Verifying a p-Adic Abelian Stark Conjecture at $s = 1$

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Abstract

In a previous paper [13], the second author developed a new approach to the abelian p-adic Stark conjecture at $s = 1$ and stated related conjectures. The aim of the present paper is to develop and apply techniques to numerically investigate one of these – the 'Weak Refined Combined Conjecture' – in fifteen cases.

1 Introduction

In the 1970's and 80's Harold Stark [14] made a series of conjectures concerning the values at $s = 1$ and $s = 0$ of complex Artin L-series attached to a Galois extension of number fields K/k. Subsequently, much theoretical and computational work has been done, extending and testing these conjectures, with particular attention paid recently to certain refined conjectures in the case where K/k is abelian ([7], [5]).

In [13], a new approach to the abelian case of the p-adic conjecture at $s = 1$ was developed and several related conjectures were stated. The main aim of the present paper is to develop and apply techniques to numerically investigate one of these – the 'Weak Refined Combined Conjecture' (Conjecture 3.6 of [13], here Conjecture 2.2) – in a number of cases.

In Section 2, we shall recall the definitions of the complex and p-adic 'twisted zeta functions'. They depend on two parameters: a proper ideal $\mathfrak{f}$ of $\mathcal{O}_k$, and a set T of primes ideals of $\mathcal{O}_k$ (which, for the purpose of p-adic interpolation, must contain the primes above p). Then the statements of the two 'combined conjectures' of [13] are given. (The term 'combined' refers to the fact that each conjecture predicts both a complex and a p-adic equality.) The main reference for this section is, of course, [13], but also [12] which contains a reformulation developed by the second author of a refined complex abelian conjecture at $s = 0$ originally made by Rubin in [7]. Briefly, the 'Weak Refined Combined Conjecture' takes the following form: we assemble all the complex (*resp.* p-adic) twisted zeta-functions for given $\mathfrak{f}$ and T into a single group-ring-valued function $\Phi_{\mathfrak{f},T}(s)$ (*resp.* $\Phi_{\mathfrak{f},T,p}(s)$, assuming

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that T contains the primes above p). Then, assuming that the primes in T do not divide $\mathfrak{f}$, the value of the latter at $s = 1$ is conjectured to be equal to the complex (*resp.* p-adic) group-ring-valued regulator of a certain element $\eta_{\mathfrak{f},T}$ multiplied by an explicit algebraic constant. The element $\eta_{\mathfrak{f},T}$ is constructed from certain S-units of the field K which in this case is simply the ray-class field $k(\mathfrak{f})$.

Section 3 develops a new formula to compute the element $\Phi_{\mathfrak{f},T,p}(1)$. We concentrate on the case where k is real quadratic although our technique should extend to other totally real fields. Relying as it does on Shintani's method and the theory of p-adic measures, this technique is very different in nature from that used to evaluate complex L-functions. (For the latter we use [1].) We stress that it passes most naturally not by the analogous p-adic L-functions but by the p-adic twisted zeta-functions themselves. Indeed, this was one of the major reasons for introducing these functions and, in preparation, their complex analogues.

Finally, Section 4 is devoted to the numerical investigation of the 'Weak Refined Combined Conjecture' over a real quadratic field. We first explain some procedures (for example a continued fraction method based on ideas of Zagier) that greatly shorten the calculation of $\Phi_{\mathfrak{f},T,p}(1)$ using the formula of the previous section. Then we explain the basis of our method for verifying the conjecture. Since $[k : \mathbb{Q}] = 2$ and K is totally real, our conjectures are 'second order' in the sense that the relevant complex L-functions have at least a double zero at $s = 0$. The corresponding fact on the 'other side' of the conjectures is that both the complex and p-adic regulators must be of rank 2. One consequence is that, unlike verifications of the (complex) first order abelian Stark Conjectures (see for example [6]), the regulators themselves do not determine S-units of K. We therefore need different methods for finding K and $\eta_{\mathfrak{f},T}$ and new criteria for affirming that the latter satisfies the combined conjecture to the precision of our computations. In fact, we use the methods of [6] (which actually rely on the first-order complex conjecture!) to independently and verifiably construct the ray-class field K. We then illustrate our methodology by numerically confirming the conjecture in fifteen different examples, using a number of different primes p in each example. The resulting data are displayed in tables at the end of the paper. We hope that they will serve to stimulate further interest in these conjectures, their possible refinements and extensions.

2 The p-adic Stark conjectures at $s = 1$

The main reference for this section are [12] (for the complex twisted functions) and [13].

2.1 Complex twisted zeta functions

Let $k \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ be any number field of finite degree over $\mathbb{Q}$ and let $\mathcal{O}$ its ring of integers. Let I be any fractional ideal of k and ξ any character on (the additive group of) I with values in $\mu(\mathbb{C})$, the complex roots of unity. The *annihilator* of ξ is the ideal $\mathfrak{f} \lhd \mathcal{O}$ given by $\mathfrak{f} = \{a \in \mathcal{O} : \xi(ax) = 1 \ \forall\, x \in I\}$. Suppose that $\mathfrak{z}$ is the formal product of some subset of the real places of k and write $\mathfrak{m}$ for the *cycle* that is the formal product $\mathfrak{f}\mathfrak{z}$. We denote by

$E_{\mathfrak{m}}$ the subgroup of finite index in $E(K) := \mathcal{O}^\times$ consisting of the units that are congruent to 1 modulo $\mathfrak{m}$ in the usual sense. For any finite set T of finite places (prime ideals) of $\mathcal{O}$, the group $E_{\mathfrak{m}}$ acts by multiplication on the following subset of I

$$\mathcal{S}(I, \mathfrak{z}, T) := \{a \in I \,:\, a \in k_{\mathfrak{z}}^\times \text{ and } (aI^{-1}, T) = 1\}$$

where $k_{\mathfrak{z}}^\times$ denotes the elements of $k^\times$ which are positive at all places dividing $\mathfrak{z}$ and the notation $(J, T) = 1$ indicates that an ideal J of $\mathcal{O}$ has support disjoint from T. For $s \in \mathbb{C}$, $\Re(s) > 1$ we consider the absolutely convergent Dirichlet series, called the 'twisted zeta-function' for these data, defined by

$$\begin{aligned} Z_T(s; \xi, I, \mathfrak{m}) := \sum_{a \in \mathcal{S}(I,\mathfrak{z},T)/E_{\mathfrak{m}}} \frac{\xi(a)}{|I : (a)|^s} &= \sum_{a \in \mathcal{S}(I,\mathfrak{z},T)/E_{\mathfrak{m}}} \frac{\xi(a)}{N(aI^{-1})^s} \\ &= NI^s \sum_{a \in \mathcal{S}(I,\mathfrak{z},T)/E_{\mathfrak{m}}} \xi(a) |\mathrm{N}_{k/\mathbb{Q}}(a)|^{-s} \end{aligned} \tag{1}$$

Let $W_{\mathfrak{f}}$ be the set of all pairs (ψ, J), where ψ is a character of annihilator $\mathfrak{f}$ on a fractional ideal J. In [13] a natural equivalence relation (depending on $\mathfrak{z}$) was defined on $W_{\mathfrak{f}}$ in such a way that $Z_T(s; \xi, I, \mathfrak{m})$ equals $Z_T(s; \xi', I', \mathfrak{m})$ if (ξ, I) and (ξ', I') are equivalent. Let $\mathfrak{W}_{\mathfrak{m}}$ denote the quotient set of $W_{\mathfrak{f}}$ by this equivalence relation. Then for any equivalence class $\mathfrak{w} \in \mathfrak{W}_{\mathfrak{m}}$ we can unambiguously define $Z_T(s; \mathfrak{w}) := Z_T(s; \xi, I, \mathfrak{m})$ for any $(\xi, I) \in \mathfrak{w}$. Let $\mathrm{Cl}_{\mathfrak{m}}(k)$ denote the ray-class group of k modulo $\mathfrak{m}$. Thus $\mathrm{Cl}_{\mathfrak{m}}(k) := \mathcal{I}_{\mathfrak{f}}(k)/P_{\mathfrak{m}}(k)$ where $\mathcal{I}_{\mathfrak{f}}(k)$ denotes the group of fractional ideals prime to $\mathfrak{f}$ and $P_{\mathfrak{m}}(k)$ the subgroup consisting of those of the form (a) for some $a \in k^\times$, $a \equiv 1 \pmod{\mathfrak{m}}$. For any $\mathfrak{c}$ in $\mathrm{Cl}_{\mathfrak{m}}(k)$ and $\mathfrak{w}$ in $\mathfrak{W}_{\mathfrak{m}}$ we let $\mathfrak{c} \cdot \mathfrak{w}$ denote the element of $\mathfrak{W}_{\mathfrak{m}}$ given by the equivalence class of the pair $(\xi|_{\mathfrak{a}I}, \mathfrak{a}I) \in W_{\mathfrak{f}}$ where (ξ, I) is any pair in the class $\mathfrak{w}$ and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{f}}(k)$ any integral ideal in the class $\mathfrak{c}$. This map is well-defined and determines an action of $\mathrm{Cl}_{\mathfrak{m}}(k)$ on $\mathfrak{W}_{\mathfrak{m}}$. One can check that this action is free and transitive.

Let $\mathcal{D} \lhd \mathcal{O}$ denote the absolute different of k and write $\xi_{\mathfrak{f}}^0$ for the character on $\mathfrak{f}^{-1}\mathcal{D}^{-1}$ which sends a to $\exp(2\pi i \,\mathrm{Tr}_{k/\mathbb{Q}}(a))$. Thus the pair $(\xi_{\mathfrak{f}}^0, \mathfrak{f}^{-1}\mathcal{D}^{-1})$ lies in $W_{\mathfrak{f}}$ and we write $\mathfrak{w}_{\mathfrak{m}}^0$ for its class in $\mathfrak{W}_{\mathfrak{m}}$. Let $k(\mathfrak{m}) \subset \mathbb{C}$ be the *ray-class field* over k modulo $\mathfrak{m}$. The Galois group $G_{\mathfrak{m}} := \mathrm{Gal}(k(\mathfrak{m})/k)$ is isomorphic to $\mathrm{Cl}_{\mathfrak{m}}(k)$ *via* the Artin map which sends $\mathfrak{c} \in \mathrm{Cl}_{\mathfrak{m}}(k)$ to $\sigma_{\mathfrak{c},\mathfrak{m}} = \sigma_{\mathfrak{c}} \in G_{\mathfrak{m}}$. We let $\mathbb{C}G_{\mathfrak{m}}$ denote the complex group-ring of $G_{\mathfrak{m}}$ and make the

Definition 2.1 *For any cycle $\mathfrak{m} = \mathfrak{f}\mathfrak{z}$ for k and any finite set T of prime ideals of $\mathcal{O}$, we write $\Phi_{\mathfrak{m},T}$ for the function*

$$\begin{aligned} \Phi_{\mathfrak{m},T} \;:\; \{s \in \mathbb{C} : \Re(s) > 1\} &\longrightarrow \mathbb{C}G_{\mathfrak{m}} \\ s &\longmapsto \sum_{\mathfrak{c} \in \mathrm{Cl}_{\mathfrak{m}}(k)} Z_T(s; \mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0)\sigma_{\mathfrak{c}}^{-1} \end{aligned}$$

The basic properties of $\Phi_{\mathfrak{m},T}(s)$ are given in [12, §3] and [13, §2]. In particular Theorem 2.2 of [13] gives a relation between $\Phi_{\mathfrak{m},T}$ and the primitive L-functions of the characters of $G_{\mathfrak{m}}$ (or $\mathrm{Cl}_{\mathfrak{m}}(k)$).

2.2 p-Adic interpolation

We turn now to the definition of the p-adic analogue of $\Phi_{\mathfrak{m},T}$ by p-adic interpolation. For this we need k to be *totally real* which we shall assume henceforth.

We choose a prime number p, write $\mathbb{C}_p$ for the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers and fix an embedding $j : \bar{\mathbb{Q}} \to \mathbb{C}_p$. We let $\mu(\mathbb{Q}_p)$ be the group of roots of unity in $\mathbb{Q}_p$, w_p its cardinality, and consider the set of rational integers defined by

$$\mathcal{M}(p) := \{m \in \mathbb{Z} : m \leq 0,\ m \equiv 1 \pmod{w_p}\}$$

Let $L(s, \psi)$ denote the complex L-function of a primitive ray-class character ψ. It is well-known that its values at the points of $\mathcal{M}(p)$ are algebraic and that their images under j may be 'interpolated' to define a p-adic L-function attached to the primitive p-adic-valued ray-class character $j \circ \psi$ (this is summarised in [13, Theorem 2.4]). It can also be shown (see [13, Theorem/Definition 2.1]) that the values of $\Phi_{\mathfrak{m},T}$ on $\mathcal{M}(p)$ lie in $\bar{\mathbb{Q}}G_{\mathfrak{m}}$ and using, for example, the p-adic L-functions one may interpolate these values whenever the following condition is satisfied

$$T \text{ contains the set } T_p \text{ of all primes dividing } p \text{ in } \mathcal{O}. \tag{2}$$

More precisely, let $D(p)$ denote the set $1 + 2\mathbb{Z}_p$, the closure of $\mathcal{M}(p)$ in $\mathbb{Q}_p$. Then under condition (2) there exists a unique p-adic valued function $\Phi_{\mathfrak{m},T,p}(s)$, defined on $D(p)$ and depending on j, such that

$$\Phi_{\mathfrak{m},T,p}(m) = j(\Phi_{\mathfrak{m},T}(m)) \quad \forall m \in \mathcal{M}(p). \tag{3}$$

(here, j has been extended to a homomorphism $\bar{\mathbb{Q}}G_{\mathfrak{m}} \to \mathbb{C}_pG_{\mathfrak{m}}$ by acting on the coefficients). Furthermore $\Phi_{\mathfrak{m},T,p}$ is p-adic meromorphic on $D(p)$. If $\mathfrak{f}$ is *not* a product of distinct primes lying in T, then $\Phi_{\mathfrak{m},T,p}$ is actually analytic on this domain. Otherwise it has at most a unique, simple pole at $s = 1$. In all cases $x\Phi_{\mathfrak{m},T,p}(s)$ is analytic in $D(p)$ for any x in the augmentation ideal $I(\mathbb{C}_pG_{\mathfrak{m}})$ of $\mathbb{C}_pG_{\mathfrak{m}}$. Note that we shall write $\Phi^{(j)}_{\mathfrak{m},T,p}$ instead of $\Phi_{\mathfrak{m},T,p}$ whenever the dependence on j needs to be made explicit.

2.3 The conjectures

We recall the combined conjectures stated in [13, §3]. These are made up of a complex and a p-adic part formulated side by side for the same field k (of degree r say, over $\mathbb{Q}$), cycle $\mathfrak{m} = \mathfrak{f}\mathfrak{z}$ and set T, but in terms of $\Phi_{\mathfrak{m},T}(1)$ and $\Phi_{\mathfrak{m},T,p}(1)$ respectively. We make the

Hypothesis 2.1

(i). k is totally real,

(ii). $\mathfrak{f}$ is not a product of distinct primes lying in T (in particular, $\mathfrak{f}$ is not trivial), and

(iii). $\mathfrak{z}$ is trivial, i.e. $\mathfrak{m} = \mathfrak{f}$

Hypothesis 2.1 will be assumed from now on unless the contrary is explicitly stated, so that, in general, we can write $\Phi_{\mathfrak{f},T}$ *etc.* in place of $\Phi_{\mathfrak{m},T}$ *etc.* We shall also write K for the ray-class field $k(\mathfrak{m}) = k(\mathfrak{f}) \subset \mathbb{C}$ (necessarily totally real) and G for $G_{\mathfrak{f}} = \mathrm{Gal}(K/k)$. Let S_∞ and $S_0 = S_0(\mathfrak{f})$ denote respectively the set of infinite (real) places of k and the set of finite places dividing $\mathfrak{f}$ in k. We let $S = S_\infty \cup S_0$. The notations $S_\infty(K)$, $S_0(K)$ and $S(K)$ refer to the sets of places of K dividing those in these three sets. We abbreviate to $U_S(K)$ or U_S the group $U_{S(K)}(K)$ of $S(K)$-units of K. Let $\iota_1, \ldots, \iota_r$ denote the embeddings of k into $\bar{\mathbb{Q}}$ (ι_1 is the inclusion). For each $i = 1, \ldots, r$ we choose an embedding $\tilde{\iota}_i : K \to \bar{\mathbb{Q}}$ extending ι_i. We write $\iota_{i,p}$ for the p-adic embedding $j \circ \iota_i$ of k into $\mathbb{C}_p$, and also $\tilde{\iota}_{i,p}$ for its extension $j \circ \tilde{\iota}_i : K \to \mathbb{C}_p$. We then define logarithmic maps $\lambda_i : U_S \to \mathbb{R}G$ and $\lambda_{i,p} : U_S \to \mathbb{C}_pG$ by setting

$$\lambda_i(u) := \sum_{\sigma\in G} \log|\tilde{\iota}_i \circ \sigma(u)|\sigma^{-1} \text{ and } \lambda_{i,p}(u) := \sum_{\sigma\in G} \log_p(\tilde{\iota}_{i,p} \circ \sigma(u))\sigma^{-1} \text{ for all } u \in U_S.$$

where '$\log_p$' denotes Iwasawa's p-adic logarithm. Both λ_i and $\lambda_{i,p}$ are clearly $\mathbb{Z}G$-linear and so 'extend' by $\mathbb{Q}$-linearity to $\mathbb{Q}U_S := \mathbb{Q} \otimes_{\mathbb{Z}} U_S$. (Henceforth, we shall often write $\mathcal{R}A$ in place of $\mathcal{R} \otimes_{\mathbb{Z}} A$ considered as an $\mathcal{R}$-module, for any commutative ring $\mathcal{R}$ and abelian group A.) These extensions in turn define unique, $\mathbb{Q}G$-linear, group-ring-valued 'regulator maps' R and R_p sending the rth exterior power $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S \cong \mathbb{Q} \otimes \bigwedge_{\mathbb{Z}G}^r U_S$ into $\mathbb{R}G$ and $\mathbb{C}_pG$ respectively and satisfying

$$R(u_1\wedge\ldots\wedge u_r) = \det(\lambda_i(u_l))_{i,l=1}^r \text{ and } R_p(u_1\wedge\ldots\wedge u_r) = \det(\lambda_{i,p}(u_l))_{i,l=1}^r \ \forall\, u_1, \ldots, u_r \in \mathbb{Q}U_S$$

We shall use an additive notation for $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$ as $\mathbb{Z}G$-module and write $\lambda_{i,p}^{(j)}$ and $R_p^{(j)}$ instead of $\lambda_{i,p}$ and R_p whenever their dependence on j (*via* the $\tilde{\iota}_{i,p}$) needs to be made explicit. Finally, we let $\sqrt{d_k} \in \mathbb{R}$ denote the positive square-root of the (positive) absolute discriminant d_k of k.

Conjecture 2.1 (Basic Combined Conjecture) *If Hypothesis 2.1 and condition (2) hold then, there exists $\eta_{\mathfrak{f},T} \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$ such that*

$$\Phi_{\mathfrak{f},T}(1) = \frac{2^r}{\sqrt{d_k}\prod_{\mathfrak{p}\in T} N\mathfrak{p}} R(\eta_{\mathfrak{f},T}) \quad \textit{in } \mathbb{C}G \tag{4}$$

and

$$\Phi_{\mathfrak{f},T,p}^{(j)}(1) = \frac{2^r}{j(\sqrt{d_k})\prod_{\mathfrak{p}\in T} N\mathfrak{p}} R_p^{(j)}(\eta_{\mathfrak{f},T}) \quad \textit{in } \mathbb{C}_pG. \tag{5}$$

Remark 2.1 It is proved in [13, Prop. 3.3] that if $\eta_{\mathfrak{f},T}$ satisfies equation (5) for one embedding $j : \bar{\mathbb{Q}} \to \mathbb{C}_p$ then it also satisfies it for any other embedding.

