INVARIANT MEASURES FOR A STOCHASTIC FOKKER-PLANCK EQUATION

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Abstract. We study the kinetic Fokker-Planck equation perturbed by a stochastic Vlasov force term. When the noise intensity is not too large, we solve the Cauchy Problem in a class of well-localized (in velocity) functions. We also show that, when the noise intensity is sufficiently small, the system with prescribed mass admits a unique invariant measure which is exponentially mixing. The proof uses hypocoercive decay estimates and hypoelliptic gains of regularity. At last we also exhibit an explicit example showing that some restriction on the noise intensity is indeed required.

Contents

1. Introduction

In this paper, we are interested in studying the invariant measures of the following stochastic Fokker-Planck equation

$$\mathrm{d}f \;+\; v\cdot\nabla_x f\;\mathrm{d}t \;+\; \lambda\nabla_v f\odot\mathrm{d}W_t \;=\; \mathcal{Q}(f)\;\mathrm{d}t. \tag{1.1}$$

The unknown f depends on the variables $t\in[0,\infty)$, $x\in\mathbb{T}^N$ and $v\in\mathbb{R}^N$. The operator $\mathcal{Q}$ is the Fokker-Planck operator whose expression is given by

$$\mathcal{Q}(f) \;=\; \Delta_v f \;+\; \mathrm{div}_v(vf).$$

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The noise term in (1.1) is white in time, coloured in space: we let Γ be a self-adjoint and non-negative operator on $L^2(\mathbb{T}^N;\mathbb{R}^N)$ with $\mathrm{Tr}(\Gamma)<\infty$. Let $(G_j)_{j\in\mathbb{N}}$ be a complete orthonormal system in $L^2(\mathbb{T}^N;\mathbb{R}^N)$ of eigenvectors of Γ with associated non-negative eigenvalues $(\gamma_j)_{j\in\mathbb{N}}$:

$$\Gamma G_j = \gamma_j G_j, \quad j\in\mathbb{N}.$$

The random perturbation $\mathrm{d}W_t$ is a Γ–Wiener process on $L^2(\mathbb{T}^N;\mathbb{R}^N)$ (see for instance [1, Section 4.1] for the complete definition). It can be written as

$$\mathrm{d}W_t(x) \;=\; \sum_j \Gamma^{\frac{1}{2}} G_j(x)\ \mathrm{d}\beta_j(t) = \sum_j \gamma_j^{\frac{1}{2}} G_j(x)\ \mathrm{d}\beta_j(t)$$

where $(\beta_j)_{j\in\mathbb{N}}$ is a family of real independent Brownian motions. In what follows, we set $F_j := \Gamma^{\frac{1}{2}} G_j$ and write the noise under the form

$$\mathrm{d}W_t(x) \;=\; \sum_j F_j(x)\ \mathrm{d}\beta_j(t).$$

The notation $\odot$ in (1.1) emphasizes the scalar product in $\mathbb{R}^N$ and the fact that we consider the stochastic term in the Stratonovich sense. The parameter $\lambda>0$ represents the size of the random perturbation. We assume the following additional regularity in space of the noise:

$$\sum_j \|F_j\|_\infty^2 + \|\nabla_x F_j\|_\infty^2 \le 1. \tag{1.2}$$

Since we fix the intensity of the noise as defined by the sum in (1.2) to the value 1, it is the parameter λ that will measure the strength of the noise term in (1.1) (see Remark 2.3 for an insight of the role of λ in our work).

From a physical point of view, this kind of equation can describe the evolution of the distribution function $f(t,x,v)$ of a cloud of particles which, at a time t, are at position x and have velocity v. The transport term $v\cdot\nabla_x f$ corresponds to the free flow of particles while the Fokker-Planck operator $\mathcal{Q}$ models interactions between particles and the surrounding medium. The noisy term $\lambda\nabla_v f\ \odot \mathrm{d}W_t$ describes the effect of a random force $\lambda\dot{W}_t$ acting on the particles.

The operator $\mathcal{Q}$ is self-adjoint in the weighted space $L^2(\mathbb{R}^N,\mathcal{M}^{-1}\mathrm{d}v)$, $\mathcal{M}$ being the Maxwellian distribution on $\mathbb{R}^N$, which is defined by

$$\mathcal{M}(v) = (2\pi)^{-N/2} e^{-|v|^2/2}, \quad v\in\mathbb{R}^N.$$

We will study Equation (1.1) in the state space $L^2(\mathbb{R}^N,\mathcal{M}^{-1}\mathrm{d}v)$. Equivalently, considering the new unknown $g=\mathcal{M}^{-\frac{1}{2}} f$ instead of f, we will try to solve the following problem in $L^2(\mathbb{R}^N,\mathrm{d}v)$:

$$\begin{cases} \mathrm{d}g \;+\; v\cdot\nabla_x g\ \mathrm{d}t \;+\; \lambda\left(\nabla_v - \dfrac{v}{2}\right) g\ \odot \mathrm{d}W_t \;=\; Lg\ \mathrm{d}t \\ g(0) = g_{\mathrm{in}} \end{cases} \tag{1.3}$$

with

$$Lg \;=\; \Delta_v g \;+\; \left(\frac{N}{2} - \frac{|v|^2}{4}\right) g$$

being a self-adjoint operator[1] on $L^2(\mathbb{R}^N)$.

[1] This operator is essentially the well-known Hamiltonian of quantum harmonic oscillators, and operators D^* and D below are associated creation and annihilation operators. However, since we have derived it from $\mathcal{Q}$, hereafter we shall abuse terminology and call L itself a Fokker-Planck operator.

The aim of this paper is twofold. First of all, we want to study existence, uniqueness and qualitative properties of solutions to problem (1.3). Then, we investigate existence and uniqueness of an invariant measure for this problem. We hope that the present contribution will participate to a growing effort to trigger a lasting interest on qualitative behaviour of solutions to kinetic equations under perturbations by random forces. With this respect, the present paper is the first of a planned series of papers on the long-time behaviour of Vlasov-Poisson-Fokker-Planck equations with stochastic force term. We wish to study the existence of invariant measure in the case of repulsive (Coulomb) forces for such equations. We also would like to show the uniqueness, ergodicity and the potential mixing properties of the invariant measure of the Vlasov-Poisson-Fokker-Planck equations with stochastic force term in the case where the Poisson kernel is regularized and small, and the noise is also small enough. The deterministic framework behind this second type of problem is the one of Section 17 in [5]: weakly self-consistent Vlasov-Fokker-Planck equations. With regard to this last problematic, the present paper deals with the linear case: except for the noise term, there is no (weakly self-consistent) Vlasov force term. We prove that, indeed, for a noise intensity small enough, there is a unique, ergodic invariant measure that is exponentially mixing: see Theorem 4.1.

We obtain the existence of the solutions to Equation (1.3) through a standard Galerkin scheme. Precisely, we project Equation (1.3) on some finite dimensional space. Doing so, we construct a sequence $(g_m)_m$ of approximate solutions to our problem. Then, one has to derive energy estimates on the sequence $(g_m)_m$ in order to pass to the limit in the approximate problem. Note that, to ensure existence, we need that the coefficient λ in front of the noise is small enough so that the random perturbation does not affect too much the dissipation of the operator L (see Remark 2.3 on that subject).

In the sequel, we also derive both hypocoercive and hypoelliptic estimates on solutions. Some possibly growing-in-time uniform energy estimates are sufficient to prove existence and uniqueness of solutions to (1.3). The refined hypocoercive and hypoelliptic estimates will be our main tool to prove existence and uniqueness of an invariant measure for problem (1.3). Therefore let us say a few words about the theory of hypocoercivity as introduced by Villani [5] in a simple context. It is particularly well-suited to providing rates of convergence to equilibrium of solutions to kinetic collisional models. For instance, consider the following class of kinetic models

$$\partial_t f + v \cdot \nabla_x f = Qf, \tag{1.4}$$

where Q is a linear collisional operator which acts on the velocity variable only, and choose some weighted-L^2 space H_v such that Q is symmetric on $L^2_x \otimes H_v$. Also suppose that, denoting by Π_{loc} the orthogonal projection on $\ker(Q)$, the following (local-in-space) weak coercivity assumption holds

$$\langle Qh, h\rangle \leq -c\|h - \Pi_{\text{loc}} h\|^2$$

for some $c > 0$. This implies that Q has a spectral gap when considered as acting on H_v, that is on functions homogeneous in space. The class of operators we have just introduced includes, among others, the cases of linearized Boltzmann, classical relaxation, Landau and Fokker-Planck equations. Note that while the global steady states of these models do belong to $\ker(Q)$, the foregoing kernel is not reduced to Maxwellians so that the above weak coercivity fails to yield convergence to equilibrium. Introducing the global projection $\bar{\Pi}$ on $\ker(-v\cdot\nabla_x + Q)$ defined by

$$\bar{\Pi} h = \int_{\mathbb{T}^N} \Pi_{\text{loc}} h(x, \cdot)\, dx.$$

we first remark that, if f is a solution to Equation (1.4), $\bar\Pi f(t) = \bar\Pi f(0)$ is independent of time. Then the piece of information that stems from hypocoercivity theory is the exponential damping of the solution f to equilibrium $\bar\Pi f(0)$:

$$\|f(t) - \bar\Pi f(0)\|_{\mathcal{H}} \leq K e^{-\tau t}, \quad t \geq 0,$$

in some Sobolev space $\mathcal{H}$ built on $L^2_x \otimes H_v$. The key-point is that (local-in-velocity) weak coercivity estimates afforded by commutators of $-v \cdot \nabla_x$ and Q may be incorporated in an energy estimate so as to ensure a full control of $\|f(t) - \bar\Pi f(t)\|_{\mathcal{H}}$. We refer the reader to the memoir of Villani [5] and references therein and also to the paper of Mouhot and Neumann [3] where the hypocoercivity is used to study the convergence to equilibrium for many kinetic models including Fokker-Planck equations. Our approach to hypoellipticity is global and mimic hypocoercive estimates as in [5].

