Université Claude Bernard Lyon 1

# Asymptotic stability and modulation of periodic wavetrains

## General theory & applications to thin film flows

**Mémoire d'Habilitation à Diriger des Recherches**

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December 2013

# Foreword

The aim of the present Habilitation thesis is to describe the author's work since his Ph.D. defense. While the latter Ph.D. thesis was focused on asymptotic stability of some special self-similar structures in fluid dynamics, Oseen vortices, the present memoir is organized around a much more general issue, the stability of periodic wavetrains, or seldom of solitary waves enclosing a family of periodic waves (see Section 3.2), as solutions of systems of partial differential equations. The memoir ends with a short chapter, Chapter 5, introducing directions of future research including some long-term important problems.

Pieces of work discussed here are fruits of collaborations with Blake Barker (Indiana, USA), Sylvie Benzoni-Gavage (Lyon 1, France), Mathew Johnson (Kansas, USA), Pascal Noble (INSA Toulouse, France) and Kevin Zumbrun (Indiana, USA). Most of them are published (or submitted) [11, 14, 185, 12, 127, 128, 13, 125, 126, 20] but some are relevant advanced parts of work in progress [10, 21].

On the 9th of December 2013 material of the present Habilitation thesis was defended in front of

Sylvie Benzoni-Gavage (Lyon 1, France)
Thierry Gallay (Grenoble 1, France)
François Hamel (Aix-Marseille, France)
David Lannes (ÉNS, France)
Guy Métivier (Bordeaux 1, France)
Guido Schneider (Stuttgart, Germany)
and Denis Serre (ÉNS de Lyon, France)

in view of reports of François Hamel, David Lannes and Robert Pego (Carnegie Mellon, USA).

We warm the reader that arguments when given are rather sketched than detailed. Also, unwittingly the terse tone of the memoir might sometimes lead to awkward formulations. It goes without telling that any such expository flaw should be attributed to the author of the present memoir himself and not to any of his collaborators.

# Presented work list

*Warning:* Quotes along the text refer to the general bibliography. Corresponding designations are given here in square brackets.


1. **[14]** Blake Barker, Mathew A. Johnson, L. Miguel Rodrigues, and Kevin Zumbrun. Metastability of solitary roll wave solutions of the St. Venant equations with viscosity. *Phys. D*, 240(16):1289–1310, 2011.

2. **[11]** Blake Barker, Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Whitham averaged equations and modulational stability of periodic traveling waves of a hyperbolic-parabolic balance law. *Journées Équations aux dérivées partielles*, pages 1–24, 6 2010. Available as http://eudml.org/doc/116384.

3. **[185]** Pascal Noble and L. Miguel Rodrigues. Whitham's modulation equations and stability of periodic wave solutions of the generalized Kuramoto-Sivashinsky equations. *Indiana Univ. Math. J.*

4. **[12]** Blake Barker, Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Stability of periodic Kuramoto-Sivashinsky waves. *Appl. Math. Lett.*, 25(5):824–829, 2012.

5. **[13]** Blake Barker, Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto-Sivashinsky equation. *Phys. D*, 258(0):11 – 46, 2013.

6. **[127]** Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Nonlocalized modulation of periodic reaction diffusion waves: nonlinear stability. *Arch. Ration. Mech. Anal.*, 207(2):693–715, 2013.

7. **[128]** Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Nonlocalized modulation of periodic reaction diffusion waves: the Whitham equation. *Arch. Ration. Mech. Anal.*, 207(2):669–692, 2013.

8. **[125]** Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Spectral stability of periodic wave trains of the Korteweg-de Vries/Kuramoto-Sivashinsky equation in the Korteweg-de Vries limit. *Trans. Amer. Math. Soc.*

9. **[126]** Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Behavior of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations. *Invent. Math.*, pages 1–99, 2013.

**10. [20]** Sylvie Benzoni-Gavage, Pascal Noble, and L. Miguel Rodrigues. Slow modulations of periodic waves in Hamiltonian PDEs, with application to capillary fluids. Submitted in 2013.

**11. [10]** Blake Barker, Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Stability of St. Venant roll-waves: from onset to the large-Froude number limit. Work in progress.

**12. [21]** Sylvie Benzoni-Gavage and L. Miguel Rodrigues. Co-periodic stability of periodic waves in some Hamiltonian PDEs. Work in progress.

# Acknowledgements

Gratitude being more easily expressed in language of tenderness, that is in general in native language of the author, a brief French parenthesis follows. May the non French-reader to which a significant part of my gratitude is ought replace the few non-understandable words that follow by the most warm thanks !

Le temps passant, la liste de ceux auprès de qui je me sens redevable s'allonge au-delà du raisonnable sans que pour autant mon sentiment de gratitude pour chacun en soit atténué. Aussi mes tous premiers mots sont pour eux, les innombrables non cités. Qu'ils reçoivent mes chaleureux remerciements !

Alors que, cher lecteur, votre périple à travers ce texte commence, j'aimerais remercier infiniment ceux qui ont accepté au préalable d'en analyser chaque détail. Ce fut un honneur pour moi de me soumettre au jugement d'expert de François Hamel, David Lannes et Bob Pego. Je suis tout autant flatté de voir figurer dans mon jury deux contributeurs décisifs à la théorie, Denis Serre et Guido Schneider. Qu'ils sachent comme j'estime leur présence au sein de mon comité ! Je suis par ailleurs touché de voir Guy Métivier accepter à nouveau cette tâche après s'être déjà prêté à ma soutenance de thèse.

Mon comité compte aussi les non moins estimables Sylvie Benzoni-Gavage et Thierry Gallay. À des périodes différentes de ma vie j'ai eu la chance de les côtoyer au quotidien. Croyant en le valeur de l'exemple et bien que je ne sois pas sûr qu'ils en soient fiers, j'aimerais qu'ils sachent que ce que je suis leur doit beaucoup et que quoi qu'il arrive leur image de rectitude et de professionnalisme m'habitera encore longtemps.

Je souhaiterais également exprimer encore une fois ma joie d'avoir trouvé à mes côtés mes compagnons d'aventure mathématique, Blake Barker, Sylvie Benzoni-Gavage, Mat Johnson, Pascal Noble et Kevin Zumbrun. Profitons-en pour reconnaître également le rôle décisif des supports financiers du projet ANR « Écoulement de fluides complexes en eaux peu profondes », porté par Pascal Noble, et de l'Université de l'Indiana qui ont rendu ces si nombreuses rencontres possibles.

J'en arrive à ceux dont le rôle est peut-être moins apparent mais tout aussi décisif, mes collègues au quotidien, à l'Institut Camille Jordan bien sûr mais aussi à l'UMPA, et principalement :

- d'une part, ceux qui ont su m'accueillir chaleureusement, m'insérer dans leur vie et m'offrir l'exemple de l'exigence, du travail acharné et de la passion, en particulier, outre les déjà cités Sylvie Benzoni-Gavage, Pascal Noble et Denis Serre, Vincent Calvez, Thierry Dumont et Francis Filbet ;

- d'autre part, mes camarades d'arrivée à Lyon, Nadine Badr-Vovelle, Sarah Delcourte, Sébastien Gauthier, Paul Vigneaux et Julien Vovelle. Notablement le tra-

vail de sape au quotidien de Sébastien et Julien aura durablement modifié mon appréhension de mes activités d'enseignant et de chercheur.

Merci à eux tous !

J'aimerais également remercier la masse anonyme des innombrables étudiants auxquels j'ai eu le plaisir d'essayer d'enseigner le peu que je savais. Leur voisinage quotidien m'a également imperceptiblement changé.

Enfin je devrais en arriver aux remerciements plus personnels encore. Mais ce mémoire est sans doute un document trop public pour cela. Aussi j'espère que ceux que j'aime sauront trouver en eux le souvenir de tous les mots doux et tendres que j'aimerais à nouveau leur adresser.

# Contents

# List of Figures

# 1. Introduction

A fundamental strategy to shed some light on real world phenomena is to try to identify coherent structures as organizing centers for the dynamics. Among these special elementary blocks, waves — and especially periodic waves — play the main role for many fields: acoustics, electromagnetism, hydrodynamics, combustion... Correspondingly periodic waves are to many partial differential systems as periodic cycles studied by Gaston Floquet and Henri Poincaré to two-dimensional dynamics. Beyond fundamental concerns, a deeper and deeper understanding of dynamics around these simple waves has also generated major technical breakthrough all along the twentieth century. Mark for instance that one revolution in telecommunications originated in the discovery that dynamics about periodic waves could support modulation processes and this way convey information. Such an impact is still reflected by the amplitude modulation (AM)/frequency modulation (FM) distinction of radio broadcasting. Accompanying both advanced laboratory experiments and design of technical devices long ago a rich theory has been developed by engineering and physicist communities.

Yet, despite its practical and fundamental importance and the enlightening formal theory, until very recently, by some respects mathematical theory for nonlinear dynamics around periodic waves was still in its infancy. Mark that the behavior is difficult to analyze precisely because it carries modulation waves, dynamics about periodic waves being rich, multiscale, and by essence infinite-dimensional. This stems from the combination of two facts: on one hand spatial variations of periodic waves occurs in infinitely-many cells and expand up to infinity; on the other hand families of periodic waves usually exhibit a large number of degrees of freedom. During a long period of time even the formulation of the problem in a suitable functional framework remained an open question. Indeed, while for similar wave patterns but with a simpler spatial structure and less degrees of freedom, such as simple-bump solitary waves, kinks or fronts, pioneering contributions to nonlinear stability date back to at least the early 1970s (with work of T. Brooke Benjamin or David H. Sattinger), similar major breakthrough for periodic waves awaited mid-1990s and contributions of Guido Schneider. And yet the latter were still restricted to cases where periodic waves could only carry one *single* modulation signal.

Original ambition of the author and of his collaborators was to try to put on a par mathematical knowledge about nonlinear dynamics around periodics with those of other much more studied patterns. It turned out that for dissipative systems the strategy worked beyond all initial hopes. In particular, for parabolic systems, we have not only proved that any periodic wave that is linearly stable — in a suitable diffusive sense — is also nonlinearly stable but also that all dynamics near such a wave are asymptotically reduced in large time — through some form of averaging built in the evolution — to slow modulation processes well-described by the formal modulation theory. Note that to do so we had to introduce a new notion of stability, adapted to dynamics around periodic waves. Going back to applications, we have also thoroughly investigated both analytically and numerically spectral (linear) stability of some hydrodynamic periodic waves, called *roll-waves*, that emerge in shallow fluid films flowing down an inclined ramp as primary hydrodynamic instabilities. As in any dissipative system, theses nonlinear patterns may be thought of as manifestations of a suitable balance between enhancing forces, here gravity, and inhibiting mechanisms, here friction and viscosity. For conservative systems, nonlinear theory is still to develop but the author and some of his collaborators already tried to prepare convenient linear grounds for it.

As suggested by the former paragraph, in our strategy, the stability issue is tackled in two steps: on one hand the study of spectral stability, on the other hand a nonlinear analysis under spectral assumptions (see Chapter 4). In the exposition of the present memoir, concerning spectral stability, we shall also distinguish essentially analytical studies (Chapter 2) from essentially numerical ones (Chapter 3). Note that while the nonlinear analysis can be carried out in an extremely general context — the one of parabolic[1] systems —, complete spectral studies — though they follow robust methods — are inherently restricted to a certain field of applications, in our case principally to the consideration of *roll-waves*.

As a guide through the rest of the memoir, we now introduce equations involved in the present text and identify the kind of stability expected about a space-periodic solution. Since the present memoir will not contain any significant result not included in published papers nor give much details about the published proofs, the author believes that a significant part of its value lies precisely in the following long contextualization.

## 1.1. Equations' types & physical models

Our results naturally organize themselves following three levels of generality:

- without doubt the most general ones examine the consequences on spectral stability of the formal modulation theory. They yield necessary conditions for (general) spectral stability and sufficient conditions for spectral stability under *sideband*[2] perturbations of modulation type. This kind of conditions is in general hard to elucidate analytically but their implications are proved both for (partially) parabolic systems — which form the heart of the present memoir — and (often dispersive) Hamiltonian systems;

- come next in generality nonlinear stability deduced from spectral assumptions and fine descriptions of asymptotic behavior. Although these results are exposed for a restricted class of systems, they do extend to parabolic systems in full generality and even to some classes of partially parabolic systems;

[1] Including some partially parabolic systems whose principal and sub-principal parts are simultaneously symmetrizable and satisfy a genuine-coupling condition. See Appendix A for a rough introduction to the field.

[2] In the terminology adopted here, this means *almost* co-periodic, that is corresponding to Bloch-waves with Floquet multipliers close to one.

- at last, conclusions of complete studies of spectral stability, though they follow from general strategies, are restricted to specified models, here describing thin films flowing down an incline.

Without entering into the detailed assumptions[3] justifying the terminology, we first distinguish the two classes of systems for which we discuss sideband spectral stability. On one hand, quasilinear systems of balance laws of parabolic type considered in the present memoir are written as[4]

$$\mathbf{U}_t + (\mathbf{f}(\mathbf{U},\dots,\mathbf{U}_{\underbrace{x\cdots x}_{2(\ell-1)}}))_x = \mathbf{g}(\mathbf{U}) + (\mathbf{B}(\mathbf{U},\dots,\mathbf{U}_{\underbrace{x\cdots x}_{2(\ell-1)}})\,\mathbf{U}_{\underbrace{x\cdots x}_{2\ell-1}})_x, \tag{1.1.1}$$

where $\ell \in \mathbf{N}^*$ and gives the order $2\ell$ of the system, $\mathbf{U} : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}^d$ is the unknown $\mathbf{R}^d$-valued field ($d \in \mathbf{N}^*$), $t \in \mathbf{R}^+$ and stands for time, $x \in \mathbf{R}$ and stands for space, and $\mathbf{f} : \Omega \to \mathbf{R}^d$, $\mathbf{g} : \Omega' \to \mathbf{R}^d$, $\mathbf{B} : \Omega \to \mathcal{M}_d(\mathbf{R})$ provide coefficients of the system and are defined respectively on open subsets of $\mathbf{R}^{d\times 2(\ell-1)}$, $\mathbf{R}^d$ and $\mathbf{R}^{d\times 2(\ell-1)}$. We shall particularly focus on cases $\ell = 1$ and $\ell = 2$. Again we should add some structural assumptions but will postpone this task for the moment.

## Hamiltonian systems

On the other hand, we will consider Hamiltonian systems of the following specific type,

$$\partial_t \mathbf{U} = \mathscr{J}(\mathsf{E}\mathscr{H}[\mathbf{U}])\,, \tag{1.1.2}$$

where $\ell \in \mathbf{N}^*$ and gives the order $2\ell + 1$ of the system, $\mathbf{U} : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}^d$ is the unknown $\mathbf{R}^d$-valued field ($d \in \mathbf{N}^*$), again $t \in \mathbf{R}^+$ and stands for time, $x \in \mathbf{R}$ and stands for space, $\mathscr{J} = \partial_x \mathbf{J}$ is a skew-symmetric differential operator, $\mathbf{J}$ being a symmetric invertible matrix (with constant coefficients), $\mathscr{H}$ is a functional involving derivatives of $\mathbf{U}$ up to, at most, order $\ell \in \mathbf{N}^*$ and $\mathsf{E}$ denotes the Euler operator. Through the text we shall mostly restrict ourselves to $\ell = 1$, in which case

$$\mathsf{E}\mathscr{H}[\mathbf{U}]_\alpha = \frac{\partial \mathscr{H}}{\partial U_\alpha}(\mathbf{U},\mathbf{U}_x) \,-\, \partial_x\left(\frac{\partial \mathscr{H}}{\partial U_{\alpha,x}}(\mathbf{U},\mathbf{U}_x)\right)\,, \quad \alpha \in \{1,\dots,d\}.$$

In any case, besides conservation of the Hamiltonian energy, associated with invariance under time translations and read on the local conservation law when $\ell = 1$

$$\partial_t \mathscr{H}(\mathbf{U}) = \partial_x(\tfrac{1}{2}\mathsf{E}\mathscr{H}[\mathbf{U}] \cdot \mathbf{J}\mathsf{E}\mathscr{H}[\mathbf{U}] + \partial_x(\mathsf{E}\mathscr{H}[\mathbf{U}]) \cdot \nabla_{\mathbf{U}_x}\mathscr{H}(\mathbf{U},\mathbf{U}_x)), \tag{1.1.3}$$

the system (1.1.2) also comes with a local conservation law of some *impulse* $\mathscr{Q}$, stemming from invariance under translation in space[5] and such that $\mathscr{J}\mathsf{E}\mathscr{Q}[\mathbf{U}] = \partial_x \mathbf{U}$. Since $\mathbf{J}$ is

[3]Including for parabolic systems at least a form of ellipticity and when the ellipticity is of partial type a coupling condition with the subprincipal part (see Appendix A).

[4]The author does not claim any effort in uniformization of notation for partial derivative of a function $f$ with respect to a variable $\alpha$ that may be denoted $f_\alpha$, or $\partial_\alpha f$, or $\frac{\partial}{\partial \alpha} f$,...

[5]Note already that when looking for *traveling* waves, the two conservation laws are necessarily tied together. At some stage a counting of equations will be needed, together they will then account for one additional equation plus one entropy for the evolution.

nonsingular, an explicit licit choice is $\mathscr{Q}(\mathbf{U}) := \frac{1}{2}\,\mathbf{U}\cdot\mathbf{J}^{-1}\mathbf{U}$. With this choice, the extra conservation law — satisfied by any smooth solution to (1.1.2) — reads

$$\partial_t \mathscr{Q}(\mathbf{U}) = \partial_x(\mathscr{S}[\mathbf{U}]) \tag{1.1.4}$$

where, when $\ell = 1$,

$$\begin{aligned} \mathscr{S}[\mathbf{U}] &:= \mathbf{U}\cdot \mathsf{E}\mathscr{H}[\mathbf{U}] + \mathsf{L}\mathscr{H}[\mathbf{U}]\,, \\ \mathsf{L}\mathscr{H}[\mathbf{U}] &:= U_x\cdot\nabla_{\mathbf{U}_x}\mathscr{H}(\mathbf{U},\mathbf{U}_x) - \mathscr{H}(\mathbf{U},\mathbf{U}_x)\,, \end{aligned} \tag{1.1.5}$$

with $\mathsf{L}$ denoting the Legendre transform (kept in original variables $(\mathbf{U},\mathbf{U}_x)$). We shall make use of other conservation laws but the latter are not stemming from Hamiltonian symmetries but from non invertibility of $\mathscr{J}$ and are merely given by each equation of the system (1.1.2). Once again we postpone to a later discussion the introduction of the actual structural assumptions that we will need in order to carry out our precise analysis.

We instead give some concrete examples. The most simple one is offered by the generalized Korteweg–de Vries equation (gKdV),

$$\partial_t v + \partial_x p(v) = -\partial_x^3 v\,, \tag{1.1.6}$$

which fits into the frame with $d = 1$ and $\mathscr{H} = f(v) + \frac{1}{2}v_x^2\,,\quad f' = -p$ and whose impulse conservation law leads to

$$\partial_t(\tfrac{1}{2}v^2) + \partial_x(f + pv + vv_{xx} - \tfrac{1}{2}v_x^2) = 0\,.$$

But the subclass that we are more interested in and that we will study thoroughly is given by the Euler–Korteweg system, written in Eulerian coordinates

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) \;=\; 0\,, \\ \partial_t u + u\partial_x u \;+\; \partial_x(\mathsf{E}_\rho\mathscr{E}) \;=\; 0\,, \end{cases} \tag{1.1.7}$$

where $\rho$ is the density field of the fluid, $u$ its velocity field and $\mathscr{E}$ provides its density of energy. It corresponds to $d = 2$,

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \mathbf{U} = \begin{pmatrix} \rho \\ u \end{pmatrix} \quad\text{and}\quad \mathscr{H} = \frac{1}{2}\rho u^2 + \mathscr{E}(\rho,\rho_x)\,,$$

leading to

$$\mathscr{Q} = \rho u\,,\quad \mathscr{S} = \rho\,\mathsf{E}_\rho\mathscr{E} + \mathsf{L}\mathscr{E}\,.$$

It is often advantageous at least in space dimension one to work rather with the mass-Lagrangian version of (1.1.7). It is obtained by interpreting the mass conservation law as the fact that some 1-form is closed hence exact, thus by introducing[6] $\mathrm{d}y = \rho\mathrm{d}x - \rho\,u\,\mathrm{d}t$ in order to perform the change of variables $(t,x)\to(t,y)$. When doing so, one receives

$$\begin{cases} \partial_t v \;=\; \partial_y u\,, \\ \partial_t u \;=\; \partial_y(\mathsf{E}_v e)\,, \end{cases} \tag{1.1.8}$$

[6]As is readily seen, the mass-Lagrangian change of variables $x \leftrightarrow y$ remains under control only as long as the density stays bounded away from zero and infinity.

where $v = 1/\rho$ denotes the specific volume of the fluid, $u$ its velocity field (but seen as a function of $(t, y)$) and $\mathscr{e}$ its specific energy — that is $\mathscr{E} = \rho \mathscr{e}$. A classical choice of energy — the one of the capillarity theory of Korteweg — is given by

$$\mathscr{E}(\rho, \rho_x) = F(\rho) + \frac{1}{2}\mathscr{K}(\rho)\,(\rho_x)^2\,, \tag{1.1.9}$$

or equivalently by

$$\mathscr{e}(v, v_y) \;=\; f(v) \;+\; \frac{1}{2}\,\kappa(v)\,(v_y)^2\,, \tag{1.1.10}$$

where $F = \rho f$, $\kappa = \rho^5 \mathscr{K}$. Positivity (or rather nonvanishing) is the main constraint on $\mathscr{K}$ and $\kappa$. As emphasized in the survey paper [19], even after having restricted generality from (1.1.2) to (1.1.7)–(1.1.9) or (1.1.8)–(1.1.10), one is still in position to cover a wealth of physical situations:

- dynamics of capillary flows (liquid-vapor mixtures, superfluids, regular fluids at small scales) often leading to the choice of a constant $\kappa$ but letting the pressure $-f'$ depend on the precise situation to analyze;
- hydrodynamical formulation of some Schrödinger equations — which themselves provide an approximate description of many situations — obtained from
$$\mathrm{i}\partial_t\psi - \tfrac{1}{2}\partial_x^2\psi = \psi\, g(|\psi|^2)$$
through the Madelung transformation[7] $\psi = |\psi|\mathrm{e}^{\mathrm{i}\phi}$, $(\rho, u) = (|\psi|^2, \partial_x\phi)$, and that fits into frame when $F' = g$ and $\mathscr{K}(\rho) = 1/(4\rho)$;
- the generalized Boussinesq systems
$$\begin{cases} \partial_t h \;=\; \partial_y u\,, \\ \partial_t u \;+\; \partial_y p(h) \;=\; -\kappa \partial_y^3 h\,, \end{cases}$$
arising initially in the approximate description of the evolution of the height $h$ of water-waves (when $p$ is a polynomial function of degree two), and that, setting $v = h$, correspond to choices $f' = -p$ and constant $\kappa$.

## Parabolic systems

We now come back to parabolic systems to provide both typical systems for which our analysis is built and specific ones that we discuss thoroughly. The class (1.1.1) is well exemplified by the Navier–Stokes equations[8]

$$\begin{cases} \partial_t\rho \;+\; \partial_x(\rho\, u) \;=\; 0, \\ \partial_t(\rho\, u) \;+\; \partial_x(\rho\, u^2 + p(\rho)) \;=\; f(\rho, u) \;+\; \partial_x(\mu(\rho)\,\partial_x u)\,, \end{cases} \tag{1.1.11}$$

[7] Unfortunately as hinted at by formal considerations they are essentially relevant when $\psi$ does not vanish, an assumption that seems natural only when it is forced by boundary conditions at infinity, for instance by $|\psi| \overset{\pm\infty}{\to} 1$ when $g(\rho) = \rho - 1$.

[8] By essence compressible in space dimension one; here also barotropic, with usual physical reserves concerning viscous barotropic flows...

where $\rho$ describes a density field (or sometimes a height, in which case it will be rather denoted by $h$), $u$ a velocity field, $p$ provides the pressure law, $\mu > 0$ and gives diffusion coefficient and $f$ accounts for forces exerted on the fluid, and the Korteweg–de Vries/Kuramoto–Sivashinsky equations[9]

$$\partial_t u \;+\; \partial_x(\tfrac{1}{2}\,u^2) \;+\; \partial_x^3 u \;=\; -\delta\,(\partial_x^2 u + R\,\partial_x^2(u^2) + \partial_x^4 u)\,, \tag{1.1.12}$$

where $\delta > 0$ measures instability and $R \in \mathbf{R}$. The latter equations are a universal[10] correction to the Korteweg–de Vries equation when dealing with weakly-nonlinear unidirectional waves evolving in a medium where constant states are weakly unstable to low-frequency perturbations but weakly stable to high-frequency perturbations. Mark that it is natural to work around this kind of instabilities when looking for the emergence of periodic patterns of length one.

However we have fully developed the nonlinear analysis only for slightly caricatural versions of (1.1.1): semilinear parabolic equations of order two, with source terms either nonsingular or vanishing everywhere. Explicitly we analyze reaction-diffusion systems

$$\mathbf{U}_t \;=\; \mathbf{g}(\mathbf{U}) \;+\; \mathbf{U}_{xx} \tag{1.1.13}$$

in [127, 128] and parabolic systems of conservation laws in the form

$$\mathbf{U}_t \;+\; \mathbf{f}(\mathbf{U})_x \;=\; \mathbf{U}_{xx} \tag{1.1.14}$$

in [126]. This expository choice is commanded by our will, in the presentation of the nonlinear analysis of the stability of periodic traveling waves, to avoid any difficulty of algebraic nature (Friedrichs symmetrizers, Kawashima compensators[11], order of analytic semigroups,...) that are essentially of the same nature as the ones encountered in the analysis of stability of constant states. Our goal is, in contrast, to emphasize obstacles that are intimately tied to the fact that we work about *periodic* solutions, that is in

[9] Strictly speaking, the Korteweg–de Vries equation reads $\partial_t u + \partial_x(\frac{1}{2}\,u^2) + \partial_x^3 u = 0$ while the Kuramoto–Sivashinsky equation is written as $\partial_t u + \partial_x(\frac{1}{2}\,u^2) + \delta\,(\partial_x^2 u + \partial_x^4 u) = 0$. Equation (1.1.12) with $R = 0$ is sometimes called the Kawahara equation.

[10] It is easily seen on a formal expansion extracted from a scalar conservation law or a system formed by a pair conservation law – relaxation system (see p.63). Indeed, starting from $\partial_t u + \partial_x(f(u;\epsilon)) + \partial_x^2(g(u;\epsilon)) + \partial_x^3(h(u;\epsilon)) + \cdots = 0$, the usual KdV renormalization

$$u \mapsto v = (u - \underline{u})/\epsilon \;(\underline{u}\text{ constant}), \qquad (x,t) \mapsto (\xi,\tau) = (\epsilon^{\frac{1}{2}}\,(x - f'(\underline{u};\epsilon)t), \epsilon^{\frac{3}{2}}\,t)$$

leads to

$$\begin{aligned}\partial_\tau v \;&+\; \partial_\xi(\tfrac{1}{2}\,f''(\underline{u};\epsilon)\,v^2) \;+\; \partial_\xi^3(h'(\underline{u};\epsilon)\,v)\\ &=\; -\epsilon^{\frac{1}{2}}\,(\partial_\xi^2((g'(\underline{u};\epsilon)/\epsilon)\,v) \;+\; \partial_\xi^2(\tfrac{1}{2}\,g''(\underline{u};\epsilon)\,v^2) + \partial_\xi^4(i'(\underline{u};\epsilon)\,v))\end{aligned}$$

which can be scaled to (1.1.12) provided that coefficients of the KdV part do not vanish and remaining coefficients have signs compatible with the instability scenario borne by (1.1.12). Note that the formal derivation required $g'(\underline{u};\epsilon) = \mathcal{O}(\epsilon)$ whereas $g'(\underline{u};\epsilon) = o(1)$ would have been sufficient to obtain the KdV first-order description.