In [13, §3] 'basic' conjectures were formulated concerning the existence of elements η separately satisfying (4) and (5). These were followed – under certain conditions – by 'refined' versions that require η to lie in a certain $\mathbb{Z}G$-lattice inside $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$. (These are 'conjectures over $\mathbb{Z}$' in the terminology of [15] and [7], indeed the complex version is closely linked to that of the latter paper, see [12], [13]). A weakened, combined version of these conjectures was then given which nevertheless refines Conjecture 2.1. It is stated simultaneously for all (eligible) primes p. Before giving this we first recall some notations: For any χ in G^* (identified with $\mathrm{Cl}_\mathfrak{f}(k)^*$) we set $r(S,\chi) := \dim_\mathbb{C}(\mathrm{e}_\chi \mathbb{C}\mathrm{U}_\mathrm{S})$. Let $\chi_0 \in G^*$ denote the trivial character and for any place v of k, let $G(v)$ denote the decomposition subgroup of G associated to each of the places w of K dividing v. It can be shown that

$$\begin{aligned} r(S,\chi) &= \begin{cases} r + |\{\mathfrak{q} : \mathfrak{q}|\mathfrak{f},\ \chi|_{G(\mathfrak{q})} = 1\}| & \text{if } \chi \neq \chi_0, \text{ and} \\ r - 1 + |\{\mathfrak{q} : \mathfrak{q}|\mathfrak{f}\}| & \text{if } \chi = \chi_0 \end{cases} \\ &= r + \mathrm{ord}_{s=1}(\chi(\Phi_{\mathfrak{m},T}(s))) \end{aligned} \tag{6}$$

where the last equation holds *provided that* $(\mathfrak{f}, T) = 1$. Because $\mathfrak{f} \neq \mathcal{O}$, it follows that $r(S,\chi) \geq r$ for every $\chi \in G^*$ and $r(S,\chi_0) = r$ if and only if $\mathfrak{f}$ is a (non-trivial) power of a prime ideal. The latter condition will be denoted simply '$\mathfrak{f} = \mathfrak{q}^l$'. If it does not hold then $r(S,\chi_0) > r$ and we shall write '$\mathfrak{f} \neq \mathfrak{q}^l$'. We set

$$e_{S,r} := \sum_{\substack{\chi \in G^* \\ r(S,\chi)=r}} e_\chi \quad \text{and} \quad e_{S,>r} := 1 - e_{S,r} = \sum_{\substack{\chi \in G^* \\ r(S,\chi)>r}} e_\chi \tag{7}$$

These idempotents actually lie in $\mathbb{Q}G$. Let g denote the cardinality of G, then $\tilde{e}_{S,r} := g e_{S,r}$ and $\tilde{e}_{S,>r} := g e_{S,>r}$ clearly lie in $\mathbb{Z}G$. For any $\mathbb{Z}G$-module A, we shall write $A^{[S,r]} := \ker \tilde{e}_{S,>r}|A$ so that $A^{[S,r]} \supset \tilde{e}_{S,r}A \supset gA^{[S,r]}$. For any $\mathbb{Z}G$-submodule M of U_S, we denote by $\overline{\bigwedge^r_{\mathbb{Z}G} M}$ the image of the exterior power $\bigwedge^r_{\mathbb{Z}G} M$ in $\bigwedge^r_{\mathbb{Q}G} \mathbb{Q}U_S$. The conjecture that will be numerically verified in this article is the following

Conjecture 2.2 (Weak Refined Combined Conjecture) *Suppose that k is totally real and $\mathfrak{f} \neq \mathcal{O}$ is any proper integral ideal. Then, in the above notations, there exists a unique element $\eta_\mathfrak{f}$ of $\left(\bigwedge^r_{\mathbb{Q}G} \mathbb{Q}U_S\right)^{[S,r]}$ with the following properties*

(i).

$$\frac{2^r}{\sqrt{d_k}} R(\eta_\mathfrak{f}) = \Phi_{\mathfrak{f},\emptyset}(1) \tag{8}$$

(ii). For every prime number p with $(p,\mathfrak{f}) = 1$ and for every embedding $j : \bar{\mathbb{Q}} \to \mathbb{C}_p$ we have

$$\prod_{\mathfrak{p} \in T_p} (1 - N\mathfrak{p}^{-1}\sigma_{\mathfrak{p},\mathfrak{f}}) \frac{2^r}{j(\sqrt{d_k})} R_p^{(j)}(\eta_\mathfrak{f}) = \Phi^{(j)}_{\mathfrak{f},T_p,p}(1) \tag{9}$$

(iii). If $\mathfrak{f} \neq \mathfrak{q}^l$ then

$$\eta_\mathfrak{f} \in \mathbb{Z}[1/g]\overline{\textstyle\bigwedge^r_{\mathbb{Z}G} U_S}^{[S,r]} = \mathbb{Z}[1/g]\overline{\textstyle\bigwedge^r_{\mathbb{Z}G} E(K)}^{[S,r]} \tag{10}$$

(iv). If $\mathfrak{f} = \mathfrak{q}^l$ then

$$\eta_\mathfrak{f} \in \frac{1}{2}\mathbb{Z}[1/g]\overline{\textstyle\bigwedge^r_{\mathbb{Z}G} U_S}^{[S,r]} \tag{11}$$

and

$$I(\mathbb{Z}G)\eta_\mathfrak{f} \subset \mathbb{Z}[1/g]\overline{\textstyle\bigwedge^r_{\mathbb{Z}G} U_S}^{[S,r]} \tag{12}$$

where $I(\mathbb{Z}G)$ is the augmentation ideal of $\mathbb{Z}G$.

Remark 2.2 The point of introducing the condition $\eta_\mathfrak{f} \in \left(\bigwedge^r_{\mathbb{Q}G} \mathbb{Q}U_S\right)^{[S,r]}$ is that, essentially without cost, it allows us to insist upon the uniqueness of $\eta_\mathfrak{f}$ (*cf.* [13, Prop. 3.8]). Given this uniqueness and the relation between $\Phi_{\mathfrak{f},\emptyset}$ and $\Phi_{\mathfrak{f},T_p}$ when $(p,\mathfrak{f}) = 1$ (see [13, eq. (29)]), it can be shown that equation (9) is actually a consequence of Conjecture 2.1 (with $T = T_p$) and (8). Moreover, the extra conditions of parts (iii) and (iv), which refine Conjecture 2.1, also follow from it *together with* the assumption of the 'refined complex conjecture' mentioned above for certain sets T. For more details, we refer to Prop. 3.10 of [13]. Note also (*cf.* the preceding remark) that for given p, (9) actually holds for all embeddings j if and only if it holds for one. Finally, for the (non-conjectural!) equality in (10), we refer to [13, Lemma 3.5].

3 An expression for $\Phi_{\mathfrak{f},T_p,p}(1)$ in the quadratic case

3.1 An application of Shintani's method

We start with a more general situation than the one suggested by the title of this section. The data, notated in the usual way as k, $\mathfrak{m} = \mathfrak{f}\mathfrak{z}$, T and p, are subject only to parts (i) and (ii) of Hypothesis 2.1 and to Condition (2). In particular, $\mathfrak{f} \neq \mathcal{O}$. For any $j : \bar{\mathbb{Q}} \to \mathbb{C}_p$, we know by Theorem/Definition 2.1 and Lemma 3.3 of [13] that $\Phi^{(j)}_{\mathfrak{m},T,p}$ is a p-adic analytic map from $D(p)$ to the group ring $\mathbb{Q}_p(\mu_f)^+G$. By taking its coefficients we obtain, for each $\mathfrak{w} \in \mathfrak{W}_\mathfrak{m}$, an analytic map $Z^{(j)}_{T,p}(\,\cdot\,;\mathfrak{w}) : D(p) \to \mathbb{Q}_p(\mu_f)^+$ (the *p-adic twisted zeta-function attached to T, $\mathfrak{w}$ and j*). More precisely, since the action of $\mathrm{Cl}_\mathfrak{m}(k)$ on $\mathfrak{W}_\mathfrak{m}$ is free and transitive, we can actually *define* the $Z^{(j)}_{T,p}(\,\cdot\,;\mathfrak{w})$ by the equation

$$\Phi^{(j)}_{\mathfrak{m},T,p}(s) =: \sum_{\mathfrak{c}\in \mathrm{Cl}_\mathfrak{m}(k)} Z^{(j)}_{T,p}(s;\mathfrak{c}\cdot\mathfrak{w}^0_\mathfrak{m})\sigma_\mathfrak{c}^{-1}$$

Thus Definition 2.1 and equation (3) give the interpolation property

$$Z^{(j)}_{T,p}(m;\mathfrak{w}) = j(Z_T(m;\mathfrak{w})) \quad \text{for all } m \in \mathcal{M}(p) \tag{13}$$

which, by density, uniquely characterises $Z^{(j)}_{T,p}(\,\cdot\,;\mathfrak{w})$ as a continuous map from $D(p)$ to $\mathbb{C}_p$.

We are interested in calculating $\Phi^{(j)}_{\mathfrak{m},T,p}(1)$ in the case $\mathfrak{z} = \emptyset$, $\mathfrak{m} = \mathfrak{f}$ and it clearly suffices to calculate $Z^{(j)}_{T,p}(1;\mathfrak{c}\cdot\mathfrak{w}^0_\mathfrak{f})$ for each $\mathfrak{c} \in \mathrm{Cl}_\mathfrak{f}(k)$. However, it is technically and conceptually a little easier to calculate $Z^{(j)}_{T,p}(1;\tilde{\mathfrak{c}}\cdot\mathfrak{w}^0_{\mathfrak{f}+})$ where the infinite cycle '+' is the formal product of *all* the real places of k and $\tilde{\mathfrak{c}}$ lies in $\mathrm{Cl}_{\mathfrak{f}+}(k)$. To get back to $\mathfrak{f}$, we use the natural surjection $\pi_{\mathfrak{f}+,\mathfrak{f}} : \mathrm{Cl}_{\mathfrak{f}+}(k) \to \mathrm{Cl}_{\mathfrak{f}}(k)$ and the fact that $Z^{(j)}_{T,p}(1; \mathfrak{c}\cdot\mathfrak{w}^0_{\mathfrak{f}}))$ equals $|\ker \pi_{\mathfrak{f}+,\mathfrak{f}}|\, Z^{(j)}_{T,p}(1;\tilde{\mathfrak{c}}\cdot\mathfrak{w}^0_{\mathfrak{f}+})$ for *any* $\tilde{\mathfrak{c}} \in \mathrm{Cl}_{\mathfrak{f}+}(k)$ such that $\pi_{\mathfrak{f}+,\mathfrak{f}}(\tilde{\mathfrak{c}}) = \mathfrak{c}$ (this follows from [13, Cor. 2.1]). We therefore fix until further notice an element $\mathfrak{w}$ of $\mathfrak{W}_{\mathfrak{f}+}$ and a pair $(\xi, I) \in W_{\mathfrak{f}}$ representing $\mathfrak{w}$. Define $f \in \mathbb{Z}_{>0}$ by $\mathfrak{f} \cap \mathbb{Z} = f\mathbb{Z}$ and denote by μ_f the group of fth roots of unity in $\mathbb{C}$. Then $\mathrm{Im}(\xi) = \mu_f$ (see [13, §3]) and when $\Re(s) > 1$, we have

$$Z_T(s;\mathfrak{w}) = Z_T(s;\xi,I,\mathfrak{f}+) = \sum_{a\in\mathcal{S}(I,+,T)/E_{\mathfrak{f}+}} \frac{\xi(a)}{|I:(a)|^s} = NI^s \sum_{a\in\mathcal{S}(I,+,T)/E_{\mathfrak{f}+}} \frac{\xi(a)}{(\iota_1(a)\dots\iota_r(a))^s} \tag{14}$$

where $\iota_1,\dots,\iota_r : k \to \bar{\mathbb{Q}} \subset \mathbb{C}$ are as in Subsection 2.3.

Shintani's method allows us to analytically continue the second factor in the fourth member of (14) and then find its value at any $m \in \mathbb{Z}_{\leq 0}$. To explain how, we shall simplify matters by assuming from now on that k is real quadratic ($r = 2$). We require the following notation. Let τ_1 and τ_2 be two elements of $I \cap k^\times_+$, linearly independent over $\mathbb{Q}$ and such that $\xi(\tau_1)$, $\xi(\tau_2) \neq 1$. Then τ_1 and τ_2 define a half-open 'parallelogram' and 'cone' in $k^\times_+$ given respectively by:

$$P(\tau_1,\tau_2) := \{\lambda\tau_1 + \mu\tau_2 : \lambda,\mu \in \mathbb{Q},\ 0 < \lambda \leq 1,\ 0 \leq \mu < 1\}$$

and

$$C(\tau_1,\tau_2) := \{\lambda\tau_1 + \mu\tau_2 : \lambda,\mu \in \mathbb{Q},\ 0 < \lambda,\ 0 \leq \mu\} = \dot{\bigcup_{n_1,n_2\in\mathbb{N}}} (P(\tau_1,\tau_2) + n_1\tau_1 + +n_2\tau_2)$$

Let $\underline{\iota}$ denote the embedding of k into $\mathbb{R}^2 \cap \bar{\mathbb{Q}}^2$ which sends $a \in k$ to $(\iota_1(a), \iota_2(a))$. Figure 1 illustrates $\underline{\iota}(P(\tau_1,\tau_2))$ and $\underline{\iota}(C(\tau_1,\tau_2))$ in the case where $\det \begin{pmatrix} \underline{\iota}(\tau_1) \\ \underline{\iota}(\tau_2) \end{pmatrix} < 0$ Clearly, $I \cap P(\tau_1,\tau_2)$ is a fundamental domain for the additive action of $\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ on I, so given a class $A \in I/(\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)$, we shall write $\tilde{a} = \tilde{a}(A)$ for the unique element of $A \cap P(\tau_1,\tau_2)$. Then $A = \tilde{a} + \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ and $A \cap C(\tau_1,\tau_2) = \tilde{a} + \mathbb{N}\tau_1 + \mathbb{N}\tau_2$. We define complex analytic functions on the set $\{s : \Re(s) > 1\}$ (see *e.g.* Theorem 3.1 for convergence and analyticity) by setting

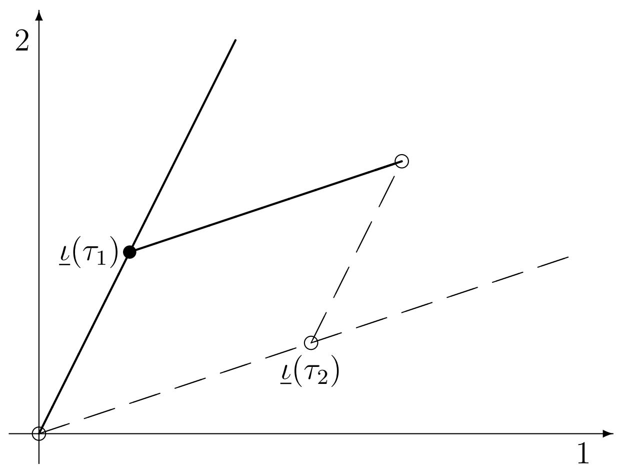
$$z(s;\xi,A,\tau_1,\tau_2) := \sum_{a\in A\cap C(\tau_1,\tau_2)} \frac{\xi(a)}{(\iota_1(a)\iota_2(a))^s} = \sum_{n_1,n_2\in\mathbb{N}} \frac{\xi(\tilde{a}+n_1\tau_1+n_2\tau_2)}{\iota_1(\tilde{a}+n_1\tau_1+n_2\tau_2)^s\iota_2(\tilde{a}+n_1\tau_1+n_2\tau_2)^s}$$

and also

$$z(s;\xi,I,\tau_1,\tau_2) := \sum_{a\in I\cap C(\tau_1,\tau_2)} \frac{\xi(a)}{(\iota_1(a)\iota_2(a))^s} = \sum_{A\in I/(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)} z(s;\xi,A,\tau_1,\tau_2)$$

Let $\mathbb{R}[[\underline{X}]]$ denote the ring of formal power series in $\underline{X} = (X_1, X_2)$, a pair of formal variables. For any pair $\underline{u} = (u_1, u_2) \in \mathbb{R}^2$ we write $(1+\underline{X})^{\underline{u}}$ for the product $(1+X_1)^{u_1}(1+X_2)^{u_2}$ of

Figure 1: $\underline{\iota}(P(\tau_1,\tau_2))$ and $\underline{\iota}(C(\tau_1,\tau_2))$



two (formal) binomial series in $\mathbb{R}[[\underline{X}]]$ and we set

$$F_{\underline{\iota}}(\underline{X};\xi,A,\tau_1,\tau_2) := \frac{\xi(\tilde{a})(1+\underline{X})^{\underline{\iota}(\tilde{a})}}{(1-\xi(\tau_1)(1+\underline{X})^{\underline{\iota}(\tau_1)})(1-\xi(\tau_2)(1+\underline{X})^{\underline{\iota}(\tau_2)})}$$

and also

$$F_{\underline{\iota}}(\underline{X};\xi,I,\tau_1,\tau_2) := \sum_{A\in I/(\mathbb{Z}\tau_1+\mathbb{Z}\tau_2)} F_{\underline{\iota}}(\underline{X};\xi,A,\tau_1,\tau_2) = \frac{\sum_{\tilde{a}\in I\cap P(\tau_1,\tau_2)}\xi(\tilde{a})(1+\underline{X})^{\underline{\iota}(\tilde{a})}}{(1-\xi(\tau_1)(1+\underline{X})^{\underline{\iota}(\tau_1)})(1-\xi(\tau_2)(1+\underline{X})^{\underline{\iota}(\tau_2)})} \qquad (15)$$

(The sum in the numerator is of course finite.) *A priori* these lie in the fraction field of $\bar{\mathbb{Q}}[[\underline{X}]]$. However the constant term $(1-\xi(\tau_1))(1-\xi(\tau_2))$ of their denominators is non-zero by hypothesis, so they actually lie in $\bar{\mathbb{Q}}[[\underline{X}]]$ itself (in fact, in $k(\mu_f)[[\underline{X}]]$, as is easily seen).

REMARK 3.1 Note that intuitively (but illegally) we could also imagine 'expanding the denominator of $F_{\underline{\iota}}(\underline{X};\xi,A,\tau_1,\tau_2)$ as an infinite series in $(1+\underline{X})$'. We could then write

$$\text{“ } F_{\underline{\iota}}(\underline{X};\xi,A,\tau_1,\tau_2) = \sum_{a\in A\cap C(\tau_1,\tau_2)} \xi(a)(1+\underline{X})^{\underline{\iota}(a)} \text{ ”}$$

We use quotation marks because the sum fails to converge in $\bar{\mathbb{Q}}[[\underline{X}]]$, but the idea is useful.

Let us write Δ for the differential operator $(1+X_1)(1+X_2)\frac{\partial^2}{\partial X_1 \partial X_2}$ acting on any power series in X_1 and X_2.

Theorem 3.1 *Let k be a real quadratic field, $\mathfrak{f} \neq \mathcal{O}$ an integral ideal with $\mathfrak{f} \cap \mathbb{Z} = f\mathbb{Z}$, ($f \in \mathbb{Z}_{>0}$) and let (ξ, I) be an element of $W_\mathfrak{f}$. Suppose that τ_1, τ_2 are two $\mathbb{Q}$-linearly independent elements of $I \cap k_+^\times \setminus \ker\xi$. Then for any element A of $I/(\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)$, we have*

(i). The function $z(s;\xi, A, \tau_1, \tau_2)$ converges absolutely for $\Re(s) > 1$ and possesses a meromorphic continuation to $\mathbb{C}$.