In the case of the deterministic Fokker-Planck equation (1.3) where $\lambda = 0$, the kernel of $-v \cdot \nabla_x + L$ is spanned by the function $\mathcal{M}^{\frac{1}{2}}$ and

$$\bar\Pi g = \rho_\infty(g) \mathcal{M}^{\frac{1}{2}},$$

where $\rho_\infty(g) := \iint g(t)\mathcal{M}^{\frac{1}{2}} \, \mathrm{d}x \, \mathrm{d}v = \iint g(0)\mathcal{M}^{\frac{1}{2}} \, \mathrm{d}x \, \mathrm{d}v$ (this quantity being time independent). And one can prove (see [3, Section 5.3]) an exponential damping for the quantity $g(t) - \rho_\infty(g)\mathcal{M}^{\frac{1}{2}}$ in a weighted $H^1(\mathbb{T}^N \times \mathbb{R}^N)$ norm.

In the present paper, we prove hypocoercive estimates on the Fokker-Planck model (1.3) which has been perturbed by a random force. To handle the stochastic term we need to incorporate accommodation of corrections from Itô formula in the roadmap of the proof of Mouhot and Neumann [3]. By doing so we achieve the following hypocoercitive estimate:

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq \; Ce^{-ct}\mathbb{E}\|g_{\mathrm{in}}\|^2_{L^2_{\nabla,D}} + K\mathbb{E}|\rho_\infty(g)|^2, \quad t \geq 0, \tag{1.5}$$

where $L^2_{\nabla,D}$ is a suitable weighted version of $H^1(\mathbb{T}^N \times \mathbb{R}^N)$ Sobolev space (see below (2.5) for the precise definition). In particular, any two solutions of the problem (1.3) g_1 and g_2 with respective initial conditions $g_{\mathrm{in},1}$ and $g_{\mathrm{in},2}$ such that $\rho_\infty(g_{\mathrm{in},1}) = \rho_\infty(g_{\mathrm{in},1})$ meet exponentially fast. A priori the latter is only guaranteed when g_{in} belongs to $L^2_{\nabla,D}$. However following the same lines we also prove a hypoelliptic regularizing effect showing that the flow instantaneously send $L^2_{x,v}$ to $L^2_{\nabla,D}$.

Concerning the proof of existence, uniqueness and mixing of the invariant measure for problem (1.3), we make the most of both hypoellipticity and hypocoercivity. Indeed existence follows from compactness of time-averages that stems from compact embedding of $L^2_{\nabla,D}$ in $L^2_{x,v}$, instantaneous regularization and uniform-in-time bounds in $L^2_{\nabla,D}$, while mixing follows from exponential convergence of stochastic trajectories.

2. Existence and uniqueness of solutions

2.1. **Preliminaries.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which is supposed to be right continuous and such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. We study the following stochastic equation in $\mathbb{T}^N \times \mathbb{R}^N$

$$\begin{cases} \mathrm{d}g \; + \; v \cdot \nabla_x g \; \mathrm{d}t \; + \; \lambda \left(\nabla_v - \dfrac{v}{2}\right) g \; \odot \mathrm{d}W_t \; = \; Lg \; \mathrm{d}t, \\ g(0) = g_{\mathrm{in}}. \end{cases} \tag{2.1}$$

Note that, writing the Stratonovich correction explicitly, the first equation then reads in Itô form

$$\begin{aligned} &\mathrm{d}g + v\cdot\nabla_x g\ \mathrm{d}t + \lambda\left(\nabla_v - \frac{v}{2}\right) g\cdot \mathrm{d}W_t - Lg\ \mathrm{d}t \\ &= \frac{\lambda^2}{2}\sum_j F_j\cdot\left(\nabla_v - \frac{v}{2}\right)\left(F_j\cdot\left(\nabla_v - \frac{v}{2}\right) g\right)\mathrm{d}t. \end{aligned} \tag{2.2}$$

Functional Spaces. In the following, we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively the scalar product and the norm of $L^2_{x,v} := L^2(\mathbb{T}^N\times\mathbb{R}^N)$. For any Hilbert space H and any $T>0$, we denote by $C_w([0,T],H)$ the space of functions on $[0,T]$ with values in H that are continuous for the weak topology of H. Let us introduce the differential operators

$$D = \nabla v + \frac{v}{2},\quad D^* = -\nabla v + \frac{v}{2},$$

where D^* is the formal adjoint of D component-wise, *i.e.* $D_k^* = (D_k)^*$, $k = 1,...,N$. Note that, for f sufficiently smooth and localized,

$$\|Df\|^2 \ = \ \|\nabla_v f\|^2 + \frac{1}{4}\|vf\|^2 \ - \ \frac{N}{2}\|f\|^2 \tag{2.3}$$

and

$$\|D^*f\|^2 \ = \ \|\nabla_v f\|^2 + \frac{1}{4}\|vf\|^2 \ + \ \frac{N}{2}\|f\|^2. \tag{2.4}$$

We introduce the space

$$L^2_D = \{f\in L^2(\mathbb{R}^N); Df\in L^2(\mathbb{R}^N)\} = \{f\in L^2(\mathbb{R}^N); D^*f\in L^2(\mathbb{R}^N)\}$$

and then define the spaces

$$L^2_{x,D} = L^2(\mathbb{T}^N; L^2_D),\quad L^2_{\nabla,D} = \{f\in L^2_{x,D}; \nabla_x f\in L^2_{x,v}\}, \tag{2.5}$$

equipped respectively with norms

$$\|f\|^2_{L^2_{x,D}} = \|D^*f\|^2,\quad \|f\|^2_{L^2_{\nabla,D}} = \|D^*f\|^2 + \|\nabla_x f\|^2.$$

Fokker-Planck Operator. For the sake of writing convenience, we define the transport operator $A = v\cdot\nabla_x$ which is skew-adjoint, that is which satisfies $A^* = -A$. Concerning the Fokker-Planck operator L, we gather hereafter some of its properties. First, we recall the expression

$$Lf \ = \ \Delta_v f \ + \ \left(\frac{N}{2} - \frac{|v|^2}{4}\right) f.$$

Alternatively L is also given by

$$Lf \ = \ -\sum_k D_k^* D_k f \ = \ Nf - \sum_k D_k D_k^* f,$$

which hereafter we denote $L = -D^*D = N\mathrm{Id} - DD^*$ for short. Note in particular that this implies the following dissipative bound

$$-\langle f, Lf\rangle \ = \ \|Df\|^2. \tag{2.6}$$

The formal adjoint of $\mathcal{Q}$ on $L^2(\mathbb{R}^N)$ is

$$\mathcal{Q}^*\colon f\mapsto \Delta_v f - v\cdot\nabla_v f.$$

The operator $\mathcal{Q}^*$ is self-adjoint on $L^2(\mathbb{R}^N, \gamma)$, where γ is the Gaussian measure with density $\mathcal{M}$ with respect to the Lebesgue measure on $\mathbb{R}^N$. The Hermite Polynomials

$$H_j(v) = \frac{(-1)^{|j|}}{\sqrt{j!}} \mathcal{M}^{-1} \partial_v^j(\mathcal{M}), \tag{2.7}$$

where

$$|j| = j_1 + \ldots + j_N, \quad j! = j_1! \cdots j_N!, \quad \partial_v^j = \partial_{v_1}^{j_1} \cdots \partial_{v_N}^{j_N},$$

form a Hilbertian basis of $L^2(\mathbb{R}^N, \gamma)$ of eigenvectors of $\mathcal{Q}^*$:

$$\mathcal{Q}^* H_j = -|j| H_j.$$

The operator L is related to the operator $\mathcal{Q}^*$ by the formula $Lf = M\mathcal{Q}^*(M^{-1}f)$. It follows that, setting $q_j = MH_j$, we obtain a Hilbertian basis of $L^2(\mathbb{R}^N)$ constituted of eigenvectors of L associated with eigenvalues $-|j|$. There is a compact expression of q_j: using the formula $\partial_v^j(Mf) = (-1)^{|j|} M \, [D^*]^j \, f$ (which can be proved by recursion on $|j|$), and the definition (2.7) of the Hermite Polynomial, we obtain

$$q_j = \frac{1}{\sqrt{j!}} \, [D^*]^j \, M. \tag{2.8}$$

The formula (2.8) gives in particular

$$D_k^* q_j = c_{k,j} q_{j+e_k}, \quad D_k q_j = d_{k,j} \mathbf{1}_{j_k > 0} q_{j-e_k}, \tag{2.9}$$

for some given coefficients $c_{k,j}$, $d_{k,j}$. The formula for $D_k q_j$ is obtained by computing the l-th coefficient $\langle D_k q_j, q_l \rangle = \langle q_j, D_k^* q_l \rangle$.