[11] Yet in Appendix A we provide the reader not familiar with the notion with a very rough introduction. When doing so, we are motivated by the fact that some familiarity with nonlinear dynamics around homogeneous solutions of hyperbolic-parabolic systems helps entering our nonlinear analysis.

particular to explain how to perform a kind of reduction to a simpler and much more studied problem, precisely the stability of constant states. Nevertheless we endeavor to point out through the text how would work an extension to the general form (1.1.1). Two detailed extensions of a nonlinear stability result [129] — less involved than the ones discussed in the present memoir — are provided in [135] and [13], respectively for a quasilinear system of partial parabolic type with a degenerate source-term ((1.1.17) below) and for some semilinear parabolic equations of order four (including (1.1.18) below). With this in mind, let us stress that the form (1.1.14) provides a class of problems involving all the specific analytical difficulties of the periodic analysis whereas, as we shall discuss thoroughly later, the form (1.1.13) exhibit the only reduction in algebraic complexity leading to a significant simplification of the analysis.

Our nonlinear results are deduced from spectral stability and, though we detail the analysis only on simplest systems (1.1.13) and (1.1.14), it is indeed for some systems of the form (1.1.11) and (1.1.12) that we investigate whether this spectral stability holds or not. We introduce now the mentioned systems that are both involved in the modeling of dynamics of thin films.

In the modeling of shallow flows down a slope, a dimension reduction[12] may lead in the vanishing thickness limit — at least formally — from the incompressible free-surface Navier–Stokes equations including gravity and bottom friction to the following compressible Navier–Stokes equations

$$\begin{cases} \partial_t h + \partial_x (h\,u) \;=\; 0\,, \\ \partial_t (h\,u) + \partial_x (h\,u^2 \;+\; \mathrm{g}\cos(\theta)\frac{1}{2}\,h^2) \;=\; \mathrm{g}\sin(\theta) h \;-\; \mathrm{C}_f |u|\,u \;+\; \mu\,\partial_x (h\,\partial_x u)\,, \end{cases}$$

called the St. Venant equations in this context. The reduced description follows the surface *via* the fluid height $h$ over the incline — which prevents the consideration of surfaces that are not given by a graph over the ramp — and retains from the speed only an averaged $u$ of its component parallel to the slope — which assumes implicitly that it is possible to build from this a good approximation of the complete velocity field —, with both $h$ and $u$ given as functions of the coordinate along the incline (here oriented downward). The above system has not been fully nondimensionalized yet, in particular it involves the standard gravity acceleration g, a turbulent friction coefficient $\mathrm{C}_f > 0$ and a viscosity coefficient $\nu > 0$. The slope angle $\theta$ between the ramp and the horizontal direction also enters into the system. Although the former system may be derived from a free-surface incompressible description [101], the modeling includes a phenomenological part (already in the higher-dimensional description), for instance in the choice of a quadratic — and not linear — friction force designed to give a better account of the presence of a turbulent boundary layer at the bottom, a choice that is usually thought as having its origin in Antoine de Chézy's work.

Aiming at nondimensionalizing, we introduce a number, referred to as the *Froude number* in this context — but which plays the role of the Mach number in the compressible

[12] Actually this is a second reduction but we will not discuss the reduction from space dimension three to space dimension two.

Navier–Stokes terminology —, by [13]

$$\mathbf{F} \;=\; \frac{[u]}{\sqrt{\mathrm{g}\cos(\theta)[h]}}\,,$$

a number that compares the fluid velocity to the speed of acoustic (gravity) waves. Afterwards, by setting $[t]^{-1} = [u]\,[x]^{-1}$, $[u] = \mu\,[x]^{-1}$, $[t]^{-1}[u] = \mathrm{g}\sin(\theta)$ and $[t]^{-1}[h] = \mathrm{C}_f[u]$, we receive

$$\left\{\begin{array}{l} \partial_t h + \partial_x(h\,u) \;=\; 0\,, \\ \partial_t(h\,u) + \partial_x(h\,u^2 \;+\; \frac{1}{2}\,\frac{h^2}{\mathbf{F}^2}) \;=\; h \;-\; |u|\,u \;+\; \partial_x(h\,\partial_x u)\,, \end{array}\right. \tag{1.1.15}$$

with

$$\mathbf{F} \;=\; \sqrt{\frac{\tan(\theta)}{\mathrm{C}_f}}. \tag{1.1.16}$$

Mark that at fixed $\mathrm{C}_f$ the Froude number is an increasing function of the slope, going to 0 and to $\infty$ respectively in the horizontal and vertical limits. Once again it may be advantageous to switch to a mass-Lagrangian description, here obeying[14]

$$\left\{\begin{array}{l} \partial_t\tau - \partial_y u \;=\; 0\,, \\ \partial_t u + \partial_y(\frac{1}{2}\,\frac{\tau^{-2}}{\mathbf{F}^2}) \;=\; 1 \;-\; \tau\,|u|\,u \;+\; \partial_y(\tau^{-2}\,\partial_y u)\,. \end{array}\right. \tag{1.1.17}$$

The main very specific feature of system[15] (1.1.17) is that all constant solutions $(\tau_0, u_0(\tau_0)) = (\tau_0, \tau_0^{-1/2})$ ($\tau_0 > 0$ arbitrary) turn unstable simultaneously: when $0 < \mathbf{F} < 2$ they are stable, while $2 < \mathbf{F}$ implies instability. The latter instability stems directly from a violation of the subcharacteristic condition: $|u_0'(\tau_0)| \leq \tau_0^{-3/2}\mathbf{F}^{-1}$, that is $\frac{1}{2} \leq \mathbf{F}^{-1}$. Accordingly, in the limit $\mathbf{F} \to 2^+$, the dynamics of (1.1.17) is expected to be well-approximated by the one of a suitable equation of type (1.1.12) with $\delta \sim \sqrt{\mathbf{F}-2}$. Reflecting the simultaneous destabilization of constant states, the relevant parameter $R$ vanishes[16]. For this reason, henceforth, we shall consider mostly

$$\partial_t u \;+\; \partial_x(\tfrac{1}{2}\,u^2) \;+\; \partial_x^3 u \;\;=\;\; -\delta\,(\partial_x^2 u + \partial_x^4 u) \tag{1.1.18}$$

rather than the general form (1.1.12). Actually we will not justify any direct relation between (1.1.18) and (1.1.15)/(1.1.17), essentially because we are mostly interested in large-time behavior and hope is few in general to commute large-time asymptotics and

[13]We denote by $[L]$ the typical size of a quantity $L$.

[14]To match with notation of corresponding publications of the author, the specific volume is here denoted $\tau$ and not $v$.

[15]For the main part of what follows — a discussion of properties of a linearization — one may keep working with (1.1.15).

[16]More on this reduction may be found in Section 2.2, p.63.

a (singular) limit in one parameter. Yet, the reader interested in a form of direct justification will find a proof of an analogous reduction in [239]. However, as it is often[17] the case in such situations, one may hope that the proof of some results for (1.1.18) is the sign that some counterpart does exist for (1.1.15)/(1.1.17). Such hope supports our interest in equation (1.1.12), a canonical reduced equation.

Let us now add a few words about models for fluids in the vanishing thickness limit. The expository choice of the present memoir, that introduces some thin film equations mostly as application fields for the main theorems, for which one may check the needed spectral assumptions, may mislead the reader and leave him with the feeling that the shallow flow approximation is a modeling process of limited range. Exactly the opposite is true, shallow fluid layers are ubiquitous, in the human body (cornea, lung,...), at the geophysical scale (ocean, atmosphere,...), in industrial processes (lubrication,...)... We do not claim to give even a rough idea of the wealth of involved phenomena but we strongly encourage the reader to choose an entering gateway among [27, 42, 140, 186, 194, 197] or some of devoted chapters in [15]. Incidentally we also point to the curiosity of the walker that the planar traveling waves of (1.1.15) that we study thoroughly in this memoir are an idealized version of *roll-waves* easily observed on sloping streets by rainy days[18]. Naturally clearer coherent structures are observed in laboratory experiments conditions. The literature on the subject is too wide to be properly quoted but we at least mention that many fine pictures and even some illuminating movies may be found among papers and webpages of Neil Balmforth (British Columbia), Jerry P. Gollub (Haverford) and Christian Ruyer-Quil (Paris 6) and of their collaborators.

### Discussion

We conclude the introduction of the equations by a (not so) short discussion of topic choices and start by justifying the fact that we consider planar traveling waves only as one-dimensional objects. This choice comes with two main consequences. On one hand nonlinear analysis is considerably harder in space dimension one since scattering mechanisms are weaker thus decay resulting from diffusive-like behavior is slower[19]. This is reflected by the fact that the first results of nonlinear stability of spectrally stable periodic waves of systems of type (1.1.1) were first proved in dimension higher than two, then in dimension two and finally in dimension one [190, 193, 192, 129], [131], [126]. This is also responsible for the asymptotically linear behavior exhibited by localized perturbations of periodic waves in dimension higher than one. On the other hand, spectral stability may seem easier to met in dimension one since in some sense spectral stability in dimension one may be interpreted as a higher-dimensional spectral stability but restricted to coplanar perturbations. Yet one may rightfully wonder what could be the physical

[17] Paradigmatic examples of equations playing the mixed role of reduced asymptotic equation and toy-model are provided by the Hopf, the Burgers and the Ginzburg–Landau equations. About the latter the reader is referred to the nice survey by Alexander Mielke [177].

[18] Certainly this strongly hints at their stability as a solution to an appropriate model.

[19] Think about decay rate $t^{-\frac{N}{2}}$ in $L^\infty$-norms for solutions to the heat equation in space-dimension $N$ starting from integrable initial data.

meaning of a situation where a planar wave would be stable as a one-dimensional object but unstable as a higher-dimensional object. An element of answer lies in the fact that our one-dimensional solutions often correspond in higher dimension not to planar waves but to genuinely multidimensional objects exhibiting some confinement mechanisms that may preserve stability[20]. In short, dimension one offers both a richer nonlinear dynamics — at the cost of substantially more involved analysis — and better hope to find stable waves.

One may also raise the issue of the absence of boundaries. Clearly our choice is commanded by our will to describe some real evolutions by idealized patterns such as solitary waves, periodic waves, fronts, self-similar solutions... The underlying motivation is classical, and the reader is referred to appropriate discussions in [46, 47, 50, 220]. This idealization offers the possibility to escape case-by-case discussions often coming with the consideration of bounded domains. Still, the general hope is that, although the precise selection of the dynamics may depend strongly on boundaries[21], this dynamics is often built — in a first-order description — from simple superpositions of elementary blocks that are better understood as coherent structures of extended systems. At a technical level, the absence of boundaries helps in getting the spectral analysis more explicit by the introduction of suitable integral transforms but, by often precluding any hope for a spectral gap, it usually makes the nonlinear analysis much more intricate than in bounded domains[22].

Our goal is to focus on the stability[23] of such elementary blocks and, thus, we take their existence as granted, either by assumption or by appealing to previous results, proved by other authors. Concerning planar traveling waves, it amounts to looking for special solutions to some ordinary differential equations, called profile equations. More, for the Hamiltonian partial differential equations that we study in detail, the existence part stems directly from the fact that the corresponding profile equations are planar Hamiltonian differential equations. However, for our parabolic systems, no such simple argument holds and to the knowledge of the author only existence results in some asymptotic limits are available[24]: close to constant states through a Hopf bifurcation, close to solitary waves through a Shilnikov homoclinic bifurcation, close to some planar Hamiltonian profiles, for instance close to the KdV waves about the threshold of instability, close to an array of discontinuous shocks in the inviscid limit... Beyond their own theoretical interest, these asymptotic limits — and especially the Hopf bifurcation — also serve as starting points for complete numerical studies based on continuation algorithms.

Actually, spectral stability and existence of special solutions are intimately tied to-

[20]Think about the reconstitution of the full velocity profile from its averaged longitudinal component in the shallow water approximation (either with a free-surface as here or between two rigid walls as in some lubrication problems).

[21]A paradigmatic situation being offered by the influence of the shape of the box on the selection of the orientation of rolls or of the form of cells in Rayleigh–Bénard convection experiments.

[22]Albeit usually less than in extended domains with boundaries.

[23]Note however that one may build — at least seemingly — stable solutions by piecing together unstable elementary blocks as we illustrate in Section 3.2.

[24]In particular there does not seem to be in the literature any existence result based on variational, topological or compactness arguments.

gether since transition to instability of a certain form of solutions often gives rise[25] to a new family of patterns whose stability may in turn be investigated... Hence the classical strategy — for equations involving some set of parameters — consisting in carrying a parametric study of stability/instability. Starting from a simple family of solutions, explicit or even trivial, known to be stable for a certain range of parameters, one varies these parameters up to a transition to instability. At this threshold emerges a new family of special solutions, whose stability is also tracked when varying parameters and that can also yield yet another family of solutions, and so on and so forth. The patterns emerging from the first transition are usually called primary instabilities, those coming next secondary instabilities. Although one may artificially build systems exhibiting an infinite number of such transitions, it seems that in most of classical physical problems the instability of secondary patterns leads rather to chaos then turbulence. An argument supporting this phenomenological rule of thumb is that the emergence of new patterns often goes with a symmetry breaking increasing the dimensional complexity: trivial solutions are one dimensional, primary instabilities two-dimensional, secondary ones three-dimensional, then comes chaos. Our investigation of thin film layers perfectly enters in this classical hydrodynamical instability framework. Our crucial parameter is the Froude number $\mathbf{F}$; constant solutions of (1.1.15) are[26] one-dimensional waves whose primary instabilities are roll-waves, described here as planar periodic traveling waves. An important goal of the present memoir is then to analyze the stability of these primary hydrodynamical instabilities. The reader interested in further developments of these notions is referred to [40, 43, 67, 169, 172, 229, 250].

One may also wonder why, as the title of the memoir suggests, we are so much interested in global-in-time stability and large-time asymptotics and not in finite time dynamics. Motivation is essentially the same as the one that led us to neglect boundaries. The hope is that even the dynamics starting from large initial data may after a transient period of time be resolved into local dynamics about a given pattern that gives rise in the large-time asymptotics to a universal structure that does not require a case-by-case study.

At last, let us explain what is the origin of our Hamiltonian/parabolic distinction, a separation that corresponds neither to a classical variational distinction as Hamiltonian/gradient, nor to a difference in type as hyperbolic/parabolic. We do use the parabolicity (even of partial type) in our nonlinear analysis of the parabolic case. Yet, since we investigate the Hamiltonian case only at the spectral level, the reason why we perform such a distinction originates from spectral considerations. Actually the common part of the spectral analysis is based on expansions along the periodic profiles manifold — the dimension of which being related to the number of local conservation laws supported by the dynamics — so that at the (relevant) spectral level this distinction mainly manifests itself in the presence/absence of extra "hidden" conservation laws, as (1.1.3) and (1.1.4) for system (1.1.2).

[25] Through usual bifurcation processes, as analyzed by Lyapunov–Schmidt methods combined with normal form reductions.

[26] Recall that system (1.1.15) is obtained, starting from free-surface equations, by performing a dimension reduction.

## 1.2. Space-modulated stability

We start now our introduction to the kind of stability that may be expected about a space-periodic wave. As a first axis of discussion, we analyze the consequences of periodicity on localization of perturbations.

Before doing so, we strongly advise the reader against the idea that the stability of a periodic solution should be investigated only under perturbations of the same period. Keeping in mind that our periodic solutions do not live on a torus but on an extended domain, one arrives readily at the conclusion that considered perturbations should also be functions defined on the full space. This is our specific line of work. As a consequence, there seems to be be no shortcut through compactness arguments, spectral gaps, Poincaré inequalities or variational principles.

We recall that even for the simplest cases where is examined the stability of a unimodal pattern (a monotone front, a single-bump solitary wave...), some routine care should be taken. Indeed when there exist close-by profiles corresponding to solutions traveling with a slightly different speed, then a small perturbation may eventually change the traveling speed, say by a factor $(\delta c)$ so that after some time $t$ appears a phase shift of typical size $(\delta c)\,t$ that results in a growing difference[27] between the perturbed solution and the reference pattern. A well-known response is to allow for a phase shift before comparison. Explicitly, if $\mathcal{H}$ is the Banach space in which one initially wants to measure distances, then instead of considering directly $\|\mathbf{U}-\underline{\mathbf{U}}\|_{\mathcal{H}}$, one introduces

$$\inf_{\text{uniform translations }\Psi} \|\mathbf{U}\circ\Psi-\underline{\mathbf{U}}\|_{\mathcal{H}}\,.$$

When the original equation is invariant under space-translations, the resulting notion of stability is called *orbital* stability, because it amounts to the stability of the orbit of the original pattern generated by the action of uniform translations. To put it in another way, by allowing for translation before comparison, we actually measure distances between equivalence classes. Of course, when there is a richer group of continuous symmetries for the dynamics, it is both possible and needed to enrich accordingly the set of allowed normalizations before comparison.

For periodic waves, the issue differs on at least two points. First, the changes in velocity that lead to phase shifts are essentially of local nature so that they can occur independently at any of the infinitely-many cells of our periodic solution. Mark that, even when the initial perturbation is localized at one cell, it usually splits into different perturbation waves traveling at different (group) velocities so that, after some time, perturbations are indeed operating in different cells. In general, after interaction, each perturbation wave leaves in its wake shifted cells that are out of phase with front cells still untouched. Second, as soon as $\mathcal{H}$ encodes some localization, say, as soon as it involves some $L^p(\mathbf{R})$-space with a finite $p$, uniform translations become almost completely useless since instead of preventing the growth of the direct difference between the reference wave

[27] Roughly: for monotone fronts, it grows in time as $t^{\frac{1}{p}}$ in $L^p(\mathbf{R})$-norm; for solitary waves, it saturates at essentially twice the norm of the solitary wave.

and the actual solution they make this difference become infinite because they move both infinities out of phase. As a consequence, some nonuniform treatment is needed.

There are at least two ways in which the problem may be tackled. One may either introduce an at-most-countable number of discrete shifts and combine them into a piece-wise linear transformation enabling us to synchronize the solution and the reference wave on an unknown number of cells, or, alternatively, allow for a continuous change of variable. Since most of people that have addressed this issue belonged to the field of analysis of partial differential equations, and therefore prefer by essence a partial differential equation to an infinite number of coupled ordinary differential equations, the latter response is overrepresented in the literature but the reader may find some instances of the former in [184, 216]. Despite our need for a nonuniform synchronization, we still aim at identifying together solutions that differ by a uniform translation and only them. Since, whatever the choice of type of synchronization — continuous change of variables or countably-many shifts — the class of allowed transformations is too wide thus generates too large orbits, we introduce a weight ensuring that transformations performing local shifts are not too far from being uniform translations. With this in mind, we define[28]

$$\delta_{\mathcal{H}}(\mathbf{U},\underline{\mathbf{U}}) \;=\; \inf_{\text{invertible }\Psi}\;\|\mathbf{U}\circ\Psi-\underline{\mathbf{U}}\|_{\mathcal{H}}\;+\;\|\partial_x(\Psi-\mathrm{Id}_{\mathbf{R}})\|_{\mathcal{H}}. \tag{1.2.1}$$

We could elaborate on this to build a topology or even a distance, but we will not follow this path. Nevertheless we henceforth misuse the standard terminology and wrongfully refer to $\delta_{\mathcal{H}}$ as a *space-modulated* distance. We call the corresponding stability, space-modulated stability. The notion is flexible enough to be easily generalized to the case where more continuous symmetries are present besides translation invariance.

We do not claim any novelty in introducing a change of variable $\Psi$ to capture defects of synchronization. This is indeed now part of the folklore of the area. Yet we believe that the sound and precise picture obtained by introducing $\delta_{\mathcal{H}}$ and a new notion of stability appeared for the first time in [126] and that it does help to unify and clarify even previously obtained results. We also point to the reader familiar with topologies on measure spaces and stochastic processes that our underlying topology is also a clear analog of the Skorokhod topologies.

It may seem at first glance that by constantly identifying functions coinciding up to a uniform translation we definitely loose any track of the original physical frame and therefore necessarily fail to provide a detailed account of spatial features of the dynamics, as involved in important classical hydrodynamical distinction between absolute and convective (in)stability [44, 49, 210, 204]. This would indeed be the case were we contenting ourselves with a space-modulated asymptotic stability result. Yet we do go much further by providing a refined description of asymptotic behavior that in particular does provide information about a good choice of $\Psi$ and how this particular $\Psi$ evolves in space-time.

We stress that by going to space-modulated versions of distances we are able not only to compensate the local defects of synchronization due to the trend of the time evolution to create local phase shifts but also to capture the possible defects of localization

[28]With usual convention that, when $f\notin\mathcal{H}$, $\|f\|_{\mathcal{H}}=\infty$.

initially present in the data hence to enlarge the class of initial data that we may consider as perturbations of our wave. However we remark already here that these two goals are somewhat antagonists and this brings some further complications in the nonlinear analysis.

We will constantly refer to perturbations belonging to the enlarged class of initial data as *nonlocalized* perturbations. Let us thus explain the terminology. First, a localized perturbation of a wave $\underline{\mathbf{U}}$ is something that may be written as $\underline{\mathbf{U}}+\mathbf{V}$ with $\mathbf{V}$ localized (and smooth), say $\mathbf{V}\in\mathcal{H}$ in our abstract description or $\mathbf{V}\in L^1(\mathbf{R})\cap H^K(\mathbf{R})$ (with $K$ large) for concreteness. In the localized perturbation framework, the most classical way to obviate the difficulty stemming from the fact that a periodic wave does not belong to our favorite functional spaces is to work with $\mathbf{V}$ only, the perturbative part. However, as we have already explained, this simple strategy measures spurious growth of distances that reflects more the inadequacy of the framework than a genuine instability. Now, once adopted the space-modulation strategy, it is quite natural to take full benefit of the possibility to relax constraints on initial data. But our motivation goes further and is partly of more concrete nature. Indeed, note that in order for the process of gluing together a left portion of the reference wave, some function on a finite interval, and the remaining right portion of the original wave to yield a localized perturbation (according to our definition), the left-hand and right-hand copies of the wave should be in phase. It is this stringent condition that we want to relax in going to nonlocalized perturbations $\mathbf{U}$ satisfying merely $\mathbf{U}\circ\Psi=\underline{\mathbf{U}}+\mathbf{V}$ for some $\mathbf{V}\in\mathcal{H}$ and some invertible $\Psi$ such that $\partial_x(\Psi-\mathrm{Id}_{\mathbf{R}})\in\mathcal{H}$.

It seems that there could be yet another classical strategy adapted to handle periodic waves and neighboring dynamics. It consists in choosing functional spaces requiring no localization, such as $L^\infty$, $\mathrm{BV}_{loc}$ or uniformly local spaces. Yet, since we aim at giving a precise description of the large-time behavior and in particular sharp decay rates, we do need information about localization. This is readily seen on scale-invariant systems since scaling symmetries obviously[29] provide some relation between neighborhoods of infinity respectively in space for the initial data and in time for the solution. But this is widely true when scattering (either of dispersive or diffusive type) plays the main role in determining decaying properties[30]. Note, by the way, that the dynamics about periodic waves of systems (1.1.13) does involve some hidden asymptotic scale-invariance, this single-scale property being a strong sign that they should be easier to analyze.

We end the present Section in a more concrete way by revisiting some of the previous points on three direct simulations not all appearing in [13] but computed to prepare its writing. We plot space-time diagrams of three time-evolution studies about some chosen waves of the Kuramoto–Sivashinsky equation $\partial_t u+\partial_x(\frac{1}{2}u^2)=-(\partial_x^2u+\partial_x^4u)$. In all cases we choose a frame moving at the speed of the wave so as to make the waves stationary. All waves exhibit single-bump profile and we show peaks (thick green) and troughs (thin blue) of the solution so that were there no perturbation we would have

[29]See [206] and references therein for further discussions.

[30]Of course this is not always the case. For a simple instance, damping may provide uniform decay. More interesting, in the dual situation where decay is brought by mixing, then *regularity* information is needed to estimate decay [180, 17].

obtained two arrays of vertical lines spaced by the period, in all numerical experiments 6.3 in the units of the figures. In all cases the initial data is obtained by adding a small well-localized Gaussian perturbation to the original wave.

In Figure 1.1, the perturbed wave is stable and the original perturbative part splits after some transient time in two waves. At first order these two waves travels along straight lines and a closer look reveals that they are diffusion-waves. We plot in solid red the corresponding straight lines and in dotted red lines the boundaries of their diffusive zone of influence as predicted at linear order by the formal slow modulation theory introduced in Section 1.4. Mark that the part of the solution lying inside the cone delimited by these principal directions of perturbation propagation and far away from these straight lines, say, outside the influence zone delimited by dotted lines, looks perfectly steady and, more, seems to have relaxed to the original wave. Yet, as expected, this inside version of the wave has already interacted with the perturbative part and is not in phase with the outside version. Since the involved inside zone is growing linearly in time, it causes that the $L^p(\mathbf{R})$-norm of the perturbative part actually grows as $t^{1/p}$ when $t \to \infty$. However, this local shift is small and may be compensated by a near-identity transformation thus seems compatible with some space-modulated asymptotic stability. Note that in the special cases where there is only one diffusion-wave involved in the process or more generally when there is only one wave carrying perturbation for local phases there is no room for a shifted version of the wave to emerge and usual (and not space-modulated) asymptotic stability under localized perturbations may be proved. This is exactly the simplification of the analysis that exemplify systems (1.1.13). At last, we point that our way of showing the evolution can reveal only parts of the perturbations having some significant impact on the local phase shifts but, for the Kuramoto–Sivashinsky equation it turns out that there is nothing more to see.

In Figures 1.2 and 1.3, to illustrate the importance — in the stability mechanism — of the diffusive character of the waves carrying perturbations we show two time-evolutions about waves that fail to be stable by not fulfilling this requirement. Straight lines given by linear group velocities continue to play an organizing role for the evolution but now inside the cone that they form some instability is slowly developing itself until it finally plays a prominent role and completely break inside the cone the periodic-cell spatial structure of the original wave. We insist on the fact that the hidden presence of a form of reversed heat equation does not lead to ill-posedness but simply to a growing parabolic instability. This is due to the fact that these *a priori* ill-posed equations are involved only in the description of fields that are by essence slow, that is, low-Fourier[31]. Obviously other types of instability may manifest themselves in very different ways.

In our refined asymptotic description of the large-time dynamics about a stable wave, our main goal will be to derive analytically the phenomena observed on Figure 1.1: eventual splitting into weakly-interacting diffusion-waves traveling with their own group velocities. We stress that the identification of this scenario provides yet another piece

[31]We hope that this somehow cryptic remark will become much clearer after the introduction of the Bloch transform and the formal derivation of the slow-modulation behavior. However we already add that this necessarily slow character is an effect of the fact that Floquet parameters $\xi$ introduced below all belongs to a compact interval, the Brillouin zone.

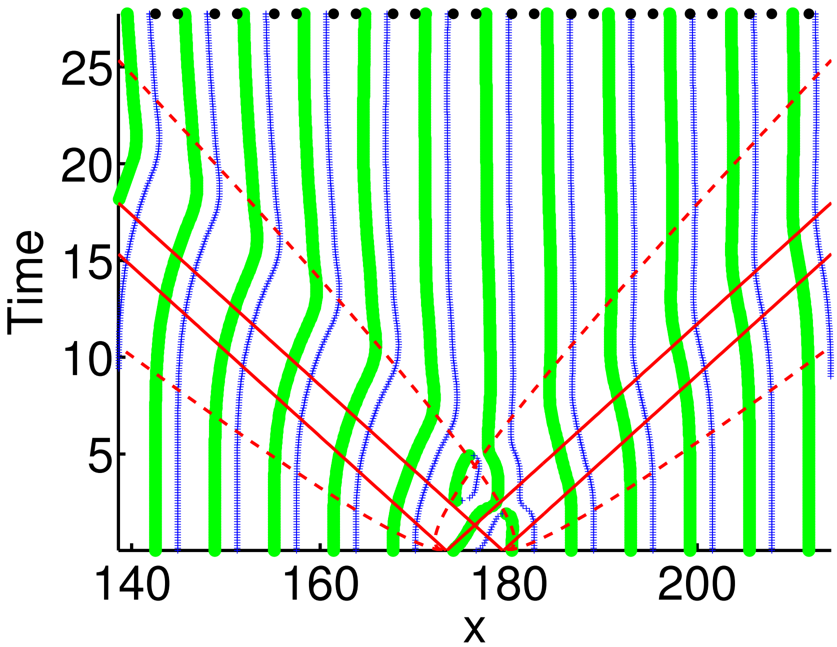



Figure 1.1.: Time-evolution about a stable wave.