(ii). For each $m \in \mathbb{Z}_{\leq 0}$ this continuation is analytic at m and for any $\underline{\iota}$, we have

$$z(m;\xi, A, \tau_1, \tau_2) = \Delta^{-m}|_{\underline{X}=0}\, F_{\underline{\iota}}(\underline{X};\xi, A, \tau_1, \tau_2) \tag{16}$$

(iii). $z(m;\xi, A, \tau_1, \tau_2) \in \mathbb{Q}(\mu_f)$ for all $m \in \mathbb{Z}_{\leq 0}$.

Proof Parts (i) and (ii) follow from [9, Prop. 1] with substitutions "r"="n"=2, "χ_i"$= \xi(\tau_i)$, $i = 1, 2$, *etc.* Because the "χ_i" are different from 1, the Laurent series defining Shintani's "$B_m(a, y, \chi)^{(1)}$" and "$B_m(a, y, \chi)^{(2)}$" are actually power series. Substituting $1 + X_1 = e^{-ut_2}$, $1+X_2 = e^{-u}$ and $1+X_1 = e^{-u}$, $1+X_2 = e^{-ut_1}$ in these two series respectively and combining them gives (16) after a little manipulation. (Strictly speaking, Shintani's condition that his "x_1" and "x_2" be strictly positive is only met if our $\tilde{a}$ lies in the *interior* of $P(\tau_1, \tau_2)$. However, his proof seems to require only that $\iota(\tilde{a})$ belong to $\mathbb{R}_+^2$! In any case, even if $\tilde{a}$ *does* lie on the ray $\mathbb{Q}_+^\times\tau_1$, Equation (16) can still be recovered from Shintani's full result (see [10, Rem. 2.2].)) As for part (iii) of the Theorem, Equation (16) already implies that $z(m;\xi, A, \tau_1, \tau_2)$ lies in $\bar{\mathbb{Q}}$ (in fact, in $k(\mu_f)$). Now any $\alpha \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts coefficientwise on $\bar{\mathbb{Q}}[[\underline{X}]]$ and it is clear from the definitions that

$$F_{\underline{\iota}}(\underline{X};\xi, A, \tau_1, \tau_2)^\alpha = F_{\alpha\circ\underline{\iota}}(\underline{X};\alpha\circ\xi, A, \tau_1, \tau_2) \tag{17}$$

where $\alpha \circ \underline{\iota}$ denotes $(\alpha \circ \iota_1, \alpha \circ \iota_2)$. Since Δ commutes with α, Equations (16) and (17) give

$$\begin{aligned} \alpha(z(m;\xi, A, \tau_1, \tau_2)) &= \Delta^{-m}|_{\underline{X}=0}\, (F_{\underline{\iota}}(\underline{X};\xi, A, \tau_1, \tau_2)^\alpha) \\ &= \Delta^{-m}|_{\underline{X}=0}\, F_{\alpha\circ\underline{\iota}}(\underline{X};\alpha\circ\xi, A, \tau_1, \tau_2) \\ &= z(m;\alpha\circ\xi, A, \tau_1, \tau_2) \quad \text{for all } m \in \mathbb{Z}_{\leq 0} \end{aligned} \tag{18}$$

and the result follows on letting α run through $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\xi)) = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_f))$. □

Remark 3.2 For $\Re(s) > 1$, the function $z(s;\xi, A, \tau_1, \tau_2)$ clearly does not depend on the ordering of the ι_i's in $\underline{\iota}$. By analytic continuation, neither does the L.H.S. of (16) and so the R.H.S. cannot either. This latter fact was used implicitly in the proof of part (iii) of the Theorem but is easy to see independently. Indeed if $\underline{\iota}' = (\iota_{\pi(1)}, \iota_{\pi(2)})$ for some $\pi \in S_2$ then clearly $F_{\underline{\iota}'}(X_1, X_2;\xi, A, \tau_1, \tau_2) = F_{\underline{\iota}}(X_{\pi^{-1}(1)}, X_{\pi^{-1}(2)};\xi, A, \tau_1, \tau_2)$. But $\Delta^{-m}|_{\underline{X}=0}$ is symmetric in X_1, X_2, so $\Delta^{-m}|_{\underline{X}=0}\, F_{\underline{\iota}'}(\underline{X};\xi, A, \tau_1, \tau_2) = \Delta^{-m}|_{\underline{X}=0}\, F_{\underline{\iota}}(\underline{X};\xi, A, \tau_1, \tau_2)$, as required.

Example 3.1 By way of illustration, we show how these methods can be used to prove facts about the values $Z_\emptyset(m;\mathfrak{w}) = Z_\emptyset(m;\xi, I, \mathfrak{f}+)$ which, in a more general context, were used in [13] (see Lemma 3.2, *ibid.*). Let ε be any generator of $E_{\mathfrak{f}+} \cong \mathbb{Z}$ and ρ any element of

$I \cap k_+^\times$ not in $\ker \xi$. Then $\varepsilon\rho$ also lies in $I \cap k_+^\times$ but not in $\mathbb{Q}\rho$ and $\xi(\varepsilon\rho) = \xi(\rho) \neq 1$. Thus we can define $z(s; \xi, I, \rho, \varepsilon\rho)$ and since it is well known that $I \cap C(\rho, \varepsilon\rho)$ is a fundamental domain for the action of $E_{\mathfrak{f}+}$ on $I \cap k_+^\times$, Equation (14) shows that

$$Z_\emptyset(s; \xi, I, \mathfrak{f}+) = NI^s z(s; \xi, I, \rho, \varepsilon\rho) = NI^s \sum_{A \in I/(\mathbb{Z}\rho + \mathbb{Z}\varepsilon\rho)} z(s; \xi, A, \rho, \varepsilon\rho) \tag{19}$$

whenever $\Re(s) > 1$. Now apply Theorem 3.1 to equation (19). Part (i) of the Theorem allows us to analytically continue the equalities to $s = m \in \mathbb{Z}_{\leq 0}$. Taking $\tau_1 = \rho$, $\tau_2 = \varepsilon\rho$ in parts (ii) and (iii), summing over $A \in I/(\mathbb{Z}\rho + \mathbb{Z}\varepsilon\rho)$ and combining with equation (19) gives the explicit formula

$$Z_\emptyset(m; \xi, I, \mathfrak{f}+) = NI^m \Delta^{-m}|_{\underline{X}=0} F_{\underline{\iota}}(\underline{X}; \xi, I, \rho, \varepsilon\rho) \qquad \forall\, m \in \mathbb{Z}_{\leq 0}$$

and shows that

$$Z_\emptyset(m; \xi, I, \mathfrak{f}+) \in \mathbb{Q}(\mu_f) \quad \text{ for all } m \in \mathbb{Z}_{\leq 0}$$

The same procedure applied to equation (18) gives

$$Z_\emptyset(m; \xi, I, \mathfrak{f}_+)^\alpha = Z_\emptyset(m; \alpha \circ \xi, I, \mathfrak{f}+). \quad \text{ for all } m \in \mathbb{Z}_{\leq 0} \text{ and } \alpha \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

Lemma 3.2 of [13] asserts that last two statements hold for more general k (totally real), T and $\mathfrak{m}$ but the suggested proof is essentially an elaboration of the above method due to Shintani.

3.2 Introduction of the prime p

We now introduce a fixed prime number p such that $(p, \mathfrak{f}) = 1$. To p-adically interpolate $\Phi_{\mathfrak{m},T}(s)$ (*i.e.* $Z_T(s; \mathfrak{w})$) in Subsection 2.2, we had to assume that T contained T_p. Therefore, taking $T = T_p$ and τ_1, τ_2, to be $\mathbb{Q}$-linearly independent and lying in $I \cap k_+^\times$ but not in $\ker \xi$, as before, we now define a complex analytic function (see Theorem 3.2) on the set $\{s : \Re(s) > 1\}$ by setting

$$z_{T_p}(s; \xi, I, \tau_1, \tau_2) := \sum_{\substack{a \in I \cap C(\tau_1, \tau_2) \\ p \nmid |I:(a)|}} \frac{\xi(a)}{(\iota_1(a)\iota_2(a))^s} \tag{20}$$

and also

$$F_{T_p, \underline{\iota}}(\underline{X}; \xi, I, \tau_1, \tau_2) := \frac{\displaystyle\sum_{\substack{\tilde{a} \in I \cap P(p\tau_1, p\tau_2) \\ p \nmid |I:(\tilde{a})|}} \xi(\tilde{a})(1+\underline{X})^{\underline{\iota}(\tilde{a})}}{(1 - \xi(p\tau_1)(1+\underline{X})^{\underline{\iota}(p\tau_1)})(1 - \xi(p\tau_2)(1+\underline{X})^{\underline{\iota}(p\tau_2)})}$$

which is again an element of $k(\mu_f)[[\underline{X}]]$, since $(p, \mathfrak{f}) = 1$ implies $\xi(p\tau_1), \xi(p\tau_2) \neq 1$.

Theorem 3.2 *We use the hypotheses and notation of Theorem 3.1. For any prime number p with $(p,\mathfrak{f})=1$ we have:*

(i). The function $z_{T_p}(s;\xi,I,\tau_1,\tau_2)$ converges absolutely for $\Re(s)>1$ and possesses a meromorphic continuation to $\mathbb{C}$.

(ii). For each $m\in\mathbb{Z}_{\le 0}$ this continuation is analytic at m and for any $\underline{\iota}$, we have

$$z_{T_p}(m;\xi,I,\tau_1,\tau_2)=\Delta^{-m}|_{\underline{X}=0}\,F_{T_p,\underline{\iota}}(\underline{X};\xi,I,\tau_1,\tau_2)$$

(iii). $z_{T_p}(m;\xi,I,\tau_1,\tau_2)\in\mathbb{Q}(\mu_f)$ for all $m\in\mathbb{Z}_{\le 0}$.

Proof The condition $p\nmid|I:(a)|$ is equivalent to $a\notin\mathfrak{p}I$ for any prime ideal $\mathfrak{p}$ of $\mathcal{O}$ dividing p. Since $\mathbb{Z}p\tau_1+\mathbb{Z}p\tau_2\subset pI\subset\bigcap_{\mathfrak{p}|p}\mathfrak{p}I$, it follows that the set of a satisfying this condition is a union of those cosets $A\in I/(\mathbb{Z}p\tau_1+\mathbb{Z}p\tau_2)$ not contained in (*i.e.* not intersecting) $\mathfrak{p}I$ for any $\mathfrak{p}|p$. Letting $\mathcal{A}'$ denote the (finite) set of all such cosets, it follows from the definitions that

$$z_{T_p}(s;\xi,I,p\tau_1,p\tau_2)=\sum_{A\in\mathcal{A}'}z(s;\xi,A,p\tau_1,p\tau_2)$$

and

$$F_{T_p,\underline{\iota}}(\underline{X};\xi,I,\tau_1,\tau_2)=\sum_{A\in\mathcal{A}'}F_{\underline{\iota}}(\underline{X};\xi,A,p\tau_1,p\tau_2)$$

But $C(\tau_1,\tau_2)=C(p\tau_1,p\tau_2)$, so $z_{T_p}(s;\xi,I,p\tau_1,p\tau_2)=z_{T_p}(s;\xi,I,\tau_1,\tau_2)$. The Proposition therefore follows from Theorem 3.1 with $p\tau_1$ and $p\tau_2$ in place of τ_1 and τ_2. □

Before proceeding with a p-adic interpolation of $z_{T_p}(s;\xi,I,\tau_1,\tau_2)$, we formulate a hypothesis and a definition and prove a lemma.

Hypothesis 3.1

(i). $(p,\mathfrak{f})=1$,

(ii). p splits in k, i.e. *$p\mathcal{O}=\mathfrak{p}_1\mathfrak{p}_2$ with $\mathfrak{p}_1\neq\mathfrak{p}_2$, and*

(iii). I is prime to p, i.e. *$\mathrm{ord}_{\mathfrak{p}_i}(I)=0$ for $i=1,2$.*

Remark 3.3 Condition (i) has already been imposed. Without it the nature of the interpolation problem would change significantly. Assuming it, and taking $T=T_p$, Condition (ii) of Hypothesis 2.1 is *equivalent to* the condition $\mathfrak{f}\neq\mathcal{O}$. Condition (ii) of Hypothesis 3.1 is not necessary for (a generalized version of) the results that follow but it simplifies their exposition and the computations based on them. Finally, Condition (iii) is no obstruction to computing $Z_{T_p}(s;\mathfrak{w})$ since any $\mathfrak{w}\in\mathfrak{W}_{\mathfrak{f}+}$ can always be represented by some (ξ,I) with I prime to p.

Definition 3.1 *Let $\{c_n\}_{n=1}^{\infty}$ be the sequence of rational integers given by*

$$c_n = c_n(p) := \sum_{\zeta^p=1} (\zeta - 1)^n = p \sum_{0 \leq r \leq n/p} (-1)^{n-pr} \begin{pmatrix} n \\ pr \end{pmatrix} \tag{21}$$

where ζ runs through the pth roots of unity in any algebraic closure of $\mathbb{Q}$.

(The second formula in (21) follows from the first by expanding $(\zeta - 1)^n$.) Let $P(X)$ denote the polynomial $((X + 1)^p - 1)/X$. Taking $n \geq p$, writing out the L.H.S. of the equation $(\zeta - 1)^{n-(p-1)} P(\zeta - 1) = 0$ as a polynomial in $\zeta - 1$ and summing over ζ, we obtain the useful recurrence relation

$$c_n = -\left(\begin{pmatrix} p \\ 1 \end{pmatrix} c_{n-1} + \begin{pmatrix} p \\ 2 \end{pmatrix} c_{n-2} + \ldots + \begin{pmatrix} p \\ p-1 \end{pmatrix} c_{n-(p-1)} \right) \quad \forall\, n \geq p \tag{22}$$

with the initial conditions $c_n = (-1)^n p$ for $1 \leq n \leq (p - 1)$ which follow from the second formula in (21). Let $|\cdot|_p$ denote the absolute value on $\mathbb{C}_p$ normalised by $|p|_p = p^{-1}$ and for $x \in \mathbb{R}$, let $\lceil x \rceil$ denote $\min\{l \in \mathbb{Z} : x \leq l\}$.

Lemma 3.1 *For all $n \geq 1$, we have $|c_n|_p \leq p^{-\lceil n/(p-1) \rceil}$ and the quotient c_n/pn is p-integral.*

PROOF The estimate follows from the fact that $|\zeta - 1|_p = p^{-1/(p-1)}$ for every ζ not equal to 1, or by induction from (22). For the p-integrality statement, define $m \in \mathbb{N}$ by $p^m \leq n < p^{m+1}$ and note that $\mathrm{ord}_p(c_n/pn)$ is at least $\lceil n/(p-1) \rceil - 1 - \mathrm{ord}_p(n)$ which is clearly zero if $m = 0$ and is otherwise at least $\lceil p^m/(p-1) \rceil - 1 - m = \sum_{i=0}^{m-1} (p^i - 1) \geq 0$. $\square$

We can now state the main result of this section.

Theorem 3.3 *We use the hypotheses and notation of Theorem 3.1. Suppose that p is any prime number satisfying Hypothesis 3.1 and $j : \bar{\mathbb{Q}} \to \mathbb{C}_p$ any embedding. We have:*

(i). There exists a unique p-adically continuous function $z^{(j)}_{T_p,p}(\,\cdot\,; \xi, I, \tau_1, \tau_2) : D(p) \longrightarrow \mathbb{C}_p$ satisfying the interpolation condition

$$z^{(j)}_{T_p,p}(m; \xi, I, \tau_1, \tau_2) = j(z_{T_p}(m; \xi, I, \tau_1, \tau_2)) \quad \forall m \in \mathcal{M}(p) \tag{23}$$

(ii). For any $\underline{\iota}$, write $(1 + \underline{X})^{-1} F_{\underline{\iota}}(\underline{X}; \xi, I, \tau_1, \tau_2)^j$ as $\sum_{i,l \geq 0} a_{i,l} X_1^i X_2^l$. Then $a_{i,l}$ lies in $\mathbb{Z}_p[\mu_f]$ for all $i, l \geq 0$ and

$$z^{(j)}_{T_p,p}(1; \xi, I, \tau_1, \tau_2) = \frac{1}{p^2} \sum_{i,l \geq 0} \frac{c_{i+1} c_{l+1} a_{i,l}}{(i+1)(l+1)} \in \mathbb{Z}_p[\mu_f] \tag{24}$$

$\square$

Note that we do mean $F_{\underline{\iota}}(\underline{X};\xi,I,\tau_1,\tau_2)^j$, not $F_{T_p,\underline{\iota}}(\underline{X};\xi,I,\tau_1,\tau_2)^j$ in part (ii) and that the exponent indicates that j has been applied to the coefficients of the power-series. The estimate of $\mathrm{ord}_p(c_n/pn)$ in the proof of Lemma 3.1 shows that the infinite sum in (24) converges (p-adically). The proof of Theorem 3.3 will provide an expression for $z^{(j)}_{T_p,p}(s;\xi,I,\tau_1,\tau_2)$ as a p-adic integral (see Equation (32)). We defer it while we deduce a result that allows us in principle to calculate $Z^{(j)}_{T_p,p}(1;\mathfrak{w})$ for all $\mathfrak{w}\in\mathfrak{W}_{\mathfrak{f}+}$ and hence $\Phi_{\mathfrak{f},T_p,p}(1)$ as explained at the beginning of this section. Let $\omega:\mathbb{Z}_p^\times\to\mu(\mathbb{Q}_p)$ be the Teichmüller character which is uniquely defined by the requirement that $\langle x\rangle := \omega^{-1}(x)x$ should lie in $1+p\mathbb{Z}_p$ for all $x\in\mathbb{Z}_p^\times$ (and in $1+4\mathbb{Z}_2$ if $p=2$).

Corollary 3.1 *Under the hypotheses of the Theorem, let $\mathfrak{w}$ be the class of (ξ,I) in $\mathfrak{W}_{\mathfrak{f}+}$, let ε be any generator of $E_{\mathfrak{f}+}$ and let ρ any element of $I\cap k_+^\times$ not in $\ker\xi$. Then*

$$Z^{(j)}_{T_p,p}(s;\mathfrak{w}) = \omega(NI)\langle NI\rangle^s z^{(j)}_{T_p,p}(s;\xi,I,\rho,\varepsilon\rho) \qquad \forall\, s\in D(p) \tag{25}$$

where $z^{(j)}_{T_p,p}(s;\xi,I,\rho,\varepsilon\rho)$ is as in part (i) of the Theorem. In particular,

$$Z^{(j)}_{T_p,p}(1;\mathfrak{w}) = \frac{NI}{p^2}\sum_{i,l\geq 0}\frac{c_{i+1}c_{l+1}a_{i,l}}{(i+1)(l+1)} \in \mathbb{Z}_p[\mu_f] \tag{26}$$

where $\mathfrak{w}$ is the class of (ξ,I) in $\mathfrak{W}_{\mathfrak{f}+}$ and the $a_{i,l}$ are defined by $(1+\underline{X})^{-1}F_{\underline{\iota}}(\underline{X};\xi,I,\rho,\varepsilon\rho)^j = \sum_{i,l\geq 0}a_{i,l}X_1^iX_2^l$ for any $\underline{\iota}$.

Note that Hypothesis 3.1 (iii) implies $(NI,p)=1$, so that $\omega(NI)$ is well-defined and the function $\langle NI\rangle^s$ is both well-defined and analytic for $s\in\mathbb{Z}_p$.