Eigenspaces. Let $(p_i)_{i \in \mathbb{Z}^N}$ denote the standard trigonometric Hilbertian basis of $L^2(\mathbb{T}^N)$ — in particular it is formed by normalized eigenfunctions for the Laplacian $-\Delta_x$ — and recall that $(q_j)_{j \in \mathbb{N}^N}$ is the spectral Hilbertian basis for the Fokker-Planck operator L in $L^2(\mathbb{R}^N)$ introduced above. We define the Hilbertian basis $(e_{k,l})_{(k,l) \in \mathbb{Z}^N \times \mathbb{N}^N}$ of $L^2_{x,v}$ by

$$e_{k,l}(x, v) := p_k \otimes q_l(x, v) = p_k(x) q_l(v), \quad (k, l) \in \mathbb{Z}^N \times \mathbb{N}^N, \; x \in \mathbb{T}^N, \; v \in \mathbb{R}^N.$$

For any $(k_0, l_0) \in (\mathbb{N} \cup \{\infty\})^2$, we set

$$E_{k_0,l_0} \; := \; \text{Closure}_{L^2_{x,v}} \big(\text{Span} \; \{ \; e_{k,l} \; ; |k| \leq k_0 \text{ and } |l| \leq l_0 \; \} \big)$$

and introduce Π_{k_0,l_0} the $L^2_{x,v}$ orthogonal projection on E_{k_0,l_0}. When $k_0 = l_0$, we simplify notation to E_{k_0} and Π_{k_0}. In particular $\bar{\Pi} = \Pi_{0,0}$ and $\mathrm{Id} = \Pi_{\infty,\infty}$. By (2.9), we have the commutation rules

$$\Pi_m D^* = D^* \Pi_{m,m-1}, \quad D \Pi_m = \Pi_{m,m-1} D \tag{2.10}$$

for all $m \geq 1$ (these identities will be used to derive the hypocoercive estimates on the approximate Galerkin solution to (2.1)).

Let us also introduce the orthogonal projector $\Pi_{\text{loc}} = \Pi_{\infty,0}$ on $L^2_x \otimes \text{Span}\{q_0\}$:

$$\Pi_{\text{loc}}(f)(x, v) \; = \; \langle \mathcal{M}^{\frac{1}{2}}, f(x, \cdot) \rangle_{L^2_v(\mathbb{R}^N)} \, \mathcal{M}^{\frac{1}{2}}(v), \quad \Pi_{\text{loc}}^\perp = I - \Pi_{\text{loc}}.$$

Then, we have

$$-\langle f, Lf \rangle \geq \|\Pi_{\text{loc}}^\perp f\|^2. \tag{2.11}$$

Using (2.6), we then deduce from (2.11) that

$$\|f\|^2 \leq \|\Pi_{\text{loc}} f\|^2 + \|Df\|^2. \tag{2.12}$$

Finally, in the sequel, we denote by $\{T,T'\} := TT' - T'T$ the commutator of two operators T and T'. We point out that one readily shows the following algebraic identities

$$\{D,A\} = \nabla_x, \qquad \{D,D^*\} = N\mathrm{Id},$$

and stress that the former identity is the cornerstone of both our hypocoercive and hypoelliptic estimates.

We are now ready to state our main result concerning existence and uniqueness of solutions to problem (2.1). The question of qualitative properties of solutions and existence and uniqueness of invariant measures will be studied later.

Theorem 2.1. *Suppose that hypothesis (1.2) holds and let $g_{\rm in} \in L^2(\Omega; L^2_{x,v})$. For any $\lambda < 1$, there exists a unique adapted process $\{g(t),\, t \geq 0\}$ on $L^2(\Omega; L^2_{x,v})$ which satisfies:*

- (*i*) *for all $T > 0$, $g \in C_w([0,T]; L^2(\Omega; L^2_{x,v}))$ and $Dg \in L^2(\Omega \times (0,T); L^2_{x,v})$;*
- (*ii*) *$g(0) = g_{\rm in}$;*
- (*iii*) *for all $m \in \mathbb{N}$, for all φ in E_m and all $t \geq 0$,*

$$\begin{aligned}\langle g(t), \varphi\rangle \;&=\; \langle g_{\rm in}, \varphi\rangle + \int_0^t \langle g(s), v\cdot\nabla_x\varphi\rangle \mathrm{d}s + \lambda \sum_{j\geq 0}\int_0^t \langle g(s), F_j\cdot D\varphi\rangle\, d\beta_j(s) \\ &\quad + \int_0^t \langle g(s), L\varphi\rangle \mathrm{d}s + \frac{\lambda^2}{2}\sum_{j\geq 0}\int_0^t \Big\langle g(s), (F_j\cdot D)^2\varphi\Big\rangle \mathrm{d}s, \quad a.s.\end{aligned} \tag{2.13}$$

Moreover for this solution the quantity $\rho_\infty(g) := \iint g\mathcal{M}^{\frac12}$ is a.s. constant in time.

Note that every $\varphi \in E_m$ is smooth in x, v and exponentially decreasing at infinity in v. In particular, all terms in (2.13) makes sense.

To prove the existence part of Theorem 2.1, we use a Galerkin projection method, that is first we project Equation (2.1) onto the finite dimensional space spanned by some finite subset of vectors from a Hilbertian basis of $L^2_{x,v}$ and look for a solution valued in this finite-dimensional subspace, then we pass to the limit when this finite subset increases up to the whole Hilbertian basis.

2.2. **The Galerkin scheme.** Looking for an approximate solution $g_m : [0,T]\times\Omega \to E_m$ to (2.1) we prove the following result.

Proposition 2.2. *Suppose that hypothesis (1.2) holds and let $g_{\rm in} \in L^2(\Omega; L^2_{x,v})$. For all $m \geq 0$ and any $T > 0$, there exists a unique adapted process $g_m \in C(0,T; L^2(\Omega; E_m))$ satisfying, for all $t \in [0,T]$, for all $\varphi \in E_m$,*

$$\begin{aligned}\langle g_m(t), \varphi\rangle \;&=\; \langle g_{\rm in}, \varphi\rangle + \int_0^t \langle g(s), v\cdot\nabla_x\varphi\rangle \mathrm{d}s + \lambda \sum_{j\geq 0}\int_0^t \langle g(s), F_j\cdot D\varphi\rangle\, d\beta_j(s) \\ &\quad + \int_0^t \langle g(s), L\varphi\rangle \mathrm{d}s + \frac{\lambda^2}{2}\sum_{j\geq 0}\int_0^t \Big\langle g(s), (F_j\cdot D)^2\varphi\Big\rangle \mathrm{d}s, \quad a.s.\end{aligned} \tag{2.14}$$

Moreover, if $\lambda < 1$, then

$$\frac12 \max_{t\in[0,T]} e^{-2N\lambda^2 t}\mathbb{E}\|g_m(t)\|^2 \;+\; (1-\lambda^2)\int_0^T e^{-2N\lambda^2 t}\mathbb{E}\|D\, g_m(t)\|^2 \mathrm{d}t \;\leq\; \frac12\mathbb{E}\|g_{\rm in}\|^2\,. \tag{2.15}$$

Proof. For $g_m \in C(0,T;L^2(\Omega;E_m))$, Equations (2.14) are equivalently written — in terms of the coefficients $d_{k,l} = \langle g_m, e_{k,l}\rangle$, $|k| \le m$ and $|l| \le m$, of g_m — as a finite-dimensional Itô system with globally Lipschitz coefficients. It follows then from standard arguments that there exists a unique adapted and continuous process $g_m \in C(0,T;L^2(\Omega;E_m))$ satisfying (2.14). Assume now $\lambda < 1$. To derive the uniform bound we multiply (2.14) by $d_{k,l}$ and sum over k and l to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|g_m\|^2 + \mathbb{E}\|Dg_m\|^2 &= \lambda^2\,\frac{1}{2}\mathbb{E}\sum_j \langle (F_j\cdot D^*)^2 g_m, g_m\rangle + \langle (F_j\cdot D^*)g, (F_j\cdot D^*)g_m\rangle \\ &= \lambda^2\,\mathbb{E}\sum_j \langle (F_j\cdot D^*)g_m, (F_j\cdot \frac{D+D^*}{2})g_m\rangle \\ &\le \lambda^2\,\mathbb{E}\|D^*g_m\|^2 \ \le\ \lambda^2\,\mathbb{E}\|Dg_m\|^2 + N\lambda^2\mathbb{E}\|g_m\|^2, \end{aligned} \tag{2.16}$$

from which the bound follows since (2.14) implies $g_m(0) = \Pi_m g_{\mathrm{in}}$ hence $\mathbb{E}\|g_m(0)\|^2 \le \mathbb{E}\|g_{\mathrm{in}}(0)\|^2$. Here above we have used both $\|Df\| \le \|D^*f\|$ and $\|D^*f\|^2 = \|Df\|^2 + N\|f\|^2$. □

Obviously, alternatively we may view g_m as belonging to $C(0,T;L^2(\Omega;L^2_{x,v}))$ and satisfying

$$\mathrm{d}g_m + \Pi_m(v\cdot\nabla_x g_m)\mathrm{d}t - \lambda\Pi_m(D^*g_m \odot \mathrm{d}W_t) = Lg_m\mathrm{d}t, \tag{2.17}$$

with initial condition

$$g_m(0) = \Pi_m g_{\mathrm{in}}.$$

This does imply that for any t, a.s. $\Pi_m g_m(t) = g_m(t)$ hence $g_m(t) \in E_m$.

2.3. **Proof of Theorem 2.1.**
In this section, we prove Theorem 2.1.