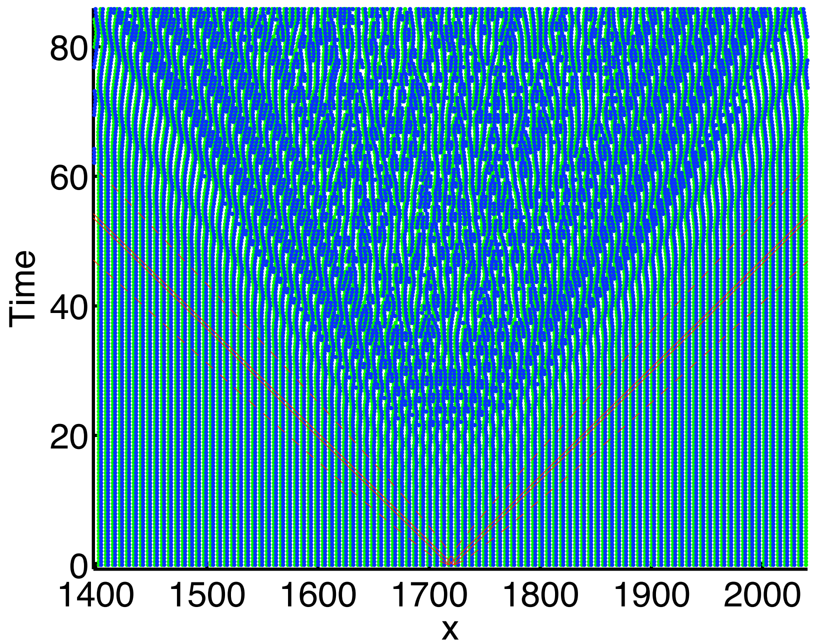



Figure 1.2.: Time-evolution illustrating effects of a loss of diffusivity.

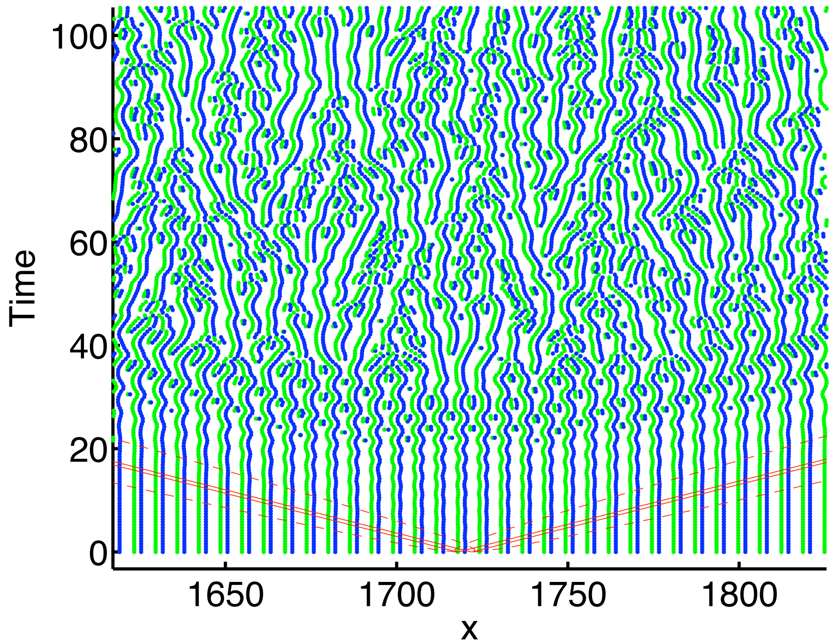



Figure 1.3.: Another time-evolution illustrating effects of a loss of diffusivity.

of motivation to introduce nonlocalized perturbations. Indeed, since in the end the diffusion-waves carrying perturbation information are well-separated in space, one may wish to analyze precisely what occurs about a single such wave. Yet, to do so, we do need to allow for nonlocalized perturbations of our original traveling wave, since each perturbative diffusion-wave is surrounded on the left and on the right by copies of waves that are out of phase[32].

[32] This is in perfect analogy with a situation that may be more familiar to some of the readers: the resolution of viscous two-dimensional incompressible flows in point-vortices. Note, as a preliminary, that, often, the origin of the dimension two behavior is also shallowness, stemming either directly from confinement or fluid scarcity, or through some stratification for instance due to strong rotation effects. It is known that the asymptotic behavior of solutions to the two-dimensional incompressible Navier–Stokes equations with well-localized vorticity is provided by superpositions of viscous point-vortices. When a finite amount of (macroscopic) kinetic energy is carried out by the flow, these point-vortices are constrained to algebraically sum to yield a zero global circulation. To analyze precisely the case where one single vortex is isolated from the rest, one is then compelled to leave the finite-energy framework exactly as we abandon in the present memoir the localized-perturbation frame that forces phase shifts to sum to zero thus prevents the consideration of an isolated single diffusion-wave in local wavenumber. We refer the reader to [206] and references therein for further discussions. We stress that the analogy goes up to technical details, at least when stated in vague terms. Indeed, in both cases, difficulties arising in the nonlinear analysis are essentially not regularity issues but are rather stemming from poor localization and inherent slow decay.

## 1.3. Spectrum of operators with periodic coefficients

A large part of the present memoir is devoted to the study either analytical or numerical of spectral stability for some periodic waves and almost all the remaining part to the derivation of nonlinear results when spectral stability is met. We start now our spectral considerations by recalling in an informal way well-known facts about the spectrum of linear operators with periodic coefficients. The reader is referred to [95, 246, 215, 39] — complemented by [146] — for both proofs and precise statements. This requires as a preliminary step the introduction of the appropriate integral transform. The relevant transform is well-known; nevertheless we choose to introduce it by slowly listing the desired properties that it should fulfill.

As usual, to define an appropriate integral transform, we are guided by the will to obtain from its inverse transform the decomposition of any function as a sum of elementary objects. As hinted at by the Floquet theory, in the periodic context, the most simple objects are functions that experience simply a multiplication by a constant when uniformly translated by one fundamental period $\Xi$. The multiplying constant is called a Floquet multiplier and requiring boundedness over the full line forces the Floquet multiplier to be of modulus one, hence written $e^{i\Xi\xi}$, with[33] $\xi \in [-\pi/\Xi, \pi/\Xi)$. These simple objects are called Bloch-waves, the corresponding $\xi$ is called a Floquet exponent[34] and the range $[-\pi/\Xi, \pi/\Xi)$ of the Floquet parameters is named the Brillouin zone[35]. From now on, for notational convenience, let us fix the period $\Xi$ to one. The natural outcome of this paragraph is that we are looking for a decomposition

$$g \;=\; \int_{-\pi}^{\pi} \tilde{g}(\xi,\cdot)\,\mathrm{d}\xi,$$

where each $\tilde{g}(\xi,\cdot)$ is a Bloch wave of Floquet exponent $\xi$, that is

$$\forall x \in \mathbf{R}, \qquad \tilde{g}(\xi, x+1) \;=\; e^{i\xi}\,\tilde{g}(\xi, x).$$

In the form written above, the decomposition would be of rather unpractical use since, because of the different boundary conditions, each of the $\tilde{g}(\xi,\cdot)$ belongs naturally to a different functional space. But there is an easy response that consists in noting that there is a one-to-one mapping between Bloch waves of a given Floquet exponent and functions of period one. This leads to look for

$$g(x) \;=\; \int_{-\pi}^{\pi} e^{i\xi x}\,\check{g}(\xi,x)\,\mathrm{d}\xi, \tag{1.3.1}$$

with each $\check{g}(\xi,\cdot)$ periodic of period one, that is

$$\forall x \in \mathbf{R}, \qquad \check{g}(\xi, x+1) \;=\; \check{g}(\xi, x).$$

[33] One should keep in mind however that $\mathbf{R}/(2\pi\Xi\mathbf{Z})$ is the natural space for $\xi$ and that all the definition of $\xi$-dependent objects that are coming below may indeed be naturally extended to the full line, these extensions possessing then some natural symmetries with respect to the canonical action of $2\pi\Xi\mathbf{Z}$.

[34] Actually the true Floquet exponent is $i\xi$.

[35] Obviously the object is more interesting in higher dimensions where the geometric structure of possible lattices of periods, primitive cells and dual primitive cells is richer.

Such an inverse formula may be obtained by summing appropriately an inverse Fourier decomposition. For this purpose we introduce direct and inverse Fourier transforms, *via*

$$\hat{g}(\xi) \;:=\; \frac{1}{2\pi}\int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}\xi x} g(x)\,\mathrm{d}x, \qquad g(x) \;=\; \int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}\xi x}\,\hat{g}(\xi)\,\mathrm{d}\xi.$$

Then the adequate integral transform, called the *Bloch transform* or the Floquet-Bloch transform, may be defined by

$$\check{g}(\xi,x) \;:=\; \sum_{j\in\mathbf{Z}} \mathrm{e}^{\mathrm{i}\,2j\pi x}\,\widehat{g}(\xi+2j\pi). \tag{1.3.2}$$

Up to now, we have been rather bold concerning summation issues and meaning of equalities. But there is actually no hidden trap so that we will safely go on with this line. Indeed, the actual resolution of these questions follows from a combination of the classical known results for the Fourier transform/series. As an illustration, mark that, up to a normalizing factor, the Bloch transform defines a total isometry between $L^2(\mathbf{R})$ and[36] $L^2([-\pi,\pi],L^2([0,1]))$.

To balance the huge number of linear estimates involved in the eventual nonlinear analysis, we shall try to give them a unified treatment, by deducing them as much as possible from simplest estimates on the Bloch transform: the Parseval equality

$$\|g\|_{L^2(\mathbf{R})}^2 \;=\; (2\pi)\;\|\check{g}\|_{L^2([-\pi,\pi],L^2([0,1]))}^2 \tag{1.3.3}$$

and the Hausdorff–Young inequalities, for any $2\leq p\leq\infty$,[37]

$$\Big\|\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\xi\cdot}\,g(\xi,\cdot)\,\mathrm{d}\xi\Big\|_{L^p(\mathbf{R})} \;\leq\; (2\pi)^{-1/p}\;\|g\|_{L^{p'}([-\pi,\pi],L^p([0,1]))} \tag{1.3.4}$$

that are classically obtained from an interpolation between the Triangle Inequality and the Parseval identity.

We stress the two fundamental simple facts

$$(g\,h)\check{}\,(\xi,x) \;=\; g(x)\;\check{h}(\xi,x) \qquad \text{if } g \text{ is one-periodic,} \tag{1.3.5}$$

$$(\partial_x h)\check{}\,(\xi,\cdot) \;=\; (\partial_x+\mathrm{i}\xi)\;\check{h}(\xi,\cdot). \tag{1.3.6}$$

As a consequence of the former observation, if $g$ is one-periodic and $h$ is slow, that is, if the Fourier transform of $h$ has support in $[-\pi,\pi]$, then

$$(g\,h)\check{}\,(\xi,x) \;=\; g(x)\;\widehat{h}(\xi). \tag{1.3.7}$$

[36] We use throughout the memoir standard notation about $L^p(A,L^q(B))$-spaces and their norms $\|g\|_{L^p(A,L^q(B))} = \|a\mapsto\|g(a,\cdot)\|_{L^q(B)}\|_{L^p(A)}$ and do similarly for composition of other functional spaces.

[37] We use standard notation for conjugation of Lebesgue indices, that is, if $1\leq p\leq\infty$, then $1\leq p'\leq\infty$ and $1/p+1/p'=1$.

This explains the crucial role of the Bloch transform in the two-scale analysis of slow-modulation behaviors. Indeed $g\,h$ is the most trivial version of a function varying on a slow-modulation scale, in this case just a periodic function times a slow amplitude function. And, at least on this paradigmatic version, the Bloch transform separates the two scales, leaving the fast periodic scale in original variables and sending the slow scale into the Fourier/Bloch dual space. This separation is a good preparation to an averaging process isolating the slow evolution. Note, however, that a large part of the analysis is devoted to estimate precisely parts of the solution that do not come in the well-prepared slow-modulation form.

Another consequence of the above simple observations is that if $L$ is a differential operator with one-periodic coefficients then so are $L_\xi$,

$$(L_\xi g)(x) \;=\; \mathrm{e}^{-\mathrm{i}\xi x}\, L(\mathrm{e}^{\mathrm{i}\xi\cdot}g(\cdot))(x), \qquad \xi \in [-\pi,\pi].$$

This leads to a Bloch diagonalization

$$(Lg)(x) \;=\; \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\xi x}\, (L_\xi \check{g}(\xi,\cdot))(x)\,\mathrm{d}\xi\,. \tag{1.3.8}$$

What is gained in the reduction is that instead of working with $L$ that acts on functions defined on the full line, we may handle a continuous family of $L_\xi$, each acting on periodic functions, thus on functions living on one periodic cell. Technical advantages stem then from compactness[38]. In particular, for each $L_\xi$ we may expect discrete spectra, spectral gaps, continuous perturbation with respect to $\xi$,...

For the operators we will consider in our stability studies, this decomposition will lead to

$$\sigma(L) \;=\; \bigcup_{\xi\in[-\pi,\pi]} \sigma_{per}(L_\xi) \tag{1.3.9}$$

that decomposes the spectrum of $L$ which is completely of essential nature in a continuous family of discrete spectra. Let us sketch a brief justification of this well-known fact in the case where[39] the underlying space is $L^2(\mathbf{R})$. Assume that $L_0$ has compact resolvents (with a nonempty resolvent set) and that $L_\xi$ are all smooth — in $\xi$ — relatively compact perturbations[40] of $L_0$. Then if $\lambda$ does not belong to any $\sigma_{per}(L_\xi)$, then from continuity stems a uniform bound for $(L_\xi - \lambda\mathrm{Id})^{-1}$ that can be used together with inverse Bloch formula (1.3.1) and the Parseval identity to obtain a bounded inverse for $L-\lambda\mathrm{Id}$. Reciprocally, if $\lambda$ belongs to $\sigma_{per}(L_{\xi_0})$ for some $\xi_0$ then, by continuity of spectral projections

[38] Compare to the Fourier transform that reduces constant-coefficients differential operators to a continuous family of matrices, hence bringing all that comes with finite dimensionality.

[39] In dimension one, the interpretation of spectral problems as some spatial dynamics problem for some ordinary differential equations combined with classical results of regularity of solutions to ordinary differential equations yield that the spectrum hardly depend on the choice of a reasonable functional space. Incidentally we note that the sketched proof for the $L^2(\mathbf{R})$ choice does not use any specificity of the dimension one case.

[40] In particular they are relatively bounded with arbitrarily small relative bound.

associated with spectrum in a given suitably small ball centered at $\lambda$, for any $\varepsilon > 0$ one may synthesize[41] a nonzero

$$\varphi_\varepsilon \;=\; \int_{\xi_0-\delta_\varepsilon}^{\xi_0+\delta_\varepsilon} \mathrm{e}^{\mathrm{i}\xi\cdot}\;\varphi_{\varepsilon,\xi}\;\mathrm{d}\xi$$

such that

$$\|(L-\lambda\mathrm{Id})\,\varphi_\varepsilon\| \;\leq\; \varepsilon\;\|\varphi_\varepsilon\|.$$

It is in this Floquet-parametrized way that we will analyze various spectra throughout the memoir. More, the mere knowledge of $\sigma(L)$ would be insufficient to perform our nonlinear analysis of stability and we will use information about the way in which the spectrum $\sigma_{per}(L_\xi)$ varies with $\xi$. To make this discussion more concrete, let us look now at the spectrum of a linearized operator associated with a spectrally stable wave.

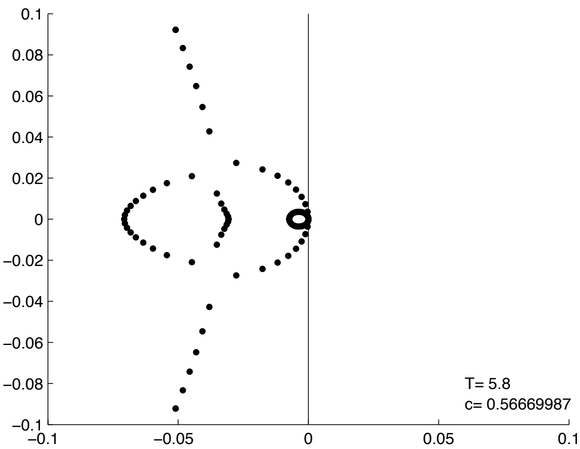



Figure 1.4.: Spectrum of a stable wave.

Figure 1.4 is extracted from [11, Figure 4]. It provides as an outcome of a numerical simulation[42] a discretized picture in the complex plane of continuous spectral curves — locally parametrized by the Floquet exponent $\xi$ — that form the spectrum of the linear operator $L$ arising from linearization about a spectrally stable wave of (1.1.17). The minimum requirement in order to claim spectral stability is

$$\sigma(L) \;\subset\; \{\lambda \mid \mathrm{Re}\lambda \leq 0\}\,.$$

[41]One way is to use Kato's theory to build dual right and left bases of the total spectral space and form the corresponding matrix $\Lambda_\xi$ (see the proof of Theorem 2.1.1 for some details). Then one obtains continuous spectral curves $\xi \mapsto \lambda_\xi$ such that $\lambda_{\xi_0} = \lambda$ and an associated measurable curve $\xi \mapsto \varphi_\xi$ of unitary eigenfunctions (for instance by applying Gaussian elimination to $\Lambda_\xi - \lambda_\xi \mathrm{I}$). Then set $\varphi_{\varepsilon,\xi} = \varphi_\xi$ and choose $\delta_\varepsilon$ such that $\sup_{|\xi-\xi_0|\leq\delta_\varepsilon} |\lambda_\xi - \lambda|$ is less than $\varepsilon$.

[42]Performed with SpectrUW, an implementation of Hill's method. Some details about this are given in Section 3.1.

For Hamiltonian systems (1.1.2), we will not require more. But, in order to conclude nonlinear stability from spectral stability in parabolic systems (1.1.1), we need a much more precise notion of stability, that is usually called *diffusive* spectral stability and consists in conditions (D1)-(D2)-(D3) below. We now try to let it emerge from the inspection of Figure 1.4. On this figure the plotted vertical line marks the imaginary axis. Notice that actually

(D1) $$\sigma(L) \subset \{\lambda \mid \mathrm{Re}\lambda < 0\} \cup \{0\}.$$

The presence of 0 in the spectrum is mandatory. It comes from the fact that the reference traveling wave is not an isolated exceptional wave but is one among a continuous family of periodic waves. By taking some variations along this family we obtain elements of the generalized kernel of $L_0$. Varying $\xi$ leads then to curves passing through the origin and spectral stability implies that, if regular[43], the curves should be quadratically tangent to the imaginary axis, that is, that they should touch the imaginary axis in a parabolic way. This is indeed the case on Figure 1.4. For spectrally stable waves, the parabolicity is generic and due to the fact that, at first order, imaginary parts of the eigenvalues about the origin depend linearly on the Floquet exponent $\xi$ while their real parts exhibit a quadratic dependence. As a consequence, then, we expect that there exists $\theta > 0$ such that

(D2) $$\sigma_{per}(L_\xi) \subset \left\{\lambda \;\middle|\; \mathrm{Re}\lambda \leq -\theta|\xi|^2\right\} \quad \text{for any } \xi \in [-\pi, \pi].$$

We stress also that the mentioned parabolicity at the origin of the critical spectral curves is in close relation with the diffusive character of the diffusion-waves observed in previous Section and will lead to algebraic decay of heat-like type.

With in mind usual strategies towards nonlinear stability, at a very rough[44] first glance, the part of the spectrum that lies far away from the imaginary axis may be thought as being asymptotically negligible since corresponding to exponential decay. Whatever we do however, this leaves at least the curves passing through the origin to handle. To be in position to do this, we will restrict ourselves to the the case where

(D3) 0 is, as an eigenvalue, of minimal algebraic multiplicity for $L_0$.

When requiring minimality, we have in mind that the presence of 0 in the spectrum of $L_0$ is forced by the existence of a nontrivial periodic profile manifold so that, under this assumption, we expect that all that we need to know about the spectral curves at the origin may be obtained by taking variations along this manifold. More, we may also hope to give a correct account of the critical evolution through some equivalent dynamics on the finite-dimensional manifold of periodic traveling waves. Note that since it must reproduce continuous spectral curves the equivalent dynamics is necessarily infinite-dimensional.

[43] Except in some exceptional cases where some coefficients that are not forced to vanish by the structure do vanish.

[44] This may only be true at the linearized level. The best that one can hope for at the nonlinear level is that inessential contributions are slaved to crucial ones.

This again hints at the introduction of *local* parameters whose time evolution approximately obeys some set of partial differential equations. One may wonder what will be gained by replacing some infinite-dimensional evolution with another infinite-dimensional evolution[45]. The point is that while the original dynamics organizes itself about a periodic solution the reduced one takes place about constant states, the reduction is therefore an averaging process.

We stress that most of the previous observations have natural counterparts in the spectral analysis of the stability of patterns with localized variations (fronts, kinks, solitary waves...). Yet, for those, since there is no equivalent to the Floquet parametrization, there is no obstruction to the existence of a spectral gap between the critical zero eigenvalue and the rest of the spectrum leading to an exponential decay toward the periodic wave manifold and the reduced dynamics may be described by *global* parameters approximately solving some set of ordinary differential equations. We have already met this global/local distinction when discussing phase shifts. Indeed, as we shall see below, one of the local/global parameters does account for spatial variations of the phase.

## 1.4. Averaged equations

Now we show how formal expansions yield candidates for the reduction and the reduced evolution. Afterwards we shall prove validity of the resulting conjectures, at the spectral level for both Hamiltonian and parabolic systems, at the nonlinear level for parabolic systems only. Therefore we do unfold the first stages of the formal theory for both types of system. For Hamiltonian systems that have a richer algebraic structure there are many possible paths to reach the same conclusion. But we choose to show a common derivation for both types of system. Nevertheless the formal derivation of the first-order systems (1.4.1)/(1.4.2) has now widely spread in both mathematicians and physicists communities and we refer in particular the reader to [249] for a pioneering source where may be found Hamiltonian/Lagrangian derivations and to [141] for a recent account oriented toward integrable Hamiltonian systems. To celebrate the decisive contribution of Gerald Whitham to the modulation theory, the first-order systems are usually called Whitham's systems and, by extension, the author and its collaborators apply this name also to their higher-order counterparts. The following modulation theory is far from being an isolated item and belongs to a circle of ideas including amplitude equations, sideband stability, nonlinear geometric optics, group velocities... We refer the reader to [255] for a nice historical survey about the emergence of these similar fundamental ideas.

As a preliminary step we need a better knowledge of the structure of periodic traveling wave profiles. We recall that a traveling wave solution $\mathbf{U}$ is a solution that preserves its shape along time but moves with some speed $c$, and is thus given by $\mathbf{U}(t,x) = \underline{\mathbf{U}}(x-ct)$, its profile $\underline{\mathbf{U}}$ being a stationary solution of the original equations written in a frame moving with speed $c$.

[45] Especially since there will be no simplification in the form of the involved systems of partial differential equations.

Traveling wave profiles $\underline{\mathbf{U}}$ of (1.1.2) satisfy

$$\mathsf{E}(\mathscr{H}+c\mathscr{Q})[\underline{\mathbf{U}}] \;=\; \boldsymbol{\lambda}\,, \qquad \mathscr{S}[\underline{\mathbf{U}}]+c\mathscr{Q}(\underline{\mathbf{U}}) \;=\; \mu\,,$$

fo some $\boldsymbol{\lambda}\in\mathbf{R}^d$ and $\mu\in\mathbf{R}$. Therefore it is natural to expect in this case that periodic profiles, identified when coinciding up to translation, form a manifold of dimension $(d+2)$, parametrized by $(c,\boldsymbol{\lambda},\mu)$. To perform a similar count for (1.1.1), we assume $\mathbf{U} = \begin{pmatrix}\mathbf{v} & \mathbf{u}\end{pmatrix}^T \in \mathbf{R}^{d'}\times\mathbf{R}^{d-d'}$ and

$$\forall\,\mathbf{U}, \qquad \mathbf{g}(\mathbf{U}) \;=\; \begin{pmatrix}0\\ \mathbf{g}^{(\mathbf{u})}(\mathbf{U})\end{pmatrix}$$

with a nondegenerate $\mathbf{g}^{(\mathbf{u})}$. Then traveling wave profiles $\underline{\mathbf{U}}$ of (1.1.1) satisfy

$$\begin{cases} \mathbf{q}-c\,\underline{\mathbf{v}}+\mathbf{f}^{(\mathbf{v})}(\underline{\mathbf{U}},\dots,\underline{\mathbf{U}}_{x\cdots x}) \;=\; \mathbf{B}^{(\mathbf{v})}(\underline{\mathbf{U}},\dots,\underline{\mathbf{U}}_{x\cdots x})\,\underline{\mathbf{v}}_{x\cdots x}\,,\\ -c\,\underline{\mathbf{u}}+\mathbf{f}^{(\mathbf{u})}(\underline{\mathbf{U}},\dots,\underline{\mathbf{U}}_{x\cdots x}) \;=\; \mathbf{g}^{(\mathbf{u})}(\underline{\mathbf{U}})+\mathbf{B}^{(\mathbf{u})}(\underline{\mathbf{U}},\dots,\underline{\mathbf{U}}_{x\cdots x})\,\underline{\mathbf{u}}_{x\cdots x}\,, \end{cases}$$

for some $\mathbf{q}\in\mathbf{R}^{d'}$. In this case, we expect a profile manifold of dimension $(d'+1)$, parametrized by $(c,\mathbf{q})$.

Note that systems (1.1.13), (1.1.14), (1.1.15)/(1.1.17) and (1.1.18) correspond respectively to $d'=0$, $d'=d$, $d'=1$ and $d'=1$. As will be clear already from the formal description below, this explains why perturbations to a stable periodic wave of (1.1.13) are resolved in a *single* diffusion-wave in local parameters, leaving no room for the emergence in an expanding zone of a shifted version of the original wave, why *two* such diffusion-waves appeared in the time-evolution about a stable wave of (1.1.18), and why the spectrum of a stable wave of (1.1.17) shows *two* critical spectral curves passing through the origin.

The above count of dimensions is indeed correct when some generic transversality is met, and our present ambition does not go beyond nondegenerate situations so that we do assume the former transversality. However, the proposed parametrization are well-adapted neither to the derivation of formal expansions that involve averaging processes nor to the use of integral transforms. Indeed, for the Bloch transform, only co-periodic functions play a special role. Therefore we need to normalize profiles so that they share the same period, here chosen to be one. To this end, we introduce wavenumbers $k$ and write periodic traveling waves as

$$\mathbf{U}(t,x) \;=\; \underline{\mathbf{U}}(k\,(x-ct))$$

with $\underline{\mathbf{U}}$ periodic of period one. Besides, we replace constants of integration $(\boldsymbol{\lambda},\mu)$ and $\mathbf{q}$ with averages[46] of the solution $(\mathbf{M},P)=(\langle\underline{\mathbf{U}}\rangle,\langle\mathscr{Q}[\underline{\mathbf{U}}]\rangle)$ and $\mathbf{M}^{(\mathbf{v})}=\langle\underline{\mathbf{v}}\rangle$. We mark the dependence accordingly on the periodic profiles $\underline{\mathbf{U}}$, phase velocity $c$ and time frequency $\omega=-k\,c$,

$$\underline{\mathbf{U}} \;=\; \underline{\mathbf{U}}^{(k,\mathbf{M},P)}\,, \qquad c \;=\; c\,(k,\mathbf{M},P)\,, \qquad \text{and} \qquad \omega \;=\; \omega\,(k,\mathbf{M},P)$$

[46]If $g$ is a periodic function of period $\Xi$ then $\langle g\rangle = \frac{1}{\Xi}\int_0^{\Xi} g$ denotes its averaged value. With our choice of period this reduces to $\langle g\rangle = \int_0^1 g$.

or[47]

$$\underline{\mathbf{U}} \;=\; \underline{\mathbf{U}}^{(\mathbf{M}^{(\mathbf{v})},k)}\,,\qquad c \;=\; c\,(\mathbf{M}^{(\mathbf{v})},k)\,,\qquad \text{and}\qquad \omega \;=\; \omega\,(\mathbf{M}^{(\mathbf{v})},k)\,.$$

Since there is no risk of confusion, we drop from now on the $^{(\mathbf{v})}$-superscript on $\mathbf{M}$.