Proof Arguing just as in Example 3.1, Equations (14) and (20) show that $Z_{T_p}(s;\mathfrak{w})$ equals $NI^s z_{T_p}(s;\xi,I,\rho,\varepsilon\rho)$ for every s such that $\Re(s)>1$ and hence, by analytic continuation, for every $m\in\mathcal{M}(p)$. But $NI^m=\omega(NI)\langle NI\rangle^m$ for such m, so part (i) of the Theorem gives

$$\omega(NI)\langle NI\rangle^m z^{(j)}_{T_p,p}(m;\xi,I,\rho,\varepsilon\rho) = NI^m j(z_{T_p}(m;\xi,I,\rho,\varepsilon\rho)) = j(Z_{T_p}(m,\mathfrak{w}))\ \ \forall\, m\in\mathcal{M}(p)$$

Since the function $\omega(NI)\langle NI\rangle^s z^{(j)}_{T_p,p}(s;\xi,I,\rho,\varepsilon\rho)$ is continuous on $D(p)$ by part (i) of the Theorem, the equality (25) now follows from the uniqueness of the interpolation in (13). Equation (26) is then a direct consequence of part (ii) of the Theorem. □

3.3 Proof of theorem 3.3

Our methods of proof generalise those employed by Lang to evaluate $L_p(1,\chi)$ in [4, Ch. 4]. They use the theory of p-adic-valued measures on $\mathbb{Z}_p^2$ and their relation to formal power series. We recall that such a measure is a $\mathbb{C}_p$-valued, bounded linear functional on the $\mathbb{C}_p$-Banach algebra $\mathrm{Cont}(\mathbb{Z}_p^2,\mathbb{C}_p)$ of all continuous, $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p^2$ under the (ultrametric) uniform norm $\|\cdot\|$. The 'boundedness' requirement on such a functional ν means that the set $\{|\nu(f)|_p/\|f\| \,:\, f \in \mathrm{Cont}(\mathbb{Z}_p^2,\mathbb{C}_p), f \neq 0\}$ is bounded. By writing $\|\nu\|$ for its supremum we define an ultrametric norm under which the set $\mathrm{Meas}(\mathbb{Z}_p^2,\mathbb{C}_p)$ of all such measures acquires the structure of a $\mathbb{C}_p$-Banach space. For $\nu \in \mathrm{Meas}(\mathbb{Z}_p^2,\mathbb{C}_p)$ and $f \in \mathrm{Cont}(\mathbb{Z}_p^2,\mathbb{C}_p)$ the value $\nu(f)$, will often be written as $\int_{\mathbf{t}\in\mathbb{Z}_p^2} f(\mathbf{t})\,d\nu$ or just $\int f\,d\nu$. Clearly, $\mathrm{Meas}(\mathbb{Z}_p^2,\mathbb{C}_p)$ is a natural $\mathrm{Cont}(\mathbb{Z}_p^2,\mathbb{C}_p)$-module where for ν in the former and g in the latter, we define the measure $g\nu$ by $\int f\,d(g\nu) := \int fg\,d\nu \ \ \forall\, f \in \mathrm{Cont}(\mathbb{Z}_p^2,\mathbb{C}_p)$. When g is the characteristic function χ_S of an open and closed subset S of $\mathbb{Z}_p^2$, we often write $\nu|_S$ for $\chi_S\nu$ ('the restriction of ν to S') and $\int_{\mathbf{t}\in S} f(\mathbf{t})\,d\nu$ instead of $\int f\,d(\chi_S\nu) = \int f\chi_S\,d\nu$.

Let us write $\mathcal{A}(\underline{X})$ for the $\mathbb{C}_p$-subspace of $\mathbb{C}_p[[\underline{X}]]$ consisting of those power-series with (p-adically) bounded coefficients. For such a power series F, we define the norm $\|F\|$ to be the supremum of the p-adic absolute values of its coefficients. The power-series/measure correspondence is then a norm-preserving isomorphism of $\mathbb{C}_p$-Banach spaces between $\mathcal{A}(\underline{X})$ and $\mathrm{Meas}(\mathbb{Z}_p^2,\mathbb{C}_p)$. In [4, Ch. 4], Lang discusses in detail a restricted correspondence $\mathbb{O}[[X]] \leftrightarrow \mathrm{Meas}^{(1)}(\mathbb{Z}_p,\mathbb{C}_p)$ where $\mathbb{O}$ denotes the ring $\{a \in \mathbb{C}_p : |a|_p \leq 1\}$ and $\mathrm{Meas}^{(1)}(\mathbb{Z}_p,\mathbb{C}_p)$ the space of measures on $\mathbb{Z}_p$ of norm ≤ 1. By simply taking $\mathbb{C}_p$-spans we get a correspondence between $\mathcal{A}(X)$ and $\mathrm{Meas}(\mathbb{Z}_p,\mathbb{C}_p)$ (see also Appendices 5 and 6 of [8]). The generalisation of this from one to two (or more) variables seems to be well known although we have been unable to find a full and detailed account in the published literature. In any case, it is very straightforward. The facts we require are as follows (see also [11] for the r-variable case). The correspondence can be characterised as associating $\nu \in \mathrm{Meas}(\mathbb{Z}_p^2,\mathbb{C}_p)$ with $F \in \mathcal{A}(\underline{X})$ if and only if

$$\int_{\mathbf{t}\in\mathbb{Z}_p^2} (1+u_1)^{t_1}(1+u_2)^{t_2}\,d\nu = F(u_1,u_2) \qquad \forall\,(u_1,u_2) \in \mathbb{C}_p^2 \text{ s.t. } |u_1|_p,\ |u_2|_p < 1 \tag{27}$$

in which case we shall write $\nu = \mathcal{N}(F)$ and $F = \mathcal{F}(\nu)$. By expanding Equation (27) as a power series in u_1 and u_2 we can deduce (see [11]) that for all $\nu \in \mathrm{Meas}(\mathbb{Z}_p^2,\mathbb{C}_p)$ and $n_1, n_2 \in \mathbb{Z}_{\geq 0}$

$$\mathcal{F}(t_1^{n_1}t_2^{n_2}\nu) = \left((1+X_1)\frac{\partial}{\partial X_1}\right)^{n_1}\left((1+X_1)\frac{\partial}{\partial X_1}\right)^{n_2}\mathcal{F}(\nu) \tag{28}$$

We denote by D^- the open p-adic bidisc $\{(x_1,x_2) \in \mathbb{C}_p^2 \,:\, |x_1|_p,\ |x_2|_p < 1\}$ and by $\mathcal{A}_1(\underline{X}) \subset \mathbb{C}_p[\underline{X}]$ the $\mathbb{C}_p$-algebra of power series convergent at every point of D^-. Thus $\mathcal{A}_1(\underline{X})$ contains $\mathcal{A}(\underline{X})$. We define an action of the group $\mu_p^2 = \mu_p(\mathbb{C}_p)^2$ on $\mathcal{A}_1(\underline{X})$ by setting $(\underline{\zeta}\bullet F)(X_1,X_2) = F(\zeta_1(1+X_1)-1, \zeta_2(1+X_2)-1)$ for any $\underline{\zeta} = (\zeta_1,\zeta_2) \in \mu_p^2$ and $F \in \mathcal{A}_1(\underline{X})$. It is easy to check that this indeed gives a well-defined $\mathbb{C}_p$-linear left action (see [10, Sec. 3.2]). Restricting to the subgroup $\mu_p \times \{1\} \subset \mu_p^2$, the idempotent corresponding to the trivial character of this group is the operator V_1:

$$V_1 \bullet F := \frac{1}{p}\sum_{\zeta_1^p=1} F(\zeta_1(1+X_1)-1, X_2)$$

An operator V_2 is defined similarly by acting on the variable X_2 and we write U for the idempotent operator $U = (1 - V_1)(1 - V_2) = (1 - V_2)(1 - V_1)$. It can be checked ([10, Sec. 3.2]) that the '$\bullet$' action preserves $\mathcal{A}(\underline{X})$. Moreover, it follows easily from (27) that given any $F \in \mathcal{A}(\underline{X})$ and $(\zeta_1, \zeta_2) \in \mu_p^2$, the measure $\mathcal{N}((\zeta_1, \zeta_2) \bullet F)$ is simply the measure $\mathcal{N}(F)$ multiplied by the continuous (locally constant) function $\mathbf{t} \mapsto \zeta_1^{t_1}\zeta_2^{t_2}$. From this it follows that $\mathcal{N}(V_1 \bullet F) = \chi_{p\mathbb{Z}_p \times \mathbb{Z}_p}\mathcal{N}(F)$, $\mathcal{N}(V_2 \bullet F) = \chi_{\mathbb{Z}_p \times p\mathbb{Z}_p}\mathcal{N}(F)$ and so

$$\mathcal{N}(U \bullet F) = \chi_{(\mathbb{Z}_p^\times)^2}\mathcal{N}(F) \tag{29}$$

We fix $\underline{\iota}$, τ_1, τ_2 and j, and abbreviate the power series $F_{\underline{\iota}}(\underline{X}, \xi, \tau_1, \tau_2)^j$ and $F_{T_p,\underline{\iota}}(\underline{X}, \xi, \tau_1, \tau_2)^j$ to F_ξ and F_ξ^* respectively.

Lemma 3.2 *F_ξ lies in $\mathbb{Z}_p[\mu_f][[\underline{X}]]$, hence in $\mathcal{A}(\underline{X})$. Furthermore $F_\xi^* = U \bullet F_\xi$.*

Proof On the R.H.S of (15) we can multiply both the numerator and denominator by the power series

$$\left(\sum_{n_1=0}^{p-1}(\xi(\tau_1)(1+\underline{X})^{\underline{\iota}(\tau_1)})^{n_1}\right)\left(\sum_{n_2=0}^{p-1}(\xi(\tau_2)(1+\underline{X})^{\underline{\iota}(\tau_2)})^{n_2}\right) = \sum_{n_1,n_2=0}^{p-1}\xi(n_1\tau_1+n_2\tau_2)(1+\underline{X})^{\underline{\iota}(n_1\tau_1+n_2\tau_2)}$$

But $I \cap P(p\tau_1, p\tau_2)$ is the disjoint union of the translates $n_1\tau_1 + n_2\tau_2 + (I \cap P(\tau_1, \tau_2))$ for $0 \le n_1, n_2 \le p - 1$, so (after applying j to (15)) we see that F_ξ can be written as

$$F_\xi = \sum_{\tilde{a} \in I \cap P(p\tau_1, p\tau_2)} \frac{\xi(\tilde{a})(1+\underline{X})^{\underline{\iota}_p(\tilde{a})}}{(1-\xi(p\tau_1)(1+\underline{X})^{\underline{\iota}_p(p\tau_1)})(1-\xi(p\tau_2)(1+\underline{X})^{\underline{\iota}_p(p\tau_2)})} \tag{30}$$

Here for any $a \in k$, the notation $(1+\underline{X})^{\underline{\iota}_p(a)}$ indicates $((1+\underline{X})^{\underline{\iota}(a)})^j = (1+X_1)^{a_1}(1+X_2)^{a_2}$, the product of two formal p-*adic* binomial series with $a_1 := j \circ \iota_1(a)$ and $a_2 := j \circ \iota_2(a)$ which are the two embeddings of a in $\mathbb{C}_p$. Now, parts (ii) and (iii) of Hypothesis 3.1 imply that a_1 and a_2 lie in $\mathbb{Z}_p$ whenever a lies in I. As is well known, this implies in turn that the series $(1 + X_1)^{a_1}$ and $(1 + X_2)^{a_2}$ have coefficients in $\mathbb{Z}_p$ for any such a, hence that the numerator and the (common) denominator of each term on the R.H.S. of (30) lie in $\mathbb{Z}_p[\mu_f][[\underline{X}]]$. The constant term of this denominator is $(1-\xi(\tau_1)^p)(1-\xi(\tau_2)^p) =: c$, say. Now, Hypothesis 3.1 (i) of implies that $(p, f) = 1$ so $\xi(\tau_1)$ and $\xi(\tau_2)$ are roots of unity of order prime to p, non-trivial by assumption, so the same is true of their pth powers. It follows that c lies in $\mathbb{Z}_p[\mu_f]^\times$, so that each term in (30) actually has denominator lying in $\mathbb{Z}_p[\mu_f][[\underline{X}]]^\times$ and hence itself lies in $\mathbb{Z}_p[\mu_f][[\underline{X}]] \subset \mathcal{A}[[\underline{X}]]$. The first statement in the Lemma follows. As for the second, it is easy to show that for any $a \in I$, the element $(\zeta_1, \zeta_2) \in \mu_p^2$ acts on $(1 + X_1)^{a_1}(1 + X_2)^{a_2}$ by multiplication by $\zeta_1^{a_1}\zeta_2^{a_2}$ (since $a_1, a_2 \in \mathbb{Z}_p$). Thus if $\tilde{a}$ is an element of $I \cap P(p\tau_1, p\tau_2)$ then $(\zeta_1, \zeta_2) \in \mu_p^2$ multiplies the corresponding term on the R.H.S. of (30) by $\zeta_1^{\tilde{a}_1}\zeta_2^{\tilde{a}_2}$ (using the fact that it acts trivially on the denominator). It follows that V_1 acts on this term by 1 or 0 respectively, according as p does or does not divide $\tilde{a}_1$ in $\mathbb{Z}_p$, and similarly for V_2, *mutatis mutandi* with the result that U acts by 0 or 1 according as p does or does not divide $\tilde{a}_1\tilde{a}_2 = |\mathcal{O} : I||I : (\tilde{a})|$. Here $|\mathcal{O} : I| \in \mathbb{Q}^\times$ is the (generalised) index and lies in $\mathbb{Z}_p^\times$ by Hypothesis 3.1 (iii). Putting this all together and applying U to Equation (30) we get

$$U \bullet F_\xi = \sum_{\substack{\tilde{a}\in I\cap P(p\tau_1,p\tau_2)\\ p\nmid|I:(\tilde{a})|}} \frac{\xi(\tilde{a})(1+\underline{X})^{\iota_p(\tilde{a})}}{(1-\xi(p\tau_1)(1+\underline{X})^{\iota_p(p\tau_1)})(1-\xi(p\tau_2)(1+\underline{X})^{\iota_p(p\tau_2)})}$$

and the R.H.S. is, by definition, the image of $F_{T_p,\underline{\iota}}(\underline{X};\xi,I,\tau_1,\tau_2)$ under j, as required. $\square$

Lemma 3.2 implies that both F_ξ and F_ξ^* lie in $\mathcal{A}(\underline{X})$ and that if we set $\nu_\xi = \mathcal{N}(F_\xi)$ and $\nu_\xi^* = \mathcal{N}(F_\xi^*)$ then $\nu_\xi^* = \chi_{(\mathbb{Z}_p^\times)^2}\nu_\xi$. For any elements m of $\mathcal{M}(p)$ and F of $\mathcal{A}(\underline{X})$, Equation (28) implies that $\mathcal{N}(\Delta^{-m}F) = (t_1t_2)^{-m}\mathcal{N}(F)$ so Equation (27) with $u_1 = u_2 = 0$ gives

$$\int_{\mathbb{Z}_p^2} (t_1t_2)^{-m} d\mathcal{N}(F) = \Delta^{-m}|_{\underline{X}=0}F$$

Applying this with $F = F_\xi^*$, and noting that Δ commutes with j, Theorem 3.2 part (ii) gives, for all $m \in \mathcal{M}(p)$:

$$\begin{aligned} j(z_{T_p}(m;\xi,I,\tau_1,\tau_2)) &= \Delta^{-m}|_{\underline{X}=0}F_\xi^* = \int_{\mathbb{Z}_p^2}(t_1t_2)^{-m}d\nu_\xi^* \\ &= \int_{(\mathbb{Z}_p^\times)^2}(t_1t_2)^{-m}d\nu_\xi = \int_{(\mathbb{Z}_p^\times)^2}\omega(t_1t_2)^{-1}\langle t_1t_2\rangle^{-m}d\nu_\xi \end{aligned} \tag{31}$$

For any $s \in D(p)$ we define

$$\begin{aligned} f_s \ : \ \mathbb{Z}_p^2 &\longrightarrow \mathbb{Z}_p \\ \mathbf{t} &\longmapsto \begin{cases} \omega(t_1t_2)^{-1}\langle t_1t_2\rangle^{-s} & \text{if } \mathbf{t} \in (\mathbb{Z}_p^\times)^2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus with our definitions, the last integral in (31) is strictly to be interpreted as $\int_{\mathbb{Z}_p^2} f_m(\mathbf{t})d\nu_\xi$. But $f_s(\mathbf{t})$ is easily seen to be uniformly continuous as a function of $(s,\mathbf{t}) \in D(p)\times\mathbb{Z}_p^2$, so it follows that on defining

$$z_{T_p,p}^{(j)}(s;\xi,I,\tau_1,\tau_2) := \int_{\mathbb{Z}_p^2} f_s(\mathbf{t})d\nu_\xi = \int_{(\mathbb{Z}_p^\times)^2}\omega(t_1t_2)^{-1}\langle t_1t_2\rangle^{-s}d\nu_\xi \tag{32}$$

we have a p-adically continuous function which, by (31), satisfies the interpolation condition (23). Its unicity follows from the density of $\mathcal{M}(p)$ in $D(p)$. This proves part (i) of Theorem 3.3 and gives

$$z^{(j)}_{T_p,p}(1;\xi,I,\tau_1,\tau_2)=\int\limits_{(\mathbb{Z}_p^\times)^2}(t_1t_2)^{-1}d\nu_\xi=\int\limits_{\mathbb{Z}_p^2}f_1(\mathbf{t})d\nu_\xi=G_\xi(0,0) \tag{33}$$

where we define $G_\xi(\underline{X})\in\mathcal{A}(\underline{X})$ to be $\mathcal{F}(f_1\nu_\xi)\in\mathcal{A}(\underline{X})$. To determine G_ξ we use the

Lemma 3.3 *If H is any element of $\mathcal{A}_1(\underline{X})$ satisfying $\Delta H=F_\xi$ then $U\bullet H=G_\xi$. (In particular, $U\bullet H$ lies in $\mathcal{A}(\underline{X})$.)*

Proof Since Δ commutes with the $\bullet$-action, the condition on H, together with Equations (28) and (29), implies that

$$\Delta(U\bullet H-G_\xi)=\Delta U\bullet H-\mathcal{F}(t_1t_2f_1(\mathbf{t})\nu_\xi)=U\bullet(\Delta H)-\mathcal{F}(\chi_{(\mathbb{Z}_p^\times)^2}\nu_\xi)=U\bullet F_\xi-U\bullet F_\xi=0$$

Since $(1+\underline{X})$ is an invertible power series, it follows from the definition of Δ that $U\bullet H-G_\xi=B_1(X_1)+B_2(X_2)$ for some single-variable power series B_1 and B_2. Since $U\bullet H-G_\xi$ lies in $\mathcal{A}_1(\underline{X})$, it is easy to see that both $B_1(X_1)$ and $B_2(X_2)$ must too, and also that $V_2\bullet B_1(X_1)=B_1(X_1)$ and $V_1\bullet B_2(X_2)=B_2(X_2)$. Thus $U\bullet B_1(X_1)=U\bullet B_2(X_2)=0$. On the other hand, $U\bullet G_\xi=G_\xi$ (since $\chi_{(\mathbb{Z}_p^\times)^2}f_1=f_1$) and since U is idempotent, we obtain

$$U\bullet H-G_\xi=U\bullet(U\bullet H-G_\xi)=U\bullet(B_1(X_1)+B_2(X_2))=0$$

proving the Lemma. □

Now let us write $(1+\underline{X})^{-1}F_\xi=\sum_{i,l\geq 0}a_{i,l}X_1^iX_2^l$ as in the statement of the Theorem. Lemma 3.2 implies that the $a_{i,l}$ lie in $\mathbb{Z}_p[\mu_f]$ and so, by easy, standard estimates, the power series H_0 defined by

$$H_0(\underline{X}):=\sum_{i,l\geq 0}\frac{a_{i,l}}{(i+1)(l+1)}X_1^{i+1}X_2^{l+1}$$

lies in $\mathcal{A}_1(\underline{X})$. Moreover, ΔH_0 equals F_ξ by construction. Therefore, Lemma 3.3 gives

$$G_\xi=U\bullet H_0 \tag{34}$$

Now, for any $\zeta\in\mu_p$ we clearly have $H_0(\zeta-1,0)=H_0(0,\zeta-1)=H_0(0,0)=0$ from which it follows in particular that $(V_1\bullet H_0)(0,0)=(V_2\bullet H_0)(0,0)=0$. Thus, combining Equations (33) and (34), and expanding U as $1-V_1-V_2+V_1V_2$ we obtain

$$z^{(j)}_{T_p,p}(1;\xi,I,\tau_1,\tau_2)=(U\bullet H_0)(0,0)=(V_1V_2\bullet H_0)(0,0)=$$
$$\frac{1}{p^2}\sum_{\zeta_1^p=\zeta_2^p=1}H_0(\zeta_1-1,\zeta_2-1)=\frac{1}{p^2}\sum_{i,l\geq 0}\frac{c_{i+1}c_{l+1}a_{i,l}}{(i+1)(l+1)} \tag{35}$$

Finally, Lemma 3.1 shows that the last member of (35) lies in $\mathbb{Z}_p[\mu_f]$. □

4 Numerical investigation of conjecture 2.2

In this section we present a number of examples in which Conjecture 2.2 is verified up to the precision of computation for a real quadratic field k. Of course, in each example, part (ii) of the conjecture will only be checked for a (small) finite set of primes! (Moreover, these primes will be subjected to certain further conditions that facilitate the calculation of $\Phi_{\mathfrak{f},T_p,p}(1)$. See below.) Before presenting the examples themselves, we explain some of our computational techniques and methods.