Existence. Let $T > 0$. We use estimate (2.15) to obtain uniform bounds on g_m in $L^\infty(0,T;L^2(\Omega;L^2_{x,v}))$ and on Dg_m in $L^2(\Omega\times(0,T);L^2_{x,v})$ by some quantities depending on N, T, λ and the norm $\mathbb{E}\|g_{\mathrm{in}}\|^2$. As a consequence, $(g_m)_m$ admits a subsequence (still denoted $(g_m)_m$) such that

$$g_m \rightharpoonup g \text{ in } L^2(\Omega\times(0,T);L^2_{x,v})$$

where g, $Dg \in L^2(\Omega\times(0,T);L^2_{x,v})$. From (2.14) and the uniform estimates on the approximate solutions g_m in $L^\infty(0,T;L^2(\Omega;L^2_{x,v}))$, we can deduce (using Ascoli's Theorem and a diagonal argument) that there is a further subsequence of $(g_m)_m$ converging to g in $C_w([0,T];L^2(\Omega;L^2_{x,v}))$. We now have all in hands to pass to the limit $m\to\infty$ in (2.14). We deduce the existence of a solution g satisfying the points (i), (ii) and (iii) of Theorem 2.1.

Uniqueness. If $g \in C_w([0,T];L^2(\Omega;L^2_{x,v}))$ is solution to (2.1) in the sense of (i), (ii) and (iii) of Theorem 2.1, then g satisfies the energy estimate

$$\frac{1}{2}\mathbb{E}\|g(t)\|^2 + \int_0^t \mathbb{E}\|Dg(s)\|^2 ds \le \lambda^2\int_0^t \mathbb{E}[\|Dg(s)\|^2 + N\|g(s)\|^2]ds + \frac{1}{2}\mathbb{E}\|g(0)\|^2, \quad t\ge 0. \tag{2.18}$$

Since $\lambda < 1$, (2.18) immediately gives, with Gronwall's lemma, that a solution with initial condition $g_{\mathrm{in}} \equiv 0$ is zero in $L^\infty(0,T;L^2(\Omega;L^2_{x,v}))$ for every $T > 0$. Hence the uniqueness by linearity of the problem. To prove (2.18), on the basis of (i), (ii) and (iii) of Theorem 2.1, we apply the weak formulation (2.13) with $\varphi = e_{k,l}$ and use Itô Formula. Note that the differential of $g\mapsto |\langle g, e_{k,l}\rangle|^2$ at g is

$$f \mapsto 2\mathrm{Re}\Big(\overline{\langle g, e_{k,l}\rangle}\langle f, e_{k,l}\rangle\Big).$$

Since $g(t)$ is real-valued and $e_{k,l}(x,v) = e^{-2\pi i k\cdot x} q_l(v)$ where $q_l(v)$ is real-valued, the term

$$\mathrm{Re}\Big(\overline{\langle g(t), e_{k,l}\rangle}\langle g(t), v\cdot\nabla_x e_{k,l}\rangle\Big)$$

vanishes and we obtain

$$\begin{aligned}\frac{1}{2}\mathbb{E}|\langle g(t), e_{k,l}\rangle|^2 =& \frac{1}{2}\mathbb{E}|\langle g(0), e_{k,l}\rangle|^2 - \mathbb{E}\int_0^t |l||\langle g(s), e_{k,l}\rangle|^2 ds \\ &+ \frac{\lambda^2}{2}\sum_j \mathrm{Re}\,\mathbb{E}\int_0^t \Big[\overline{\langle g(s), e_{k,l}\rangle}\langle g(s), (F_j\cdot D)^2 e_{k,l}\rangle + |\langle g(s), (F_j\cdot D)e_{k,l}\rangle|^2\Big] ds.\end{aligned}$$

We sum the result over k, l and use Property (i) of Theorem 2.1 and Bessel Identity to obtain

$$\frac{1}{2}\mathbb{E}|\langle g(t), e_{k,l}\rangle|^2 + \int_0^t \mathbb{E}\|Dg(s)\|^2 ds = \frac{1}{2}\mathbb{E}|\langle g(0), e_{k,l}\rangle|^2 + \lambda^2\sum_j \mathbb{E}\int_0^t \|(F_j\cdot D)g(s)\|^2 ds.$$

The estimate (2.18) then follows from Hypothesis (1.2) on the size of the coefficients of the noise.

Properties of the solution g. The fact that the quantity $\rho_\infty(g)$ is constant in time follows from taking $(k,l) = (0,0)$ in (2.14) and passing to the limit $m\to\infty$. $\square$

Remark 2.3. *With essentially the same proof one may obtain existence and uniqueness of solutions to the original formulation* (1.1) *when $f_{\mathrm{in}} \in L^2(\Omega; L^2_{x,v})$* without *any restriction on λ since contribution of the noise to the corresponding L^2 estimate vanishes in the underlying estimate. Yet some piece of information concerning extra localization property in the v variable of f_{in}, here expressed as $g_{\mathrm{in}} := M^{-1}f_{\mathrm{in}} \in L^2(\Omega; L^2_{x,v})$, is indeed required to derive decay rates even for the evolution generated by $\mathcal{Q}$ on (space) homogeneous functions ; see e.g.* [2, Appendix A]. *The constraint $\lambda < 1$ then arises to ensure that the localization property is not altered by the stochastic force term of Equation* (1.1). *See also Remark 4.3 on that size condition.*

3. Regularization and decay

We prove now extra properties for solutions provided by Theorem 2.1 summarized in the following theorem.

Theorem 3.1. *Suppose that hypothesis* (1.2) *holds. There exists $0 < \lambda_0(N) < 1$ such that, for all $\lambda < \lambda_0$ and any $g_{\mathrm{in}} \in L^2(\Omega; L^2_{x,v})$, the solution g given by Theorem 2.1 satisfies the following properties. The solution g gains regularity instantaneously : for any $t_0 > 0$, there exists a constant $C(N, t_0) > 0$ such that*

$$\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} \leq C\mathbb{E}\|g_{in}\|^2. \tag{3.1}$$

Moreover, for any $t_0 > 0$, there exist positive constants c, C and K depending on N only such that g satisfies, for $t \geq t_0$, the bound

$$\begin{aligned}\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \;&+\; c\,\mathbb{E}\int_{t_0}^t \|g(s)\|^2_{L^2_{\nabla,D}} + \|D\nabla_x g(s)\|^2 + \|D^2 g(s)\|^2 \mathrm{d}s \\ &\leq\; C\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} + C\mathbb{E}|\rho_\infty|^2(t-t_0),\end{aligned} \tag{3.2}$$

and the hypocoercive estimate

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq\; Ce^{-c(t-t_0)}\mathbb{E}\|g(t_0)\|^2_{L^2_{\nabla,D}} + K\mathbb{E}|\rho_\infty(g)|^2. \tag{3.3}$$

3.1. **Termwise estimates.** In this subsection, we derive some estimates on various functionals of the approximate solutions $(g_m)_m$. Next, we shall combine these termwise bounds to deduce hypocoercive estimates (see Section 3.2) and follow a similar strategy to obtain regularization properties through hypoelliptic estimates (see Section 3.3).

3.1.1. *Heuristics.* Our aim will be to evaluate $\mathbb{E}\Phi(g)$ where Φ is a quadratic functional of the form

$$\Phi(g) = \langle Sg, Tg \rangle,$$

where S and T are operators in the variables x or v of order at most one. In particular, S and T are *linear*. The rigorous procedure that we follow hereafter is to bound $\mathbb{E}\Phi(g_m)$ and pass to the limit, since all computations are readily fully justified when applied to the finite-dimensional system satisfied by g_m.

However, for exposition purpose, proceeding in a formal way, we first explain the spirit of our computations on Equation (2.1) satisfied by g. Apply S to (2.1) and then test against Tg, and do the same with the roles of S and T exchanged, to obtain

$$d\Phi(g) \;=\; -\langle SAg, Tg\rangle dt \;+\; \lambda \sum_j \langle S(F_j \cdot D^*)g, Tg\rangle \circ d\beta_j(t) \;+\; \langle SLg, Tg\rangle dt \;+\; \mathsf{sym}, \tag{3.4}$$

where by "$B(S,T)+\mathsf{sym}$" in the right-hand side of (3.4), we mean $B(S,T)+B(T,S)$. Switching to Itô form and taking expectation in (3.4) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\Phi(g) = -\mathbb{E}\langle SAg, Tg\rangle + \mathbb{E}\langle SLg, Tg\rangle + \frac{\lambda^2}{2}\mathbb{E}\mathcal{N}_{S,T}(g) + \mathsf{sym}. \tag{3.5}$$

where we have introduced the piece of notation

$$\mathcal{N}_{S,T}(g) := \sum_j \langle S(F_j \cdot D^*)^2 g, Tg\rangle + \langle S(F_j \cdot D^*)g, T(F_j \cdot D^*)g\rangle.$$

Note also, in the case $S = T$, that, by (2.6),

$$\mathbb{E}\langle SLg, Sg\rangle = \mathbb{E}\langle LSg, Sg\rangle + \mathbb{E}\langle \{S, L\}g, Sg\rangle = -\mathbb{E}\|DSg\|^2 + \mathbb{E}\langle \{S, L\}g, Sg\rangle, \tag{3.6}$$

thus, modulo a commutator, the term $\mathbb{E}\langle SLg, Sg\rangle$ in (3.5) provides the part $-\mathbb{E}\|DSg\|^2$ whose contribution helps to set up our hypocoercive estimates. In contrast control on space derivatives is gained by examining the case $S = \nabla_x$, $T = D$ and noticing that

$$-\mathbb{E}\langle DAg, \nabla_x g\rangle \;=\; -\mathbb{E}\langle \{D, A\}g, \nabla_x g\rangle \;-\; \mathbb{E}\langle ADg, \nabla_x g\rangle \;=\; -\mathbb{E}\|\nabla_x g\|^2 \;-\; \mathbb{E}\langle ADg, \nabla_x g\rangle$$

provides the missing $\mathbb{E}\|\nabla_x g\|^2$.