We are now in position to introduce formal descriptions. Our *ansatz* encodes a two-scale phenomenon, fast evolution is supposed to be of oscillatory type and scaled to occur on a characteristic length one while slow evolution scale is denoted by $1/\varepsilon$. It leads to (formally) analyzing the behavior of an $\varepsilon$-family of solutions expanding as

$$\mathbf{U}(t,x) \;=\; (\mathbf{U}_0+\varepsilon\mathbf{U}_1)\Big(\underbrace{\varepsilon t}_{T},\underbrace{\varepsilon x}_{X};\underbrace{\frac{\Psi(\varepsilon t,\varepsilon x)}{\varepsilon}}_{\theta}\Big) + o(\varepsilon)\,,$$

with $\mathbf{U}_0$ and $\mathbf{U}_1$ one-periodic in $\theta$. By doing so, we are following the "two-timing" method already used by Whitham, a nonlinear analog of the famous Wentzel–Kramers–Brillouin method. The *ansatz* may be partly motivated by the fact that it includes periodic traveling waves as a special case: $\mathbf{U}=\mathbf{U}_0$ constant in slow variables $(T,X)$, linear phase $\Psi(T,X) \;=\; k\,(X-cT)$ for some $k$, $c$.

Having inserted the former *ansatz*, identification of the leading-order in $\varepsilon$ of (1.1.2) or (1.1.1) brings a periodic profile equation in variable $\theta$ with time frequency $\Omega=\Psi_T$ and spatial wavenumber $\kappa=\Psi_x$. Nonlinear dispersion relation of the periodic profile equation forces respectively for some $(\boldsymbol{\mathcal{M}},\mathcal{P})$, $\Omega=\omega(\kappa,\boldsymbol{\mathcal{M}},\mathcal{P})$ hence

$$\Psi_T \;=\; \omega(\kappa,\boldsymbol{\mathcal{M}},\mathcal{P})\,,\qquad \kappa_T - (\omega(\kappa,\boldsymbol{\mathcal{M}},\mathcal{P}))_X \;=\; 0\,,$$

and, for some $\boldsymbol{\mathcal{M}}$, $\Omega=\omega(\boldsymbol{\mathcal{M}},\kappa)$ hence

$$\Psi_T \;=\; \omega(\boldsymbol{\mathcal{M}},\kappa)\,,\qquad \kappa_T - (\omega(\boldsymbol{\mathcal{M}},\kappa))_X \;=\; 0\,.$$

The equation on the time-evolution of the local wavenumber $\kappa$ is usually called equation of *conservation of waves*. It needs to be completed with equations for the time-evolution of $(\boldsymbol{\mathcal{M}},\mathcal{P})$ and $\boldsymbol{\mathcal{M}}$ respectively, in order to form a closed system. To obtain complementary equations, we consider the next order in $\varepsilon$ that yields an affine equation in $\mathbf{U}_1$,

$$\begin{array}{rcl} L_0\mathbf{U}_1 &=& (\mathbf{U}_0)_T-\mathbf{J}(\mathsf{E}\mathscr{H}_\kappa[\mathbf{U}_0])_X + (\cdots)_\theta \\ &\text{and}& \\ L_0\mathbf{U}_1 &=& \begin{pmatrix}(\mathbf{U}_0)_T+(\mathbf{f}^{(\mathbf{v})}(\mathbf{U}_0,\dots,\kappa^{2\ell-1}(\mathbf{U}_0)_{\theta\cdots\theta}))_X\\ \cdots\end{pmatrix} \\ && -\begin{pmatrix}(\mathbf{B}^{(\mathbf{v})}(\underline{\mathbf{U}}_0,\dots,\kappa^{2\ell-2}(\underline{\mathbf{U}}_0)_{\theta\cdots\theta})\,\kappa^{2\ell-1}(\underline{\mathbf{v}}_0)_{\theta\cdots\theta})_X\\ \cdots\end{pmatrix} + (\cdots)_\theta\end{array}$$

whose linear part is given by the restriction to co-periodic functions $L_0$ of the operator $L$ generating the linearized evolution about the periodic profile $\underline{\mathbf{U}}_0$. We stress

[47] Discrepancy on listing order of parameters is maintained to match respectively [20] and [126].

that $L_0$ should be thought as a family of operators acting on functions of the variable $\theta$ but parametrized by $(T, X)$. Note also that the subscript $_\kappa$ marks the modifications associated with scalings performed to normalize period, for instance if, for $\mathbf{U} = \begin{pmatrix}\mathbf{v} & \mathbf{u}\end{pmatrix}^T \in \mathbf{R}^{d'} \times \mathbf{R}^{d-d'}$, $\mathscr{H}[\mathbf{U}] = \mathscr{I}(\mathbf{v}, \mathbf{u}) + \mathscr{E}(\mathbf{v}, \mathbf{v}_\theta)$, then

$$\mathscr{H}_k[\mathbf{U}] \ = \ \mathscr{I}(\mathbf{v}, \mathbf{u}) + \mathscr{E}(\mathbf{v}, k\,\mathbf{v}_\theta).$$

Missing equations then emerge from solvability conditions for the above affine equations. To this purpose, we remark that with each of the local conservation laws supported by the evolution comes an element of the kernel of the adjoint of $L_0$. For instance, when the $j$-th equation of the original system is a conservation law, the constant function with value $\mathrm{e}_j$ (the $j$-th vector of the canonical basis) belongs to this kernel. As a consequence for our formal expansions, we obtain

$$\boldsymbol{\mathcal{M}}_T - (\mathbf{J}\,\mathbf{G}(\kappa, \boldsymbol{\mathcal{M}}, \mathcal{P}))_X \ = \ 0 \quad \text{with} \quad \mathbf{G}(k, \mathbf{M}, P) = \langle \mathsf{E}\mathscr{H}_k[\underline{\mathbf{U}}^{(k,\mathbf{M},P)}]\rangle$$

and

$$\begin{aligned} \boldsymbol{\mathcal{M}}_T \quad + \quad & (\mathbf{F}(\boldsymbol{\mathcal{M}}, \kappa))_X \ = \ 0 \\ \text{with} \quad & \mathbf{F}(\mathbf{M}, k) = \langle \mathbf{f}^{(\mathbf{v})}(\underline{\mathbf{U}}^{(\mathbf{M},k)}, \dots, k^{2\ell-1}(\underline{\mathbf{U}}^{(\mathbf{M},k)})_{\theta\cdots\theta})\rangle \\ & -\langle \mathbf{B}^{(\mathbf{v})}(\underline{\mathbf{U}}^{(\mathbf{M},k)}, \dots, k^{2\ell-2}(\underline{\mathbf{U}}^{(\mathbf{M},k)})_{\theta\cdots\theta})\, k^{2\ell-1}(\underline{\mathbf{v}}^{(\mathbf{M},k)})_{\theta\cdots\theta}\rangle\,. \end{aligned}$$

Observe that in the latter when system (1.1.1) is actually semilinear in $\mathbf{v}$, that is, when $\mathbf{B}^{(\mathbf{v})}$ is constant, then it turns out that $\mathbf{B}$ does not contribute to $\mathbf{F}$. For Hamiltonian systems (1.1.2), the conservation of impulse $\mathscr{Q}$ yields also that the profile itself belongs to the kernel of the adjoint linearized operator. To be able to use it directly, we would need to explicit the $(\cdots)$-parts of the above affine system in $\mathbf{U}_1$. A shortest indirect way leading to the same conclusion is to expand and average directly the local conservation law (1.1.4). This provides us with

$$\mathcal{P}_T - (S(\kappa, \boldsymbol{\mathcal{M}}, \mathcal{P}))_X \ = \ 0 \quad \text{with} \quad S(k, \mathbf{M}, P) = \langle \mathscr{S}_k[\underline{\mathbf{U}}^{(k,\mathbf{M},P)}]\rangle.$$

As an outcome, the leading-order part of the expansion is[48]

$$\mathbf{U}(t, x) \ = \ \underline{\mathbf{U}}^{(\kappa,\boldsymbol{\mathcal{M}},\mathcal{P})(\varepsilon t, \varepsilon x)}\Big(\frac{\Psi(\varepsilon t, \varepsilon x)}{\varepsilon}\Big) \ + \ \mathcal{O}(\varepsilon)$$

with $\kappa = \Psi_X$ and

$$\left\{ \begin{aligned} \kappa_T \ &= \ (\omega(\kappa, \boldsymbol{\mathcal{M}}, \mathcal{P}))_X \\ \boldsymbol{\mathcal{M}}_T \ &= \ \mathbf{J}\,(\mathbf{G}(\kappa, \boldsymbol{\mathcal{M}}, \mathcal{P}))_X \\ \mathcal{P}_T \ &= \ (S(\kappa, \boldsymbol{\mathcal{M}}, \mathcal{P}))_X \end{aligned} \right. \tag{1.4.1}$$

[48] We discard the possibility of a uniform phase shift depending on slow variables, as being a high-order correction to the phase.

for (1.1.2); and

$$\mathbf{U}(t,x) \;=\; \underline{\mathbf{U}}^{(\boldsymbol{\mathcal{M}},\kappa)(\varepsilon t,\varepsilon x)}\Big(\frac{\Psi(\varepsilon t,\varepsilon x)}{\varepsilon}\Big) + \mathcal{O}(\varepsilon)$$

with $\kappa = \Psi_X$ and

$$\text{(1.4.2)} \qquad \begin{cases} \boldsymbol{\mathcal{M}}_T + (\mathbf{F}(\boldsymbol{\mathcal{M}},\kappa))_X &= 0 \\ \kappa_T - (\omega(\boldsymbol{\mathcal{M}},\kappa))_X &= 0 \end{cases}$$

for (1.1.1). Mark in the latter case that, when $d' = 0$, system (1.4.2) is reduced to

$$\text{(1.4.3)} \qquad \kappa_T \;=\; (\omega(\kappa))_X\,.$$

A wealth of information may be extracted from (1.4.1)/(1.4.2). As a preliminary, note that our assumption on the parametrization of periodic profiles corresponds to evolutionarity of the systems considered as equations on the manifold of periodic profiles. Linearization of the above systems about a constant state corresponding to a spectrally stable wave must be weakly hyperbolic, that is, they must have real characteristics. Hence they may be used as a predictor of instability. For a stable wave, in the non-degenerate situation where characteristics are distinct, they are strictly hyperbolic and provide us with a large part of the elements necessary to describe the diffusion-wave resolution of perturbations mentioned in Section 1.2: mode-by-mode separation, linear group velocities, coefficients of the quadratic[49] interaction. For the simplest case leading to (1.4.3) one recovers the classical result that linear group velocity of perturbations over a periodic wave of wavenumber $\underline{k}$ is given by $\omega'(\underline{k})$.

Mark that starting from a slow/oscillatory two-scale *ansatz*, we end with a slow modulation *ansatz*. Looking at scale one about a particular point of space one observes a periodic traveling wave of the full system but its parameters evolve themselves on a much slower scale $1/\varepsilon$. The slow evolution of local parameters is then expected to approximately obey averaged modulation systems.

The main goal of Section 2.1 is to examine at the spectral level all the interconnections between the original problem and the averaged modulation systems. Most of them seem to be part of the folklore in the physicist community. Yet to the knowledge of the author at this level of generality they are proved for the first time in [222] for (1.1.1) and in [20] for (1.1.2).

Although it already provides some valuable insight, the above first-order expansion is still insufficient to offer a satisfactory picture of the nonlinear behavior about a spectrally stable wave of (1.1.1). A diffusive correction is needed. It may be derived by enhancing the order of the expansions

$$\text{(1.4.4)} \qquad \mathbf{U}(t,x) = \big(\mathbf{U}_0 + \varepsilon\mathbf{U}_1 + \varepsilon^2\mathbf{U}_2\big)\Big(\underbrace{\varepsilon t}_{T}, \underbrace{\varepsilon x}_{X}; \underbrace{\frac{(\Psi_0+\varepsilon\Psi_1)(\varepsilon t,\varepsilon x)}{\varepsilon}}_{\theta}\Big) + o(\varepsilon^2)\,,$$

[49]Higher order contributions are too weak to be significant. Likewise, nonlinear contributions of diffusive corrections are asymptotically irrelevant.

with $\mathbf{U}_0$, $\mathbf{U}_1$ and $\mathbf{U}_2$ one-periodic in $\theta$. Results of the step-by-step identification may then be grouped into

$$\mathbf{U}(t,x) \;\sim\; \underline{\mathbf{U}}^{(\boldsymbol{\mathcal{M}},\kappa)(t,x)}\Big(\,\Psi(t,x)\,\Big) \tag{1.4.5}$$

with $\kappa=\Psi_x$, $(\boldsymbol{\mathcal{M}},\kappa)$ (approximately) solving

$$\left\{\begin{array}{rcl}\boldsymbol{\mathcal{M}}_t \;+\; (\mathbf{F}(\boldsymbol{\mathcal{M}},\kappa))_x &=& (d_{11}(\boldsymbol{\mathcal{M}},\kappa)\boldsymbol{\mathcal{M}}_x+d_{12}(\boldsymbol{\mathcal{M}},\kappa)\kappa_x)_x\\ \kappa_t \;-\; (\omega(\boldsymbol{\mathcal{M}},\kappa))_x &=& (d_{21}(\boldsymbol{\mathcal{M}},\kappa)\boldsymbol{\mathcal{M}}_x+d_{22}(\boldsymbol{\mathcal{M}},\kappa)\kappa_x)_x\end{array}\right. \tag{1.4.6}$$

and $\Psi$ (approximately) satisfying

$$\Psi_t \;-\; \omega(\boldsymbol{\mathcal{M}},\kappa) \;=\; d_{21}(\boldsymbol{\mathcal{M}},\kappa)\boldsymbol{\mathcal{M}}_x+d_{22}(\boldsymbol{\mathcal{M}},\kappa)\kappa_x. \tag{1.4.7}$$

We stress that it is by going back to physical variables that we removed any trace of $\varepsilon$ even in second-order averaged modulation equations. Yet one should keep in mind that the resulting systems are only characterized by the fact that they provide by insertion of a slow *ansatz*

$$(\boldsymbol{\mathcal{M}},\kappa)(x,t) \;=\; \Big((\boldsymbol{\mathcal{M}}_0,\kappa_0)+\varepsilon(\boldsymbol{\mathcal{M}}_1,\kappa_1)\Big)\Big(\,\underbrace{\varepsilon t}_{T},\underbrace{\varepsilon x}_{X}\,\Big)\;+\;o(\varepsilon)$$

the correct set of equations for a two-scale slow/oscillatory expansion (1.4.4). We will not provide in the present memoir full details about the derivation of the higher-order corrections nor formulas for the coefficients. As usual in homogenization processes, the formulas are obtained by averaging quantities involving solutions of some affine problems $L_0(\cdots)=\cdots$ posed on a periodic cell. To perform the actual derivation which is an expansion about a modulated wave and hence involves the detailed structure of linearized problems about a full family of periodic waves, we need to assume that this structure is known and common in an open subset of the periodic profile manifold. This assumption is rather natural when the common structure is a consequence of nondegeneracy or of a certain type of degeneracy forced by symmetries of the system[50]. For the purpose of the present memoir — stability about a given wave —, there is an easy way to relax this constraint. Indeed choosing one wave $\underline{\mathbf{U}}^{(\underline{\mathbf{M}},\underline{k})}$ and expanding about it also produces an outcome of type (1.4.5)-(1.4.6). But now the result is characterized by the fact that it provides by insertion of a slow *ansatz* of type

$$(\boldsymbol{\mathcal{M}},\kappa)(x,t) \;=\; (\underline{\mathbf{M}},\underline{k})+\Big(\varepsilon(\boldsymbol{\mathcal{M}}_1,\kappa_1)+\varepsilon^2(\boldsymbol{\mathcal{M}}_2,\kappa_2)\Big)\Big(\,\underbrace{\varepsilon t}_{T},\underbrace{\varepsilon x}_{X}\,\Big)\;+\;o(\varepsilon^2)$$

the correct set of equations for a two-scale slow/oscillatory expansion of type

$$\begin{array}{rcl}\mathbf{U}(t,x) &=& \underline{\mathbf{U}}^{(\underline{\mathbf{M}},\underline{k})}\Big(\frac{\Psi^\varepsilon(\varepsilon t,\varepsilon x)}{\varepsilon}\Big)\\ &+& \big(\varepsilon\mathbf{U}_1+\varepsilon^2\mathbf{U}_2+\varepsilon^3\mathbf{U}_3\big)\Big(\varepsilon t,\varepsilon x\,;\frac{\Psi^\varepsilon(\varepsilon t,\varepsilon x)}{\varepsilon}\Big)\;+\;o(\varepsilon^3)\,,\\ \text{with} & \Psi^\varepsilon(T,X) \;=\; \underline{k}\,X+\omega(\underline{\mathbf{M}},\underline{k})\,T+\varepsilon\Psi_1(T,X)+\varepsilon^2\Psi_2(T,X)+\;o(\varepsilon^3) & \end{array}$$

[50]For instance reflection symmetry may induce near a given even-symmetric standing wave that any traveling wave is actually a standing wave. As we shall see this in turn have deep implications on the structure of the linearized operator. Actually the former yields full phase-decoupling, in the sense of Section 4.2.

and $\mathbf{U}_0$, $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$ one-periodic in $\theta$.

The latter derivation is perfectly tailored to match our nonlinear stability analysis. But both predictions coincide at our level of validation. More, at this stage it should be clear that system (1.4.6) is far from being uniquely determined[51] by the fact that it provides the correct elements for our diffusion-wave scenario. Worst, even when the reference wave is diffusively spectrally stable and system (1.4.2) is strictly hyperbolic, there seems to be no reason why any formal process should lead to a well-posed system (1.4.6). At least two questions are then in order. For the strictly-hyperbolic diffusively-stable case, is there a canonical way to select a well-posed[52] system (1.4.6) ? Is it unique in some class of systems ? Both questions receive a positive answer.

As we shall see, for the nonlinear stability analysis of a strictly-hyperbolic diffusively-stable wave, it is sufficient to retain a semilinear, (fully) parabolic, symmetrizable version of (1.4.6) with a diffusion matrix commuting with the linearization of the hyperbolic part at the parameters of the wave. Here is the recipe that yields the desired diffusion matrix[53]: linearize any version of (1.4.6) about the parameters vector $(\underline{\mathbf{M}}, \underline{k})$ of interest, move to a basis diagonalizing the hyperbolic part and retain from the obtained diffusion matrix only the diagonal part, then by going back to the original basis one obtains the appropriate diffusion matrix. That any formally obtained system (1.4.6) leads to the same canonical form and that this canonical form is indeed fully parabolic follow from spectral relations with the original problem as established in [185], higher-order versions of the connections proved in [222]. There are more than formal grounds to found the reduction. It is indeed known for quasilinear symmetrizable (strictly) hyperbolic-(partially) parabolic systems satisfying a genuine-coupling Kawashima condition that the asymptotic behavior about a given constant state is well-approximated by a suitable combination of weakly-interacting diffusion-waves and that for this purpose one may replace the original system with a semilinear parabolic system whose linearized first-order and second-order are simultaneously diagonalizable [148, 170]. The role of the diffusion matrix is to provide the diffusion coefficient for each of the diffusion-waves, hence a diagonal matrix is sufficient. The systems resulting from applying this general strategy are sometimes referred to as effective *artificial viscosity*[54] systems. A further reduction — albeit not required to yield well-posedness — may be carried out since it is also possible to replace the first-order nonlinear hyperbolic part by a quadratic approximation and, in the strictly hyperbolic case, up to lowering the order of approximation of the asymptotic behavior, to work with a fully diagonalized semilinear quadratic system exhibiting decoupled diffusion-waves

[51] This is a common feature of higher-order descriptions whose most simple and most famous instance is probably the Korteweg–de Vries/Benjamin–Bona–Mahony alternative.

[52] Of course this kind of issue arises in many other asymptotic analyses. As an example, we note that an extremely (sophisticated and) interesting instance may be found when approximating water-wave problems [3]. Incidentally we mention to the reader the asymptotics leading to the Burnett equation and to the Prandtl layer equation as two famous intricate situations where the ill-posedness of the asymptotic systems seems to have not received yet a fully satisfactory definitive answer.

[53] See Appendix A and [126, Appendix B] for more details.

[54] The terminology originates from the fact that a large part of the theory was designed to handle precisely the compressible Navier–Stokes systems. We refer the interested reader to [119, Section 6] for a general multidimensional version and [208] for a simple two-dimensional implementation.

instead of weakly-interacting ones. We emphasize that the classification above has been so far as we know verified only for symmetrizable hyperbolic-parabolic systems satisfying a Kawashima structural condition. Again, although it is unclear to us whether our formally obtained Whitham systems all satisfy such conditions, we know that, by applying to them, on formal grounds, the asymptotic-equivalence reduction, we do obtain a system satisfying such conditions and providing the correct asymptotic behavior. Since we may safely ignore these technical details, henceforth we will not recall that when performing nonlinear validations we choose one of the systems (1.4.6) that satisfies suitable conditions ensuring well-posedness.

Though it is plausible that one may build family of solutions satisfying the explored expansions it may be unclear to the reader why we expect that *any* solution starting in a neighborhood of a stable periodic wave will match such expansions in the large-time limit. A similar justification is well-known for parabolic[55] systems where any solution starting near a stable constant state is eventually slow. Obviously this remark directly applies to the modulation system (1.4.6). Likewise about stable waves exhibiting a spectrum similar to the one plotted in Figure 1.4 it is natural to expect that nearby solutions will be eventually of slow modulation type. Yet we warm the reader that this behavior is expected *in the end* and that formal asymptotics based on some slowness assumption fail to describe how a given initial data incorporating fast modes yield an effective[56] initial data for the eventual asymptotics.

To conclude and before starting a precise account of the rest of the memoir, we recall that we will use the above formal expansions to provide

- necessary stability conditions and spectral critical expansions corresponding to side-band — that is, low-Floquet — perturbations of modulation type;
- large-time nonlinear asymptotic behavior for diffusively spectrally stable waves, through a slow-modulation evolution with local parameters developing weakly-interacting diffusion-waves.

In particular, we will not even try to give any direct nonlinear justification of the modulation systems. Nevertheless the interested reader may found various such justifications in [66] for reaction-diffusion systems (1.1.13) yielding (1.4.3) as a first-order Whitham equation and in [69] for a nonlinear Schrödinger equation.

## 1.5. Outline of the content

Now that we have gathered all the pieces of the picture, we outline the content of the rest of the memoir. It is divided in four chapters and two appendices.

[55] Again this includes symmetrizable hyperbolic-parabolic systems satisfying a genuine-coupling Kawashima condition. Indeed, for linearizations of such systems, at time $t$ Fourier modes of frequency $\xi$ are bounded by $M\,\exp(-\eta\,t\frac{|\xi|^2}{1+|\xi|^2})$, for some positive $M$ and $\eta$. See Appendix A

[56] This is the classical formulation of scattering problems. However, as we shall see, here the behavior albeit not eventually linear is simple enough to support explicit answers.

Next chapter, Chapter 2, is devoted to the presentation of [185, 20, 21, 125, 10] and concerns analytical results about spectral stability. It contains two sections. In the first some details are given about [20] and some comments are made about [185, 21], common feature being that they examine at the spectral level interconnections between modulation averaged systems and original systems. The second section focuses on KdV-limits near threshold of constant-state instability (and emergence of periodic waves) and the obtention of a simple criterion determining diffusive spectral stability for families of near-KdV waves. There, is detailed such a result for the KdV-KS equation (1.1.18) in the limit $\delta \to 0^+$ [125] and mentioned a similar result for the St. Venant system (1.1.17) in the limit $\mathbf{F} \to 2^+$ [10].

Chapter 3 offers a glimpse at [11, 10, 12, 13, 14] and is also split in two sections. The first one introduces the principles of numerical methods that enable us to perform numerical investigation of diffusive spectral stability of periodic waves of both (1.1.18) in [12, 13] and (1.1.17) in [11, 10]. Its second section briefly discusses connections, numerically investigated in [14], between convective nature of instability of solitary waves, stability of arrays of solitary waves and stability of (not so) long periodic traveling waves.

Chapter 4 shows how to use diffusive spectral stability and obtain nonlinear results. It introduces last stage of the theory as developed in [127, 128, 126]. Details are given for results on systems (1.1.14) [126] but are contrasted and put in parallel with similar results for simplest systems (1.1.13) [127, 128]. Once in a while, when we give some clues about how would work an extension of recent results to a larger class of systems, we also refer to [12, 13], that implements for equation (1.1.18) an earlier stage of the theory. Chapter 4 is divided in two sections, one dedicated to proof of nonlinear space-modulated stability and the other one devoted to nonlinear validation of the modulation theory and its consequences.

Chapter 5 is a prospective chapter. Appendix A is a terse introduction to the Kawashima condition for hyperbolic-parabolic systems and stability of constant states. Appendix B recalls topological nature of Evans' functions winding numbers. Strictly speaking none of the appendices is vital but we hope that by shedding some light on adjacent topics they may facilitate a good understanding of the main body of the memoir.

# 2. Spectral stability: analytical approaches

We report now on spectral investigations based on analytical arguments.

First, we discuss various stability criteria in relation with the Whitham systems introduced in Section 1.4. We report mostly on consequences for systems of type (1.1.2) that are derived in [20, 21]. Most of analogs for (1.1.14) were previously obtained by Denis Serre in [222]. Note however that, in order to prepare both nonlinear analysis and spectral inspections of KdV limits, various extensions that we will address only at further places of the memoir were thoroughly investigated in [185].

Second, we analyze KdV limits — $\delta \to 0^+$ for (1.1.18) and $\mathbf{F} \to 2^+$ for (1.1.15)/(1.1.17) — to yield evaluable stability criteria for near-KdV periodic traveling waves.

## 2.1. About averaged equations

### Perturbations of slow modulation type

We now make explicit a set of assumptions that makes possible the use of standard perturbations arguments [146] to study (1.1.2) in the Floquet-Bloch framework. For concreteness' sake we restrict ourselves to $\ell = 1$ but *mutatis mutandis* arguments do apply to the general case. Therefore we focus on

$$\partial_t \mathbf{U} = \mathscr{J}\left(\mathsf{E}\mathscr{H}[\mathbf{U}]\right), \tag{2.1.1}$$

where $\mathscr{J} = \partial_x \mathbf{J}$ is a skew-symmetric differential operator, $\mathbf{J}$ being a symmetric, nonsingular matrix with constant coefficients, $\mathscr{H}$ is a functional involving first order derivatives only, and $\mathsf{E}$ denotes again the Euler operator. Moreover inspired by our examples (1.1.7)–(1.1.9), (1.1.8)–(1.1.10), and (1.1.6), we write the $\mathbf{U}$-space as $\mathbf{R}^d = \mathbf{R}^{d'} \times \mathbf{R}^{d-d'}$ for some integer $d'$, $0 \leq d' \leq d$, require that

$$\mathbf{U} = \begin{pmatrix} \mathbf{v} \\ \mathbf{u} \end{pmatrix}, \quad \mathscr{H}(\mathbf{U}) = \mathscr{I}(\mathbf{v}, \mathbf{u}) + \mathscr{E}(\mathbf{v}, \mathbf{v}_x),$$

and assume that $\mathscr{H} + c\mathscr{Q}$ is uniformly strongly convex in both $\mathbf{v}_x$ and $\mathbf{u}$ on the range of $(\mathbf{U}, \mathbf{v}_x)$-values and speeds $c$ under consideration. A simple way to make this assumption independent of $c$ is to assume that $\mathbf{J}^{-1}$ has a block structure of the form

$$\mathbf{J}^{-1} = \left(\begin{array}{c|c} * & * \\ \hline * & 0_{(d-d')\times(d-d')} \end{array}\right),$$

as is the case for the Euler–Korteweg system or the generalized Korteweg–de Vries equation.