4.1 Remarks on computational methods

To compute $\Phi_{\mathfrak{f},\emptyset}(1)$, we use the decomposition given by [13, eq. (12)] and take $s=1$ in [13, eq. (10)], making use of the functional equation of $L(s,\tilde{\chi})$, to obtain the expression

$$\Phi_{\mathfrak{f},\emptyset}(1) = \frac{4}{\sqrt{d_k}} \sum_{\substack{\chi\in G^* \\ \chi\neq\chi_0}} \prod_{\substack{\mathfrak{p}\ \textit{prime} \\ \mathfrak{p}|\mathfrak{f},\mathfrak{p}\nmid\mathfrak{f}(\chi)}} (1-\tilde{\chi}(\mathfrak{p})^{-1}) L^{(2)}(0,\tilde{\chi}^{-1}) e_\chi + \begin{cases} 0 & \text{if } \mathfrak{f}\neq\mathfrak{q}^l, \\ -\log(N\mathfrak{q})\frac{2h_k R_k}{\sqrt{d_k}} e_{\chi_0} & \text{if } \mathfrak{f}=\mathfrak{q}^l \end{cases}$$

where $L^{(2)}(0,\tilde{\chi}) := \lim_{s\to 0} s^{-2} L(s,\tilde{\chi})$. (Note in particular that the Gauss Sums $g_{\mathfrak{m}(\chi)}(\tilde{\chi})$ disappear. For more details, see Lemma 5.1 of [12].) The values of $L^{(2)}(0,\tilde{\chi})$ can then be computed using the method of [1].

As for the p-adic computations, since k is quadratic, the results of Section 3 can be used to calculate $\Phi_{\mathfrak{f},T_p,p}(1)$ for any prime p such that *p is prime to* $\mathfrak{f}$ and *p splits in k*. Suppose also that *f divides $p-1$*. This means that the additive character $j\circ\xi$ takes values in μ_{p-1} hence in $\mathbb{Z}_p^\times$ for any $(\xi,I)\in W_\mathfrak{f}$. Consequently, Corollary 3.1 implies that $\Phi_{\mathfrak{f},T_p,p}(1)\in\mathbb{Z}_pG$. Moreover, the coefficients of the formal power series $F(\underline{X};\xi,I,\rho,\varepsilon\rho)^j$ *etc.* lie in $\mathbb{Z}_p$ (since I will be prime to p and p splits). The assumption $f|(p-1)$ therefore speeds up the calculations considerably, although it is unnecessary from the theoretical viewpoint and places a major restriction on p. We shall assume from now on that p satisfies *the three conditions above.*

The remarks at the beginning of Section 3 show that Corollary 3.1 now suffices in principle for the numerical calculation $\Phi^{(j)}_{\mathfrak{f},T_p,p}(1)$. In practice, however, the computation of the formal power series $F(\underline{X};\xi,I,\rho,\varepsilon\rho)$ by means of (15) can still be prohibitively lengthy. This is because the number of points $\tilde{a}$ in $I\cap P(\rho,\varepsilon\rho)$ equals the index $|I:\mathbb{Z}\rho+\mathbb{Z}\varepsilon\rho|$ which in turn is proportional to the coefficients of ε in a $\mathbb{Z}$-basis $\{1,b\}$ of $\mathcal{O}$ (for fixed, optimal ρ and I). But these coefficients can be extremely large, even for k of moderate discriminant and (especially) $\mathfrak{f}$ of moderate norm. (Recall that $E_{\mathfrak{f}+}=\langle\varepsilon\rangle$.) To tackle this problem the approach of Corollary 3.1 can be refined as follows. Suppose that $\rho_0,\dots,\rho_L$ lie in $I\cap k_+^\times$ but not in $\ker\xi$, with $\rho_0=\rho$, $\rho_L=\varepsilon\rho$ and

$$\operatorname{sgn}\left(\det\begin{pmatrix}\underline{\iota}(\rho_{t-1})\\ \underline{\iota}(\rho_t)\end{pmatrix}\right) = \operatorname{sgn}\left(\det\begin{pmatrix}\underline{\iota}(\rho)\\ \underline{\iota}(\varepsilon\rho)\end{pmatrix}\right) (= \operatorname{sgn}(\iota_2(\varepsilon)-\iota_1(\varepsilon))) \text{ for } t=1,\dots,L. \tag{36}$$

This condition means that the cone on ρ and $\varepsilon\rho$ is a 'fan' of the cones on successive pairs (ρ_{t-1}, ρ_t). More precisely, $C(\rho, \varepsilon\rho)$ is the disjoint union $C(\rho_{t-1}, \rho_t)$ for $t = 1, \ldots, L$ and it follows from (20) that $z_{T_p}(s; \xi, I, \rho, \varepsilon\rho)$ is the sum of the $z_{T_p}(s; \xi, I, \rho_{t-1}, \rho_t)$ for $\Re(s) > 1$, hence for all $s \in \mathbb{C}$ by analytic continuation. Thus the uniqueness of the interpolation in Theorem 3.3 (i) implies that

$$z^{(j)}_{T_p,p}(s; \xi, I, \rho, \varepsilon\rho) = \sum_{t=1}^{L} z^{(j)}_{T_p,p}(s; \xi, I, \rho_{t-1}, \rho_t) \tag{37}$$

for all $s \in D(p)$, and in particular for $s = 1$. (In fact, the power series $F(\underline{X}; \xi, I, \rho, \varepsilon\rho)$ equals $\sum_{t=1}^{L} F(\underline{X}; \xi, I, \rho_{t-1}, \rho_t)$ but we do not need to know this.) We can therefore calculate $Z^{(j)}_{T_p,p}(1; \mathfrak{w})$ by means of Equations (25), and (37), using (24) to determine $z^{(j)}_{T_p,p}(1; \xi, I, \rho_{t-1}, \rho_t)$ for each $t = 1, 2, \ldots, L$, once the corresponding $F(\underline{X}; \xi, I, \rho_{t-1}, \rho_t)$ has been calculated.

Following work of Zagier ([16]) Stark and others in similar contexts, we now explain briefly how *continued fractions* may be used to obtain a sequence $\{\rho_t\}_{t=0}^{L}$ such that the index $|I : \mathbb{Z}\rho_{t-1} + \mathbb{Z}\rho_t|$ is small for all t. Most of the details can also be found in [3]. Without loss of generality we can assume that $\iota_1(\varepsilon) < 1 < \iota_2(\varepsilon)$. Consider the following conditions on a pair of points $(x, y) \in (I \cap k_+^\times)^2$

$$\text{(a)}\;\; \mathbb{Z}x + \mathbb{Z}y = I, \quad \text{(b)}\;\; \iota_1(x) > \iota_1(y) \quad \text{and (c)}\;\; \det\begin{pmatrix} \underline{\iota}(x) \\ \underline{\iota}(y) \end{pmatrix} > 0$$

It is easy to find a pair $(x, y) = (\tilde{\rho}_0, \tilde{\rho}_1)$, say, satisfying these conditions and to see that they must then also hold with $(x, y) = (\tilde{\rho}_1, \tilde{\rho}_2)$ where

$$\tilde{\rho}_2 := -\tilde{\rho}_0 + b_1\tilde{\rho}_1 \;\text{ and }\; b_1 := \lceil \iota_1(\tilde{\rho}_0/\tilde{\rho}_1) \rceil \geq 2$$

hence also for $(x, y) = (\tilde{\rho}_2, \tilde{\rho}_3)$ where $\tilde{\rho}_3 := -\tilde{\rho}_1 + b_2\tilde{\rho}_2 := -\tilde{\rho}_1 + \lceil \iota_1(\tilde{\rho}_1/\tilde{\rho}_2) \rceil \tilde{\rho}_2$ and so on inductively. In this way we produce an infinite sequence $\tilde{\rho}_0, \tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3, \tilde{\rho}_4, \ldots$ such that Conditions (a)—(c) are obeyed for each successive pair $(x, y) = (\tilde{\rho}_{n-1}, \tilde{\rho}_n)$ for $n = 1, 2, 3, \ldots$, and $\tilde{\rho}_{n+1} = -\tilde{\rho}_{n-1} + b_n\tilde{\rho}_n$ where $b_n := \lceil \iota_1(\tilde{\rho}_{n-1}/\tilde{\rho}_n) \rceil \geq 2$. In fact, we have a *'type II continued fraction'* expansion converging to $\iota_1(\tilde{\rho}_0/\tilde{\rho}_1)$:

$$\iota_1(\tilde{\rho}_0/\tilde{\rho}_1) = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \ldots}}$$

The discreteness of $\underline{\iota}(I)$ implies that one cannot have both $\iota_1(\tilde{\rho}_{n-1}) > \iota_1(\tilde{\rho}_n) > 0$ and also $\iota_2(\tilde{\rho}_{n-1}) > \iota_2(\tilde{\rho}_n) > 0$ indefinitely. Thus there exists $N \geq 1$ such that $\iota_2(\tilde{\rho}_{N-1}) < \iota_2(\tilde{\rho}_N)$ and one can show inductively that this property too is inherited from then on: for each $n \geq N$ the pair $(x, y) = (\tilde{\rho}_{n-1}, \tilde{\rho}_n)$ must satisfy Condition (a) together with the following strengthening of Conditions (b) and (c)

$$\text{(b}'\text{)}\;\; \iota_1(x) > \iota_1(y) \;\text{ and }\; \iota_2(x) < \iota_2(y)$$

In the terminology of [3], $\tilde{\rho}_{n-1}$ and $\tilde{\rho}_n$ are successive points of the *'convexity polygon of I'*. But the group $E_+(k)$, hence also $E_{\mathfrak{f}+}$, acts on this polygon and it follows that there exists $M > 0$ such that $\tilde{\rho}_{n+M} = \varepsilon\tilde{\rho}_n$ for all $n \geq N$. (This reflects the fact that the sequence $\{b_n\}_{n=1}^{\infty}$ is eventually periodic.) Thus, choosing any $n_0 \geq N$ we obtain a finite sequence $\{\rho'_m := \tilde{\rho}_{n_0+m}\}_{m=0}^{M}$ with $\rho'_M = \varepsilon\rho'_0$ and $|I : \mathbb{Z}\rho'_{m-1} + \mathbb{Z}\rho'_m| = 1$ for $m = 1, \ldots, M$. Unfortunately, we may have $\rho'_m \in \ker(\xi)$ for some values of m, but, since ξ is non-trivial on I, such 'bad' terms must at least be *non-consecutive*. We can therefore choose n_0 such that $\rho'_0 \notin \ker(\xi)$ (so $\rho'_M \notin \ker(\xi)$) and by simply skipping the bad terms and renumbering, we finally arrive at a subsequence $\rho_0 = \rho'_0, \ldots, \rho_t, \ldots, \rho_L = \rho'_M = \varepsilon\rho_0$ in $(I \cap k_+^\times) \setminus \ker(\xi)$ satisfying (36). Moreover, the indices $|I : \mathbb{Z}\rho_{t-1} + \mathbb{Z}\rho_t| = |I \cap P(\rho_{t-1}, \rho_t)|$ are equal either to 1 or to some *'partial quotient'* b_n of the continued fraction. Hence (empirically at least) they are still relatively small. In fact one easily checks the explicit formula:

$$I \cap P(\rho_{t-1}, \rho_t) = \begin{cases} \{\rho'_{m-1}\} & \text{if } (\rho_{t-1}, \rho_t) = (\rho'_{m-1}, \rho'_m) \text{ for some } m,\ 1 \leq m \leq M \\ \{\rho'_{m-1}\} \,\dot{\cup}\, \{\rho'_m, 2\rho'_m, \ldots, (b_{n_0+m} - 1)\rho'_m\} & \text{if } (\rho_{t-1}, \rho_t) = (\rho'_{m-1}, \rho'_{m+1}) \text{ for some } m,\ 2 \leq m \leq M-1 \end{cases}$$

which serves to calculate each $F(\underline{X}; \xi, I, \rho_{t-1}, \rho_t)$ and hence $Z^{(j)}_{T_p,p}(1; \mathfrak{w})$, as explained above. In practice this leads to massive time-savings compared with the use of ρ_0 and $\rho_L = \varepsilon\rho_0$ alone: The effect of the smaller indices greatly outweighs that of having L such calculations instead of one.

We need to know how to ensure the accuracy of our calculated value of $\Phi_{\mathfrak{f},T_p,p}(1)$ to a given number $N \geq 0$, say, of p-adic places. By Equations (37) and (25), it suffices to calculate each value $z^{(j)}_{T_p,p}(1; \xi, I, \rho_t, \rho_{t-1})$, $t = 0, \ldots, L$ with an error less than p^{-N} in p-adic absolute value. For this, we fix t and write $(1 + \underline{X})^{-1}F(\underline{X}; \xi, I, \rho_t, \rho_{t-1})$ as $\sum_{i,l\geq 0} a_{i,l}X_1^i X_2^l$. Consider the real function $f_p(x)$ defined for all $x > -2$ by $f_p(x) := \frac{x+2}{p-1} - \frac{2}{\log p}\log(x/2 + 1) - 2$. We note that f_p is monotonic increasing and unbounded on the interval $\left[\frac{2(p-1)}{\log p} - 2, \infty\right)$. The following result therefore solves our error-control problems.

Proposition 4.1 *With the above notations, choose $M \geq \frac{2(p-1)}{\log p} - 2$ such that $f_p(M) > N$ and suppose that for each pair (i, l) with $i, l \geq 0$ and $i + l < M$ we have computed an element $\tilde{a}_{i,l}$ of $\mathbb{Z}_p[\mu_f]$ $(= \mathbb{Z}_p)$ such that $|\tilde{a}_{i,l} - a_{i,l}|_p < p^{-N}$. Then*

$$\left| z^{(j)}_{T_p,p}(1; \xi, I, \rho_t, \rho_{t-1}) - \frac{1}{p^2} \sum_{\substack{0\leq i,l \\ i+l<M}} \frac{c_{i+1}c_{l+1}\tilde{a}_{i,l}}{(i+1)(l+1)} \right|_p < p^{-N} \tag{38}$$

PROOF Lemma 3.1 shows that $c_{i+1}c_{l+1}/p^2(i+1)(l+1)$ is p-integral for all $i, l \geq 0$. It is therefore enough to show that Equation (38) holds with $\tilde{a}_{i,l}$ replaced by $a_{i,l}$ and by (24) it suffices to prove that $|c_{i+1}c_{l+1}a_{i,l}/p^2(i+1)(l+1)|_p < p^{-N}$ for any $i, l \geq 0$ with $i + l \geq M$.

But the $a_{i,l}$ are p-integral so the estimate of Lemma 3.1 together with the obvious estimate $|1/(i+1)(l+1)|_p \leq (i+1)(l+1) \leq ((i+l+2)/2)^2$ shows that, for such i,l we have $|c_{i+1}c_{l+1}a_{i,l}/p^2(i+1)(l+1)|_p \leq p^{-f_p(i+l)} \leq p^{-f_p(M)} < p^{-N}$ as required. $\square$

In performing the calculations to compute (the approximations $\tilde{a}_{i,j}$ to) the $a_{i,j}$, it is a good idea to represent all power series $A(\underline{X}) \in \mathbb{Z}_p[[\underline{X}]]$ as the sum of their homogeneous components $A_\nu(\underline{X})$ with $\nu \geq 0$. Indeed, if we are only interested in the coefficients of $X_1^i X_2^l$ with $i+l<M$ then we can simply ignore the components $A_\nu(\underline{X})$ with $\nu \geq M$. Moreover, if $B(\underline{X})$ is another power series similarly represented as $\sum_\nu B_\nu(\underline{X})$, then the homogeneous components of the sum, product and (assuming $p \nmid B_0(\underline{X})$) the quotient of $A(\underline{X})$ by $B(\underline{X})$ can be calculated easily in terms of the $A_\nu(\underline{X})$ and $B_\nu(\underline{X})$. For instance, in the last case, if $B_0(\underline{X}) = b \in \mathbb{Z}_p^\times$ then $(A/B)_0(\underline{X}) = b^{-1}A_0(\underline{X})$ and for $\nu \geq 1$ there is the simple recurrence $(A/B)_\nu(\underline{X}) = b^{-1}[A_\nu(\underline{X}) - B_1(\underline{X})(A/B)_{\nu-1}(\underline{X}) - \ldots - B_\nu(\underline{X})(A/B)_0(\underline{X}))]$.