To proceed with the actual proof we modify the definition of $\mathcal{N}_{S,T}$ to

$$\mathcal{N}^{(m)}_{S,T}(g) := \sum_j \langle S(\Pi_m(F_j \cdot D^*))^2 g, Tg\rangle + \langle S\Pi_m(F_j \cdot D^*)g, T\Pi_m(F_j \cdot D^*)g\rangle$$

so as to accommodate the presence of a projector in (2.17).

3.1.2. *First estimate:* $\mathbb{E}\|g_m\|^2$. We have already showed along the proof of Proposition 2.2 that by taking $S = T = \mathrm{Id}$, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|g_m\|^2 + \mathbb{E}\|Dg_m\|^2 \leq \lambda^2 \mathbb{E}\|D^* g_m\|^2\,. \tag{3.7}$$

3.1.3. *Second estimate:* $\mathbb{E}\|\nabla_x g_m\|^2$. By choosing $S = T = \nabla_x$, we obtain, due to the fact that A is skew-symmetric,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2 \leq \frac{\lambda^2}{2}\mathbb{E}\mathcal{N}^{(m)}_{\nabla_x,\nabla_x}(g_m),$$

where $\mathcal{N}^{(m)}_{\nabla_x,\nabla_x}(g_m)$ is also written as

$$\begin{aligned}\frac{1}{2}\mathcal{N}^{(m)}_{\nabla_x,\nabla_x}(g_m) &= \sum_j \langle \Pi_m(F_j\cdot D^*)\nabla_x g_m, (F_j\cdot \tfrac{D+D^*}{2})\nabla_x g_m\rangle \\ &+ \sum_j \langle \Pi_m(\nabla_x(F_j)\cdot D^*)g_m, (F_j\cdot \tfrac{D+D^*}{2})\nabla_x g_m\rangle \\ &+ \sum_j \langle \Pi_m(F_j\cdot D^*)g_m, (\nabla_x(F_j)\cdot \tfrac{D+D^*}{2})\nabla_x g_m\rangle \\ &+ \frac{1}{2}\sum_j \|\Pi_m(\nabla_x(F_j)\cdot D^*)g_m\|^2 .\end{aligned}$$

As a result, using (2.10), we obtain

(3.8)
$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2 \\ &\quad\leq \frac{\lambda^2}{2}\mathbb{E}\big[\|D^*g_m\|^2 + (2\|D^*g_m\| + \|D^*\Pi_{m,m-1}\nabla_x g_m\|)\,(\|D\nabla_x g_m\| + \|D^*\Pi_{m,m-1}\nabla_x g_m\|)\big]\end{aligned}$$

3.1.4. *Third estimate:* $\mathbb{E}\|Dg_m\|^2$. Recalling $\{A, D\} = -\nabla_x$ and $\{D, L\} = -ND$, by choosing $S = T = D$, we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|Dg_m\|^2 = -\mathbb{E}\langle\nabla_x g_m, Dg_m\rangle - \mathbb{E}\|D^2 g_m\|^2 - N\mathbb{E}\|Dg_m\|^2 + \frac{\lambda^2}{2}\mathbb{E}\mathcal{N}^{(m)}_{D,D}(g_m).$$

Furthermore, we have

$$\begin{aligned}\mathcal{N}^{(m)}_{D,D}(g_m) &= \sum_j \|D\Pi_m(F_j\cdot D^*)g_m\|^2 \\ &+ \sum_j \langle (F_j\cdot D^*)\Pi_m(F_j\cdot D^*)g_m, D^*Dg_m\rangle\end{aligned}$$

and, by (2.10), $\langle\nabla_x g_m, Dg_m\rangle = \langle \Pi_{m,m-1}\nabla_x g_m, Dg_m\rangle$. It follows then (using some inequalities like $\|D^*Dh\| \leq \|(D^*)^2h\|$ and $\|\Pi h\| \leq \|h\|$ with $\Pi = \Pi_m$ or $\Pi = \Pi_{m,m-1}$) that

(3.9)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|Dg_m\|^2 + \mathbb{E}\|D^2g_m\|^2 \leq \mathbb{E}\|\Pi_{m,m-1}\nabla_x g_m\|\|Dg_m\| + \frac{\lambda^2}{2}\mathbb{E}\big[\|(D^*)^2g_m\|^2 + \|DD^*g_m\|^2\big].$$

3.1.5. *Fourth estimate:* $\mathbb{E}\langle\nabla_x g_m, Dg_m\rangle$. We apply (3.5) with $S = \nabla_x$ and $T = D$. It yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\langle\nabla_x g_m, Dg_m\rangle &= -\mathbb{E}\langle\nabla_x\Pi_m Ag_m, Dg_m\rangle - \mathbb{E}\langle D\Pi_m Ag_m, \nabla_x g_m\rangle \\ &+ \mathbb{E}\langle\nabla_x Lg_m, Dg_m\rangle + \mathbb{E}\langle DLg_m, \nabla_x g_m\rangle \\ &+ \frac{\lambda^2}{2}\mathbb{E}[\mathcal{N}^{(m)}_{\nabla_x,D}(g_m) + \mathcal{N}^{(m)}_{D,\nabla_x}(g_m)].\end{aligned}$$

For the total equation, *i.e.* when there is no projector Π_m, one would use

$$-\langle\nabla_x Ag, Dg\rangle - \langle DAg, \nabla_x g\rangle = -\|\nabla_x g\|^2. \tag{3.10}$$

A way to derive[2] (3.10) is to write

$$-\langle \nabla_x A g, Dg\rangle - \langle DAg, \nabla_x g\rangle = \langle (D^* - D)Ag, \nabla_x g\rangle$$

and to use the identity $A = (D + D^*) \cdot \nabla_x$. This gives

$$-\langle \nabla_x A g, Dg\rangle - \langle DAg, \nabla_x g\rangle = \langle \{D^*, D\}\nabla_x g \nabla_x g\rangle,$$

and one concludes by use of the identity $\{D^*, D\} = -\mathrm{Id}$. For the terms with projectors, the same kind of computations gives

$$-\langle \nabla_x \Pi_m A g_m, Dg_m\rangle - \langle D\Pi_m A g_m, \nabla_x g_m\rangle = \langle \{D^*\Pi_m, D\Pi_m\}\nabla_x g \nabla_x g\rangle.$$

By (2.10) and the identity $\Pi_m \Pi_{m,m-1} = \Pi_{m,m-1}$, it follows that

$$-\langle \nabla_x \Pi_m A g_m, Dg_m\rangle - \langle D\Pi_m A g_m, \nabla_x g_m\rangle = -\|\Pi_{m,m-1}\nabla_x g_m\|^2 + \langle D\nabla_x g_m, D(\Pi_m - \Pi_{m,m-1})\nabla_x g_m\rangle$$

Besides, identity $L = -D^*D = N\mathrm{Id} - DD^*$ provides

$$\begin{aligned}
\mathbb{E}\langle \nabla_x L g_m, Dg_m\rangle + \mathbb{E}\langle DLg_m, \nabla_x g_m\rangle &= -\mathbb{E}\langle D^*D\nabla_x g_m, Dg_m\rangle - \mathbb{E}\langle DD^*Dg_m, \nabla_x g_m\rangle \\
&= -\mathbb{E}\langle D\nabla_x g_m, D^2 g_m\rangle - \mathbb{E}\langle D^*DDg_m, \nabla_x g_m\rangle - N\mathbb{E}\langle Dg_m, \nabla_x g_m\rangle \\
&= -2\mathbb{E}\langle D\nabla_x g_m, D^2 g_m\rangle - N\mathbb{E}\langle Dg_m, \Pi_{m,m-1}\nabla_x g_m\rangle.
\end{aligned}$$

Concerning terms $\mathcal{N}^{(m)}_{\nabla_x, D}(g_m)$ and $\mathcal{N}^{(m)}_{D,\nabla_x}(g_m)$, we write them as

$$\begin{aligned}
\mathcal{N}^{(m)}_{\nabla_x, D}(g_m) + \mathcal{N}^{(m)}_{D,\nabla_x}(g_m) = &-\sum_j \langle (\Pi_m F_j \cdot D^*)^2 g_m, D \cdot \nabla_x g_m\rangle + \langle \nabla_x \Pi_m (F_j \cdot D^*) g_m, D\Pi_m (F_j \cdot D^*) g_m\rangle \\
&+ \sum_j \langle (\Pi_m F_j \cdot D^*)^2 g_m, D^* \cdot \nabla_x g_m\rangle + \langle D\Pi_m (F_j \cdot D^*) g_m, \nabla_x \Pi_m (F_j \cdot D^*) g_m\rangle
\end{aligned}$$

to bound them proceeding as before by

$$\|(D^*)^2 g_m\|\|D\nabla_x g_m\| + 2(\|D^*\Pi_{m,m-1}\nabla_x g_m\| + \|D^* g_m\|)\|DD^* g_m\| + \|(D^*)^2 g_m\|\|D^*\Pi_{m,m-1}\nabla_x g_m\|\,.$$

