^6-12X^5+96X^4-189X^3+130X^2-687X+871$	$3^2\cdot 61^2$	$(76,38,2^4)$
$X^8+26X^6+208X^4+520X^2+144$	277^2	$(140^2,4^3)$
$X^8+25X^6+187X^4+506X^2+441$	2311^2	(95^2)
$X^8-X^7+41X^6+53X^5+346X^4+1018X^3+1475X^2+478X+2425$	$3^2\cdot 61^2$	$(18^2,2^5)$
$X^8+30X^6+143X^4+205X^2+49$	$13^2\cdot 199^2$	$(159,53)$
$X^8-X^7+10X^6+50X^5+92X^4-12X^3+842X^2-202X+361$	547^2	$(164^2,4,2)$
$X^8+20X^6+140X^4+396X^2+361$	$2^8\cdot 43^2$	(14^4)
$X^8+29X^6+263X^4+774X^2+81$	2803^2	(117^2)
$X^8+22X^6+47X^4+23X^2+1$	$19^2\cdot 37^2$	$(924,154,2^4)$
$X^8-3X^7+29X^6-66X^5+375X^4-738X^3+3084X^2+2050X+2027$	277^2	$(1612,806,2)$
$X^8+19X^6+79X^4+98X^2+9$	$7^2\cdot 109^2$	$(1302,434,2^2)$
$X^8-X^7-7X^6+24X^5+363X^4-1144X^3+518X^2-244X+1471$	$7^2\cdot 97^2$	$(546,182,2^3)$
$X^8+25X^6+167X^4+330X^2+49$	3727^2	(207^2)
$X^8+54X^6+828X^4+3024X^2+1296$	163^2	$(592,296,8,4,2^3)$
$X^8-3X^7+42X^6-97X^5+701X^4-1329X^3+6436X^2+3965X+5215$	277^2	$(140,70,14^2,2^3)$
$X^8+34X^6+284X^4+752X^2+400$	547^2	$(900^2,4,2)$
$X^8-2X^7+30X^6-5X^5-192X^4+56X^3+339X^2+1411X+2843$	$19^2\cdot 37^2$	$(3108,518,2^3)$
$X^8+19X^6+113X^4+204X^2+16$	$31^2\cdot 43^2$	$(1974,658,2^4)$
$X^8+65X^6+1300X^4+8125X^2+5625$	277^2	$(540^2,4^2,2^6)$
$X^8+45X^6+707X^4+4430X^2+8649$	$37^2\cdot 151^2$	$(8769,2923)$
$X^8-2X^7+32X^6-93X^5+267X^4-1782X^3+9004X^2-8936X+14864$	$3^4\cdot 7^2\cdot 19^2$	$(804^2,4,2)$
$X^8+20X^6+130X^4+280X^2+36$	$2^6\cdot 7^2\cdot 19^2$	$(198^2,6,3^2)$
$X^8+90X^6+2300X^4+14000X^2+10000$	163^2	$(3686^2,2^7)$
$X^8+51X^6+639X^4+2538X^2+2025$	547^2	$(1118^2,2^7)$

TABLE 2. Extensions of type II

$$-2264522611330X^2+1176633991400X+2202625519192.$$

The discriminant of K is $37^{16}\,151^{16}$, thus we take $S=\{\infty,37,151\}$ where ∞ is the infinite place of $\mathbb{Q}$. We have

$$\mathrm{Cl}_K\simeq\mathbb{Z}/8\,769\mathbb{Z}\times\mathbb{Z}/2\,923\mathbb{Z}. \tag{6.9}$$

Observe that $8\,769=3\cdot 37\cdot 79$ and $2\,923=37\cdot 79$. The first step is to compute the expression of $\theta^{(>1)}_{K/k,S}$. For that, we use the expression given by Theorem 4.6. We have $h_K=25\,631\,787$, $h_{K^+}=3$, $h_F=79$ and $h_{F^+}=1$. One checks that some units of K are not totally real and thus, using Lemma 4.7, we compute that

$$\frac{h_KR_K}{h_{K^+}R_{K^+}}=93\,536^2\quad\text{and}\quad\frac{h_FR_F}{h_{F^+}R_{F^+}}=316.$$

It remains to compute the value of $L_{K/F^+}(0,\nu)$. For this, we use the function `bnrL1` that computes approximations of the first non-zero coefficient of the Taylor expansion at $s=0$ of Hecke L-functions. We find that

$$L_{K/F^+}(0,\nu)\approx-1184.00000000+5126.87039040i.$$

Although we have only a numerical approximation, we can still deduce an exact value in the following way. Let S_0 be the set of places consisting of the infinite places of F^+ and of the finite places that ramify in K/F^+. One can check easily using the formula [15, Prop. I.3.4] for the order of vanishing of L-functions at $s=0$ that $L_{K/F^+,S_0}(0,\nu^t)=0$ for $t\in\{0,2,3,4\}$. Thus, we have

$$\begin{aligned}\theta_{K/F^+,S_0} &= L_{K/F^+}(0,\nu)\,e_{\bar\nu} + L_{K/F^+}(0,\nu^5)\,e_{\bar\nu^5}\\ &= L_{K/F^+}(0,\nu)\,e_{\bar\nu} + L_{K/F^+}(0,\bar\nu)\,e_{\nu} = L_{K/F^+}(0,\nu)\,e_{\bar\nu} + \overline{L_{K/F^+}(0,\nu)}\,e_{\nu}\\ &\approx -2368.00 - 10064.00\,a - 7696.00\,a^2 + 2368.00\,a^3 + 10064.00\,a^4 + 7696.00\,a^5.\end{aligned}$$