We fix a wave profile $\underline{\mathbf{U}}$ of parameters $(\underline{k}, \underline{\mathbf{M}}, \underline{P})$ with wavenumber $\underline{k}$ and averages $(\underline{\mathbf{M}}, \underline{P}) = (\langle \underline{\mathbf{U}} \rangle, \langle \mathscr{Q}[\underline{\mathbf{U}}] \rangle)$ and define the operator

$$L \ := \ \underline{k}\,\mathscr{J}\,\mathbf{L}, \quad \text{with} \quad \mathbf{L} := \mathsf{Hess}(\mathscr{H}_{\underline{k}} + c\mathscr{Q})[\underline{\mathbf{U}}],$$

on $L^2(\mathbf{R};\mathbf{R}^d)$ with domain $H^3(\mathbf{R};\mathbf{R}^{d'}) \times H^1(\mathbf{R};\mathbf{R}^{d-d'})$. The Hessians here above are given by

$$\mathsf{Hess}\mathscr{Q}[\mathbf{U}] = \mathbf{J}^{-1}$$

whatever $\underline{\mathbf{U}}$, and[1]

$$\begin{aligned}(\mathsf{Hess}\mathscr{H}_{\underline{k}}[\underline{\mathbf{U}}]\mathbf{U})_\alpha \ &= \ \frac{\partial^2 \mathscr{H}_{\underline{k}}}{\partial U_\alpha \partial U_\beta} U_\beta \ + \ \frac{\partial^2 \mathscr{H}_{\underline{k}}}{\partial U_\alpha \partial U_{\beta,x}} U_{\beta,x} \\ &- \ \partial_x \left( \frac{\partial^2 \mathscr{H}_{\underline{k}}}{\partial U_{\alpha,x} \partial U_\beta} U_\beta \ + \ \frac{\partial^2 \mathscr{H}_{\underline{k}}}{\partial U_{\alpha,x} \partial U_{\beta,x}} U_{\beta,x} \right) ,\end{aligned}$$

where all second order derivatives of $\mathscr{H}_{\underline{k}}$ are evaluated at $(\underline{\mathbf{U}}, \underline{\mathbf{U}}_x)$. Correspondingly, for any $\xi \in [-\pi, \pi)$, $L_\xi$ is an operator on $L^2(\mathbf{R}/\mathbf{Z};\mathbf{R}^d)$ with domain $H^3(\mathbf{R}/\mathbf{Z};\mathbf{R}^{d'}) \times H^1(\mathbf{R}/\mathbf{Z};\mathbf{R}^{d-d'})$. We recall that in the present memoir for waves of systems of type (2.1.1) spectral stability means simply

$$\sigma(L) \ \subset \ \{ \, z \in \mathbf{C} \, | \, \mathrm{Re}\, z \ \leq \ 0 \, \} \ .$$

We now state the first form of justification of averaged modulation systems contained in the present memoir. Albeit not surprising in view of previously known case-by-case results, for instance those of [134] for the generalized Korteweg–de Vries equations, the following result is, to the knowledge of the author, the first attempt to put on a par the spectral modulation theory for (1.1.2) with the one for (1.1.14) where results from [222] do apply.

**Theorem 2.1.1** ([20])**.** *If*

- *the set of periodic profiles near* $\underline{\mathbf{U}}$ *is a* $(d+2)$*-dimensional manifold parametrized by* $(k, \mathbf{M}, P)$*,*
- *and the generalized kernel of* $L_0$ *is* $(d+2)$*-dimensional,*

*the eigenvalue* $\lambda = 0$ *of* $L_\xi$ *at* $\xi = 0$ *bifurcates into* $(d+2)$ *continuous spectral curves* $\lambda_1$*, ...,* $\lambda_{d+2}$ *that are differentiable at* $\xi = 0$

$$\lambda_\alpha(\xi) = -i\underline{k}\xi\ a_\alpha + o(\xi), \quad \alpha = 1, \ldots, d+2, \tag{2.1.2}$$

*where* $a_\alpha$*s are the eigenvalues of* $\partial_{(k,\mathbf{M},P)}(\omega, \mathbf{JG}, S)|_{(\underline{k},\underline{\mathbf{M}},\underline{P})} + c\mathrm{I}_{d+2}$*. In particular, a* necessary condition *for* $\underline{\mathbf{U}}$ *to be stable is that the modulated system (1.4.1) be* weakly hyperbolic[2] *at* $(\underline{k}, \underline{\mathbf{M}}, \underline{P})$*, that is, that all characteristic speeds* $a_\alpha + c$ *of (1.4.1) be real.*

[1] Here we are using Einstein's convention of summation over repeated indices, and we shall do so repeatedly in the sequel.

[2] Full hyperbolicity requiring of course also semisimplicity.

In general there is a Jordan block at $\xi = 0$ so that the differentiability at $\xi = 0$ would be rather unexpected without the connection to an averaged modulation system that is itself uncovered by a desingularization process.

Assumptions of Theorem 2.1.1 may be reformulated and recasted into

- the set of periodic profiles near $\underline{\mathbf{U}}$ is a $(d+2)$-dimensional manifold;
- the Whitham system (1.4.1), considered as an equation on the manifold of periodic profiles, is of evolution type;
- at the linearized level any steady, co-moving[3] and co-periodic perturbation is of modulation type.

Yet, as we shall see, assuming the first item, the last ones are equivalent. Since a count of dimensions also proves that the second item yields the first, there is indeed only one assumption in Theorem 2.1.1. Our numerical investigations on the Euler–Korteweg system shows the generic character of the present assumptions.

We prove a nonlinear connection between the averaged modulation systems and the original problem only around *stable* waves. However, at the linearized level, we also observe that, in the case that characteristic speeds are complex, that is, weak hyperbolicity fails for (1.4.1) at $(\underline{k}, \underline{\mathbf{M}}, \underline{P})$, then constant solutions of (1.4.1) are time-exponentially unstable under perfectly nice[4] perturbations, and, correspondingly, background periodic waves are spectrally unstable, so information from (1.4.1) is still in some sense consistent with behavior.

We now sketch main lines of the proof.

*Proof.* It is based on a perturbation calculation, which relates the matrix of (1.4.1) at $(\underline{k}, \underline{\mathbf{M}}, \underline{P})$ to the one of $L_\xi$ restricted by spectral projection to a $(d+2)$-dimensional invariant subspace.

We first introduce the expansion

$$L_\xi \,=\, L^{(0)} \,+\, \mathrm{i}\,\underline{k}\xi\, L^{(1)} \,+\, (\mathrm{i}\,\underline{k}\xi)^2\, L^{(2)} \,+\, (\mathrm{i}\,\underline{k}\xi)^3\, L^{(3)}\,,$$

where $L^{(0)} = L_0 = \underline{k}\mathbf{J}\partial_x\mathbf{L}^{(0)}$ is just $L$ viewed as an operator acting on 1-periodic functions, as well as $\mathbf{L}^{(0)}$ is just $\mathbf{L}$ acting on 1-periodic functions. Now differentiating profile equation

$$k\mathbf{J}\partial_x\left(\nabla_U\mathscr{H}(\underline{\mathbf{U}}, k\partial_x\underline{\mathbf{U}}) - \underline{k}\partial_x\left(\nabla_{U_x}\mathscr{H}(\underline{\mathbf{U}}, k\partial_x\underline{\mathbf{U}})\right) + c\,\nabla_U\mathscr{Q}(\underline{\mathbf{U}})\right) = 0\,,$$

with respect to parameters yield $L^{(0)}\underline{\mathbf{U}}_x \,=\, 0$,

(2.1.3) $$L^{(0)}\underline{\mathbf{U}}_{M_\alpha} \,=\, -\,\underline{k}\partial_{M_\alpha}c\;\underline{\mathbf{U}}_x\,,\ \alpha\in\{1,\dots,d\}\,,\qquad L^{(0)}\underline{\mathbf{U}}_P \,=\, -\,\underline{k}\partial_P c\;\underline{\mathbf{U}}_x\,,$$

and

(2.1.4) $$L^{(0)}\underline{\mathbf{U}}_k \,+\, L^{(1)}\underline{\mathbf{U}}_x \,=\, -\,\underline{k}\partial_k c\;\underline{\mathbf{U}}_x\,,$$

[3]Equations are written in the frame of the wave.

[4]Say, localized as a Gaussian in both physical and Fourier spaces.

where relations are evaluated at $(k,\mathbf{M},P)=(\underline{k},\underline{\mathbf{M}},\underline{P})$. Even more directly each conservation law comes with an element of the kernel of the adjoint operator $(L^{(0)})^*$, hence

$$(L^{(0)})^*\mathrm{e}_\alpha\,=\,0\,,\ \alpha\in\{1,\dots,d\}\,,\qquad (L^{(0)})^*\mathbf{J}^{-1}\underline{\mathbf{U}}\,=\,0\,,$$

where $\mathrm{e}_\alpha$ denotes the constant function with value the $\alpha$th vector of the canonical basis of $\mathbf{R}^d$. We set

$$\mathbf{q}_0^0=\underline{\mathbf{U}}_x\,,\qquad \mathbf{q}_\alpha^0=\underline{\mathbf{U}}_{M_\alpha}\,,\ \alpha\in\{1,\dots,d\}\,,\quad\text{and}\quad \mathbf{q}_{d+1}^0=\underline{\mathbf{U}}_P$$

on one hand, and

$$\widetilde{\mathbf{q}}_\alpha^0=\mathrm{e}_\alpha\,,\ \alpha\in\{1,\dots,d\}\,,\quad\text{and}\quad \widetilde{\mathbf{q}}_{d+1}^0=\mathbf{J}^{-1}\underline{\mathbf{U}}$$

on the other hand. In particular by definition of the parametrization $\langle\widetilde{\mathbf{q}}_\alpha^0,\mathbf{q}_\beta^0\rangle=\delta_{\alpha,\beta}$ for any $(\alpha,\beta)$. Hence $(\mathbf{q}_0^0,\mathbf{q}_1^0,\dots,\mathbf{q}_d^0,\mathbf{q}_{d+1}^0)$ spans the generalized kernel of $L^{(0)}$ and we can add in a function $\widetilde{\mathbf{q}}_0^0$ such that $(\mathbf{q}_0^0,\mathbf{q}_1^0,\dots,\mathbf{q}_d^0,\mathbf{q}_{d+1}^0)$ be dual to the basis $(\widetilde{\mathbf{q}}_0^0,\widetilde{\mathbf{q}}_1^0,\dots,\widetilde{\mathbf{q}}_d^0,\widetilde{\mathbf{q}}_{d+1}^0)$ of the generalized kernel of $(L^{(0)})^*$.

Since our structural assumptions ensure that $L_\xi$ is a relatively compact perturbation of $L^{(0)}$ depending analytically on $\xi$ (see [20, Appendix B.1]), in a neighborhood of the origin there exist an analytic mapping $\xi\mapsto\Pi(\xi)$ where $\Pi(\xi)$ is a spectral projector for $L_\xi$ of finite rank $d+2$, associated with the spectrum of $L_\xi$ in some fixed neighborhood of the origin. By Kato's perturbation method [146, pp. 99-100], we thus extend previous bases and construct dual bases $(\mathbf{q}_0(\xi),\mathbf{q}_1(\xi),\dots,\mathbf{q}_d(\xi),\mathbf{q}_{d+1}(\xi))$ and $(\widetilde{\mathbf{q}}_0(\xi),\widetilde{\mathbf{q}}_1(\xi),\dots,\widetilde{\mathbf{q}}_d(\xi),\widetilde{\mathbf{q}}_{d+1}(\xi))$ of, respectively, $\mathsf{R}(\Pi(\xi))$ and $\mathsf{R}(\Pi(\xi)^*)$, which depend analytically on $\xi$ in a real neighborhood of zero. The part of $L_\xi$ on the finite dimensional subspace $\mathsf{R}(\Pi(\xi))$ is determined by the matrix

$$\Lambda_\xi:=(\langle\widetilde{\mathbf{q}}_\alpha(\xi),L_\xi\mathbf{q}_\beta(\xi)\rangle)_{0\leq\alpha,\beta\leq d+1}\,.$$

Similarly as $L_\xi$, this matrix has an expansion

$$\Lambda_\xi=\Lambda^{(0)}\,+\,\mathrm{i}\,\underline{k}\xi\,\Lambda^{(1)}\,+\,(\mathrm{i}\,\underline{k}\xi)^2\,\Lambda^{(2)}+o(\xi^2)\,.$$

Inspired by the derivation of the Whitham system that proceeds by differentiating the equation for the local phase while keeping equations for local means untouched, we introduce

$$\Sigma(\xi)=\left(\begin{array}{c|ccc}\mathrm{i}\underline{k}\xi & 0 & \dots & 0\\\hline 0 & & & \\ \vdots & & \mathrm{I}_{d+1} & \\ 0 & & & \end{array}\right),$$

and set

$$\widetilde{\Lambda}_\xi=\frac{1}{\mathrm{i}\underline{k}\xi}\,\Sigma(\xi)^{-1}\Lambda_\xi\Sigma(\xi).$$

Our goal is then to prove that $\widetilde{\Lambda}_\xi$ is still an analytic function of $\xi$ and that $\widetilde{\Lambda}_0$ is $\partial_{(k,\mathbf{M},P)}(\omega,\mathbf{JG},S)|_{(\underline{k},\underline{\mathbf{M}},\underline{P})}+c\mathrm{I}_{d+2}$. This would achieve the proof of Theorem 2.1.1. Now the first point follows from

$$\Lambda^{(0)} \;=\; \left(\begin{array}{c|cccc} 0 & -\underline{k}\,\partial_{M_1}c & \dots & -\underline{k}\,\partial_{M_d}c & -\underline{k}\,\partial_P c \\ \hline \vdots & & & 0 & \\ 0 & & & & \end{array}\right),$$

$$\Lambda^{(1)} \;=\; \left(\begin{array}{c|ccc} * & * & \dots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array}\right),$$

which stem from averaging (2.1.3) and (2.1.4).

To achieve the proof we need a better knowledge of $\partial_\xi \mathbf{q}_0(0)$. We first observe that by expanding

$$\Pi(\xi)\, L_\xi \mathbf{q}_0(\xi) = L_\xi \mathbf{q}_0(\xi)$$

and using (2.1.4) and $\mathbf{q}_0(0)\in\mathsf{R}(\Pi(0))$ we obtain

$$L^{(0)}\,(\partial_\xi \mathbf{q}_0(0)\;-\;\mathrm{i}\underline{k}\,\mathbf{U}_k)\in\mathsf{R}(\Pi(0))\;=\;\ker((L^{(0)})^2)\;=\;\ker((L^{(0)})^3)\,,$$

hence

$$\partial_\xi \mathbf{q}_0(0)\;-\;\mathrm{i}\underline{k}\,\mathbf{U}_k\in\mathsf{R}(\Pi(0))\,.$$

Up to harmless modifications of $\mathbf{q}_0(\xi)$ and $\widetilde{\mathbf{q}}_\alpha(\xi)$, $\alpha\in\{0,\dots,d+1\}$, one may then ensure

$$\partial_\xi \mathbf{q}_0(0)\;=\;\mathrm{i}\underline{k}\,\mathbf{U}_k.$$

With this in hands, the rest of the proof follows by lengthy and tedious identifications. □

It should be clear that in some cases more may be obtained from the proof of Theorem 2.1.1. For instance, for $\xi\neq0$, if $\widetilde{\Lambda}_\xi$ is diagonalizable then so is the part of $L_\xi$ on $\mathsf{R}(\Pi(\xi))$ and there is a one-to-one mapping. This is in particular the case for $\xi$ small enough when (1.4.1) is strictly hyperbolic at $(\underline{k},\underline{\mathbf{M}},\underline{P})$, that is, when $a_\alpha$s are distinct. In the latter case, identification of higher-order expansions of $\lambda_\alpha$ with low-frequency expansions of eigenvalues of higher-order versions of (1.4.1) may be continued. See [185] for details illustrated on equation (1.1.18). All these points turn to be crucial in nonlinear validation of (1.4.6) as a modulation average system for (1.1.1).

As appears in the proof, the instability provided by failure of weak hyperbolicity of (1.4.1) is not arbitrary but sideband, that is, due to Bloch-waves of Floquet arbitrary small but nonzero, and of modulation type. Such instabilities are called modulational instabilities. For more on these subjects, we refer the reader to the historical review [255] and detailed discussion and references in the proof of the famous Benjamin–Feir instability of Stokes water-waves [34].

## Co-periodic perturbations

Although Theorem 2.1.1 extends some results of [222], its proof is not an extension of the proof given therein for systems of conservation law. Such an extension is possible and our choice of a different proof is only commanded by our will to get a result ready to serve nonlinear purposes. However we note also incidentally that proofs following from Bloch-wave decompositions are more likely[5] to be adapted to higher-dimensional situations, for instance for multiperiodic waves, than those stemming from Evans function computations as developed in [222]. For the generalized Korteweg–de Vries equations both strategies have been fully implemented, the one of Denis Serre in [134] and the one by Bloch inspections in [130].

We confess though that obtained results are not exactly of the same nature and something is lost when choosing the Bloch-transform way. Indeed the proof requires a knowledge of the dimension of the generalized kernel of $L_0$. In contrast, alternative statement that shows some expansion for an Evans function for small eigenvalues and small Floquet exponent $\xi$ only assumes that the set of periodic profiles near $\underline{\mathbf{U}}$ have the expected dimension[6].

Motivated by these considerations, we know discuss the extra items of our set of assumptions, parametrization by natural modulation parameters and minimal dimension of the generalized co-periodic kernel, and prove that they are equivalent. In the meantime, we provide the reader with a glimpse at what would be an alternative statement and proof by Evans function computations of which the following proof is an instance specialized to $\xi = 0$.

Since we make no use of any Bloch transform, we undo the normalization by wavenumber of periodic profiles. To set things on a more formal ground, we also define on some open neighborhood $\mathcal{U}$ of wave values $(\underline{\Xi}, \underline{c}, \underline{\mathbf{v}}(0), \underline{\mathbf{v}}_x(0), \underline{\boldsymbol{\lambda}})$ the map

$$\mathcal{R} : \qquad \begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathbf{R}^{2d'}, \\ (\Xi, c, \mathbf{v}_0, \mathbf{v}_{0,x}, \boldsymbol{\lambda}) & \longmapsto & ([\mathbf{v}]_0^\Xi, [\mathbf{v}_x]_0^\Xi) \end{array}$$

where $[\,\cdot\,]_0^\Xi$ denotes the jump $[f]_0^\Xi = f(\Xi) - f(0)$, and $\mathbf{U}$ is the solution of

$$\mathsf{E}(\mathscr{H} + c\mathscr{Q})[\mathbf{U}] = \boldsymbol{\lambda}, \quad \mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{v}_x(0) = \mathbf{v}_{0,x},$$

and we identify in the usual way nearby periodic traveling wave profiles with elements of the zero set of $\mathcal{R}$.

**Theorem 2.1.2.** *Assume that $\underline{\mathbf{U}}$ is a non trivial periodic wave and that jump map $\mathcal{R}$ has constant rank $2d' - 1$.*
*Then the generalized kernel of $L_0$ is of dimension $d + 2$ if and only if, up to translation, nearby periodic traveling wave profiles may be regularly parametrized by $(k, \mathbf{M}, P)$.*

---

[5] Though many higher-dimensional extensions of Evans functions — for instance by Fredholm determinants [63, 64] or by Galerkin approximations [187] — exist, their practical use is harder to implement than for their one-dimensional counterparts.

[6] For (1.1.2) $d + 2$, for (1.1.14) $d + 1$. The precise assumption is that the corresponding jump map have the maximal rank — respectively $2d' - 1$ and $d$ — allowed by local conservations supported by the system.

*Proof.* Our proof is based upon the fact that the dimension of the generalized kernel of $L_0$ is the algebraic multiplicity of zero as a root of some Evans function $D(\,\cdot\,)$ (see [95]). Indeed, viewing spectral problem

$$z\,\mathbf{V} \;=\; L\mathbf{V} \tag{2.1.5}$$

for $(z,\mathbf{V}) = (z,(\mathbf{v},\mathbf{u})^T)$ as a system of coupled differential equations of third-order in $\mathbf{v}$ and first-order in $\mathbf{u}$, we may introduce its fundamental solution $R(z;\cdot)$ normalized by $R(z;0) = \mathrm{Id}_{\mathbf{R}^{3d'}\times\mathbf{R}^{(d-d')}}$ and define

$$D(z) \;=\; \det([R(z;\,\cdot\,)]_0^{\Xi}).$$

Then the condition on the dimension of the generalized kernel of $L_0$ reads

$$D(z) \;=\; a\,z^{d+2} \;+\; \mathcal{O}(z^{d+3})$$

for some nonzero $a$ [95].

Let us denote by $\mathbf{V}^j(z;\cdot)$ the solution to (2.1.5) corresponding to the $j$-th column of the matrix $R(z;\cdot)$, that is $\mathbf{V}^j(z;\cdot)$ solves (2.1.5) and the vector $(\mathbf{v}^j(z;0),\mathbf{v}^j_x(z;0),\mathbf{v}^j_{xx}(z;0),\mathbf{u}^j(z;0))^T$ is the $j$-th vector of the canonical basis of $\mathbf{R}^{3d'}\times\mathbf{R}^{(d-d')}$. The Evans function is then written

$$D(z) \;=\; \begin{vmatrix} [\,\mathbf{v}^1] & \cdots & [\,\mathbf{v}^{d+2d'}] \\ [\,\mathbf{v}^1_x] & \cdots & [\,\mathbf{v}^{d+2d'}_x] \\ [\,\mathbf{v}^1_{xx}] & \cdots & [\,\mathbf{v}^{d+2d'}_{xx}] \\ [\,\mathbf{u}^1] & \cdots & [\,\mathbf{u}^{d+2d'}] \end{vmatrix}$$

where we have dropped the marks 0 and $\Xi$ on jumps.

Integrating (2.1.5) from 0 to $\Xi$ yields

$$z\int_0^{\Xi}\mathbf{V}^j \;=\; \mathbf{J}\begin{pmatrix} \sigma_{\mathbf{v}}(0)[\mathbf{v}^j_{xx}] + \;*[\mathbf{v}^j_x] + *[\mathbf{v}^j] + *[\mathbf{u}^j] \\ \sigma_{\mathbf{u}}(0)[\mathbf{u}^j] + \;*[\mathbf{v}^j] \end{pmatrix}.$$

where

$$\sigma_{\mathbf{v}} \;:=\; \left(-\frac{\partial^2\mathscr{H}}{\partial v_{\alpha,x}\partial v_{\beta,x}}(\underline{\mathbf{U}},\underline{\mathbf{U}}_x)\right)_{\alpha,\beta} \quad\text{and}\quad \sigma_{\mathbf{u}} \;:=\; \left(\frac{\partial^2\mathscr{H}}{\partial u_\alpha u_\beta}(\underline{\mathbf{U}},\underline{\mathbf{U}}_x)\right)_{\alpha,\beta}$$

are uniformly invertible. Similarly equation (2.1.5) comes with an impulse equation that integrates to

$$z\int_0^{\Xi}\frac{\partial\mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\;(\mathbf{V}^j_\alpha) \;=\; (\sigma_{\mathbf{v}}\underline{\mathbf{v}}_x)(0)\cdot[\mathbf{v}^j_x] + \;*\;[\mathbf{v}^j] + \;*\;z\int_0^{\Xi}\mathbf{V}^j.$$

To check that it is not a trivial relation, observe that, since $\underline{\mathbf{U}}$ is non trivial, $\underline{\mathbf{v}}_x(0)$ may be assumed to be nonzero by translation invariance.

Now let us pick $\ell$ such that the $\ell$-th component of $\sigma_{\mathbf{v}}(0)\underline{\mathbf{v}}_x(0)$ is nonzero and, for any $\mathbf{V} = (\mathbf{v},\mathbf{u})^T \in \mathbf{R}^d = \mathbf{R}^{d'}\times\mathbf{R}^{(d-d')}$, denote by $\mathbf{v}_*$ the vector of $\mathbf{R}^{(d'-1)}$ obtained from

$\mathbf{v}$ by deleting the $\ell$-th component. Then, up to a nonzero multiplicative constant, by elementary row operations $D(z)$ is transformed to

$$z^{d+1}\begin{vmatrix} [\mathbf{v}^1] & \cdots & [\mathbf{v}^{d+2d'}] \\ [(\mathbf{v}_*^1)_x] & \cdots & [(\mathbf{v}_*^{d+2d'})_x] \\ \int_0^\Xi \frac{\partial \mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\,(\mathbf{V}_\alpha^1) & \cdots & \int_0^\Xi \frac{\partial \mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\,(\mathbf{V}_\alpha^{d+2d'}) \\ \int_0^\Xi \mathbf{V}^1 & \cdots & \int_0^\Xi \mathbf{V}^{d+2d'} \end{vmatrix}.$$

Up to a change of basis we may also assume that $\mathbf{V}^1(0\,;\cdot) = \underline{\mathbf{U}}_x$, and for instance $(\mathbf{V}^1(0\,;\cdot),\cdots,\mathbf{V}^{2d'}(0\,;\cdot))$ is a basis of

$$\mathrm{Span}\,(\{\mathbf{U}_{(\mathbf{v}_0)_1},\cdots,\mathbf{U}_{(\mathbf{v}_0)_{d'}}\mathbf{U}_{(\mathbf{v}_{0,x})_1},\cdots,\mathbf{U}_{(\mathbf{v}_{0,x})_{d'}}\})$$

and $\mathbf{V}^j(0\,;\cdot) = \mathbf{U}_{\lambda_{j-2d'}}$ for $2d'+1 \leq j \leq d+2d'$. Now the first column vanishes at $z=0$ and $\mathbf{V}_z^1(0;\cdot)$ differs from $\mathbf{U}_c$ by an element of the kernel of $L$ which is spanned by $(\mathbf{V}^1(0\,;\cdot),\cdots,\mathbf{V}^{d+2d'}(0\,;\cdot))$. Hence, by expanding the first column and performing elementary column operations, $D(z)$ is written, up to a multiplicative nonzero constant and an additive remainder $\mathcal{O}(z^{d+3})$, as

$$z^{d+2}\begin{vmatrix} [\mathbf{v}_c] & [\mathbf{v}^2] & \cdots & [\mathbf{v}^{d+2d'}] \\ [((\mathbf{v}_*)_c)_x] & [(\mathbf{v}_*^2)_x] & \cdots & [(\mathbf{v}_*^{d+2d'})_x] \\ \int_0^\Xi \frac{\partial \mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\,((\mathbf{U}_c)_\alpha) & \int_0^\Xi \frac{\partial \mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\,(\mathbf{V}_\alpha^2) & \cdots & \int_0^\Xi \frac{\partial \mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\,(\mathbf{V}_\alpha^{d+2d'}) \\ \int_0^\Xi \mathbf{U}_c & \int_0^\Xi \mathbf{V}^2 & \cdots & \int_0^\Xi \mathbf{V}^{d+2d'} \end{vmatrix}$$

or as $z^{d+2}$ times

$$\begin{vmatrix} \underline{\mathbf{v}}_x(0) & [\mathbf{v}_c] & [\mathbf{v}^2(0\,;\cdot)] & \cdots & [\mathbf{v}^{d+2d'}(0\,;\cdot)] \\ (\underline{\mathbf{v}}_*)_{xx}(0) & [((\mathbf{v}_*)_c)_x] & [(\mathbf{v}_*^2)_x(0\,\cdot)] & \cdots & [(\mathbf{v}_*^{d+2d'})_x(0\,;\cdot)] \\ 1 & 0 & 0 & \cdots & 0 \\ \mathscr{Q}(\underline{\mathbf{U}})(0) & \int_0^\Xi \frac{\partial \mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\,((\mathbf{U}_c)_\alpha) & \int_0^\Xi \frac{\partial \mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\,(\mathbf{V}_\alpha^2(0\,;\cdot)) & \cdots & \int_0^\Xi \frac{\partial \mathscr{Q}}{\partial U_\alpha}(\underline{\mathbf{U}})\,(\mathbf{V}_\alpha^{d+2d'}(0\,;\cdot)) \\ \underline{\mathbf{U}}(0) & \int_0^\Xi \mathbf{U}_c & \int_0^\Xi \mathbf{V}^2(0\,;\cdot) & \cdots & \int_0^\Xi \mathbf{V}^{d+2d'}(0\,;\cdot) \end{vmatrix}.$$

We are ready to complete the proof by observing the latter determinant. Indeed our assumption on $\mathcal{R}$ implies that the $(2d'-1)$-st rows of the above matrix are linearly independent. Furthermore, the kernel of the corresponding linear map is the tangent space at $\underline{\mathbf{U}}$ of the profiles manifold (profiles being identified when equal up to translation). Thus the differential map of $\mathbf{U} \mapsto (\Xi, \int_0^\Xi \mathscr{Q}(\mathbf{U}), \int_0^\Xi \mathbf{U})$ is invertible on this tangent space if and only if the above determinant is non zero. Consequently, this map is full-rank if and only if the generalized kernel of $L_0$ is of dimension $d+2$. □

Theorem 2.1.2 is focused on co-periodic perturbations and its proof is build around a co-periodic Evans function. To obtain an analog of Theorem 2.1.1 based on Evans functions, one needs to handle the full version of the Evans function $D(z,\xi)$ where co-periodic jumps are replaced with $[\,\cdot\,]_{0,\xi}^{\Xi}$ where $[f]_{0,\xi}^{\Xi} = f(\Xi) - \mathrm{e}^{\mathrm{i}\,\Xi\,\xi} f(0)$, $\xi \in [-\pi/\Xi, \pi/\Xi]$, and to expand in $(z,\xi)$ small. Up to a multiplicative nonzero constant and an additive remainder $\mathcal{O}(|z|^{d+3}+|\xi|^{d+3})$, this yields the dispersion relation of system (1.4.1) linearized at the wave profile, which is a homogeneous polynomial in $(z,\xi)$ of degree $d+2$.