We next explain the basis of our method for working in $\bigwedge^2_{\mathbb{Q}G}\mathbb{Q}U_S$. Let $\mathbf{X} = \{X_1,\ldots,X_m\}$, say, be the set of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-conjugacy classes of characters in G^*. Then $\mathbf{X}$ can also be identified with the set of isomorphism classes of irreducible rational representations of G, a given conjugacy class $X_i \in \mathbf{X}$ corresponding to the unique isomorphism class of representations with character $\sum_{\chi\in X_i}\chi$. We write e_i for the rational idempotent $\sum_{\chi\in X_i}\chi \in \mathbb{Q}G$ of this character and $\mathbb{Q}(X_i)$ for $e_i\mathbb{Q}G$. Considered as a $\mathbb{Q}G$-module, the latter is a representation lying in X_i. Considered as a ring, $\mathbb{Q}(X_i)$ is a field and $\mathbb{Q}G$ is the direct product $\prod_{i=1}^m \mathbb{Q}(X_i)$. Thus we obtain a decomposition as $\mathbb{Q}G$-module

$$\mathbb{Q}U_S = \bigoplus_{i=1}^{m} e_i\mathbb{Q}U_S \cong \bigoplus_{i=1}^{m} \mathbb{Q}(X_i)^{r_i} \tag{39}$$

where r_i denotes the common value of $r(S,\chi)$ for all $\chi \in X_i$. Clearly, for each i, there exist $\mathbb{Q}(X_i)$-bases of $e_i\mathbb{Q}U_S$ consisting of S-units (more precisely, elements of $1\otimes U_S$). For small examples at least it is not hard to find such a basis by using the idempotents e_i. Let $v_{i,1},\ldots,v_{i,r_i}$ be such a basis, we then say that the elements $v_{i,1},\ldots,v_{i,r_i}$ *realize* the decomposition (39). Passing to the exterior square, the product $v_{i,j}\wedge v_{i',j'}$ is zero in $\bigwedge^2_{\mathbb{Q}G}\mathbb{Q}U_S$ unless $i=i'$, so we obtain a decomposition as $\mathbb{Q}G$-module

$$\textstyle\bigwedge^2_{\mathbb{Q}G}\mathbb{Q}U_S = \bigoplus_{i=1}^m e_i \bigwedge^2_{\mathbb{Q}G}\mathbb{Q}U_S = \bigoplus_{i=1}^m \bigoplus_{1\leq j<j'\leq r_i}\mathbb{Q}(X_i)(v_{i,j}\wedge v_{i,j'})$$

Let d_i denote the common order of each character $\chi \in X_i$. The linear extension of any character $\chi \in X_i$ defines an isomorphism $\chi : \mathbb{Q}(X_i) \to \mathbb{Q}(\mu_{d_i})$ of fields and of $\mathbb{Q}G$-modules (G acting on $\mathbb{Q}(\mu_{d_i})$ *via* χ). It follows that $\mathbb{Q}(X_i)$ has a $\mathbb{Q}$-basis of form $\{e_i, \sigma_i e_i\ \ldots, \sigma_i^{\phi(d_i)-1}e_i\}$ where $\sigma_i \in G$ is any chosen element of G such that $\chi(\sigma_i)$ is a primitive d_ith root of unity. We thus obtain $\mathbb{Q}$-bases $\mathcal{E}$ of $\mathbb{Q}U_S$ and $\mathcal{B}$ of $\bigwedge^2_{\mathbb{Q}G}\mathbb{Q}U_S$, of the forms $\{\sigma_i^k v_{i,j}\}$ and $\{\sigma_i^k v_{i,j}\wedge v_{i,j'}\}$ respectively, where i,j,j',k satisfy $1\leq i\leq m$, $1\leq j<j'\leq r_i$ and $0\leq k\leq \phi(d_i)-1$. A basis for the subspace $\bigwedge^2_{\mathbb{Q}G}\mathbb{Q}U_S^{[S,2]}$ consists of the subset $\mathcal{B}_2 = \{\sigma_i^k v_{i,1}\wedge v_{i,2} : r_i = 2\}$ of $\mathcal{B}$. Having fixed these bases, we now explain how to find a $\mathbb{Z}$-basis for the lattice $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$, expressed as column vectors in the basis $\mathcal{B}_2$. First, let $\{u_1,\ldots,u_t\}$ be any $\mathbb{Z}$-basis of the

lattice $1 \otimes U_S \cong U_S/\{\pm 1\}$ in $\mathbb{Q}U_S$. We easily express each u_i as a rational linear combination of the basis $\mathcal{E}$. By distributivity, each product $u_l \wedge u_{l'}$ is then expressed in the basis $\mathcal{B}$. (For $k + k' \geq \phi(d_i)$, the product $\sigma_i^{k+k'} v_{i,j} \wedge v_{i,j'}$ can be reexpressed in the basis $\mathcal{B}$ by using the relations $P(\sigma_i) v_{i,j} \wedge v_{i,j'} = 0$ for any multiple P in $\mathbb{Z}[X]$ of the d_ith cyclotomic polynomial.) These products generate $\overline{\bigwedge_{\mathbb{Z}G}^2 U_S}$ over $\mathbb{Z}$. We form the matrix M of their rational column vectors in the basis $\mathcal{B}$, with the coefficients of $\mathcal{B}_2$ written first. Standard column operations on M reduce it to an Hermite Normal Form from which we can read off a $\mathbb{Z}$-basis of the intersection $\bigwedge_{\mathbb{Q}G}^2 \mathbb{Q}U_S^{[S,2]} \cap \overline{\bigwedge_{\mathbb{Z}G}^2 U_S} = \overline{\bigwedge_{\mathbb{Z}G}^2 U_S}^{[S,2]}$ written in the $\mathbb{Q}$-basis $\mathcal{B}_2$. Exactly the same method can be used to find a $\mathbb{Z}$-basis for the lattice $\overline{\bigwedge_{\mathbb{Z}G}^2 E(K)}^{[S,2]}$.

4.2 The method of verification

In this section, we explain the method used to numerically verify Conjecture 2.2. We illustrate this method using the first example. Data on the verification of the conjecture in all the examples (including the first one) are summarised in several tables given at the end of this section. The first column of each table contains the number of the example, the meaning of the other columns of these tables is explained in the following subsections.

All the examples have been verified using the PARI/GP system [2].

4.2.1 The extension K/k

The data concerning the extension K/k are summarised in Table 1. First, we list the ground field k, its class number h_k, the integral ideal $\mathfrak{f}$ and the conductor $\mathfrak{f}(K/k)$ dividing $\mathfrak{f}$ of the extension K/k. (This is the minimal cycle modulo which K is the ray-class field to k.) The two integral ideals $\mathfrak{f}$ and $\mathfrak{f}(K/k)$ are given as products of prime ideals in k, with $\mathfrak{q}_q$, $\mathfrak{q}'_q$ denoting prime ideals in k above the prime q (if q is inert, we write $q\mathcal{O}_k$ instead). *In the first example, we have* $k := \mathbb{Q}(\sqrt{37})$, $\mathfrak{f} := 2\mathcal{O}_k$ *(*i.e. 2 *is inert in* k*) and* $\mathfrak{f}(K/k) = \mathfrak{f}$.

Next, we give the monic irreducible polynomial $P_\theta(X) \in \mathbb{Z}[X]$ of an algebraic integer θ such that $K = \mathbb{Q}(\theta)$ (these polynomials have been computed using the method of [6]), the factorisation of the discriminant of $K/\mathbb{Q}$, the class-number h_K of K and the structure of the Galois group G as a product of cyclic groups. Actually, in all examples but the last, the group G is cyclic (it is isomorphic to $C_3 \times C_3$ in the last example) and we let σ denote a (fixed) generator of G (σ_1 and σ_2 are two (fixed) generators in the last example). The next column of the table gives the action of this generator σ (*resp.* of σ_1 and σ_2) on the algebraic integer θ. In some examples, the expression for $\sigma(\theta)$ is too long to be conveniently included in the table. Finally, the last entry of the table is the degree n_c of K^c/K where K^c is the Galois closure of $K/\mathbb{Q}$. Thus $K/\mathbb{Q}$ is Galois if and only if $n_c = 1$.

In the first example, we have $P_\theta(X) = X^6 - 3X^5 - 2X^4 + 9X^3 - 5X + 1$, $d_K = 2^4 \cdot 37^3$, $h_K = 1$, $G \simeq C_3$, $\sigma(\theta) = -\theta^5 + 2\theta^4 + 4\theta^3 - 6\theta^2 - 4\theta + 3$ *and* $[K^c : K] = 1$.

4.2.2 The modules $\mathbb{Q}U_S$ and $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$

The corresponding data are summarised in Tables 2 and 3. We start with the columns of Table 2. Whenever space allows it, the second column gives a $\mathbb{Z}$-basis of $1 \otimes U_S \subset \mathbb{Q}U_S$. We abuse notation by writing u_i both for an element of such a basis and for the corresponding element of U_S itself (unique up to sign). The ranks of $1 \otimes E(K)$ and $1 \otimes U_S \cong U_S/\{\pm 1\}$ are $[K:\mathbb{Q}]-1$ and $[K:\mathbb{Q}]-1+|S_0|$ respectively. The first $[K:\mathbb{Q}]-1$ elements of the given basis lie in $1 \otimes E(K)$.

In the first example, the $\mathbb{Z}$-rank of U_S is 6 and the rank of $E(K)$ is 5. A $\mathbb{Z}$-basis of $1 \otimes E(K)$ is given by: $u_1 := \theta^3 - 2\theta^2 - \theta + 1$, $u_2 := \theta$, $u_3 := \theta^5 - 2\theta^4 - 3\theta^3 + 4\theta^2 + 2\theta - 1$, $u_4 := \theta^5 - 3\theta^4 - \theta^3 + 7\theta^2 - 2\theta - 1$ and $u_5 := \theta^5 - 2\theta^4 - 3\theta^3 + 5\theta^2 + 2\theta - 2$. To get a basis of $1 \otimes U_S$ we add to this system the element $u_6 := \theta^3 - 2\theta^2 - 2\theta + 3$.

Next, we give characters χ generating the group G^* of irreducible complex characters of G. These characters are defined by their values on the specified generators of G. Finally, we give the set $\mathbf{X} = \{X_1, \ldots, X_m\}$ considered as irreducible rational characters of G. Thus each X_i is written as a sum of the elements of the corresponding $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-conjugacy class in G^*. In all the examples, the character χ_0 denotes the trivial character.

In the first example, G^* is generated by the character χ with $\chi(\sigma) := e^{2i\pi/3}$ and there are two irreducible rational characters $X_1 := \chi_0$ and $X_2 := \chi + \chi^2$.

We now look at the columns of Table 3. The second column contains the structure of $\mathbb{Q}U_S$ as $\mathbb{Q}G$-module as represented in Equation (39). For the examples in which Table 2 lists the $\mathbb{Z}$-basis of $1 \otimes U_S$ used, the third column of Table 3 gives an isotypic $\mathbb{Q}$-basis $\{v_{i,j}\}$ of $\mathbb{Q}U_S$, written relative to this base, which 'realizes' the decomposition in the first column (see Section 4.1). Explicitly, the (integral) vector $(a_1, \ldots, a_l)$ represents the image of the S-unit $\pm u_1^{a_1} \cdots u_l^{a_l}$ in $1 \otimes U_S$. The fourth column contains the idempotent $\tilde{e}_{S,>2}$. Note that $\overline{\bigwedge^2_{\mathbb{Z}G} U_S} = \overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$ if and only if $\tilde{e}_{S,>2} = 0$.

Finally, we give an element γ generating a $\mathbb{Z}G$-submodule of finite index in $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$, the index being given in the last column. This element will be used below to verify the conjecture. It can be found by looking among 'small', random integral combinations of the $\mathbb{Z}$-basis of $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$ which in turn is found as described in the last part of Section 4.1. (Such a combination generates a submodule of finite index if and only if, when it is expressed in the $\mathbb{Q}$-basis '$\mathcal{B}_2$' of $\bigwedge^r_{\mathbb{Q}G} \mathbb{Q}U_S^{[S,2]}$ (see *idem*), the coefficient of $\sigma_i^k v_{i,1} \wedge v_{i,2}$ is non-zero for some $k = k(i)$ for each i with $r_i = 2$.) When the index is greater than 1, it is of course possible that there exists an element γ' generating a $\mathbb{Z}G$-submodule of smaller index in $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$. We have expressed the element γ as a sum of terms $(a_1, \ldots, a_l) \wedge (b_1, \ldots, b_l)$ where each term $(a_1, \ldots, a_l) \wedge (b_1, \ldots, b_l)$ represents the image of the element $u_1^{a_1} \cdots u_l^{a_l} \wedge u_1^{b_1} \cdots u_l^{b_l}$ in $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}$.

In the first example, we have $\mathbb{Q}U_S \cong \mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$, isotypic elements realizing this decomposition are $v_{1,1} := (-2, 2, -1, -1, -3, 0)$, $v_{1,2} := (-1, 2, -1, -2, 0, 3)$, $v_{2,1} := (-2, 5, -4, -1, 0, 0)$ and $v_{2,2} := (3, -3, 3, 3, 0, 0)$. The idempotent $\tilde{e}_{S,>2}$ vanishes so $\overline{\bigwedge^2_{\mathbb{Z}G} U_S} = \overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$ and we find that $\gamma := (0, 0, 0, 0, 1, 0) \wedge (0, 0, 0, 0, 0, 1)$ generates $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$ over

$\mathbb{Z}G$.

4.2.3 Verification of conjecture 2.2

The data concerning the numerical verification of Conjecture 2.2 are contained in Tables 4, 5, 6 and 7. More precisely, Tables 4 and 5 refer to parts (i), (iii) and the first statement of part (iv) of the conjecture while Tables 7 and 8 refer to part (ii) and the second statement of part (iv).

Tables 4 and 5 give the same data for the first eight and the last seven examples respectively. Their first two columns give computed approximations to $\frac{4}{\sqrt{d_k}}R(\gamma)$ and to $\Phi_{\mathfrak{f},\emptyset}(1)$ (see the beginning of Section 4.1 for the computation of the latter. To save some space, these values are given to a smaller precision than that to which they were actually computed.) Now, $\left(\bigwedge^2_{\mathbb{Q}G}\mathbb{Q}U_S\right)^{[S,2]}$ is free of rank 1 over $\mathbb{Q}G^{[S,2]}$, generated by γ. It follows that if a solution $\eta_{\mathfrak{f}}$ of part (i) of Conjecture 2.2 exists – that is, if there exists $\eta_{\mathfrak{f}} \in \left(\bigwedge^2_{\mathbb{Q}G}\mathbb{Q}U_S\right)^{[S,2]}$ such that $\frac{4}{\sqrt{d_k}}R(\eta_{\mathfrak{f}}) = \Phi_{\mathfrak{f},\emptyset}(1)$ – then it must be of the form $\eta_{\mathfrak{f}} = A\gamma$ for some unique $A \in \mathbb{Q}G^{[S,2]}$. So the equation to be solved becomes

$$A\frac{4}{\sqrt{d_k}}R(\gamma) = \Phi_{\mathfrak{f},\emptyset}(1) \qquad (40)$$

We solve this by first finding any solution $\tilde{A}$ of (40) in $\mathbb{R}G$ to a high real precision. This can be done using an obvious matrix method. Applying $e_{S,2}$ to $\tilde{A}$ gives the solution in $\mathbb{R}G^{[S,2]}$ which always turns out to be the approximation to the working precision of an 'obvious' element A of $\mathbb{Q}G^{[S,2]}$. The latter is listed in the fourth column of Tables 4 and 5.

It is important to note that, whether or not the conjecture holds, the non-vanishing of R_K implies that $\frac{4}{\sqrt{d_k}}R(\gamma)$ is *invertible in* $\mathbb{R}G^{[S,2]}$ and hence that (40) *always has a unique solution in* $\mathbb{R}G^{[S,2]}$, namely $(\sqrt{d_k}/4)R(\gamma)^{-1}\Phi_{\mathfrak{f},\emptyset}(1)$. On the other hand, parts (iii) and (iv) of the conjecture predict that A actually lies in $\frac{1}{b}\mathbb{Z}[1/g]G^{[S,2]}$ if $\mathfrak{f} \neq \mathfrak{q}^l$ (*resp.* $\frac{1}{2b}\mathbb{Z}[1/g]G^{[S,2]}$ if $\mathfrak{f} = \mathfrak{q}^l$) where b denotes the index of $\mathbb{Z}G\gamma$ in $\overline{\bigwedge^2_{\mathbb{Z}G}U_S}^{[S,2]}$. Even if the conjecture failed, such a solution could always be 'faked' to any desired real precision, simply by approximating coefficients of $(\sqrt{d_k}/4)R(\gamma)^{-1}\Phi_{\mathfrak{f},\emptyset}(1)$ sufficiently closely by elements of $\frac{1}{b}\mathbb{Z}[1/g]$ (or $\frac{1}{2b}\mathbb{Z}[1/g]$). Therefore, in order to verify the conjecture in a *significant* way, we must find that the 'obvious' element $A \in \mathbb{Q}G^{[S,2]}$ determined above has coefficients that lie naturally in $\frac{1}{b}\mathbb{Z}[1/g]G^{[S,2]}$ (or $\frac{1}{2b}\mathbb{Z}[1/g]G^{[S,2]}$). Moreover, their *numerators and denominators should be relatively small and stabilise rapidly as the precision increases.* This is indeed what we have observed in all our examples.

We set $\eta_{\mathfrak{f}} := A\gamma$ as a solution of part (i) of the conjecture. The next column of Tables 4 and 5 indicates whether the condition $\mathfrak{f} = \mathfrak{q}^l$ applies in this example and the last column gives the smallest positive integer $d_{\mathfrak{f}}$ such that $\eta_{\mathfrak{f}}$ belongs $\frac{1}{d_{\mathfrak{f}}}\overline{\bigwedge^2_{\mathbb{Z}G}U_S}^{[S,2]}$. This is determined using the $\mathbb{Z}$-basis of $\overline{\bigwedge^2_{\mathbb{Z}G}U_S}^{[S,2]}$ found as described in Section 4.1. According to part (iii) of the conjecture, we should have $d_{\mathfrak{f}}|g^e$ for some $e \geq 0$ if $\mathfrak{f} \neq \mathfrak{q}^l$ (*i.e.* if the answer in the previous column is 'No'), and by the first statement of part (iv) of the conjecture $d_\mathfrak{f}|2g^e$ for some $e \geq 0$ if $\mathfrak{f} = \mathfrak{q}^l$ (*i.e.* if the answer in the previous column is 'Yes'). Actually, in all examples, we have found that $d_\mathfrak{f} = 1$ if $\mathfrak{f} \neq \mathfrak{q}^l$ and $d_\mathfrak{f} = 2$ if $\mathfrak{f} = \mathfrak{q}^l$. (In the former case, therefore, $\eta_\mathfrak{f}$ lies in $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$. Indeed, using instead a $\mathbb{Z}$-basis of $\overline{\bigwedge^2_{\mathbb{Z}G} E(K)}^{[S,2]}$ we have actually checked that it lies in this latter module, see [13, Rem. 3.4].)

We now illustrate this discussion using the first example. We have computed

$$\frac{4}{\sqrt{d_k}} R(\gamma) \simeq 1.485950583942376625274365476 84 + 0.392804821642563902132940516 02(\sigma + \sigma^2)$$

and

$$\Phi_{\mathfrak{f},\emptyset}(1) \simeq 0.35017047032862441050424222240 - 0.74297529197118831263718273842(\sigma + \sigma^2)$$

A solution of (40) in $\mathbb{R}G$ is then

$$\tilde{A} \simeq 0.50000000000000000000000000000 - 0.50000000000000000000000000000(\sigma + \sigma^2)$$

Since $e_{S,2}$=1 in this example, we take A to be the element $\frac{1}{2}(1 - \sigma - \sigma^2)$ of $\frac{1}{2}\mathbb{Z}[1/3]G$ and we set $\eta_\mathfrak{f} := A\gamma$. In this example, $\mathfrak{f} = \mathfrak{q}^l$ and we find that $\eta_\mathfrak{f}$ belongs to $\frac{1}{2}\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$ but not $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$ so $d_\mathfrak{f} = 2$. Hence, part (i) and the first statement of part (iv) of Conjecture 2.2 are numerically verified for this example up to the precision of the computation.