As a result, we finally obtain

(3.11)
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\langle \nabla_x g_m, Dg_m\rangle + \mathbb{E}\|\Pi_{m,m-1}\nabla_x g_m\|^2 & \\
\leq \mathbb{E}\|D\nabla_x g_m\|^2 &+ 2\mathbb{E}\|D\nabla_x g_m\|\|D^2 g_m\| + N\mathbb{E}\|Dg_m\|\|\Pi_{m,m-1}\nabla_x g_m\| \\
&+ \frac{\lambda^2}{2}\mathbb{E}\big[\|(D^*)^2 g_m\| \; (\|D\nabla_x g_m\| + \|D^*\Pi_{m,m-1}\nabla_x g_m\|) \\
&\qquad + 2\|DD^* g_m\| \; (\|D^*\Pi_{m,m-1}\nabla_x g_m\| + \|D^* g_m\|)\big].
\end{aligned}$$

3.1.6. *Closed form of the estimates.* In this section, we gather estimates (3.7), (3.8), (3.9) and (3.11) — derived above — in a closed form with respect to g_m, $\nabla_x g_m$, Dg_m, $D\nabla_x g_m$ and $D^2 g_m$. Note in particular that we need to replace all occurrences of the operator D^* using formula

$$\|D^* f\|^2 = \|Df\|^2 + N\|f\|^2 \qquad (3.12)$$

proved by (2.3) and (2.4). In what follows C denotes a positive constant that depends only on the dimension N.

[2] Obviously one may also use concrete definitions of differential operators but the abstract way shown here has a clearer counterpart at the Galerkin level.

First estimate. The first bound (3.7) can now be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|g_m\|^2 + \mathbb{E}\|Dg_m\|^2 \le C\,\lambda^2\mathbb{E}[\|Dg_m\|^2 + \|g_m\|^2]. \tag{3.13}$$

Second estimate. The second one (3.8) becomes

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|\nabla_x g_m\|^2 + \mathbb{E}\|D\nabla_x g_m\|^2\\ &\quad\le C\,\lambda^2\mathbb{E}\big[\|g_m\|^2 + \|Dg_m\|^2 + \|\Pi_{m,m-1}\nabla_x g_m\|^2 + \|D\nabla_x g_m\|^2\big].\end{aligned} \tag{3.14}$$

Third estimate. Concerning the third one (3.9), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\|Dg_m\|^2 + \mathbb{E}\|D^2 g_m\|^2 \le \mathbb{E}\|Dg_m\|\|\Pi_{m,m-1}\nabla_x g_m\| + C\,\lambda^2\mathbb{E}\big[\|g_m\|^2 + \|Dg_m\|^2 + \|D^2 g_m\|^2\big]. \tag{3.15}$$

Fourth estimate. Finally, likewise, the fourth bound (3.11) writes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\langle\nabla_x g_m, Dg_m\rangle + \mathbb{E}\|\Pi_{m,m-1}\nabla_x g_m\|^2\\ &\le \mathbb{E}\|D\nabla_x g_m\|^2 + 2\mathbb{E}\|D\nabla_x g_m\|\|D^2 g_m\| + N\mathbb{E}\|Dg_m\|\|\Pi_{m,m-1}\nabla_x g_m\|\\ &\quad + C\,\lambda^2\mathbb{E}\big[\|g_m\|^2 + \|Dg_m\|^2 + \|\Pi_{m,m-1}\nabla_x g_m\|^2 + \|D\nabla_x g_m\|^2 + \|D^2\nabla_x g_m\|^2\big].\end{aligned} \tag{3.16}$$

3.2. **Hypocoercive estimates.** In this section, we derive hypocoercive estimates (3.2) and (3.3). Without loss of generality we assume $t_0 = 0$ and $g_{\mathrm{in}} \in L^2(\Omega; L^2_{\nabla,D})$. Our strategy is to prove uniform bounds on the approximate solutions $(g_m)_m$ and pass to the limit.

3.2.1. *Balance of the estimates.* To prove an exponential damping we shall combine (3.13), (3.14), (3.15) and (3.16) of Section 3.1 to identify a functional bounded by its own dissipation. The first step is to explain how to bound $\|g_m\|$. Mark that when $m \ge 1$

$$\begin{aligned}\|g_m\|^2 &= \sum_{\substack{|k|\le m\\ |l|\le m}} |\langle e_{k,l}, g_m\rangle|^2\\ &\le \sum_{\substack{|k|\le m\\ 0<|l|\le m}} |l|^2\,|\langle e_{k,l}, g_m\rangle|^2 + \sum_{0<|k|\le m} (2\pi|k|)^2|\langle e_{k,0}, g_m\rangle|^2 + |\langle e_{0,0}, g_m\rangle|^2\\ &\le \|Dg_m\|^2 + \|\Pi_{m,m-1}\nabla_x g_m\|^2 + |\rho_\infty(g_m)|^2\,.\end{aligned} \tag{3.17}$$

Now we look for a suitable functional in the form

$$\mathcal{F}(g) = \|g\|^2 + \alpha\|\nabla_x g\|^2 + \beta\|Dg\|^2 + 2\gamma\langle\nabla_x g, Dg\rangle.$$

where α, β, γ are some positive coefficients. First we require $\gamma^2 < \alpha\beta$ so as to ensure

$$C_1\|g\|^2_{L^2_{\nabla,D}} \le \mathcal{F}(g) \le C_2\|g\|^2_{L^2_{\nabla,D}}, \tag{3.18}$$

for some positive constants C_1, C_2.

Now by adding (3.13), (3.14), (3.15) and (3.16), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m) \;+\; \mathbb{E}\Big[\|Dg_m\|^2+\alpha\|D\nabla_x g_m\|^2+\beta\|D^2 g_m\|^2+\gamma\|\Pi_{m,m-1}\nabla_x g_m\|^2\Big]$$
$$\leq (1+\alpha+\beta+\gamma)\,C\lambda^2\mathbb{E}|\rho_\infty(g_{\mathrm{in}})|^2$$
$$+\;(\beta+N\gamma)\mathbb{E}\|Dg_m\|\|\Pi_{m,m-1}\nabla_x g_m\|+\gamma\mathbb{E}\|D\nabla_x g_m\|^2+2\gamma\mathbb{E}\|D\nabla_x g_m\|\|D^2 g_m\|$$
$$+\;(1+\alpha+\beta+\gamma)\,2C\lambda^2\mathbb{E}\big[\|Dg_m\|^2\;+\;\|\Pi_{m,m-1}\nabla_x g_m\|^2\;+\;\|D\nabla_x g_m\|^2\;+\;\|D^2\nabla_x g_m\|^2\big]$$

from which follows

(3.19)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m) \;+\; K\mathbb{E}\Big[\|Dg_m\|^2+\|D\nabla_x g_m\|^2+\|D^2 g_m\|^2+\|\Pi_{m,m-1}\nabla_x g_m\|^2\Big]$$
$$\leq K'\lambda^2\mathbb{E}|\rho_\infty(g_{\mathrm{in}})|^2\;+\;K'\lambda^2\mathbb{E}\big[\|Dg_m\|^2+\|\Pi_{m,m-1}\nabla_x g_m\|^2+\|D\nabla_x g_m\|^2\;+\;\|D^2\nabla_x g_m\|^2\big]$$

for some positive K, K' depending only on N, α, β and γ, provided that $\gamma\leq\alpha/2$ and both $(\beta+N\gamma)/\sqrt{1\times\gamma}$ and $\gamma/\sqrt{\alpha\times\beta}$ are sufficiently small. The latter constraints may be satisfied jointly with $\gamma^2<\alpha\beta$ by setting[3] $\alpha=1$, $\beta=\gamma$ and choosing γ sufficiently small.

Having picked suitable parameters α, β, γ, we now require λ to be sufficiently small — in a way that depends only on N — to derive

(3.20)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m) \;+\; K''\mathbb{E}\Big[\|g_m\|^2+\|Dg_m\|^2+\|D\nabla_x g_m\|^2+\|D^2 g_m\|^2+\|\Pi_{m,m-1}\nabla_x g_m\|^2\Big]$$
$$\leq K'''\mathbb{E}|\rho_\infty(g_{\mathrm{in}})|^2$$

for some positive constants K'', K''' depending only on N.

3.2.2. *Exponential damping.* Integrating (3.20) from 0 to t and passing to the limit $m\to\infty$ yields (3.2) (for $t_0=0$).

To prove (3.3) we first stress that proceeding as in the proof of (3.17) gives

$$\|\nabla_x g_m\|^2\;\leq\;\|\Pi_{m,m-1}\nabla_x g_m\|^2\;+\;\|D\nabla_x g_m\|^2$$

and conclude then from (3.20) and (3.18) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\mathcal{F}(g_m)\;+\;c\,\mathbb{E}\mathcal{F}(g_m)\;\leq\;C\,\mathbb{E}|\rho_\infty(g_{\mathrm{in}})|^2$$

for some positive constants c and C. This yields

$$\forall t\geq 0\,,\quad \mathbb{E}\mathcal{F}(g_m)(t)\;\leq\;\mathbb{E}\mathcal{F}(g_{\mathrm{in}})e^{-2\,ct}\;+\;\frac{C}{2c}\,\mathbb{E}|\rho_\infty(g_{\mathrm{in}})|^2$$

through a multiplication by e^{-2ct} and an integration in time. Using again (3.18) and passing to the limit $m\to\infty$ achieves the proof of (3.3) (for $t_0=0$).