An analysis of the co-periodic spectrum in a neighborhood of the origin can not yield alone any stability/instability criterion since 0 is isolated in the spectrum of $L_0$. Yet since $D(z)$ is real when $z$ is real, such a criterion is obtained by inspecting signs of $D(z)$ both when $z\to 0^+$ and $z\to+\infty$ ($z\in\mathbf{R}$). This strategy is by now well-known for many kind of patterns, see for instance [199, 2, 14] for applications to the study of point spectrum of solitary waves. It is implemented for periodic waves of the generalized Korteweg–de Vries equation in [37].

Part of [21] is devoted to the implementation of the strategy for periodic waves of (1.1.2) in the case where the reduced profile equation is actually a planar Hamiltonian ordinary differential equation ($d'=1$), a case that covers both the generalized Korteweg–de Vries equations and the Euler–Korteweg systems. There, obtained criteria are also compared with classical variational criteria [111, 112, 5, 202]. Observe that to determine the sign of $D(z)$ when $z\to 0^+$ we only need to change computations involved in the proof of Theorem 2.1.2 so as to obtain a fully explicit expression. In particular, to this end, in [21] we use the fact that $\mathbf{v}$ is scalar to apply the jump relations stemming from impulse conservation precisely at a point where $\underline{\mathbf{v}}_x$ vanishes, that is, we take benefit from the fact that we may assume $\underline{\mathbf{v}}_x(0)=0$. Concerning the limit $z\to+\infty$, observe that since we allow original equations to be quasilinear, passing to the limit in the profile differential equations does not lead to an autonomous differential system, in contrast with what occurs for the generalized Korteweg–de Vries equations. We instead rely on uniform spectrum localization (see [20, Appendix B.1]) to arrive by homotopy at an autonomous system.

### Eulerian/mass-Lagrangian intertwining

We now discuss concrete application to the Euler–Korteweg systems (1.1.7)–(1.1.9) and (1.1.8)–(1.1.10). A first natural task is to determine what remains of the Eulerian/mass-Lagrangian duality at the level of averaged modulation systems. We stress that, since the involved changes of variables are implicit in the sense that they depend on the solutions of nonlinear equations themselves, the question is far from trivial. To reinforce this conviction we point to the reader the fact that mass-Lagrangian changes of variables do not preserve the crucial property determining whether only one modulation wave may alter local phase/local wavenumber and hence deciding whether one may expect usual asymptotic behavior under localized perturbations or one should go to space-modulated asymptotic stability (see Section 4.2 and [126, Subsection 1.4] for details).

However, denoting by $(EKe)$ the Euler–Korteweg system in Eulerian formulation and by $(EK\ell)$ its mass-Lagrangian counterpart, we have the following elegant result.

**Theorem 2.1.3.** *The following diagram is commutative.*

$$\begin{array}{cccc} & & \textit{mass Lagrangian} & \\ & & \textit{change of coordinates} & \\ & (EKe) & \longrightarrow & (EK\ell) \\ \textit{Whitham's} & & & \\ \textit{averaging} & \downarrow & & \downarrow \\ & \langle EKe\rangle & \longrightarrow & \langle EK\ell\rangle \end{array}$$

For concision's sake, we do not give any clue about its proof. Yet we explain below what it means, by making explicit both what is the relation between periodic waves of both formulations (see [19]) and what is the mass-Lagrangian change of variable at the level of Whitham systems (see [20] for details). We also point that this nice intertwining extends to include the averaged version of energy conservation law (1.1.3). As already mentioned, the averaged energy equation provides an entropy for system (1.4.1). This can yield symmetrizability hence hyperbolicity when some convexity is met. Such a fact seems to have been pointed out and used for the first time in [100]. An extension of Theorem 2.1.3 includes that the corresponding convexity criterion is met simultaneously in Eulerian and mass-Lagrangian formulations.

Since we want to discuss parametrization from scratch, once again we undo wavenumber renormalization of profiles and we come back to natural parametrization by speed and constants of integration. Periodic traveling wave solutions to (1.1.7) and (1.1.8) are respectively of the form $(\rho, u) = (R, U)(x - \sigma t)$ and $(v, u) = (V, W)(y + jt)$, with a one-to-one mapping between the two frameworks encoded by

$$R(\xi)V(Z(\xi)) \,=\, 1\,,\; U(\xi) = W(Z(\xi))\,, \frac{\mathrm{d}Z}{\mathrm{d}\xi} \,=\, R \,=\, \frac{1}{V(Z)}\,.$$

Up to translations, these periodic traveling waves generically arise as four-parameter families. Natural parameters are

- their speed, that is $\sigma$ in Eulerian coordinates, and $-j$ in mass-Lagrangian coordinates ;
- a mass/volume constant of integration, which turns out to be $j$ in Eulerian coordinates, and $\sigma$ in mass-Lagrangian coordinates ;
- two other constants of integration/Lagrange multipliers, which we denote by $\lambda$ and $\mu$, in the profile equations.

To be more precise about the role of $j$ and $\sigma$, we note that

$$R\,(U\,-\,\sigma) \,\equiv\, j\,, \qquad W\,-\,j\,V \,\equiv\, \sigma\,.$$

A similar role interchange occurs for $\lambda$ and $\mu$. Moreover, there is a simple relationship between the mean values of Eulerian profiles and of mass-Lagrangian profiles. Indeed, if

$\Xi$ is the period of a traveling wave in Eulerian coordinates, the period of its counterpart in mass-Lagrangian coordinates is $Z(\Xi)$ (if $Z$ is chosen so that $Z(0) = 0$), and we have

$$\begin{aligned}\langle R\rangle &:= \frac{1}{\Xi}\int_0^{\Xi} R(\xi)\,\mathrm{d}\xi \;=\; \frac{Z(\Xi)}{\Xi}\,,\\ \langle U\rangle &:= \frac{1}{\Xi}\int_0^{\Xi} U(\xi)\,\mathrm{d}\xi \;=\; \frac{Z(\Xi)}{\Xi}\,\langle V\,W\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle V\rangle &:= \frac{1}{Z(\Xi)}\int_0^{Z(\Xi)} V(\zeta)\,\mathrm{d}\zeta \;=\; \frac{\Xi}{Z(\Xi)}\,,\\ \langle W\rangle &:= \frac{1}{Z(\Xi)}\int_0^{Z(\Xi)} W(\zeta)\,\mathrm{d}\zeta \;=\; \frac{\Xi}{Z(\Xi)}\,\langle R\,U\rangle\,,\end{aligned}$$

hence the remarkable identities

$$\langle R\rangle \;=\; \frac{1}{\langle V\rangle}\,,\quad \langle U\rangle \;=\; \frac{\langle V\,W\rangle}{\langle V\rangle}\,,\quad \langle W\rangle \;=\; \frac{\langle R\,U\rangle}{\langle R\rangle}\,,$$

on which is read the interchange between mean-value of the solution and mean-value of the impulse.

Concerning mass-average-Lagrangian change of variables, it is given by $(T, X) \leftrightarrow (T, Y)$ with

$$\mathrm{d}Y \;=\; \langle\rho_0\rangle\mathrm{d}X \;-\; \langle\rho_0 u_0\rangle\mathrm{d}T\,, \qquad \mathrm{d}X \;=\; \langle v_0\rangle\mathrm{d}Y \;+\; \langle w_0\rangle\mathrm{d}T\,,$$

where capitals letters $T$, $X$, $Y$, and the subscript $_0$ have the same meaning as in Section 1.4.

## A few numerical investigations

We go on with applications to the Euler–Korteweg systems. Having elucidated the relationship between Eulerian and mass-Lagrangian averaged systems, we now investigate directly their weak hyperbolicity.

Much is known about the Whitham systems either when original system (1.1.2) is completely integrable[7] or at both ends of the periodic family, that is, in the solitary-wave and small-amplitude limits[8] (see for instance [20, Subsection 2.3]). Incidentally we point to the reader that the above observations play a crucial role in the approximate asymptotic description of dispersive shocks [86, 164, 165, 166, 163, 106, 243, 242, 244, 82, 62, 230, 150, 110, 107, 108, 109, 45, 141, 81].

Outside these regimes various numerical investigations of hyperbolicity of the Whitham system (1.4.1) were performed to prepare [20] and some are reported therein. In particular are investigated both simplest cases where pressure law is strictly convex and cases

[7] In particular, in this case, the Whitham systems are rich in the sense of Denis Serre [221] or semi-Hamiltonian in the terminology of Sergueï Tsarëv [236, 237, 238].

[8] There the part of the systems that describes the time-evolution of $\mathcal{M}$ uncouples from the rest and, to obtain a nontrivial limit, $P = \frac{1}{2}\langle\underline{\mathbf{U}}\cdot\mathbf{J}^{-1}\underline{\mathbf{U}}\rangle$ should be replaced with a rescaled version of $\frac{1}{2}\langle\underline{\mathbf{U}}\cdot\mathbf{J}^{-1}\underline{\mathbf{U}}\rangle - \frac{1}{2}\langle\underline{\mathbf{U}}\rangle\cdot\mathbf{J}^{-1}\langle\underline{\mathbf{U}}\rangle$.

allowing for phase transitions, as associated with van der Waals pressure law. In the present memoir, we only offer two snapshots at these numerical experiments to illustrate that anything may indeed occur. They concern waves of equations with a pressure law obtained from a shallow thickness approximation $p(v) = -f'(v) = v^{-2}$ and constant $\kappa$. In each case, we work with mass-Lagrangian formulation, fix the three[9] constants of integration and let the period vary. We plot real parts of the characteristic eigenvalues of linearized modulation systems (1.4.1). When the four of them are distinct then the system is not only weakly hyperbolic but actually strictly hyperbolic. In Figure 2.1 the zone where two characteristic eigenvalues are conjugate indeed corresponds to a failure of weak hyperbolicity of the linearized Whitham system hence indicates instability of the underlying periodic waves. To compute the characteristics we invert the matrix in front of time-derivatives in the Whitham systems, therefore vertical line of Figure 2.2 that follows from non-invertibility shows a case where evolutionarity of the Whitham system fails, that is, a case where not all steady co-moving co-periodic perturbations are of modulation type. As expected it happens at an isolated point.

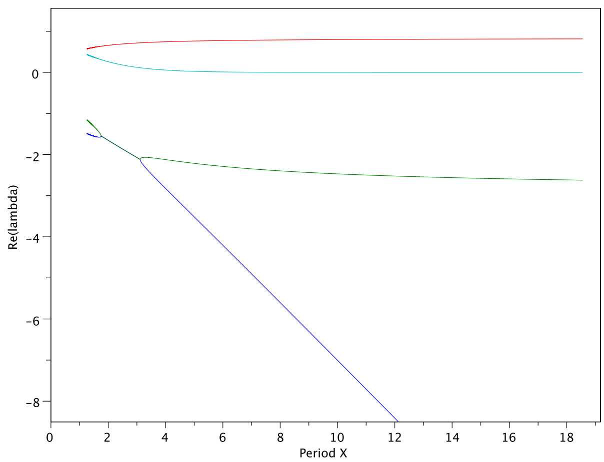



Figure 2.1.: Failure of hyperbolicity.

## 2.2. KdV limits

In determining spectral stability/instability, assets of strategies explicitly discussed above [20] or just mentioned [185, 21] are that they apply to very general classes of systems and have often nice formal interpretations. Yet, although specified areas of the Bloch spectrum that they investigate turn to be the zone determining asymptotic behavior

[9]Yet by Galilean invariance only two of them are relevant.

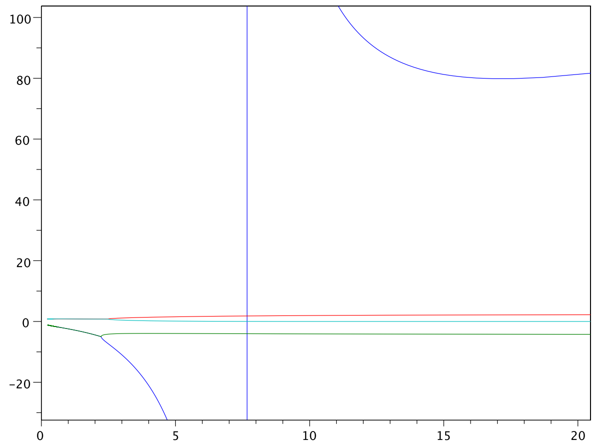



Figure 2.2.: Failure of evolutionarity.

about a diffusively-stable wave, they are able to provide a definitive answer only when indicating instability (or sideband stability, see [185]).

Now we want to derive a criterion able to preclude *any* possible instability and to provide diffusive stability as needed in our nonlinear analysis and still practical enough to be numerically decided. In our case, more is true. Prior to our rigorous proof, the obtained criterion have been already numerically investigated [8] and, following it, a *numerical proof* of spectral stability — proceeding by interval arithmetics — is in preparation [9].

Again, we stress that the goal of the present memoir is to introduce personal work of the author and the reader should not think that because we leave now the realm of spectral stability of periodic waves of Hamiltonian systems all interesting results of the field are attached to one of the families of results contained in [20, 21]. This is far from being true and, before going on, we point explicitly to the reader two other types of results, those examining spectral stability under subharmonic perturbations by looking at Krein signatures, as developed by Todd Kapitula and his collaborators [142, 115, 38], and those proving spectral stability of small solutions, as in work of Mariana Hărăguş and her collaborators [90, 116].

From now on we study specifically the spectral stability of a family of periodic wave-trains of (1.1.18) and (1.1.17) in the Korteweg–de Vries limits $\delta \to 0^+$ and $\mathbf{F} \to 2^+$. This is the limit in which (1.1.18) is expected to appropriately approximate (1.1.17) and indeed stability criteria match for associated families of waves. In doing so, we obtain definitive answers but restricted to specified models and limits, although the underlying strategy seems fairly general.

The study of the weakly unstable limit, *i.e.* near the onset of instability of spatially homogeneous solutions, naturally compares with similar results in the reaction-diffusion setting and related classical incompressible fluid flow problems by Alexander Mielke and many others [46, 175, 174, 176, 156, 157, 158, 139, 181], following the formative work of Wiktor Eckhaus [76, 77, 151]. However, there are three additional difficulties present here beyond what is faced therein. The first is — remembering that the analysis of [175, 174, 176] consists of *both* general analytical framework based on Bloch decomposition and Lyapunov reduction and delicate case-by-case computations for the resulting reduced problem, *each nontrivial* — that the presence of multiple critical modes makes the latter, matrix spectral bifurcation problem, considerably more complicated. The second and more daunting problem is that the limiting scenario is not a constant-coefficient eigenvalue problem obtained as in [175, 174, 176] by linearization around a spatially homogeneous state, but rather a periodic-coefficient eigenvalue equation obtained by linearization about Cnoidal solutions of the Korteweg–de Vries equation. At last, the third and probably not the least is that "dangerous" eigenvalues of the limiting problem that have to move to the left spectral half-plane are not finitely many but cover the full imaginary axis.

Now we introduce obtained results and details of the proof only for the limit $\delta \to 0^+$ of (1.1.18) treated in [125] but provide some formal elements of the analogous result for (1.1.15) contained in [10].

### From the KdV-KS equation

#### Statement

To state our main result more precisely, once again we first discuss parametrization. As we have already seen profiles of periodic traveling waves of the Korteweg–de Vries equation

$$\partial_t u \,+\, \partial_x(\tfrac{1}{2}\,u^2) \,+\, \partial_x^3 u \;=\; 0 \tag{2.2.1}$$

are expected to form when identified up to translation a three-dimensional manifold parametrized by period $\Xi$, mean $M = \langle \underline{u} \rangle$ and average value of the impulse $P = \frac{1}{2}\langle \underline{u}^2 \rangle$. Yet not all KdV waves yield near-KdV families of KdV-KS waves. Indeed, as easily seen by averaging against the profile itself, any periodic wave of (1.1.18) satisfies $\langle (\underline{u}')^2 \rangle = \langle (\underline{u}'')^2 \rangle$. It yields readily a selection criterion for possible KdV waves persisting as KdV-KS waves for $\delta$ small enough. It is proved in [83] that there is indeed a two-dimensional manifold of Cnoidal profiles of (2.2.1) that continues as profiles for (1.1.18), and parametrization is achieved by period and mean. The proof in [83] relies on geometric singular perturbation theory [85, 137] to build normally hyperbolic invariant manifolds and normal forms to study the reduced dynamics. As a consequence, smoothness of the profile/speed map $\delta \mapsto (\underline{u}_\delta, c_\delta)$ is limited albeit arbitrary. However, since our proof uses a finite amount of smoothness in $\delta$, one may safely ignore this technical detail.

Cnoidal waves are known to be spectrally stable [228, 159, 25] and, as expected from Hamiltonian time reversibility[10], the spectrum consists of the whole imaginary axis. In

[10]More precisely from Galilean invariance plus invariance of the equation and of suitable translates of

a first approach to the problem, one may fix a spectral couple eigenvalue – Floquet exponent $(\lambda_0,\xi)$ of the limiting KdV spectral problem and try to perform a perturbation analysis at fixed $\xi$ of the finite-multiplicity eigenvalue $\lambda_0$ as thoroughly developed in [146]. When $\lambda_0$ is simple, one may expect an expansion

$$\lambda(\xi,\lambda_0,\delta)=\lambda_0+\delta\lambda_1(\xi,\lambda_0)+\mathcal{O}(\delta^2)\,, \tag{2.2.2}$$

and, since $\lambda_0\in\mathbf{i}\mathbf{R}$,

$$\mathrm{Re}(\lambda(\xi,\lambda_0,\delta))\;=\;\delta\,\mathrm{Re}(\lambda_1(\xi,\lambda_0))+\mathcal{O}(\delta^2).$$

An important point is that, assuming (2.2.2), one may explicitly obtain $\lambda_1(\xi,\lambda_0)$ as a quotient of two integrals involving elliptic functions [125, Appendix A]. Hence the possibility to check whether, for any such $(\xi,\lambda_0)$, $\mathrm{Re}(\lambda_1(\xi,\lambda_0))\leq 0$. Arguing on formal grounds, Doron Bar and Alexander Nepomnyashchy derived the above-mentioned explicit formula and performed a numerical investigation of the corresponding — at the time formal — criteria determining which KdV wave generates a family of stable KdV-KS waves.

Beyond the natural task of proving (2.2.2) for simple couples $(\xi,\lambda_0)$, there are two major difficulties in making the full argument leading to spectral stability rigorous. The first is that $(0,0)$ is of algebraic multiplicity three so that hopes for simple expansions fall down there. The second originates in the need, to deduce a full stability for a given $\delta$-wave, of a form of uniformity in $\delta$ in the spectral expansions, hence the need to compactify the set of $(\xi,\lambda_0)$ to investigate, both at infinity and at the origin. Although to the knowledge of the author there is no proof of this simple fact, numerical investigations [25, 125] support that there is indeed no other simplicity breaking. To make this concrete, we plot in Figure 2.3 imaginary part of a portion of the KdV spectrum unfolded by Floquet exponents, here chosen in $[0,2\pi/\Xi]$. Three spectral curves passes through the origin $(0,0)$ but the rest is simply covered. Observe that on the imaginary axis in a neighborhood of the origin each eigenvalue corresponds to three different Floquet exponents, that then there are two symmetric turning points that correspond to two Floquet exponents and that each element of the rest is coupled with a simple Floquet exponent.

Let us now introduce notation enabling a statement of our main result concerning the KdV limit of (1.1.18). Due to Galilean invariance, spectral stability of a KdV-KS wave does not depend on its mean but only on its period. Hence everything is stated in terms of the period alone. Let $\mathcal{W}$ be the set of periods $\Xi$ such that there exist $(M,P)$ such that the KdV wave corresponding to $(\Xi,M,P)$ generates a family of KdV-KS wave. For each $\Xi\in\mathcal{W}$ we introduce a set of three conditions that when holding simultaneously lead to spectral stability. The two first conditions assume nondegeneracy of the KdV spectrum, the last one is the main one, and states that the (formal) stability criterion is satisfied.

We say that $\Xi\in\mathcal{W}$ satisfies (A1) if any nonzero spectral couple $(\xi,\lambda_0)$ is simple and eigenvalue 0 is attained only for $\xi=0$. The origin $(0,0)$ is known to be of algebraic multiplicity three and, as we have proved at the abstract level in Theorem 2.1.1, the three

the zero-mean profiles under $(t,x)\to(-t,-x)$.

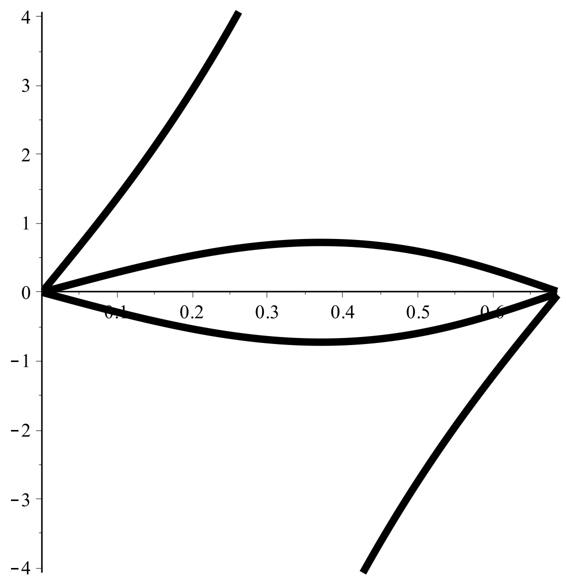



Figure 2.3.: KdV spectrum unfolded by Floquet exponents.

curves passing through zero are differential at 0 with respect to the Floquet exponent $\xi$. We say that $\Xi \in \mathcal{W}$ satisfies (A2) if the three derivatives are distinct. Note that since we already know that the linearized Whitham system is of evolution type and is weakly hyperbolic, it corresponds to assuming that it is strictly hyperbolic. This assumption ensures that at $\delta = 0$ the bifurcation around $(\xi, \lambda_0) = (0, 0)$ is analytic and offers a uniform control on the distance separating the three spectral curves as $\xi \to 0$.

If $\Xi \in \mathcal{W}$ satisfies (A1), on may define[11]

$$\mathrm{Ind}(\Xi) := \sup_{\substack{\lambda_0 \in \sigma(L_{\mathrm{KdV},\xi}) \setminus \{0\} \\ \xi \in [-\pi/\Xi, \pi/\Xi)}} \mathrm{Re}\left(\lambda_1(\xi, \lambda_0)\right) .$$

A $\Xi \in \mathcal{W}$ for which (A1) holds satisfies condition (A3) if $\mathrm{Ind}(\Xi) < 0$. The subset of periods $\Xi \in \mathcal{W}$ satisfying (A1), (A2) and (A3) is denoted by $\mathcal{W}_s$.

Note that $\mathcal{W}_s$ is open. Furthermore, it is natural to expect, based on the aforementioned numerical evidence [8], that the set $\mathcal{W}_s$ is an interval $(\Xi_1, \Xi_2)$ with $\Xi_1 \approx 8.49$ and $\Xi_2 \approx 26.17$. Now, we can state precisely the main result of our analysis.

**Theorem 2.2.1** ([125])**.** *For each $\Xi \in \mathcal{W}_s$, there exists a positive real number $\delta_0(\Xi)$ such that for each $0 < \delta < \delta_0(\Xi)$, $\Xi$-periodic traveling wave solutions of (1.1.18) constructed in [83] are diffusively spectrally stable. Moreover, $\delta_0(\cdot)$ can be taken to be uniform on compact subsets of $\mathcal{W}_s$.*

[11] With obvious notation for $L_{\mathrm{KdV}}$ and $L_{\mathrm{KdV},\xi}$. Note that the index does not depend on the particular choice of $(M, P)$...

Despite the possibility to choose $\delta_0$ locally uniform on $\mathcal{W}_s$, as mentioned above, we expect that $\mathcal{W}_s = (\Xi_1, \Xi_2)$ in which case one would have $\delta_0(\Xi) \to 0$ as $\Xi \to \Xi_1$ or $\Xi \to \Xi_2$.

Our proof yields another form of uniformity. If $\mathcal{K}$ is a compact subset of the subset of $\mathcal{W}$ on which (A1)-(A2) hold, then there exist $C > 0$ such that, for any $\Xi \in \mathcal{K}$, condition (A3) is equivalent to

$$\sup_{\substack{\lambda_0 \in \sigma(L_{\mathrm{KdV},\xi})\setminus\{0\} \\ \xi \in [-\pi/\Xi, \pi/\Xi) \\ |\lambda| \leq C}} \mathrm{Re}\,(\lambda_1(\xi, \lambda_0)) \;<\; 0,$$

Since $\lambda_1(\xi, \lambda_0)$ converges, as $(\xi, \lambda_0) \to (0, 0)$, to one among three possible limits, depending on the spectral curve followed by $(\xi, \lambda_0)$, the validation of (A3) is indeed uniformly reduced to the sign evaluation of an explicit function on a compact set.

In Chapter 4 we will come back on the precise definition of diffusive spectral stability that we have already introduced as conditions (D1)–(D3) in Section 1.3. Yet let us recall in vague words that besides usual (weak) spectral stability it includes that the critical spectrum is minimal hence reduce to Floquet 0 and eigenvalue 0 (conditions (D1) and (D2)), that the spectrum touches 0 quadratically with respect to the Floquet exponent (condition (D2)) and that the dimension of the critical spectral space is minimal hence equals 2 for (1.1.18) (condition (D3)). Moreover in Theorem 2.2.1 it is also true that the bifurcation of the critical spectrum is nondegenerate at $(0, 0)$ or equivalently that the linearized Whitham system has distinct characteristics. This is proved by showing that away from the origin all the spectrum lies far away from the imaginary axis and closely examining what occurs near the origin.

### Outline of the proof

We now sketch a plan of the proof. The goal is to reduce the question of the spectral stability to the inspection of a compact region independent of $\delta$ where Bloch eigenvalues are simple and thus expansion (2.2.2) may be proved and made uniform.