Remark 4.1 In this first example, as well as in numbers 6, 7, 10, 13 and 15, it will be noticed that certain pairs of coefficients coincide in $\Phi_{\mathfrak{f},\emptyset}(1)$. The explanation is as follows. Suppose for a moment that k is any *Galois* extension of $\mathbb{Q}$ with $\Gamma = \mathrm{Gal}(k/\mathbb{Q})$ and that the cycle $\mathfrak{m}$ and the set T are Γ-stable in the obvious sense. This implies in particular that $k(\mathfrak{m})$ is also Galois over $\mathbb{Q}$ and Γ acts by 'extension and conjugation' on $G_\mathfrak{m}$ and $\mathbb{R}G_\mathfrak{m}$. Explicitly, $\gamma(\sum_{g\in G_\mathfrak{m}} a_g g) := \sum_{g\in G_\mathfrak{m}} a_g \tilde{\gamma} g \tilde{\gamma}^{-1}$ for any $\tilde{\gamma} \in \mathrm{Gal}(k(\mathfrak{m})/\mathbb{Q})$ lifting any $\gamma \in \Gamma$. In this situation, one can show that

$$\Phi_{\mathfrak{m},T}(s) = \gamma(\Phi_{\mathfrak{m},T}(s)) \quad \forall\, \gamma \in \Gamma,\ \forall\, s \in \mathbb{C},\ \Re(s) > 1 \tag{41}$$

By Theorem 2.3 of [13], this equation can be analytically continued to all $s \in \mathbb{C} \setminus \{1\}$ and even to $s = 1$ if Hypothesis 2.1 (ii) holds. All these conditions are clearly met in the above-mentioned examples with $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ and (41) therefore explains the coincidence of coefficients in $\Phi_{\mathfrak{f},\emptyset}(1)$, since in each case Γ acts by *inversion* on $G = G_\mathfrak{f}$. (A similar coincidence in those of $\Phi_{\mathfrak{f},T_p,p}(1)$ will follow by interpolation from (the analytic continuation of) Equation (41) for $s = m \in \mathcal{M}(p)$.) To prove Equation (41), one uses obvious actions of Γ on $\mathfrak{W}_\mathfrak{m}$ and on $\mathrm{Cl}_\mathfrak{m}(k)$ in this situation (the latter corresponds by the Artin map to the action on $G_\mathfrak{m}$) noting that $\gamma(\mathfrak{w}^0_\mathfrak{m}) = \mathfrak{w}^0_\mathfrak{m}$ and that for all $\gamma \in \Gamma$, $\mathfrak{c} \in \mathrm{Cl}_\mathfrak{m}(k)$, $\mathfrak{w} \in \mathfrak{W}_\mathfrak{m}$ and $\Re(s) > 1$ we have $\gamma(\mathfrak{c} \cdot \mathfrak{w}) = \gamma(\mathfrak{c})\gamma(\mathfrak{w})$ and, crucially, $Z_T(s; \gamma(\mathfrak{w})) = Z_T(s; \mathfrak{w})$.

We now look at the entries of Tables 6 and 7 which concern part (ii) and the last statement of part (iv) of the conjecture. First, we list the prime numbers p for which part (ii) of the conjecture has been tested. These prime numbers p must split in k and satisfy $f|(p-1)$ (which implies that $(p, \mathfrak{f}) = 1$). Furthermore they must be relatively small for the computation of $\Phi_{\mathfrak{f},T_p,p}(1)$ to be feasible, *cf.* Proposition 4.1. Let $C_{p,\mathfrak{f}}$ be the constant appearing on the L.H.S. of part (ii) of the conjecture, *i.e.* $C_{p,\mathfrak{f}} := 4(1-p^{-1}\sigma_{\mathfrak{p}_1,\mathfrak{f}})(1-p^{-1}\sigma_{\mathfrak{p}_2,\mathfrak{f}})j(\sqrt{d_k})^{-1}$ where $\mathfrak{p}_1$, $\mathfrak{p}_2$ are the two prime ideals in k above p. The next column contains the values of $C_{p,\mathfrak{f}}R_p(\eta_\mathfrak{f})$ for the $\eta_\mathfrak{f}$ computed above and for each of the primes p. (The p-adic precision to which these are given is smaller in most examples than the one that was used to verify the conjecture.) Each value of $C_{p,\mathfrak{f}}R_p(\eta_\mathfrak{f})$ is checked against the computed value of $\Phi_{\mathfrak{f},T_p,p}(1)$. The two always turn out to be equal, again up to the precision of the computations. Thus, part (ii) of the conjecture is satisfied to this precision for these primes.

Note that p-adic numbers are written using the expansion to the base p with digits in the set $\{0, 1, 2, \ldots, p-1\}$. The digits before the 'decimal point' correspond to negative powers of p. If p is larger than 10 then we use the letters $A = 10, B = 11, \ldots$ to denote the extra digits. (The largest p occurring is 41 for which we use the notation $(36), \ldots, (40)$ to denote the remaining digits.) The subscript at the end of the number is simply p.

In the first example, part (ii) of Conjecture 2.2 has been numerically verified for $p = 3$, 7 *and* 11. *As mentioned above, we have found for each value of* p *that* $C_{p,\mathfrak{f}}R_p(\eta_\mathfrak{f}) = \Phi_{\mathfrak{f},T_p,p}(1)$ *(up to our fixed* p*-adic precision). We have found the following values*

$$\begin{aligned} C_{3,\mathfrak{f}}R_3(\gamma) = \; & 0.2020212220012020220111222010012121212 01_3 \\ & + 0.0021122222121101202020102101100110000 11_3(\sigma+\sigma^2) \end{aligned}$$

$$C_{7,\mathfrak{f}}R_7(\gamma) = 0.23203400342215530616416 3_7 + 0.62421446204116266010633 1_7(\sigma+\sigma^2)$$

$$C_{11,\mathfrak{f}}R_{11}(\gamma) = 0.859AA8491A4592272_{11} + 0.593A1A1A496337044_{11}(\sigma+\sigma^2)$$

The next column contains the smallest positive integer $d_{\mathfrak{f},\sigma-1}$ such that $(\sigma-1)\eta_\mathfrak{f}$ belongs to $d_{\mathfrak{f},\sigma-1}^{-1}\overline{\bigwedge_{\mathbb{Z}G}^2 U_S}^{[S,2]}$ (in the last example, the smallest positive integers $d_{\mathfrak{f},\sigma_1-1}$ and $d_{\mathfrak{f},\sigma_2-1}$ such that $(\sigma_i-1)\eta_\mathfrak{f}$ belongs to $d_{\mathfrak{f},\sigma_i-1}^{-1}\overline{\bigwedge_{\mathbb{Z}G}^2 U_S}^{[S,2]}$ for $i = 1, 2$). Indeed, the second statement of part (iv) of the conjecture asserts that if $\mathfrak{f} = \mathfrak{q}^l$ then $I(\mathbb{Z}G)\eta_\mathfrak{f} \subset \mathbb{Z}[1/g]\overline{\bigwedge_{\mathbb{Z}G}^2 U_S}^{[S,2]}$. It is easy to prove that if G is generated by σ (*resp.* σ_1, σ_2 in the last example) then $I(\mathbb{Z}G) = (\sigma-1)\mathbb{Z}G$ (*resp.* $I(\mathbb{Z}G) = (\sigma_1-1)\mathbb{Z}G + (\sigma_2-1)\mathbb{Z}G$). Therefore, this statement is true if and only if $(\sigma-1)\eta_\mathfrak{f}$ (*resp.* $(\sigma_1-1)\eta_\mathfrak{f}$ and $(\sigma_2-1)\eta_\mathfrak{f}$) belongs to $\mathbb{Z}[1/g]\overline{\bigwedge_{\mathbb{Z}G}^2 U_S}^{[S,2]}$, that is, if $d_{\mathfrak{f},\sigma-1}$ (*resp.* $d_{\mathfrak{f},\sigma_1-1}$ and $d_{\mathfrak{f},\sigma_2-1}$) divide g^e for some $e \geq 0$. In fact, in all examples with $\mathfrak{f} = \mathfrak{q}^l$ we have found that $d_{\mathfrak{f},\sigma-1}$ (*resp.* $d_{\mathfrak{f},\sigma_1-1}$ and $d_{\mathfrak{f},\sigma_2-1}$) is actually 1. In other words, Condition (49) of [13] is verified for these examples. Indeed, we have actually checked that [13, eq. (50)] is verified (see Remark 3.4, *ibid.*).

Finally, the last column gives the index of $\mathbb{Z}G\eta_\mathfrak{f}$ in $d_\mathfrak{f}^{-1}\overline{\bigwedge_{\mathbb{Z}G}^2 U_S}^{[S,2]}$. In each case, we have found that it is a small power of 2 if $\mathfrak{f} = \mathfrak{q}^l$ and that $\eta_\mathfrak{f}$ actually generates $\overline{\bigwedge_{\mathbb{Z}G}^2 U_S}^{[S,2]}$ if $\mathfrak{f} \neq \mathfrak{q}^l$. We do not expect this last fact to generalise. There is no reason to expect $\overline{\bigwedge_{\mathbb{Z}G}^2 U_S}^{[S,2]}$ to be cyclic over $\mathbb{Z}G$ in general, and even when it is, the index of $\mathbb{Z}G\eta_{\mathfrak{f}}$ should reflect the class number h_K of K as is the case with cyclotomic units, for $k = \mathbb{Q}$. Thus this index might well be non-trivial in 'larger' examples. Nevertheless, the mere fact in all our examples this index is always very small (if not trivial) *is* significant: we certainly would not expect this if $(\sqrt{d_k}/4)R(\gamma)^{-1}\Phi_{\mathfrak{f},\emptyset}(1)$ were a random element of $\mathbb{R}G^{[S,2]}$ and A a 'faked' approximation to the given precision, lying in $\frac{1}{b}\mathbb{Z}[1/g]G^{[S,2]}$ or $\frac{1}{2b}\mathbb{Z}[1/g]G^{[S,2]}$ (see discussion above).

In the first example, $\mathbb{Z}G\eta_{\mathfrak{f}}$ is of index 4 in $\frac{1}{2}\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$.

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Table 1: *The extension K/k*

#	k	h_k	$\mathfrak{f}$	$\mathfrak{f}(K/k)$	$P_\theta(X)$	d_K	h_K	G	$\sigma(\theta)$	n_c
1	$\mathbb{Q}(\sqrt{37})$	1	$2\mathcal{O}_k$	$2\mathcal{O}_k$	$X^6 - 3X^5 - 2X^4 + 9X^3 - 5X + 1$	$2^4 \cdot 37^3$	1	C_3	$-\theta^5 + 2\theta^4 + 4\theta^3 - 6\theta^2 - 4\theta + 3$	1
2	$\mathbb{Q}(\sqrt{43})$	1	$\mathfrak{q}_3^2$	$\mathfrak{q}_3^2$	$X^6 - 16X^4 - 12X^3 + 21X^2 + 10X - 7$	$2^6 \cdot 3^4 \cdot 43^3$	1	C_3	$\frac{1}{19}(7\theta^5 - 3\theta^4 - 108\theta^3 - 35\theta^2 + 105\theta - 13)$	3
3	$\mathbb{Q}(\sqrt{82})$	4	2	$\mathcal{O}_k$	$X^8 + 2X^7 - 21X^6 - 78X^5 - 53X^4 + 88X^3 + 114X^2 + 24X - 4$	$2^{12} \cdot 41^4$	1	C_4	not given (too big)	1
4	$\mathbb{Q}(\sqrt{89})$	1	$\mathfrak{q}_5$	$\mathfrak{q}_5$	$X^4 + 2X^3 - 8X^2 - 9X - 2$	$5 \cdot 89^2$	1	C_2	$-\theta - 1$	2
5	$\mathbb{Q}(\sqrt{321})$	3	$\mathfrak{q}_2$	$\mathcal{O}_k$	$X^6 + 2X^5 - 18X^4 - 55X^3 - 26X^2 + 21X + 3$	$3^3 \cdot 107^3$	1	C_3	$\frac{1}{18}(-\theta^5 + 3\theta^4 + 15\theta^3 - 26\theta^2 - 48\theta + 3)$	1
6	$\mathbb{Q}(\sqrt{349})$	1	$2\mathcal{O}_k$	$2\mathcal{O}_k$	$X^6 + 3X^5 - 36X^4 - 77X^3 + 200X^2 + 239X - 205$	$2^4 \cdot 349^3$	1	C_3	$\frac{1}{385}(5\theta^5 + 2\theta^4 - 216\theta^3 + 38\theta^2 + 1856\theta - 335)$	1
7	$\mathbb{Q}(\sqrt{401})$	5	$\mathfrak{q}_2\mathfrak{q}_2'$	$\mathcal{O}_k$	$X^{10} + 2X^9 - 20X^8 - 2X^7 + 69X^6 + X^5 - 69X^4 - 2X^3 + 20X^2 + 2X - 1$	401^5	1	C_5	$\frac{1}{27}(-7\theta^9 - 8\theta^8 + 151\theta^7 - 106\theta^6 - 473\theta^5 + 359\theta^4 + 427\theta^3 - 220\theta^2 - 67\theta + 7)$	1
8	$\mathbb{Q}(\sqrt{401})$	5	$\mathfrak{q}_5$	$\mathfrak{q}_5$	$X^{20} + 2X^{19} - 27X^{18} - 58X^{17} + 272X^{16} + 639X^{15} - 1245X^{14} - 3339X^{13} + 2469X^{12} + 8464X^{11} - 1650X^{10} - 9965X^9 + 827X^8 + 6081X^7 - 914X^6 - 1796X^5 + 510X^4 + 151X^3 - 63X^2 + X + 1$	$5^5 \cdot 401^{10}$	1	C_{10}	not given (too big)	2
9	$\mathbb{Q}(\sqrt{577})$	7	$\mathfrak{q}_2$	$\mathcal{O}_k$	$X^{14} + 2X^{13} - 25X^{12} - 69X^{11} + 161X^{10} + 632X^9 - 147X^8 - 2146X^7 - 1171X^6 + 2669X^5 + 2682X^4 - 667X^3 - 1466X^2 - 336X + 49$	577^7	1	C_7	not given (too big)	1
10	$\mathbb{Q}(\sqrt{709})$	1	$2\mathcal{O}_k$	$2\mathcal{O}_k$	$X^6 - 56X^4 + 784X^2 - 2836$	$2^4 \cdot 709^3$	1	C_3	$\frac{1}{212}(-9\theta^4 + 420\theta^2 - 106\theta - 3136)$	1
11	$\mathbb{Q}(\sqrt{709})$	1	$2\mathfrak{q}_5$	$2\mathfrak{q}_5$	$X^{12} - 53X^{10} + 970X^8 - 7657X^6 + 25350X^4 - 29025X^2 + 6125$	$2^8 \cdot 5^3 \cdot 709^6$	1	C_6	$\frac{1}{405650}(-231\theta^{11} + 109530\theta^9 - 1648250\theta^7 + 9203670\theta^5 - 13254450\theta^3 - 7332250\theta)$	2
12	$\mathbb{Q}(\sqrt{1021})$	1	$\mathfrak{q}_5$	$\mathfrak{q}_5$	$X^4 + 2X^3 - 32X^2 - 33X + 17$	$5 \cdot 1021^2$	1	C_2	$-\theta - 1$	2
13	$\mathbb{Q}(\sqrt{2069})$	1	$2\mathcal{O}_k$	$2\mathcal{O}_k$	$X^6 - 84X^4 + 1764X^2 - 8276$	$2^4 \cdot 2069^3$	1	C_3	$\frac{1}{60}(\theta^4 - 70\theta^2 - 30\theta + 784)$	1
14	$\mathbb{Q}(\sqrt{2069})$	1	$2\mathfrak{q}_5$	$2\mathfrak{q}_5$	$X^{12} - 71X^{10} - 134X^9 + 1128X^8 + 3138X^7 - 2847X^6 - 12804X^5 - 2686X^4 + 13110X^3 + 9935X^2 + 2150X + 125$	$2^8 \cdot 5^3 \cdot 2069^6$	1	C_6	not given (too big)	2
15	$\mathbb{Q}(\sqrt{9897})$	3	$\mathfrak{q}_3^2$	$\mathfrak{q}_3^2$	$X^{18} - 204X^{16} + 15822X^{14} - 590238X^{12} + 11246949X^{10} - 110721114X^8 + 550866177X^6 - 1324310688X^4 + 1327290624X^2 - 364843008$	$3^{21} \cdot 3299^9$	3	C_3^2	G has two generators σ_1 and σ_2 which are not given (too big)	1

Table 2: *The modules* $\bigwedge^2_{\mathbb{Z}G} U_S$ *and* $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$

#	S-units	G^*	$\mathbf{X}$
1	$u_1 := \theta^3 - 2\theta^2 - \theta + 1,\ \ u_2 := \theta$ $u_3 := \theta^5 - 2\theta^4 - 3\theta^3 + 4\theta^2 + 2\theta - 1,\ \ u_4 := \theta^5 - 3\theta^4 - \theta^3 + 7\theta^2 - 2\theta - 1$ $u_5 := \theta^5 - 2\theta^4 - 3\theta^3 + 5\theta^2 + 2\theta - 2,\ \ u_6 := \theta^3 - 2\theta^2 - 2\theta + 3$	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/3}$	$X_1 := \chi_0$ $X_2 := \chi + \chi^2$
2	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/3}$	$X_1 := \chi_0$, $X_2 := \chi + \chi^2$
3	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := i$	$X_1 := \chi_0$, $X_2 := \chi^2$, $X_3 = \chi + \chi^3$
4	$u_1 := \theta^3 + \theta^2 - 10\theta - 3,\ \ u_2 := \theta^3 + \theta^2 - 8\theta - 3$ $u_3 := \theta^3 + 6\theta^2 + 5\theta + 1,\ \ u_4 := 2\theta + 1$	$\langle\chi\rangle$ with $\chi(\sigma) := -1$	$X_1 := \chi_0$ $X_2 := \chi$
5	$u_1 := \frac{1}{12}(\theta^5 - \theta^4 - 19\theta^3 - 2\theta^2 + 48\theta + 9),\ \ u_2 := \frac{1}{36}(\theta^5 + 3\theta^4 - 27\theta^3 - 58\theta^2 + 84\theta - 3),$ $u_3 := \frac{1}{18}(2\theta^5 + 3\theta^4 - 30\theta^3 - 101\theta^2 - 111\theta - 15),\ \ u_4 := \frac{1}{2}(\theta^4 - 17\theta^2 - 21\theta - 1),$ $u_5 := \frac{1}{18}(\theta^5 - 21\theta^3 - 25\theta^2 + 3\theta + 6),\ \ u_6 := \frac{1}{4}(\theta^5 + \theta^4 - 19\theta^3 - 36\theta^2 + 10\theta + 11)$	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/3}$	$X_1 := \chi_0$ $X_2 := \chi + \chi^2$
6	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/3}$	$X_1 := \chi_0$, $X_2 := \chi + \chi^2$
7	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/5}$	$X_1 := \chi_0$ $X_2 := \chi + \chi^2 + \chi^3 + \chi^4$
8	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/10}$	$X_1 := \chi_0$, $X_2 := \chi^5$ $X_3 := \chi^2 + \chi^4 + \chi^6 + \chi^8$ $X_4 := \chi + \chi^3 + \chi^7 + \chi^9$
9	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/7}$	$X_1 := \chi_0$ $X_2 := \chi + \chi^2 + \chi^3 + \chi^4 + \chi^5 + \chi^6$
10	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/3}$	$X_1 := \chi_0$, $X_2 := \chi + \chi^2$
11	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/6}$	$X_1 := \chi_0$, $X_2 := \chi^3$ $X_3 := \chi^2 + \chi^4$, $X_4 := \chi + \chi^5$
12	$u_1 := \frac{1}{9}(5\theta^3 + 3\theta^2 - 166\theta + 62),\ \ u_2 := \frac{1}{9}(4\theta^3 + 24\theta^2 - 14\theta - 53)$ $u_3 := \frac{1}{3}(938\theta^3 - 4029\theta^2 - 5458\theta + 2600),\ \ u_4 := \frac{1}{9}(8\theta^3 + 57\theta^2 + 44\theta - 25)$	$\langle\chi\rangle$ with $\chi(\sigma) := -1$	$X_1 := \chi_0$ $X_2 := \chi$
13	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/3}$	$X_1 := \chi_0$, $X_2 := \chi + \chi^2$
14	not given (too big)	$\langle\chi\rangle$ with $\chi(\sigma) := e^{2i\pi/6}$	$X_1 := \chi_0$, $X_2 := \chi^3$ $X_3 := \chi^2 + \chi^4$, $X_4 := \chi + \chi^5$
15	not given (too big)	$\langle\chi_1, \chi_2\rangle$ with $\chi_1(\sigma_1) := e^{2i\pi/3}$, $\chi_1(\sigma_2) := 1$, $\chi_2(\sigma_1) := 1$ and $\chi_2(\sigma_2) := e^{2i\pi/3}$	$X_1 := \chi_0$ $X_2 := \chi_1 + \chi_1^2$, $X_3 := \chi_2 + \chi_2^2$ $X_4 := \chi_1\chi_2 + \chi_1^2\chi_2^2$, $X_5 := \chi_1\chi_2^2 + \chi_1^2\chi_2$