By the properties of the abelian Brumer-Stickelberger element, we know that $2\theta_{K/F^+,S_0}$ has integral coefficients and thus we find its exact value

$$\theta_{K/F^+,S_0} = -2368 - 10064\,a - 7696\,a^2 + 2368\,a^3 + 10064\,a^4 + 7696\,a^5.$$

Coming back to the expression of $\theta_{K/F^+,S_0}$ given above, we deduce that

$$L_{K/F^+}(0,\nu) = 1776 + 5920j$$

where $j := e^{2i\pi/3}$ is a third root of unity. Note that this method provides a proof that this is the exact value of $L_{K/F^+}(0,\nu)$. The prime numbers ramified in $K/\mathbb{Q}$ are 37 and 151. We compute their index of ramification and residual degree, we find that $e_{37}=e_{151}=3$, $f_{37}=f_{151}=2$, and therefore $\delta_{K/k}=4$. From these computations, we obtain that

$$\begin{aligned}\theta^{(>1)}_{K/k,S} &= \pm 93\,536\left(\frac{1}{316}e_\chi + 8\,\mathrm{Re}\Big(\frac{1}{1776+5920j}e_{\overline{\lambda\chi}}\Big)\right)\\ &= \pm\frac{1-c}{12}(592 + 296[a] - 296[a^2] - 64 - 104[a] - 136[a^2])\\ &= \pm\frac{(1-c)}{3}(132 + 48[a] - 108[a^2]).\end{aligned}$$

Recall that $d_G = 24$ and, therefore, we have

$$d_G\,\theta^{(>1)}_{K/k,S} = \pm(1-c)(1056 + 384[a] - 864[a^2]).$$

The first statement of the Refined Galois Brumer-Stark conjecture is satisfied: $d_G\,\theta^{(>1)}_{K/k,S}$ lies in $\mathbb{Z}[G]$. Observe, in fact, that $d_G\,\theta^{(>1)}_{K/k,S}\in 2\mathbb{Z}[G]$. This is actually true in all the examples that we have computed. This property makes it slightly easier to check the conjecture as we do not have to worry about finding generators that are anti-units. Indeed, we use the following lemma whose proof is left to the reader and is very similar to the reasoning used in the previous section, see the discussion after (5.8).

Lemma 6.2. *Let $\mathfrak{A}$ be a fractional ideal of K. Assume that $d_G\,\theta^{(>1)}_{K/k,S}$ is in $2\mathbb{Z}[G]$ and that $\mathfrak{A}^{(d_G/2)\,\theta^{(>1)}_{K/k,S}}$ is principal. Then, there exists an anti-unit α of K such that $\mathfrak{A}^{d_G\,\theta^{(>1)}_{K/k,S}} = (\alpha)$.* $\square$

Therefore, it is enough to check that $(1-c)(528+192[a]-432[a^2])$ annihilates Cl_K. We compute two classes $\mathcal{C}_1$ and $\mathcal{C}_2$ in Cl_K that realize the isomorphism in (6.9). That is, these classes are respectively of order 8769 and 2923, and the map from $\mathbb{Z}/8769\mathbb{Z}\times\mathbb{Z}/2923\mathbb{Z}$ to Cl_K that sends (s_1,s_2) to $s_1\mathcal{C}_1 + s_2\mathcal{C}_2$ is an isomorphism. We do not give explicit expressions for ideals generating these classes since these expressions are too big. We compute the action of a and b on these generating classes. We find that these are given by the following matrices

$$M_a := \begin{pmatrix}988 & 1797\\ 1597 & 2195\end{pmatrix}\quad\text{and}\quad M_b := \begin{pmatrix}1513 & 3741\\ 782 & 1410\end{pmatrix}.$$

These matrices are to be understood with the first row defined modulo 8769 and the second one modulo 2923. The first column of M_a says that $a\,\mathcal{C}_1 = 988\mathcal{C}_1 + 1597\mathcal{C}_2$, and so on. Using these matrices and the definitions of a_1, a_2 and a_3, we compute the action of $[a]$ and $[a^2]$ on Cl_K. We find that $[a]$ acts on Cl_K as the multiplication by 520 and $[a^2]$ as the multiplication by 5 104. From that we compute that $(528 + 192[a] - 432[a^2])$ acts on Cl_K as the multiplication by 0. (Note that the factor $(1 - c)$ is not needed.) Hence, $(d_G/2)\,\theta^{(>1)}_{K/k,S}$ annihilates the class group of K and $\mathbf{RBS}_{\mathrm{Gal}}(K/\mathbb{Q}, S)$ is proved.

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