We first exclude large eigenvalues. For $\Xi \in \mathcal{W}$, a direct energy estimate proves that for any $\eta \geq 0$ there exists $C > 0$ such that for $\delta > 0$ sufficiently small, there is no $\lambda \in \sigma(L_\delta)$ such that

$$\mathrm{Re}(\lambda) \;\geq\; -\eta \qquad \text{and} \qquad |\mathrm{Re}(\lambda)| \;+\; \delta^{\frac{3}{4}}\,|\mathrm{Im}(\lambda)| \;\geq\; C\,.$$

In particular the size of the possibly unstable spectrum does not grow faster than $\mathcal{O}(\delta^{-\frac{3}{4}})$ and is thus $o(\delta^{-1})$. This is what is needed to make the KdV approximation relevant also at the spectral level. For $\Xi \in \mathcal{W}$, by using it, we then prove that for any $\eta \geq 0$ there exists $C > 0$ such that for $\delta > 0$ sufficiently small, there is no $\lambda \in \sigma(L_\delta)$ such that

$$\mathrm{Re}(\lambda) \;\geq\; -\eta \qquad \text{and} \qquad \delta^{-1}\,|\mathrm{Re}(\lambda)| \;+\; |\mathrm{Im}(\lambda)| \;\geq\; C\,.$$

The proof of this claim is far from being trivial and involves lengthy Fenichel-type computations similar to those performed in [83]. The main point is that one may prepare spectral problems to better energy estimates by using the classical Floquet change of variables of the KdV spectral problem. This is made possible by the fact that besides knowing that KdV waves are stable we also have in hands explicit Bloch-resolvents [25].

This is enough to show that all the possibly dangerous spectrum converges to a compact region of the KdV spectrum. The second major difficulty is the analysis of the spectrum near the origin. We do not expect regular expansions there so that we instead work with dispersion relations directly, that is, with Evans functions. We prove that by dropping a nonvanishing factor, classical Evans function may be turned in a reduced Evans function $\widetilde{D}$ satisfying

$$\widetilde{D}_\delta(\lambda,\xi) = \prod_{j=1}^{3}(\lambda - \mathrm{i}\alpha_j(\xi)\xi) + \gamma\delta\prod_{k=1}^{2}(\lambda - \mathrm{i}\beta_k^0\xi) + \delta^2\widetilde{D}_{2,2}(\lambda,\xi,\delta) + \delta\widetilde{D}_{1,3}(\lambda,\xi,\delta),$$

with, for $\delta > 0$ sufficiently small,

$$\begin{aligned}\partial_\lambda^l\partial_\xi^m\widetilde{D}_{2,2}(0,0,\delta) &= 0 \quad \text{if } l+m \leq 2\\ \partial_\lambda^l\partial_\xi^m\widetilde{D}_{1,3}(0,0,\delta) &= 0 \quad \text{if } l+m \leq 3\,.\end{aligned}$$

Regularity — analyticity in $(\lambda,\xi)$ and finite albeit arbitrary smoothness in $\delta$ — is inherited from the fact that the proof again follows the lines of classical geometric singular perturbation theory. Simple observations yield key properties for the coefficients of the above crucial expansion: spectral stability of KdV waves implies that $\alpha_j$s are real-valued; the fact that $\widetilde{D}_\delta(\lambda,0)$ is real when $\lambda$ is real implies that $\gamma$ is real; a similar symmetry argument yields that $\beta_1^0$ and $\beta_2^0$ are either distinct and real or conjugate and that $\alpha_j$s are even functions; and, setting $\alpha_j^0 = \alpha_j(0)$, $j = 1,2,3$, condition (A2) states that $\alpha_j^0$s are distinct. With this in hands, spectrum near the origin is then analyzed in three steps described in Figure 2.4, Zone 1 being what is left for simple perturbation arguments leading to (2.2.2).

By using condition (A2), one may obtain in Zone 2 an extension of expansion (2.2.2). Indeed, if $\Xi \in \mathcal{W}$ satisfies (A2), for $\xi \neq 0$ and $\delta > 0$ with $|\xi| + \frac{\delta}{|\xi|}$ sufficiently small, $\widetilde{D}_\delta(\cdot,\xi)$ has exactly three small roots and they expand as[12]

$$\begin{aligned}\lambda_k(\xi,\delta) &= \mathrm{i}\alpha_k(\xi)\,\xi \;-\; \gamma\delta\,\frac{\prod_{j=1}^2(\alpha_k(\xi)-\beta_j^0)}{\prod_{j\neq k}(\alpha_k(\xi)-\alpha_j(\xi))} \;+\; \mathcal{O}(\delta\xi),\\ \mathrm{Re}(\lambda_k(\xi,\delta)) &= \delta A_k + \mathcal{O}(\delta\xi) \quad \text{with} \quad A_k = -\gamma\frac{\prod_{j=1}^2(\alpha_k^0-\beta_j^0)}{\prod_{j\neq k}(\alpha_k^0-\alpha_j^0)}\end{aligned}$$

when $|(\xi,\delta/\xi)| \to 0$. Moreover, one has $A_k < 0$ for all $k = 1,2,3$ if and only if the following conditions are satisfied:

**(S1)** $\beta_1^0, \beta_2^0 \in \mathbf{R}$ and $\beta_1^0 \neq \beta_2^0$;

**(S2)** $\alpha_1^0 < \beta_1^0 < \alpha_2^0 < \beta_2^0 < \alpha_3^0$ (once we have fixed $\beta_1^0 < \beta_2^0$);

**(S3)** $\gamma > 0$.

[12]Actually each $\lambda_k/\xi$ is a smooth function of $(\xi,\delta/\xi)$ in Zone 2.

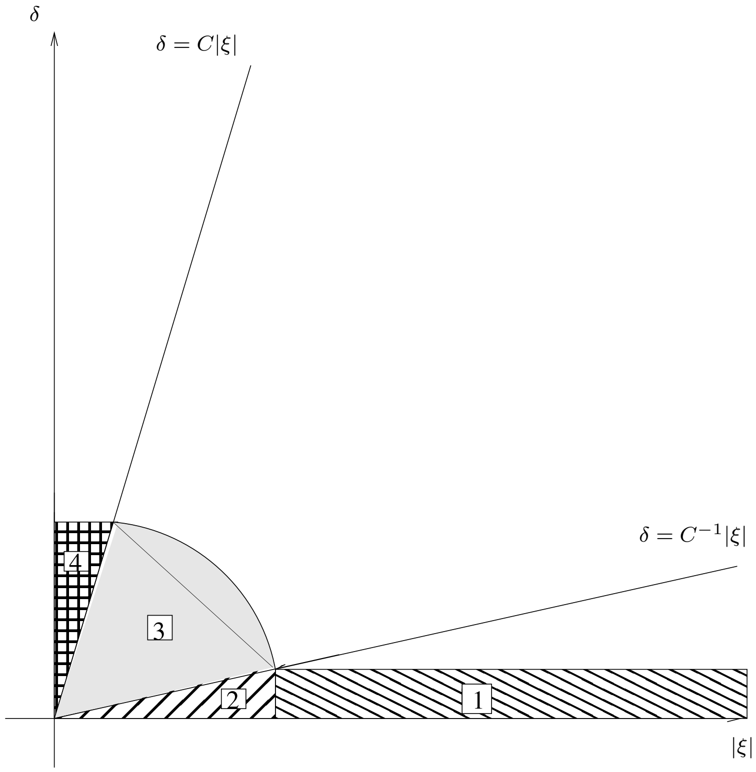



Figure 2.4.: A three-zones proof near the origin.

In particular, provided that $\Xi \in \mathcal{W}$ satisfies condition (A2),

- if (S1)–(S3) hold then, there exist $\eta > 0$ and $C > 0$ such that, for all $0 \leq \delta \leq C^{-1}|\xi|$, $|\xi| \leq \eta$, for any $\Xi$-periodic wave as in [83],
$$\sigma(L_{\delta,\xi}) \cap B(0,\eta) \ \subset \ \{ \ \lambda \mid \mathrm{Re}(\lambda) \leq -\theta_0\delta \ \} \,.$$
- condition (A3) implies conditions (S1)–(S3).

For a reason discussed below, we call conditions (S1)–(S3) subcharacteristic conditions. They are sufficient to handle Zones 2-3-4. That they are implied by (A1)–(A3) follows from the fact that $A_1$, $A_2$, $A_3$ are the three possible path-limits of $\mathrm{Re}(\lambda_1(\xi,\lambda_0))$ when $(\xi,\lambda_0) \to (0,0)$ along one of the three spectral curves.

Under conditions (A2) and (S1)–(S3), nothing occurs in Zone 3. In more explicit terms, under these conditions, if $C$ is fixed (sufficiently large to deal with Zones 2 and 4), then there exists $\eta > 0$ such that for all $C^{-1}|\xi| \leq \delta \leq C|\xi|$, $|(\xi,\delta)| \leq \eta$, for any $\Xi$-periodic wave as in [83], there is no eigenvalue of $L_{\delta,\xi}$ crossing the imaginary axis in $B(0,\eta)$.

Concerning Zone 4, under conditions (A2) and (S1)–(S3), there exists $C > 0$ and $\delta_0 > 0$ such that, for any $(\xi,\delta)$ such that $0 \leq |\xi| \leq C^{-1}\delta$ and $0 < \delta \leq \delta_0$, there are exactly three small roots $\{\lambda_j(\xi,\delta)\}_{j=1,2,3}$ of the associated Evans function $\widetilde{D}_\delta(\cdot,\xi)$, and they expand as[13]

$$\begin{array}{llcl}
j=1,2, & \lambda_j(\xi,\delta) & = & \mathrm{i}\beta_j^0\,\xi + B_j\dfrac{\xi^2}{\delta} + \mathcal{O}\left(\delta|\xi| + \dfrac{|\xi|^3}{\delta^2}\right), \\
 & \mathrm{Re}(\lambda_j(\xi,\delta)) & = & B_j\dfrac{\xi^2}{\delta} + \mathcal{O}\left(|\xi|^2 + \dfrac{|\xi|^3}{\delta^2}\right), \\
\text{and} & \lambda_3(\xi,\delta) & = & B_3\,\delta + o(\delta), \\
 & \mathrm{Re}(\lambda_3(\xi,\delta)) & = & B_3\,\delta + o(\delta)
\end{array}$$

as $|(\delta,\xi/\delta)| \to 0$, with

$$B_j \ = \ \frac{\prod_{j=1}^{3}(\beta_2^0-\alpha_j^0)}{\gamma\prod_{k\neq j}(\beta_j^0-\beta_k^0)}, \quad j=1,2, \qquad \text{and} \qquad B_3 \ = \ -\gamma.$$

Moreover under the same assumptions

$$\max_{j=1,2,3} B_j \ < \ 0.$$

In particular, there exist $\eta > 0$, $C > 0$ and $\theta > 0$ such that, for all $0 < \delta \leq \delta_0$, for any $\Xi$-periodic wave as in [83], the generalized kernel of $L_{\delta,0}$ has dimension 2, the eigenvalue 0 of $L_{\delta,0}$ breaks into two curves with distinct derivatives at $\xi = 0$ and, for $0 \leq |\xi| \leq C^{-1}\delta$,

$$\sigma(L_{\delta,\xi}) \cap B(0,\eta) \ \subset \ \left\{ \ \lambda \ \middle| \ \mathrm{Re}(\lambda) \leq -\theta\tfrac{|\xi|^2}{\delta} \ \right\}.$$

By piecing together all these elements (and filling the missing details), one achieves the proof of Theorem 2.2.1.

[13] $\lambda_1/\xi$, $\lambda_2/\xi$ and $\lambda_3/\delta$ are smooth functions of $(\delta,\xi/\delta)$ in Zone 4.

**Singular modulation**

We end this exposition by providing the reader with a formal interpretation for subcharacteristic conditions (S1)–(S3) and expansion of the Evans function from which they are derived. Actually conditions (S1)–(S3) were first derived in [185] by arguing on formal grounds from the singular modulation theory that we discuss now and the above expansion of $\widetilde{D}_\delta(\lambda,\xi)$ is an *a posteriori* linear validation of the formally-obtained modulation system essentially as Theorem 2.1.1 is a linear validation of system (1.4.1).

A direct modulation approach that would consider $\delta$ as fixed may shed some light only in a zone comparable to Zone 4 in Figure 2.4, that is $|\xi|\leq\epsilon(\delta)$ where $\epsilon(\delta)$ shrinks to zero as $\delta\to0^+$. In [185], instead, the modulation process is carried out with $|\xi|/\delta$ held fixed, that is, in the terminology of Section 1.4, with the ratio $\bar{\delta}=\delta/\varepsilon$ held fixed. This leads to

$$(2.2.3)\qquad \left\{\begin{array}{rcccl} \kappa_T & - & (\omega_0(\kappa,\mathcal{M},\mathcal{P}))_X & = & 0\,,\\ \mathcal{M}_T & + & \mathcal{P}_T & = & 0\,,\\ \mathcal{P}_T & - & (S(\kappa,\mathcal{M},\mathcal{P}))_X & = & \bar{\delta}\,R(\kappa,\mathcal{M},\mathcal{P}),\end{array}\right.$$

where $\omega_0$ is the time-frequency function of KdV waves, $S$ is the average impulse flux of KdV waves $S=\langle(\underline{u}_0)^3-\frac{3}{2}(\underline{u}_0')^2\rangle$ and $R$ implements the selection criterion of KdV waves that yield KdV-KS waves,

$$R\;=\;\langle(\underline{u}_0')^2-(\underline{u}_0'')^2\rangle\,.$$

Setting $\bar{\delta}=0$ recovers the Whitham system for (2.2.1) whose characteristic velocities are $\alpha_j^0$, $j=1,2,3$. The relaxation limit $\bar{\delta}\to\infty$ leads to a two-by-two system which is the limit as $\delta\to0$ of the Whitham system of (1.1.18), and whose characteristic velocities $\beta_j^0$, $j=1,2$, are also the limits of the linear group velocities of (1.1.18). Therefore condition (A2) ensures that the KdV Whitham system is strictly hyperbolic and then conditions (S1)–(S3) are classical subcharacteristic conditions of system (2.2.3). A weak form of them is necessary for stability of the constant state $(\underline{k},\underline{M},\underline{P})$ satisfying $R(\underline{k},\underline{M},\underline{P})=0$, a strong form as required here ensures stability in the strong sense encoded by Kawashima conditions for hyperbolic-relaxation systems. We refer the reader to [249, 147, 223, 170, 251, 252, 253, 149, 254, 209, 114, 23, 173, 16, 52, 53] for classical theory of hyperbolic-dissipative systems and its extensions.

Our inspection of Zones 2-3-4 also follows classical lines of the stability theory of hyperbolic-relaxation systems. Zone 4 corresponds to high-frequencies where the hyperbolic part dominates and spectral expansions start from them and leads to an exponential damping. Zone 2 corresponds to low-frequencies where the system uncouples into an exponentially damped mode accounting for pure relaxation and two diffusively-damped modes whose expansions start from those of the relaxed two-by-two system. By a continuity argument based on noncrossing, proved as in treatment of Zone 3, one may check that mean frequencies are also exponentially damped.

With this interpretation also comes formula enabling us to check (S1)–(S3). As a result Figure 2.5 showing characteristics *versus* period is obtained in [125]. Recall that in [8] a full numerical investigation of condition (A3) lead to the conclusion that the set $\mathcal{W}_s$ of periods of stable near-KdV waves is an interval $(\Xi_1,\Xi_2)$ with $\Xi_1\approx8.49$ and

$\Xi_2 \approx 26.17$. From Figure 2.5 we see that at the lower boundary the loss of stability is due to a modulational instability while the instability at the upper boundary is not.

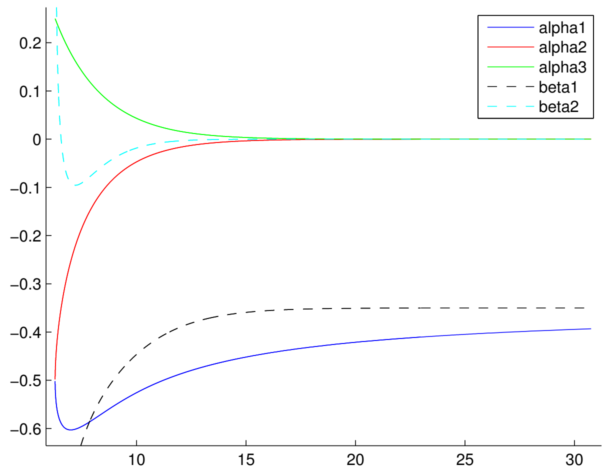



Figure 2.5.: Inspection of subcharacteristic conditions.

### From the St. Venant system

While the proof of Theorem 2.2.1 applies *mutatis mutandis* to the general (1.1.12) and the corresponding stability condition (A3) has already been investigated in [8], we have restricted a full description to (1.1.18), *i.e.* to $R = 0$, because we were motivated by (1.1.15)/(1.1.17) in the limit $\mathbf{F} \to 2^+$. Accordingly the extension of Theorem 2.2.1 to the St. Venant system proved in [10] involves exactly the same indices as for (1.1.18).

But its proof differs in essentially two points. First, the KdV limit is not a singular limit for the St. Venant system, which makes regularity issues easier to handle. Second, energy estimates and algebraic preparations to uncover the KdV limit are more involved and more tedious. As the rest of the structure of the proof is essentially similar, we will not report on this result but will only show how to prepare system (1.1.17). On the road the role of (1.1.18) will be uncovered.

To do so, we fix a constant solution $(\tau_0, u_0)$, $\tau_0 > 0$ and $u_0 = \tau_0^{-1/2}$, and set $\epsilon = \mathbf{F}^2 - 4$. Then we insert the *ansatz*

$$\begin{pmatrix}\tau\\ u\end{pmatrix}(t,x) \;=\; \begin{pmatrix}\tau_0\\ u_0\end{pmatrix} \;+\; \epsilon\,\begin{pmatrix}\tilde{\tau}\\ \tilde{u}\end{pmatrix}\Big(\underbrace{\epsilon^{\frac{3}{2}}t}_{T}\,,\,\underbrace{\epsilon^{\frac{1}{2}}(x-c_0\,t)}_{X}\Big)$$

where $c_\epsilon \;=\; \tau_0^{-\frac{3}{2}}\,\mathbf{F}^{-1} \;=\; \tau_0^{-\frac{3}{2}}/\sqrt{4+\epsilon}$ is the reference sound speed. This turns system (1.1.17) in

$$\begin{cases} \tilde{\tau}_T - \epsilon^{-1}(\tilde{u} + c_0\tilde{\tau})_X &= 0, \\ \tilde{u}_T + \epsilon^{-1}\left(\dfrac{(\tau_0+\epsilon\tilde{\tau})^{-2} - \tau_0^{-2}}{2\,\epsilon\,(4+\epsilon)} - c_0\tilde{u}\right)_X &= \epsilon^{-\frac{5}{2}}\left(\tau_0\,u_0^2 - (\tau_0+\epsilon\tilde{\tau})(u_0+\epsilon\tilde{u})^2\right) \\ &\quad + \epsilon^{-\frac{1}{2}}\left((\tau_0+\epsilon\tilde{\tau})^{-2}\tilde{u}_X\right)_X. \end{cases}$$

Now set $w = \epsilon^{-1}(\tilde{u} + c_0\tilde{\tau})$. Inserting $\tilde{u} = -c_0\tilde{\tau} + \epsilon w$ yields

$$\begin{cases} \tilde{\tau}_T - w_X &= 0, \\ \epsilon w_T - c_0 w_X &+ \epsilon^{-1}\left(\dfrac{(\tau_0+\epsilon\tilde{\tau})^{-2} - \tau_0^{-2}}{2\,\epsilon(4+\epsilon)} + c_0^2\tilde{\tau} - c_0\epsilon w\right)_X \\ &= \epsilon^{-\frac{5}{2}}\left(\tau_0\,u_0^2 - (\tau_0+\epsilon\tilde{\tau})(u_0+\epsilon(-c_0\tilde{\tau}+\epsilon w))^2\right) \\ &\quad - c_0\epsilon^{-\frac{1}{2}}\left((\tau_0+\epsilon\tilde{\tau})^{-2}\tilde{\tau}_X\right)_X + \epsilon^{\frac{1}{2}}\left((\tau_0+\epsilon\tilde{\tau})^{-2}\tilde{w}_X\right)_X. \end{cases}$$

or equivalently

$$\begin{cases} \tilde{\tau}_T &- \quad w_X = 0, \\ \epsilon w_T &+ \quad \left(\dfrac{(\tau_0+\epsilon\tilde{\tau})^{-2} - \tau_0^{-2} + 2\tau_0^{-3}\,\epsilon\tilde{\tau}}{2\,\epsilon^2(4+\epsilon)} - \dfrac{c_\epsilon^2 - c_0^2}{\epsilon}\tilde{\tau} - 2\,c_0\,w\right)_X \\ &= -\epsilon^{-\frac{1}{2}}\left(2\tau_0 u_0 w - (2u_0c_0 - \tau_0 c_0^2)\tilde{\tau}^2 + c_0\,\tau_0^{-2}\tilde{\tau}_{XX}\right) + \epsilon^{\frac{1}{2}}\,g_\epsilon(\tilde{\tau}, w) \\ &\quad - c_0\epsilon^{\frac{1}{2}}\left(\dfrac{(\tau_0+\epsilon\tilde{\tau})^{-2} - \tau_0^{-2}}{\epsilon}\tilde{\tau}_X\right)_X + \epsilon^{\frac{1}{2}}\left((\tau_0+\epsilon\tilde{\tau})^{-2}\tilde{w}_X\right)_X. \end{cases}$$

with

$$g_\epsilon(\tilde{\tau}, w) \;=\; 2\tau_0 c_0\tilde{\tau}w - \epsilon\tau_0\,w^2 - \epsilon^{-2}\tilde{\tau}\left((u_0+\epsilon(-c_0\tilde{\tau}+\epsilon w))^2 - u_0^2 + 2u_0c_0\tilde{\tau}\right).$$

On a first-order approximation is already read the Korteweg-de Vries equation

$$\begin{cases} \tilde{\tau}_T - w_X &= 0, \\ w &= \frac{1}{2}\tau_0^{-\frac{1}{2}}\left(\frac{3}{4}\tau_0^{-2}\,\tilde{\tau}^2 - \frac{1}{2}\tau_0^{-\frac{7}{2}}\,\tilde{\tau}_{XX}\right), \end{cases}$$

and at the second order is obtained the Korteweg–de Vries/Kuramoto–Sivashinsky equation

$$\begin{cases} \tilde{\tau}_T - w_X &= 0, \\ w &= \frac{1}{2}\tau_0^{-\frac{1}{2}}\left(\frac{3}{4}\tau_0^{-2}\,\tilde{\tau}^2 - \frac{1}{2}\tau_0^{-\frac{7}{2}}\,\tilde{\tau}_{XX}\right) \\ &\quad -\epsilon^{\frac{1}{2}}\;\frac{1}{2}\tau_0^{-\frac{1}{2}}\left(\frac{1}{16}\tau_0^{-3}\,\tilde{\tau} + \frac{1}{4}\tau_0^{-\frac{11}{2}}\,\tilde{\tau}_{XX}\right)_X. \end{cases}$$

One may then scale $T$, $X$, $\tilde{\tau}$, $w$ in terms of $\tau_0$ to obtain coefficients of (1.1.18) for an appropriate $\delta \sim \epsilon^{\frac{1}{2}}$.

# 3. Spectral stability: numerical approaches

Generally, far from asymptotic limits, it seems that only numerical investigations are available to determine spectral stability, especially in the restricted sense of diffusive spectral stability. With this in mind we have performed numerical studies as complete as possible of periodic waves stability in (1.1.18) and (1.1.17) [12, 13, 11, 10].

On the road we confirm the classical observation that observed stable periodic waves of the St. Venant system are often quite close to be a train of solitary waves. This contrasts with the fact that single solitary waves are always unstable and we have also tried to collect numerical evidence supporting a heuristic scenario that elucidates this weak paradox [14].

## 3.1. Hill's method and Evans functions

Our numerical investigations mostly processed by first reducing possible unstable spectrum to a bounded region of the spectral plane through energy estimates then determining the presence of unstable spectrum in the obtained region by numerically computing winding numbers of Evans' functions. Yet we have also completed our studies by spectral pictures obtained by relying instead on so-called Hill's method.

We describe here briefly the principles underlying both strategies — Hill's method and Evans' function methods. Then we give some details about computing difficulties and implementation. At last we discuss numerical outcome for the KdV-KS equation and the St. Venant system.

### Principles of the methods

Any method is preceded by a preliminary step involving a continuation/shooting algorithm computing periodic waves and their speeds. In our case they were started by initial guesses coming from Hopf bifurcations of homogeneous solutions. Mark that as a result of this preliminary step coefficients of periodic operators $L$ whose spectrum is investigated are known only in an approximate and discretized way. This stated, we introduce now Hill's method and Evans' function methods.

Hill's method is directly based upon Floquet-Bloch decomposition (1.3.8) resulting in Floquet parametrization of the spectrum (1.3.9). From (1.3.9) two successive approximations are performed to yield a finite number of finite-dimensional problems. In the first, is discretized the Brillouin zone where Floquet exponents $\xi$ vary. At this stage one still needs to compute the spectra of a finite number of periodic operators $L_\xi$ acting on functions defined on a periodic cell. The second approximation replaces each operator by a matrix by applying the standard Galerkin approximation using trigonometric bases, that is, by truncating Fourier series approximation. This leaves to evaluate the spectra of a finite number of matrices. Each of them is then approximately computed by standard algorithms, for instance by the celebrated QR algorithm. Hill's method is known to converge for a wide class of operators [240, 195, 26, 51, 133] and its advantages and drawbacks over direct finite-differences approximations are thoroughly discussed in [60].

While Hill's method aims at providing full[1] spectrum from which one may decide stability, Evans' function methods as we use them are build to answer stability issues. One of the approximations is as for Hill's method that the Brillouin zone is discretized. Then for a finite number of Floquet exponents $\xi$ we want to decide whether there are unstable roots of $D(\cdot,\xi)$, that is, whether there is any $\lambda$ such that $\mathrm{Re}(\lambda) > 0$ and $D(\lambda,\xi) = 0$. Since for each $\xi$, $D(\cdot,\xi)$ is an analytic function, provided possible unstable eigenvalues are confined to a given region bounded by simple curves contained in the resolvent set, we may obtain an answer by evaluating winding numbers of each $D(\cdot,\xi)$ along the boundary curves. Assuming that we are in such a position, then, boundary curves should be discretized. To explain the next stage of discretization — leading to numerical computation of each Evans' function — we recall here the definition of Evans' function $D$. First the eigenvalue problem $\lambda\mathbf{V} = L\mathbf{V}$ is written as a system of first-order differential equations and its fundamental solution $R(\lambda;\cdot)$ is introduced. Then one sets

$$D(\lambda,\xi) \;=\; \det(R(\lambda,\Xi) - \mathrm{e}^{\mathrm{i}\Xi\xi}\,\mathrm{I})$$

where $\Xi$ is the involved period. As a next step in the approximation process, for a finite number of $\lambda$, the spectral differential system is discretized and an approximation of $R(\lambda;\cdot)$ is computed. As for Hill's method, at first glance this is a $\xi$-fixed method combined with a discretization in $\xi$. With this respect, observe that since we apply the strategy with curves of $\lambda$ that essentially do not depend on $\xi$ and that $R(\lambda;\cdot)$ do not depend on $\xi$ either, the solving part is carried out once and used for all $\xi$.