Table 3: *The modules* $\bigwedge^2_{\mathbb{Z}G} U_S$ *and* $\overline{\bigwedge^2_{\mathbb{Z}G} U_S}^{[S,2]}$

#	$\mathbb{Q}U_S$	generators of $\bigwedge^2_{\mathbb{Q}G} \mathbb{Q}U_S$	$\tilde{e}_{S,>2}$	γ	index of $\mathbb{Z}G\gamma$
1	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	$v_{1,1} := (-2,2,-1,-1,-3,0)$, $v_{1,2} := (-1,2,-1,-2,0,3)$, $v_{2,1} := (-2,5,-4,-1,0,0)$, $v_{2,2} := (3,-3,3,3,0,0)$	0	$(0,0,0,0,1,0) \wedge (0,0,0,0,0,1)$	1
2	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	not given	0	not given	1
3	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^3 + \mathbb{Q}(X_3)^2$	not given	$1 - \sigma + \sigma^2 - \sigma^3$	not given	1
4	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	$v_{1,1} := (1,0,-2,0),\ v_{1,2} := (0,0,0,2)$ $v_{2,1} := (1,-2,0,0),\ v_{2,2} := (0,-2,0,0)$	0	$(0,1,0,0) \wedge (0,0,-1,0)$ $+ (0,0,1,0) \wedge (0,0,0,-1)$	2
5	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	$v_{1,1} := (0,0,-1,-1,-3,0)$, $v_{1,2} := (1,-2,0,0,0,3)$, $v_{2,1} := (0,3,0,0,0,0)$, $v_{2,2} := (0,0,3,0,0,0)$	0	$(0,0,0,0,1,0) \wedge (0,0,0,0,0,1)$	1
6	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	not given	0	not given	1
7	$\mathbb{Q}(X_1)^3 + \mathbb{Q}(X_2)^2$	not given	$1 + \sigma + \sigma^2 + \sigma^3 + \sigma^4$	not given	1
8	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2 + \mathbb{Q}(X_3)^2 + \mathbb{Q}(X_4)^2$	not given	0	not given	$2 \cdot 41$
9	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	not given	0	not given	1
10	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	not given	0	not given	1
11	$\mathbb{Q}(X_1)^3 + \mathbb{Q}(X_2)^2 + \mathbb{Q}(X_3)^2 + \mathbb{Q}(X_4)^2$	not given	$1 + \sigma + \sigma^2 + \sigma^3 + \sigma^4 + \sigma^5$	not given	1
12	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	$v_{1,1} := (-1,-1,-2,0),\ v_{1,2} := (-1,-1,0,2)$ $v_{2,1} := (1,1,0,0),\ v_{2,2} := (0,2,0,0)$	0	$(1,0,0,0) \wedge (0,0,-1,0)$ $+ (0,0,1,0) \wedge (0,0,0,1)$	2
13	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2$	not given	0	not given	1
14	$\mathbb{Q}(X_1)^3 + \mathbb{Q}(X_2)^2 + \mathbb{Q}(X_3)^3 + \mathbb{Q}(X_4)^2$	not given	$3 + 3\sigma^3$	not given	1
15	$\mathbb{Q}(X_1)^2 + \mathbb{Q}(X_2)^2 + \mathbb{Q}(X_3)^3$	not given	$(1 + \sigma_1 + \sigma_1^2)(2 - \sigma_2 - \sigma_2^2)$	not given	1

Table 4: *Verification of Conjecture 2.2* - part (i), (iii) and (iv)

#	$4\sqrt{d_k}^{-1}R(\gamma)$	$\Phi_{\mathfrak{f},\emptyset}(1)$	A	is $\mathfrak{f}=\mathfrak{q}^l$?	$d_{\mathfrak{f}}$
1	$1.4859505839423766252743654$ $+ 0.3928048216425639021329405(\sigma+\sigma^2)$	$0.3501704703286244105042422$ $- 0.7429752919711883126371827(\sigma+\sigma^2)$	$\frac{1}{2}(1-\sigma-\sigma^2)$	Yes	2
2	$-1.443448363709350198869637$ $+ 0.327469144766235117326033\sigma$ $- 1.848922699899164820861579\sigma^2$	$0.366471740478024869658988$ $- 0.039002595711789752332954\sigma$ $- 1.809920104187375068528625\sigma^2$	$\frac{1}{2}(1-\sigma+\sigma^2)$	Yes	2
3	$0.990736966647953569158853(1+\sigma^3)$ $- 0.104818803994323556683935(\sigma+\sigma^2)$	$0.326298344657731059802665$ $- 0.221479540663407503118729(\sigma+\sigma^3)$ $- 0.769257425984546066040123\sigma^2$	$\frac{1}{4}(-1+2\sigma-3\sigma^2)$	Yes	2
4	$-4.1759835935184954553812374$ $- 0.5378714331652445939211068\sigma$	$-0.2689357165826222969605534$ $- 2.0879917967592477276906187\sigma$	$\frac{1}{2}\sigma$	Yes	2
5	$0.3647664814623851156843183$ $+ 0.9383748471668418510324386\sigma$ $+ 1.5119832128712985863805590\sigma^2$	$0.1044209421210358098319009$ $- 1.0427957892878776608643396\sigma$ $- 0.4691874235834209255162193\sigma^2$	$\frac{1}{2}(-1+\sigma-\sigma^2)$	Yes	2
6	$0.3903032175535131898951365(1+\sigma^2)$ $+ 2.1345841304087725230472693\sigma$	$0.6769888476508730716284981$ $- 1.0672920652043862615236346(\sigma+\sigma^2)$	$\frac{1}{2}(-1-\sigma+\sigma^2)$	Yes	2
7	$-0.5430424606759486694736326(1+\sigma)$ $+ 0.8649249218235797385747707(\sigma^2+\sigma^4)$ $- 0.6437649222952621382022762\sigma^3$	$1.0860849213518973389472652$ $- 0.3218824611476310691011381(\sigma+\sigma^4)$ $- 0.2211599995283176003724945(\sigma^2+\sigma^3)$	$\frac{1}{5}(-3+2\sigma+2\sigma^2+2\sigma^3-3\sigma^4)$	No	1
8	$-1.3494436538740630114284692$ $- 1.7116028784482896128599650\sigma$ $- 0.3796444360048927748408130\sigma^2$ $- 0.8340983749986581978191287\sigma^3$ $- 0.1806103940721755799158468\sigma^4$ $+ 0.7063623627724616272858469\sigma^5$ $- 1.2043707169764199179058313\sigma^6$ $+ 1.0240929894938156671027431\sigma^7$ $- 2.4037576815736824020478056\sigma^8$ $+ 0.4024540247941539343479953\sigma^9$	$0.4769621163349386255017257$ $- 1.3960582877976217506952332\sigma$ $- 0.6175505522167333606273630\sigma^2$ $- 0.4964278267523899302068686\sigma^3$ $- 0.2291274044079765136578312\sigma^4$ $- 0.5269815315477649828362206\sigma^5$ $+ 0.2599539512328980544134731\sigma^6$ $- 0.1966713232003592665532588\sigma^7$ $- 0.0966340491363850966012588\sigma^8$ $- 0.1427744719524809127778017\sigma^9$	$\frac{1}{82}(44+29\sigma-34\sigma^2+13\sigma^3+30\sigma^4+3\sigma^5$ $-12\sigma^6+7\sigma^7-28\sigma^8-11\sigma^9)$	Yes	2

Table 5: *Verification of Conjecture 2.2* - part (i), (iii) and (iv)

#	$4\sqrt{d_k}^{-1}R(\gamma)$	$\Phi_{\mathfrak{f},\emptyset}(1)$	A	is $\mathfrak{f}=\mathfrak{q}^l$?	$d_\mathfrak{f}$
9	$1.2006136993027519158356429$ $+ 0.5601281150225709497028443\sigma$ $+ 0.4468810365792052749965755\sigma^2$ $+ 0.3336339581358396002903067\sigma^3$ $- 0.3068516261443413658424917\sigma^4$ $+ 0.9126508166528979185513609\sigma^5$ $- 0.0188887434944873685582098\sigma^6$	$0.3555763402274556807627663$ $- 0.8024573768066609557593419\sigma$ $- 0.3366875967329683122045565\sigma^2$ $- 0.3981563224960909600763009\sigma^3$ $- 0.2234405182896026374982877\sigma^4$ $- 0.0487247140831143149202745\sigma^5$ $- 0.1101934398462369627920190\sigma^6$	$\frac{1}{2}(-1-\sigma+\sigma^2-\sigma^3-\sigma^4+\sigma^5+\sigma^6)$	Yes	2
10	$0.7234352393016752990818922(1+\sigma)$ $+ 2.1793574714245388818858141\sigma^2$	$0.3662434964105941418610148$ $- 1.0896787357122694409429070(\sigma+\sigma^2)$	$\frac{1}{2}(-1+\sigma-\sigma^2)$	Yes	2
11	$0.7895116790170794653416756$ $- 1.1652625156938787944693039\sigma$ $- 1.8085639683003889670456318\sigma^2$ $- 0.7895116790170794653416756\sigma^3$ $+ 2.6211847478167423772732258\sigma^4$ $+ 0.3526417361775253842417098\sigma^5$	$2.6211847478167423772732258$ $+ 0.3526417361775253842417098\sigma$ $+ 0.7895116790170794653416756\sigma^2$ $- 1.1652625156938787944693039\sigma^3$ $- 1.8085639683003889670456318\sigma^4$ $- 0.7895116790170794653416756\sigma^5$	$\frac{1}{6}(-1-\sigma+5\sigma^2-\sigma^3-\sigma^4-\sigma^5)$	No	1
12	$-0.2877586687247090106420884$ $+3.9680836522391984256974575\sigma$	$0.1438793343623545053210442$ $- 1.9840418261195992128487287\sigma$	$-\frac{1}{2}$	Yes	2
13	$2.3349046592276594288814979$ $- 1.1674523296138297144407489(\sigma+\sigma^2)$	$1.3358165556295184849328475$ $- 0.1683642260156887704920986(\sigma+\sigma^2)$	$\frac{1}{2}(1-\sigma-\sigma^2)$	Yes	2
14	$0.4165560795603681099566393$ $+ 0.7153308793910213046570109\sigma$ $- 1.2682208798620716367535322\sigma^2$ $- 0.4165560795603681099566393\sigma^3$ $- 0.7153308793910213046570109\sigma^4$ $+ 1.2682208798620716367535322\sigma^5$	$-0.4165560795603681099566393$ $- 0.7153308793910213046570109\sigma$ $+ 1.2682208798620716367535322\sigma^2$ $+ 0.4165560795603681099566393\sigma^3$ $+ 0.7153308793910213046570109\sigma^4$ $- 1.2682208798620716367535322\sigma^5$	$\frac{1}{6}(-5+\sigma^3)$	No	1
15	$-1.2218884525300797394860547(1+\sigma_1^2)$ $+ 3.4685136586674160517352232\sigma_1$ $- 0.6973303591485227158417690(\sigma_2+\sigma_1^2\sigma_2^2)$ $+ 2.2802292524382017551015625(\sigma_1^2\sigma_2+\sigma_2^2)$ $- 0.5581621396824224664966796(\sigma_1\sigma_2+\sigma_1\sigma_2^2)$	$2.9561452818637877653536663$ $- 1.7342568293337080258676116(\sigma_1+\sigma_1^2)$ $- 1.0705305164860507528782365(\sigma_2+\sigma_2^2)$ $+ 1.7678608756345734687200056(\sigma_1\sigma_2+\sigma_1^2\sigma_2^2)$ $- 1.2096987359521510022233259(\sigma_1^2\sigma_2+\sigma_1\sigma_2^2)$	$\frac{1}{18}(1+\sigma_1+\sigma_1^2-11\sigma_2+7\sigma_1\sigma_2$ $+7\sigma_1^2\sigma_2+\sigma_2^2+\sigma_1\sigma_2^2+\sigma_1^2\sigma_2^2)$	Yes	2

Table 6: *Verification of Conjecture 2.2* - part (ii) and (iv)

#	primes	$C_{p,\mathfrak{f}} R_p(\eta_\mathfrak{f}) = \Phi_{\mathfrak{f},T_p,p}(1)$	$d_{\mathfrak{f},\sigma-1}$	index of $\mathbb{Z}G\eta_\mathfrak{f}$
1	3, 7, 11	$0.2020212220012020220111222010012121212013$ $+ 0.00211222221211012020201021011001100001 1_3(\sigma+\sigma^2)$ $0.23203400342215530616416 3_7 + 0.62421446204116266010633 1_7(\sigma+\sigma^2)$ $0.859AA8491A459227 2_{11} + 0.593A1A1A49633704 4_{11}(\sigma+\sigma^2)$	1	2^2
2	19	$0.37A267_{19} + 0.IFBDF1_{19}\sigma + 0.7C1858_{19}\sigma^2$	1	2^2
3	3, 11	$0.0220002011001002010210010201220202212 01_3$ $+ 0.0112211011102010211020200111010210201 22_3(\sigma+\sigma^3)$ $+ 0.0001201011200121022120100021122121122 01_3\sigma^2$ $0.5281109A901147AA7_{11} + 0.065022A740283901 8_{11}(\sigma+\sigma^3) + 0.692A2405AA243022 8_{11}\sigma^2$	1	2^2
4	11	$0.9627359501683452_{11} + 0.00637222A6760375_{11}\sigma$	1	2
5	5, 13	$0.4420301130404124023140 2_5 + 0.4144332442222014223300 1_5\sigma + 0.4010124214040141440003 1_5\sigma^2$ $0.5811AA04_{13} + 0.C408A99C_{13}\sigma + 0.201B3AB1_{13}\sigma^2$	1	2^2
6	3, 17	$0.1022122012102222020020202221222120220 10_3$ $+0.0110011101122221111010111102021221102 22_3(\sigma+\sigma^2)$ $0.F44A49F_{17} + 0.8218BE5_{17}(\sigma+\sigma^2)$	1	2^2
7	5, 7, 11	$0.3233210340311430413102022_5 + 0.0311412013321131241321400_5(\sigma+\sigma^4)$ $+ 0.1314122043432120343022033_5(\sigma^2+\sigma^3)$ $0.302602322532556032416053 2_7 + 0.016561466106050455310454 6_7(\sigma+\sigma^4)$ $+ 0.223130165546652233443220 6_7(\sigma^2+\sigma^3)$ $0.329784702951_{11} + 0.889019601695_{11}(\sigma+\sigma^4) + 0.7670050A8599_{11}(\sigma^2+\sigma^3)$	1	1
8	11, 41	$0.8806785A3211A8230_{11} + 0.0660061534277237 9_{11}\sigma + 0.6051479349339616 5_{11}\sigma^2$ $+ 0.05A986A29673A92A3_{11}\sigma^3 + 0.3496587A010380A79_{11}\sigma^4$ $+ 0.4940071278130917 5_{11}\sigma^5 + 0.9909949658566112 7_{11}\sigma^6$ $+ 0.6637455738542124A_{11}\sigma^7 + 0.5619A04905357615 7_{11}\sigma^8 + 0.6060356205174261 8_{11}\sigma^9$ $0.SI_{41} + 0.5(37)_{41}\sigma + 0.BL_{41}\sigma^2 + 0.1Q_{41}\sigma^3 + 0.3W_{41}\sigma^4 + 0.WL_{41}\sigma^5$ $+ 0.Y(36)_{41}\sigma^6 + 0.VN_{41}\sigma^7 + 0.SA_{41}\sigma^8 + 0.D(37)_{41}\sigma^9$	1	2^9

Table 7: *Verification of Conjecture 2.2* - part (ii) and (iv)

#	primes	$C_{p,\mathfrak{f}}R_p(\eta_\mathfrak{f}) = \Phi_{\mathfrak{f},T_p,p}(1)$	$d_{\mathfrak{f},\sigma-1}$	index of $\mathbb{Z}G\eta_\mathfrak{f}$
9	3, 11, 17	$0.222022110111002222112202221101101011211_3 + 0.002001201222221221212201112021202201020_3\sigma$ $+ 0.002001220120020121010102102021122102221_3\sigma^2$ $+ 0.001211110201122210102210100110102201221_3\sigma^3$ $+ 0.112010220020002220212200202222100011101_3\sigma^4$ $+ 0.220202201100201201000000011112201011010_3\sigma^5$ $+ 0.222022120210010100121111202101210110010_3\sigma^6$ $0.78817332A270_{11} + 0.581648097113_{11}\sigma + 0.36370A254025_{11}\sigma^2 + 0.234186663552_{11}\sigma^3$ $+ 0.68A306753247_{11}\sigma^4 + 0.A26635843A21_{11}\sigma^5 + 0.9A6002162469_{11}\sigma^6$ $0.092998_{17} + 0.411756_{17}\sigma + 0.BC80F6_{17}\sigma^2 + 0.DA572A_{17}\sigma^3$ $+ 0.25AGFF_{17}\sigma^4 + 0.8GE8C4_{17}\sigma^5 + 0.AEBFG7_{17}\sigma^6$	1	2^6
10	3, 5, 7, 11	$0.12212010110010120120110200211112211001 02_3$ $+0.0101020110202021121221000112102002112 21_3(\sigma + \sigma^2)$ $0.403003202113332322014031 21_5 + 0.2414211241000343134411211 3_5(\sigma + \sigma^2)$ $0.6135622345466460143243 20_7 + 0.6443122423512434624165 04_7(\sigma + \sigma^2)$ $0.89816292686A4601_{11} + 0.20928A9982220855_{11}(\sigma + \sigma^2)$	1	2^2
11	11	$0.4279553895127000_{11} + 0.3502739301069659_{11}\sigma + 0.4778387406583A34_{11}\sigma^2$ $+ 0.273038669A268856_{11}\sigma^3 + 0.27A95423246775A5_{11}\sigma^4 + 0.73327236A4527076_{11}\sigma^5$	1	1
12	11, 41	$0.23A9227A0541A025_{11} + 0.25A583269A5537A1_{11}\sigma$ $0.6SP_{41} + 0.5KN_{41}\sigma$	1	2
13	7, 11	$0.5605361401660032563563_7 + 0.1610430464366153334222_7(\sigma + \sigma^2)$ $0.2398796701A1346A_{11} + 0.288A2880995A2406_{11}(\sigma + \sigma^2)$	1	2^2
14	11	$0.290A17A7368678838_{11} + 0.8A649AA3287793065_{11}\sigma + 0.7663619146A559404_{11}\sigma^2$ $+0.91A093037424322 72_{11}\sigma^3 + 0.30461007823317A45_{11}\sigma^4 + 0.444749196405516A6_{11}\sigma^5$	1	1
15	13, 19	$0.37861615758_{13} + 0.B7962270777_{13}(\sigma_1 + \sigma_1^2) + 0.6853599605C_{13}(\sigma_2 + \sigma_2^2)$ $+ 0.09A939C0279_{13}(\sigma_1\sigma_2 + \sigma_1^2\sigma_2^2) + 0.65B6A46B581_{13}(\sigma_1^2\sigma_2 + \sigma_1\sigma_2^2)$ $0.BC66483F_{19} + 0.G0B815H6_{19}(\sigma_1 + \sigma_1^2) + 0.7BB0F5H3_{19}(\sigma_2 + \sigma_2^2)$ $+ 0.66709GBE_{19}(\sigma_1\sigma_2 + \sigma_1^2\sigma_2^2) + 0.BF932F8A_{19}(\sigma_1^2\sigma_2 + \sigma_1\sigma_2^2)$	$d_{\mathfrak{f},\sigma_1-1} = 1$ $d_{\mathfrak{f},\sigma_2-1} = 1$	2^6