[3]There is of course no uniqueness in this choice. For instance setting $\alpha=1$, $\beta=\gamma^\theta$, any $\frac{1}{2}<\theta<2$ would work provided γ is small enough.

3.3. **Hypoelliptic estimates.** In this part, we conclude the proof of Theorem 3.1 by showing that the solution g to Equation (2.1) with initial condition $g_{\rm in}$ in $L^2(\Omega; L^2_{x,v})$ gains regularity instantaneously. Precisely, we prove the following result.

Proposition 3.2. *Let $t_0 > 0$. There exist positive constants λ^* and C such that for any $g_{\rm in} \in L^2(\Omega; L^2_{x,v})$ and $\lambda < \lambda^*$, the corresponding solution g satisfies for any $t \in (0, t_0]$*

$$\mathbb{E}\|g(t)\|^2 \leq C\mathbb{E}\|g_{in}\|^2, \quad \mathbb{E}\|Dg(t)\|^2 \leq \frac{C}{t}\mathbb{E}\|g_{in}\|^2, \quad \mathbb{E}\|\nabla_x g(t)\|^2 \leq \frac{C}{t^3}\mathbb{E}\|g_{in}\|^2. \tag{3.21}$$

By a simple approximation argument one may reduce the proof of the proposition to the proof of estimates (3.21) starting from $g_{\rm in} \in L^2_{\nabla,D}$. For writing convenience, we assume $t_0 = 1$, modifications to obtain the proof of the general case being mostly notational.

Though the proof of Proposition 3.2 has some similarities with the proof of exponential damping, constraints on functionals leading to global hypoelliptic estimates are a lot more stringent[4] and we have not been able to produce them entirely at the level of the Galerkin approximation. Instead we directly derive estimates on g by examining the equations satisfied by $\Pi_m g$ and passing to the limit $m \to \infty$ using the already established propagation of regularity. The key gain is that terms analogous to $\mathbb{E}\|D\nabla_x g_m\|^2$ in (3.16) that arises from failure of commutativity of Π_m and D^* disappear when applied to g in the limit $m \to \infty$ because $\{\Pi_m, D^*\} = -D^*(\Pi_m - \Pi_{m,m-1})$.

We introduce the family of functionals parametrized by $t \in [0, 1]$,

$$\mathcal{K}_t(g) := \|g\|^2 + at^3\|\nabla_x g\|^2 + bt\|Dg\|^2 + 2ct^2\langle \nabla_x g, Dg\rangle$$

where a, b and c are some positive constants to be chosen later on. By requiring $c^2 < ab$, we ensure

$$\|g\|^2 + C_1(t^3\|\nabla_x g\|^2 + t\|Dg\|^2) \leq \mathcal{K}_t(g) \leq \|g\|^2 + C_2(t^3\|\nabla_x g\|^2 + t\|Dg\|^2) \tag{3.22}$$

for some positive C_1, C_2. Proceeding as explained above we derive[5] for any $0 \leq t \leq 1$

$$\mathcal{K}_t(g(t)) \ + \ C\int_0^t (\mathbb{E}\|g(s)\|^2 + s^3\mathbb{E}\|D\nabla_x g(s)\|^2 + s\mathbb{E}\|D^2 g(s)\|^2 + s^2\|\nabla_x g(s)\|^2){\rm d}s \ \leq \ \mathcal{K}_0(g_{\rm in})$$

for some positive C, provided first that a, b and c are chosen such that both $(b+c)/\sqrt{1\times c}$ and $c/\sqrt{a \times b}$ are sufficiently small and then that λ is sufficiently small. As above constraints on a, b, c may be fulfilled by choosing $a = 1$, $b = c$ and c small enough. By appealing to (3.22) we achieve the proofs.

4. Invariant measure

In this section, we prove the following result about existence, uniqueness and mixing properties of the invariant measure to problem (2.1).

Theorem 4.1. *Suppose that hypothesis (1.2) is satisfied and $\lambda < \lambda_0$ where λ_0 is as in Theorem 3.1. Let $m \in \mathbb{R}$ and introduce the space*

$$X_m \ := \ \left\{ g \in L^2_{x,v}\,;\, \langle g, \mathcal{M}^{\frac{1}{2}}\rangle = m \right\}.$$

[4] This may be seen on the fact that in the strategy hereafter estimates should be compatible with chosen powers of t.

[5] The reader is referred to the treatment of a similar case in [5, Appendix A.21] for omitted details concerning algebraic manipulations.

Then the problem

$$(\mathrm{P}_m)\qquad \begin{cases} \mathrm{d}g \;+\; v\cdot\nabla_x g\;\mathrm{d}t \;+\; \lambda\left(\nabla_v - \dfrac{v}{2}\right) g \;\odot \mathrm{d}W_t \;=\; Lg\;\mathrm{d}t \\ g(0) = g_{in} \in X_m \end{cases}$$

admits a unique invariant measure μ_m *on* X_m*. Besides, there exists some constants* $C \geq 0$*,* $\kappa > 0$*, such that*

$$|\mathbb{E}\Psi(g(t)) - \langle \Psi, \mu_m\rangle| \leq Ce^{-\kappa t}\|g_{in}\|, \tag{4.1}$$

for every $\Psi \colon X_m \to \mathbb{R}$ *which is* 1*-Lipschitz continuous.*

Proof. We fix $m \in \mathbb{R}$ and assume $\lambda < \lambda_0$.

Proof of existence. Let $g_{\text{in}} \in L^2_{x,v}$ be a deterministic initial data in X_m. We consider the unique solution g to problem (P_m) given by Theorem 2.1. First of all, using regularizing bound (3.1) of Theorem 3.1, we deduce that there exists a positive constant C such that

$$\mathbb{E}\|g(1)\|^2_{L^2_{\nabla,D}} \leq C\mathbb{E}\|g_{\text{in}}\|^2. \tag{4.2}$$

We also recall damping estimate (3.3) of Theorem 2.1: for $t \geq 1$,

$$\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq\; Ce^{-c(t-1)}\mathbb{E}\|g(1)\|^2_{L^2_{\nabla,D}} + K\mathbb{E}|\rho_\infty(g)|^2.$$

It implies, with (4.2),

$$\sup_{t\geq 1}\mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}} \leq\; C\mathbb{E}\|g_{\text{in}}\|^2 + Km^2. \tag{4.3}$$

We introduce the family $(\mu_T)_{T>0}$ of probability measures on X_m defined by

$$\mu_T := \frac{1}{T}\int_1^{1+T} \mathscr{L}(g(t))\,\mathrm{d}t,$$

where $\mathscr{L}(g(t))$ denotes the law of $g(t)$, and show that the family $(\mu_T)_{T>0}$ is tight. Since the embedding $L^2_{\nabla,D} \subset L^2_{x,v}$ is compact, balls of radius $R > 0$

$$K_R := \{f \in X_m;\; \|f\|_{L^2_{\nabla,D}} \leq R\}$$

are compact in X_m. Furthermore, thanks to Markov's inequality and (4.3),

$$\begin{aligned} \mu_T(K_R^c) \;&=\; \frac{1}{T}\int_1^{1+T} \mathbb{P}(\|g(t)\|_{L^2_{\nabla,D}} > R)\,\mathrm{d}t \\ &\leq\; \frac{1}{TR^2}\int_1^{1+T} \mathbb{E}\|g(t)\|^2_{L^2_{\nabla,D}}\,\mathrm{d}t \\ &\leq\; \frac{1}{R^2}(C\mathbb{E}\|g_{\text{in}}\|^2 + Km^2). \end{aligned}$$

This readily implies tightness of $(\mu_T)_{T>0}$. By Prohorov's Theorem, we obtain that $(\mu_T)_{T>0}$ admits a subsequence (still denoted (μ_T)) such that μ_T converges to some probability measure μ on X_m as $T \to \infty$. Furthermore, it is classical to show that this limit measure μ is indeed an invariant measure for problem (P_m), see for instance [1, Proposition 11.3].

Proof of the mixing property. Let $g_{\text{in},1}$ and $g_{\text{in},2} \in X_w$ and denote by g_1 and g_2 the solutions to (P_m) with respective initial conditions $g_{\text{in},1}$ and $g_{\text{in},2}$. For $t \geq 0$ we set $r(t) := g_1(t) - g_2(t)$ and remark that r solves (P_0) on X_0. Combining again (3.2) and (3.3) and recalling that (2.4) yields $\frac{N}{2}\|f\|^2 \leq \|f\|^2_{L^2_{\nabla,D}}$, we deduce that there exists positive constants c and C such that, for $t \geq 1$,

$$\mathbb{E}\|r(t)\|^2 \leq Ce^{-c(t-1)}\mathbb{E}\|g_{\text{in},1} - g_{\text{in},2}\|^2. \tag{4.4}$$

Let $\Psi : X_m \to \mathbb{R}$ be 1-Lipschitz continuous, let $g_{\text{in}} \in X_m$ and $s > 0$. We apply (4.4) with $g_{\text{in},1} = g_{\text{in}}$ and $g_{\text{in},2} = g(s)$ to obtain

$$\begin{aligned}\left|\mathbb{E}\Psi(g(t)) - \frac{1}{T}\int_1^{T+1} \mathbb{E}\Psi(g(t+s))ds\right|^2 &\leq \frac{1}{T}\int_1^{T+1} \mathbb{E}\|g(t) - g(t+s)\|^2 ds \\ &\leq Ce^{-c(t-1)}\frac{1}{T}\int_1^{T+1} \mathbb{E}\|g_{\text{in}} - g(s)\|^2 ds.\end{aligned} \tag{4.5}$$

By (3.1), we have $\sup_{s\in[1,T]}\|g(s)\| \leq C\|g_{\text{in}}\|$ and we deduce from (4.5) (for possibly different values of the constants) that

$$\left|\mathbb{E}\Psi(g(t)) - \frac{1}{T}\int_1^{T+1} \mathbb{E}\Psi(g(t+s))ds\right|^2 \leq Ce^{-c(t-1)}\|g_{\text{in}}\|^2,$$

Taking the limit $[T \to +\infty]$ gives the mixing estimate (4.1). □

If $m = 0$, then μ_0 is the Dirac mass on the solution 0. There is another — nontrivial — case in which we can explicitly compute the invariant measure μ_m and in particular check that some smallness condition on λ is indeed necessary.

Proposition 4.2. *Assume $m \neq 0$. Assume that W_t is an N-dimensional Brownian motion, i.e. $F_j = 0$ for $j > N$ and $F_j = \text{cst}$, F_j being the j-th vector of the canonical basis of $\mathbb{R}^N$ for $j = 1, \dots, N$. Let $V^{\text{stat}}(t)$, normally distributed with variance 1, denote the stationary solution to the Langevin equation*

$$\mathrm{d}V(t) = -V(t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t.$$

Then μ_m is the law of the function $(x, v) \mapsto mf^{\text{stat}}(t, x, v)$, where

$$f^{\text{stat}}(t, x, v) = \mathcal{M}\left(v - \frac{\lambda}{\sqrt{2}}V^{\text{stat}}(t)\right), \tag{4.6}$$

and where $\mathcal{M}$ is the Gaussian on $\mathbb{R}^N$.

Proof. It is clear that f^{stat} defined by (4.6) is a stationary solution to (1.1) when W_t is an N-dimensional Brownian motion. Let us develop the proof here to show how f^{stat} arises in the resolution of (1.1) and that exponential convergence occurs. Let $t_0 \in \mathbb{R}$. Let $(\hat{B}_t)_{t\geq t_0}$ be an N-dimensional Brownian motion on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{P}})$. Let $E(t)$, $t \geq t_0$ be a given force term that we will not specify for the moment. The law $\hat{\mu}_t$ of the process $(X_t, V_t)_{t\geq t_0}$ defined by the stochastic differential system

$$\begin{cases}\mathrm{d}X(t) &= V(t)\mathrm{d}t, \\ \mathrm{d}V(t) &= -V(t)\mathrm{d}t - \lambda E(t)\mathrm{d}t + \sqrt{2}\mathrm{d}\hat{B}_t\end{cases} \tag{4.7}$$

satisfies the (forward Kolmogorov equation) $\partial_t\hat{\mu}_t + \text{div}_x(v\hat{\mu}_t) = \mathcal{Q}\hat{\mu}_t - \text{div}_v(\lambda E(t)\hat{\mu}_t)$. If, at $t > t_0$, $\hat{\mu}_t$ has a density f_t with respect to the Lebesgue measure on $\mathbb{T}^N \times \mathbb{R}^N$, then f_t solves the equation

$$\partial_t f_t + v \cdot \nabla_x f + \lambda E(t) \cdot \nabla_v f_t = \mathcal{Q}f_t, \quad t > t_0.$$

We can compute explicitly f_t thanks to this probabilistic interpretation (see [4] for an analogous result). Let us now take $E(t) = \partial_t(W_t - W_{t_0})$. This means that we consider now W_t as a Brownian motion on $\mathbb{R}$, obtained by gluing two independent Brownian motions, one being run backward, at time $t = 0$. Then $W_{t_0,t} := W_t - W_{t_0}$ is a Brownian motion started at t_0. Similarly, we will assume that $\hat{B}_t$ is actually $\hat{B}_{t_0,t} := \hat{B}_t - \hat{B}_{t_0}$ for a given Brownian motion defined on $\mathbb{R}$. One can solve (4.7) with initial condition (X_0, V_0) to obtain

$$V_{t_0}(t) = e^{-(t-t_0)}V_0 + \hat{V}_{t_0}(t) + \frac{\lambda}{\sqrt{2}}V^\sharp_{t_0}(t),$$

where

$$\hat{V}_{t_0}(t) = \sqrt{2}\int_{t_0}^t e^{-(t-s)}d\hat{B}_{t_0,s}, \quad V^\sharp_{t_0}(t) = \sqrt{2}\int_{t_0}^t e^{-(t-s)}dW_{t_0,s},$$

and

$$X_{t_0}(t) = X_0 + \int_{t_0}^t \bar{V}(s)ds = X_0 + (1 - e^{-(t-t_0)})V_0 + \hat{X}_{t_0}(t) + \frac{\lambda}{\sqrt{2}}X^\sharp_{t_0}(t),$$

where

$$\hat{X}_{t_0}(t) = \sqrt{2}\int_{t_0}^t (1 - e^{-(t-s)})d\hat{B}_{t_0,s}, \quad X^\sharp_{t_0}(t) = \sqrt{2}\int_{t_0}^t (1 - e^{-(t-s)})dW_{t_0,s},$$

and where (X_0, V_0) is $\hat{\Omega}$-random. With respect to the alea $\hat{\omega} \in \hat{\Omega}$, the process $(\hat{X}_{t_0}(t), \hat{V}_{t_0}(t))$ is a Gaussian process with covariance matrix

$$Q_{t_0,t} = \begin{pmatrix} \hat{\mathbb{E}}|\hat{X}_{t_0}(t)|^2 & \hat{\mathbb{E}}[\hat{X}_{t_0}(t)\hat{V}_{t_0}(t)] \\ \hat{\mathbb{E}}[\hat{X}_{t_0}(t)\hat{V}_{t_0}(t)] & \hat{\mathbb{E}}|\hat{V}_{t_0}(t)|^2 \end{pmatrix}.$$

More precisely, using the Itô Isometry, we compute $Q_{t_0,t} = Q_{0,t-t_0}$, where

$$Q_{0,t} = 2\begin{pmatrix} \int_0^t (1 - e^{-s})^2 ds & \int_0^t e^{-s}(1 - e^{-s})ds \\ \int_0^t e^{-s}(1 - e^{-s})ds & \int_0^t e^{-2s}ds \end{pmatrix} \otimes \mathrm{I}_N,$$

where I_N is the $N \times N$ identity matrix. It follows that, for $\varphi \in C_b(\mathbb{T}^N \times \mathbb{R}^N)$,

$$\hat{\mathbb{E}}\varphi(X_{t_0}(t), V_{t_0}(t)) = \iint \varphi(y, w)f_{t_0}(t, y, w)dydw,$$

where the density $f_{t_0}(t, y, w)$ is given by

$$f_{t_0}(t, y, w) = \hat{\mathbb{E}}\exp\left[-\frac{1}{2}\left\langle Q^{-1}_{t_0,t}\begin{pmatrix} Y_{t_0}(t, y, w) \\ W_{t_0}(t, y, w)\end{pmatrix}, \begin{pmatrix} Y_{t_0}(t, y, w) \\ W_{t_0}(t, y, w)\end{pmatrix}\right\rangle\right] \tag{4.8}$$

where

$$Y_{t_0}(t, y, w) = y - X_0 - (1 - e^{-(t-t_0)})V_0 - \frac{\lambda}{\sqrt{2}}X^\sharp_{t_0}(t),$$
$$W_{t_0}(t, y, w) = w - e^{-(t-t_0)}V_0 - \frac{\lambda}{\sqrt{2}}V^\sharp_{t_0}(t).$$

We compute

$$Q_{0,t} = \begin{pmatrix} 2t - 3 & 1 \\ 1 & 1 \end{pmatrix} \otimes \mathrm{I}_N + \mathcal{O}(e^{-t}), \quad Q^{-1}_{0,t} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{I}_N + \mathcal{O}(t^{-1}).$$

We have also, in probability, with exponential convergence,

$$\lim_{t_0\to-\infty} V^\sharp_{t_0}(t) = V^{\mathrm{stat}}(t).$$

By taking the limit $t_0 \to -\infty$ in (4.8), we obtain the stationary density f^{stat} defined by (4.6) and exponential convergence. □

Remark 4.3. *Note that the corresponding invariant solution $g^{\mathrm{stat}}(t)$ to (2.1) is $g^{\mathrm{stat}}(t) = \mathcal{M}^{-1/2} f^{\mathrm{stat}}(t)$. We compute*

$$\|g(t)\|^2_{L^2_v} = (2\pi)^{N/2} e^{\frac{\lambda^2}{2}|V^{\mathrm{stat}}(t)|^2}.$$

In particular, we have

$$\mathbb{E}\|g(t)\|^2_{L^2_v} = \int_{\mathbb{R}^N} e^{\frac{\lambda^2}{2}|w|^2-\frac{1}{2}|w|^2} dw.$$

This is finite if, and only if, $\lambda < 1$: we recover the necessity of this restriction on the size of the noise made in the statement of Theorem 2.1. Note however that, here, no further restriction of the type $\lambda < \lambda_0$ as in the statement of Theorem 4.1 is necessary to obtain an invariant measure with mixing properties.

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