We will not even try to give a proper account of the well-developed theory around Evans functions in pattern stability, neither at the analytical level [84, 136, 1, 98, 97, 93, 94, 199, 95, 96, 99, 22, 87, 143, 144, 145, 102, 103, 161, 54, 162, 30, 31] nor at the numerical stage [35, 36, 122, 121, 123, 257, 6, 171, 167, 168, 33, 32, 7, 160]. But we do endeavor to sketch briefly why the periodic case also fits in the general common framework. This point is not so easily extracted from the literature [95] and its exposition will serve our purposes also in the description of numerical implementations. One common point to various contexts is that one starts by writing the fact that $\lambda$ is an eigenvalue as the fact that two linear spaces whose dimension sum to the dimension of the whole space intersect

[1]Obviously, yet, a stability-focused variant could be obtained by replacing the last stage of the process that determines the full spectrum by an algorithm approximating eigenvalues with largest real part through a variant of the power iteration method. Incidentally, we observe that there is at least one other strategy that instead of tackling the problem as a collection of $\xi$-fixed problems handles it as a continuation problem in $\xi$ [205, 204] offering both the possibility to determine the full spectrum or to determine part of it emerging by varying Floquet parameter $\xi$ from the co-periodic spectrum contained in a given region. The latter is implemented over the continuation software AUTO [65].

non transversally. The latter condition is then reduced to the vanishing of a determinant, Evans' function. This picture also holds in the periodic setting but through a tortuous path. Say that $\mathbf{X}' = \mathbf{A}(\lambda;\cdot)\mathbf{X}$ is the differential system encoding $\lambda\mathbf{V} = L\mathbf{V}$ and enlarge it to

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}' = \begin{pmatrix}\mathbf{A}(\lambda;\cdot) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}$$

so that the the fundamental solution of the extended system is

$$\widetilde{R}(\lambda;\cdot) = \begin{pmatrix}R(\lambda;\cdot) & 0\\ 0 & \mathrm{I}\end{pmatrix},$$

$R(\lambda,\cdot)$ being the one for the original system. For any Floquet exponent $\xi$, denote by $\Delta_\xi$ the twisted diagonal

$$\Delta_\xi = \left\{ \begin{pmatrix}\mathrm{e}^{\Xi\xi}\mathbf{X}\\ \mathbf{X}\end{pmatrix} \;\middle|\; \mathbf{X} \right\}.$$

In particular $\Delta_0$ is the canonical diagonal space. Then $\lambda$ is an eigenvalue of $L_\xi$ if and only if $\widetilde{R}(\lambda;\Xi)\Delta_0$ intersects non-trivially $\Delta_\xi$ which amounts to the vanishing of

$$\det\left(\begin{pmatrix}R(\lambda;\Xi) & \mathrm{e}^{\mathrm{i}\Xi\,\xi}\mathrm{I}\\ \mathrm{I} & \mathrm{I}\end{pmatrix}\right) = D(\lambda,\xi)\,. \tag{3.1.1}$$

In Appendix B we also briefly sketch in the periodic framework how to interpret the winding number of Evans' function as the first Chern number of some linear-space bundle over a topological sphere. The latter point does not play any explicit role in our discussion but it strongly underlies the general theory.

## In practice

In our application of Hill's method, we have benefited from SpectrUW[2] [59] a mathematical black-box software implementing the method over Maple or Mathematica, and developed at the University of Washington with collaboration from the Seattle University [59, 60, 51]. We mark that choice of its developers is to apply Hill's method with the periodic wave considered as a periodic function of twice its fundamental period. In particular, Floquet-zero perturbations actually correspond to combinations of co-periodic perturbations and principal subharmonic perturbations, sometimes called semi-co-periodic perturbations. This is motivated by the fact that in applications to self-adjoint second-order scalar operators, the Floquet-zero spectrum will then provide edges of spectral bands.

Concerning Evans' function methods, at first glance the strategy may seem deceptively simple to code as compared to the homoclinic or front stability investigation that requires to solve a differential system on the whole line with boundary conditions at both infinities. Yet a closer look shows that a naive algorithm would be essentially useless when handling either large periods $\Xi$ or large spectral parameters $\lambda$. Instead our investigation relies and builds upon the algorithms implemented in STABLAB, an interactive MATLAB-based

[2] Pronounced "spectrum".

toolbox for Evans' function computation, developed at both the Indiana University and the Brigham Young University by two collaborators of the author of the present memoir, Blake Barker and Kevin Zumbrun, and one of their collaborators, Jeffrey Humpherys. The package and its documentation is available at "http://impact.byu.edu/stablab/" and should in the end also include a version of Hill's algorithm and interval arithmetics routines involved in [9].

We describe now briefly the nature of some of the difficulties and solutions. More details may be found in [13, Section 2 & Appendix D]. The first easily detected difficulty is that Evans' function usually grows exponentially with the period or the real part of the spectral parameter. That it is so with the respect to the period follows from known convergence results for some rescaled version of the periodic Evans' function towards homoclinic Evans' function [96, 211]. These results also suggest a remedy, rescaling Evans' functions. Yet another issue stems from the fact in the same limits some parts of the solutions of the differential equations are growing while other are damped resulting in bad conditioning of computations. This issue is usually present when considering Evans' functions for fronts, homoclinic or shocks. And a large number of tools — for instance evolving exterior product methods [35, 36] or polar coordinate methods [122] that involve evolving orthogonal subspaces — resulting in some features of STABLAB have been developed to solve this problem. To apply similar techniques to the periodic setting, one needs to work with the lifted version of Evans' function (3.1.1) that unravels its similarities with other well-studied stability problems.

Now we come to a point that we have not discussed so far: how can we obtain that possible unstable spectrum is confined to a given region ? Obviously one could just take a large region and hope that it is large enough or, better, rely fully on numerics and get a first rough approximation of the spectrum, say by Hill's method, and then choose a region large enough to contain this first approximation. But our goal was to obtain an answer as close to certainty as numerics can get without going to a numerical proof. Instead, we thus turn to analytical techniques and observe that we have already solved similar problems in the proof of Theorem 2.2.1 where the two first steps were precisely dedicated to similar tasks. As there[3], we obtain that for some positive constant $R$ there is no spectrum in

$$\{\ \lambda \in \mathbf{C}\ |\ \mathrm{Re}(\lambda) \geq 0 \text{ and } |\lambda| \geq R\ \}\ .$$

Such a bound may in principle be obtained by energy estimates (as in first step of the proof of Theorem 2.2.1 or as explained in Appendix A) but it could be of poor practical use. Indeed recall that most of the trouble comes from the consideration of large $\lambda$s so that it is crucial to obtain a $R$ as small as possible. As in second step of the proof of Theorem 2.2.1, a suitable bound may be obtained by performing energy estimates after a preparation that performs an asymptotic diagonalization, here when $|\lambda|$ goes to $\infty$. The preparation could in principle be carried to an arbitrary order in $\lambda$ but computations rapidly grow and become tedious and lengthy. In [14, Section 4 & Appendix A] such an approximate diagonalization is performed for the St. Venant system (1.1.17) and

[3] Up to the fact that for computational reasons we here write the results in terms of the Euclidean norm.

applied[4] to the similar issue occurring when trying to determine the location of possible unstable point spectrum for a solitary wave. Subsequently the same diagonalization was used in the periodic context [11, 10]. In any case obtained $R$s are explicitly given in terms of the profile.

Then comes the problem that zero always belongs to the spectrum and therefore we need to treat a neighborhood of the origin separately, and leave for direct Evans' function method described above to exclude unstable spectrum in

$$\Omega_{r,R} \;=\; \{\ \lambda \in \mathbf{C} \mid \mathrm{Re}(\lambda) \geq 0 \text{ and } r \leq |\lambda| \leq R\ \ \}\ .$$

for some small positive $r$. For the method to apply, the boundary of $\Omega_{r,R}$ must not intersect the spectrum but the exclusion of nonzero marginally stable spectrum is in any case, as condition (D1), part of the verification of diffusive spectral stability. Actually only for small Floquet exponent should we handle spectrum around the origin with some care. Hence we shall indeed pick a positive $\epsilon$ sufficiently small and when $|\xi| \geq \epsilon$ compute directly the winding number of (a rescaled version of) $D(\cdot, \xi)$ along the boundary of

$$\Omega_{R} \;=\; \{\ \lambda \in \mathbf{C} \mid \mathrm{Re}(\lambda) \geq 0 \text{ and } |\lambda| \leq R\ \ \}\ .$$

to exclude unstable spectrum there. To handle the crucial remaining part we choose $r'$ such that $r < r' < R$ and that the number of roots of $D(\cdot, \xi)$ in $B(0, r')$ — in the algebraic sense given by the winding number — remains constant for all $|\xi| \leq \epsilon$. Then Taylor expansions of critical spectral curves are computed by Taylor expanding Evans' function $D$ about $(0, 0)$ and a consistency check is performed to ensure that $(r', \epsilon)$ is small enough to guarantee stability of critical curves. Both Taylor expansions and bounds on Taylor remainder are computed by numerical evaluation of contour integrals around $B(0, r') \times B(0, \epsilon)$. This requires a complexification of the Floquet parameter. An important implementation detail is that whereas contour integrals around $\Omega_{r,R}$ are computed with rescaled balance Evans' function those around $B(0, r')$ are evaluated with original Evans' function to avoid a loss of analyticity near the origin.

Some comments are in order concerning the small-eigenvalue/sideband stability investigation. First, this leads to take $r$ small but of course if $r$ gets too small then the last step consisting in evaluation of a winding number around $\Omega_{r,R}$ may become badly conditioned because contour gets to close from the origin. This happens when curvatures of critical curves are too small, in particular in the KdV limit studied in Section 2.2. This also occurs when a non critical curve passes close to the origin, as is also the case in the KdV limit. When needed, to avoid the latter issue as much as possible, we have relaxed the full check by giving up the consistency check on Taylor remainder since it involves the stringent condition that the winding number around $B(0, r')$ stays constant not only for real $\xi \in [-\epsilon, \epsilon]$ but for any complex $\xi \in B(0, \epsilon)$.

We emphasize that, when fully carried out, Evans' function method reaches a high-level of certainty since it relies on numerical solving of differential equations and numerical

[4] With tracking estimates replacing energy estimates, although energy estimates would also work in the homoclinic context.

quadrature that can be performed with a prescribed error tolerance and uses them either to evaluate integers or to Taylor expand with controlled remainder. Moreover, it provides a wealth of information on small sideband eigenvalues, that plays the main role in determining nonlinear asymptotic behavior when stability is met.

## Spectral stability diagrams

We try now to illustrate main outcomes of spectral studies. Much more tables and numerical reports are available in original papers [12, 13, 11, 10].

We start with KdV-KS equation for which we've tried to investigate the full range $\delta \in (0, +\infty)$. To compactify the region to investigate, the equation is scaled to

$$\partial_t u + \partial_x(\tfrac{1}{2}u^2) + \varepsilon\partial_x^3 u + \delta(\partial_x^2 u + \partial_x^4 u) = 0 \tag{3.1.2}$$

with $\varepsilon^2 + \delta^2 = 1$. Then the rescaled $\delta$ varies in $(0, 1)$ and may even reach the value one.

Recall that by Galilean invariance, diffusive spectral stability depends only on period $\Xi$. Figure 3.1 shows a stability diagram, period $\Xi$ *versus* parameter $\varepsilon = \sqrt{1-\delta^2}$, where shaded regions correspond to stability as determined by our application of numerical Evans' function method. Only in darker hatched regions were we able to perform the full verification including estimation of Taylor remainders. In the Korteweg–de Vries limit $\varepsilon \to 1$, there is only a single band of diffusively spectrally stable periodic traveling waves whose limits agree with numerical analysis in [8] of the stability criteria for near-KdV waves rigorously justified by Theorem 2.2.1. The Kuramoto–Sivashinsky limit $\varepsilon \to 0$ also displays stability of a single band of waves, but this band is much narrower[5]. It agrees in this limit with numerical investigations of the KS waves in[6] [88] that were based on a combination of a spectral method as in Hill's method with computations of critical Taylor expansions by direct spectral expansions. For intermediate values, however, several bands emerge and the stability picture becomes much more complicated. The full diagram is in excellent agreement with previous results in [41] where a similar stability diagram is obtained, again by Galerkin-type methods.

For the St. Venant system, such a diagram where $\mathbf{F}$ would cover $(2, \infty)$ would be much harder to obtain since Galilean invariance is lost, making it a three-parameters study. Moreover while the KdV limit $\mathbf{F} \to 2^+$ is well-captured by the analogous of Theorem 2.2.1, the vertical limit $\mathbf{F} \to \infty$ exhibit a wider diversity of possible scalings that are still under investigation [10]. Let us only give as a typical example a stability diagram along a slice obtained by fixing the Froude number and a constant of integration. To match with [11] from which pictures are extracted, we leave (1.1.17) in a not fully nondimensionalized form

$$\begin{cases} \partial_t \tau - \partial_x u \;=\; 0\,, \\ \partial_t u + \partial_x(\frac{1}{2}\,\frac{\tau^{-2}}{\mathbf{F}^2}) \;=\; 1 \;-\; \tau\,|u|\,u \;+\; \nu\,\partial_x(\tau^{-2}\,\partial_x u)\,. \end{cases}$$

[5] This observation already made there explains the title of [41].

[6] There is also carried out a formal modulation averaging process. Hence the title, since the linearized Whitham system for the KS equation may be written as a wave equation for the local wavenumber.

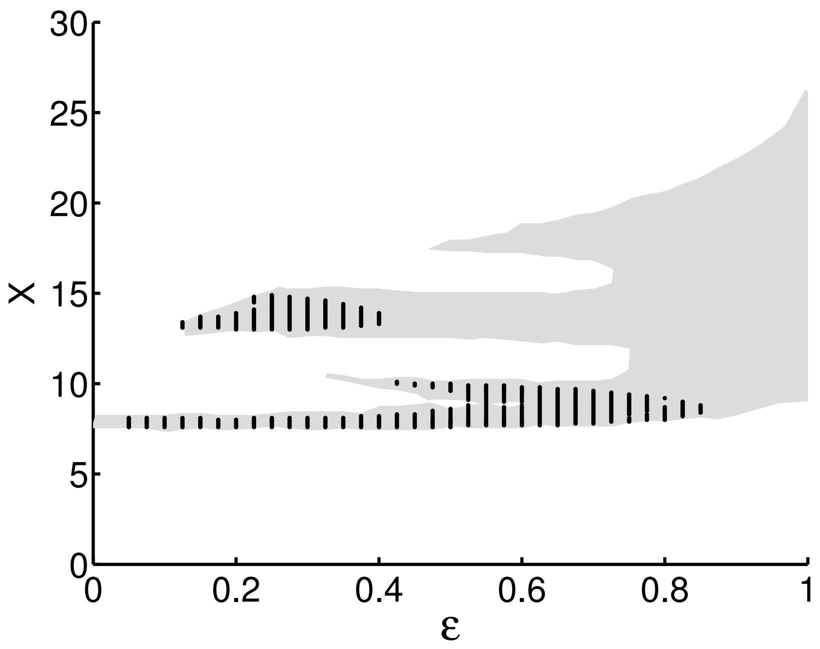



Figure 3.1.: Stability diagram for KdV-KS.

Looking for traveling waves $(\tau, u)(t, x) = (\underline{\tau}, \underline{u})(x - ct)$ leads to solve profile system

$$\begin{cases} q - c\underline{\tau} = \underline{u}\,, \\ c^2\underline{\tau}' + (\frac{1}{2}\frac{\underline{\tau}^{-2}}{\mathbf{F}^2})' = 1 - \underline{\tau}\,(q - c\,\underline{\tau})^2 - c\,\nu\,\partial_x(\underline{\tau}^{-2}\,\underline{\tau}')'\,. \end{cases}$$

Figures 3.2 and 3.3 correspond to $\mathbf{F} = \sqrt{6}$, $\nu = 0.1$ and $q$ fixed[7] while $c$ is varied. The Froude number is still sufficiently close to KdV limit $\mathbf{F} \to 2^+$ so that a single band of stable waves is observed. On Figure 3.2, examined periodic orbits are superimposed in a $(\underline{\tau}, \underline{\tau}')$-plane. Waves whose orbit lies between red orbit (lower boundary) and blue boundary (upper boundary) have been found numerically to be spectrally stable. The upper blue boundary is indistinguishable from the homoclinic orbit surrounding the wave family. A typical intermediate stable wave orbit is also colored in green. In Figure 3.3 corresponding wavespeeds and periods are shown. Period at Hopf bifurcation equals approximately 3.927, stable periodic waves have period lying approximately between 5.2 (represented by a diamond) and 20.4 (not represented on the graph), period of the typical green stable wave is around 6.2 (signified by a circle). From period 10 on, periodic waves match quite closely their solitary wave limit. Though it is not visible on Figures, both small — near Hopf bifurcation — and long — near homoclinic — periodic waves are unstable. This is a consequence of instability of all homogeneous solutions when $\mathbf{F} > 2$.

[7] At 1.5745099609375.

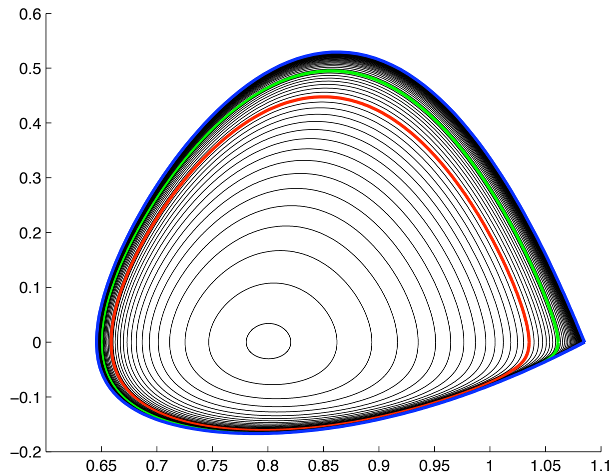



Figure 3.2.: Slice of a stability diagram for SV : superimposed phase portraits.

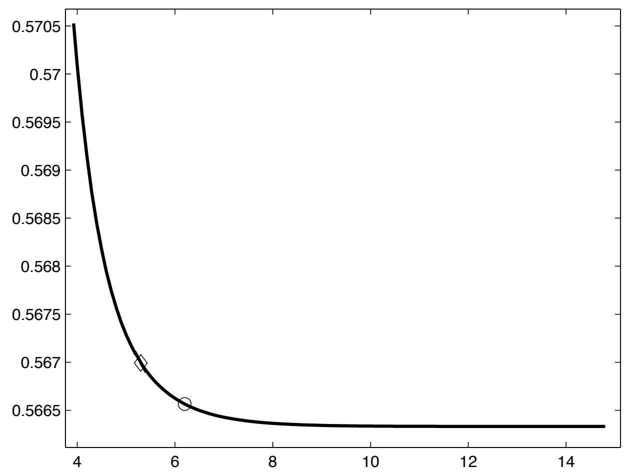



Figure 3.3.: Slice of a stability diagram for SV : speed *vs.* period.

## 3.2. Trains of solitary waves

A section of [14] is devoted to the discussion of the observation made in the previous section of the present memoir that while single solitary waves and thus also sufficiently-long periodic waves are always spectrally unstable there are stable periodic waves that look like a periodic array of solitary waves. We briefly display now this part of [14], the rest of it[8] lying slightly aside the main focus of the present memoir.

We endeavor to explain observed stable behavior of trains of solitary waves that are in isolation exponentially unstable by looking at time evolution about a single pulse as shown in Figures 3.4. All figures here are plotted as $\tau$ and $u$ *vs.* space variable $x$ according to the indicated labelings. We see the perturbation, initially to the right of the profile solution in Figure 3.4 (a), moving left and emerging de-amplified from the hump in Figure 3.4 (b) then it grows along the constant wake as oscillatory wave packets with Gaussian envelopes in Figures 3.4 (c)-(d). Locally, after interaction with the perturbation, the solution seems to settle to a translate of the original profile (kept for reference in thin lines). Here $\mathbf{F} = 3$, $\nu = 0.1$, $c$ is approximately 0.7849, and $q = 1 + c$. Again we are still close to primary instability threshold $\mathbf{F} = 2^+$.

Now the question essentially answers itself: it must be that the local dynamics of the waves are such that convected perturbations are diminished as they cross each solitary pulse, counterbalancing the growth experienced as they traverse the interval between pulses, on which they behave as perturbations of an unstable constant solution. This diminishing effect is clearly apparent visually in time-evolution studies. Nevertheless the decay we are expecting is of diffusive type and is therefore not readily encoded by the point spectrum that we usually think of as determining local dynamics of the wave but could lead by itself only to exponential decay.

To capture the diminishing property — if there is one — of pulses that stabilizes arrays by de-amplifying convective instabilities shed from their neighbors' wakes, we compute a spectrum, that we call *dynamic spectrum*, defined as the spectrum of the periodic-coefficient linearized operator about a periodic wave obtained by gluing together copies of a suitably truncated solitary pulse. Here, the choice of truncation is not uniquely specified, but should intuitively be at a point where the wave profile has *almost converged* to its limiting endstate for, otherwise, either we may have not fully captured all the stabilizing capacities of pulses, or we are missing effects of strong interactions on profiles. This dynamic spectrum should govern the behavior of a maximally closely spaced array of solitary pulses. For comparison we display for the same solitary wave both essential and dynamic spectra in Figures 3.5. Stability hinted at by the dynamic spectrum computed with period 10 is visually confirmed by time-evolution in Figures 3.6 of a perturbation of a corresponding periodic array of period 19.8.

When period of the pulse array is increased, spectrum continuously moves from dy-

[8] Including computation of essential spectrum by simple Fourier analysis, numerical evaluation of possible unstable discrete spectrum by Evans' function methods for solitary waves — requiring an analysis of consistent splitting in stable and unstable spaces and a quantitative tracking estimate obtained from asymptotic diagonalization of spectral problems —, a stability index obtained from analysis of Evans' function near the origin and near $+\infty$,...

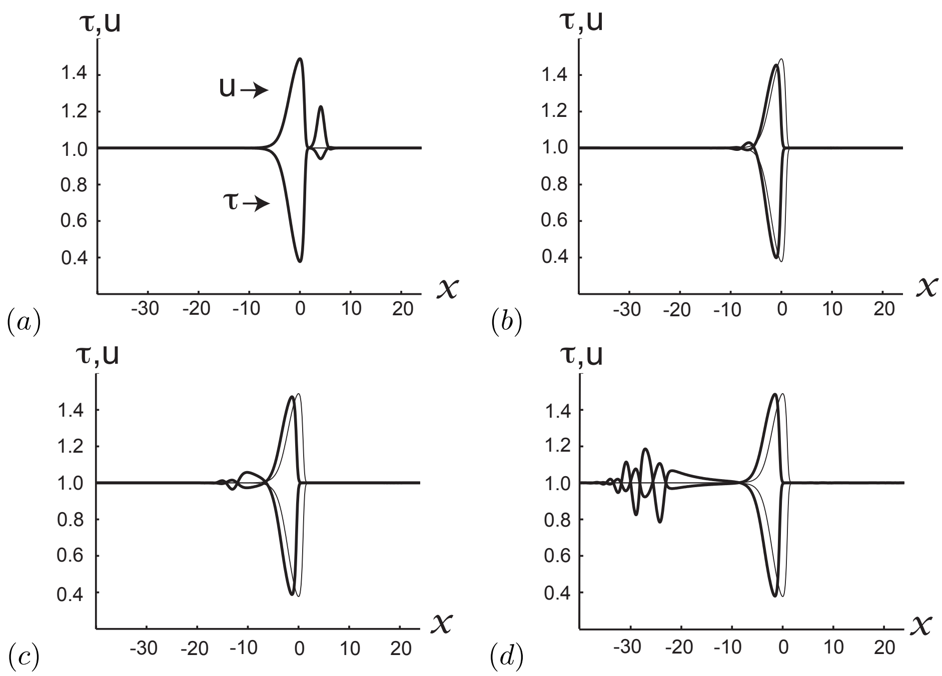



Figure 3.4.: Unstable solitary wave: time-evolution snapshots.

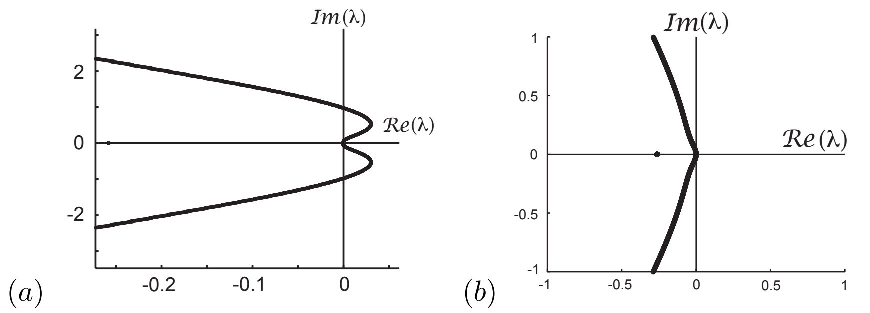



Figure 3.5.: Unstable solitary wave: essential and dynamic spectrum.

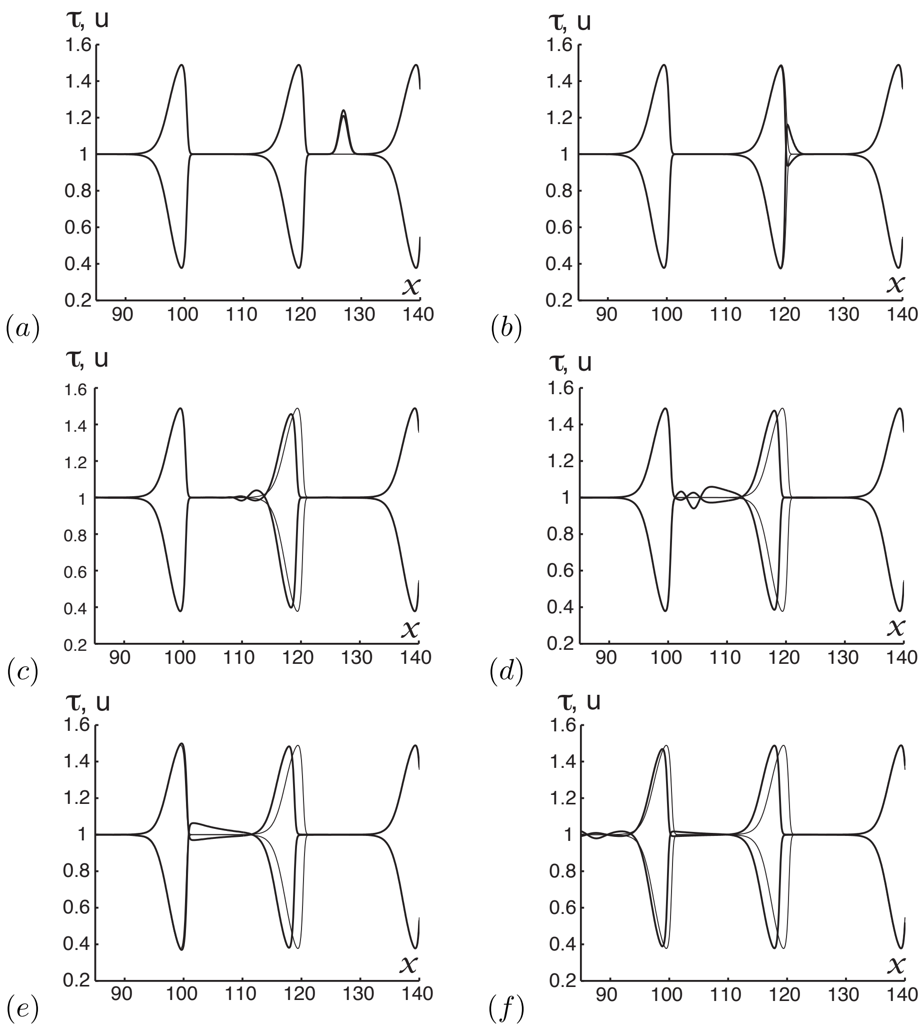



Figure 3.6.: Stable periodic array: time-evolution snapshots.

namic spectrum to essential spectrum and small spectral loops around point spectrum. Hence as already mentioned a transition to instability necessarily occurs. This is illustrated by time-evolution snapshots and spectrum in Figures 3.7 for a periodic array of period 30. Observe on this time-evolution study that in the end even the periodic structure is broken by the perturbation growth.

All computations closely match stability computations for associated periodic traveling waves. Corresponding periodic family which emerges from Hopf bifurcation at period around 3.9 exhibits a band of stable waves with periods lying approximately between 5.4 and 20.6. Mark that by continuity we expect that spectral stability illustrated by dynamic spectrum extends below the spacing threshold of weak interaction, here 10.

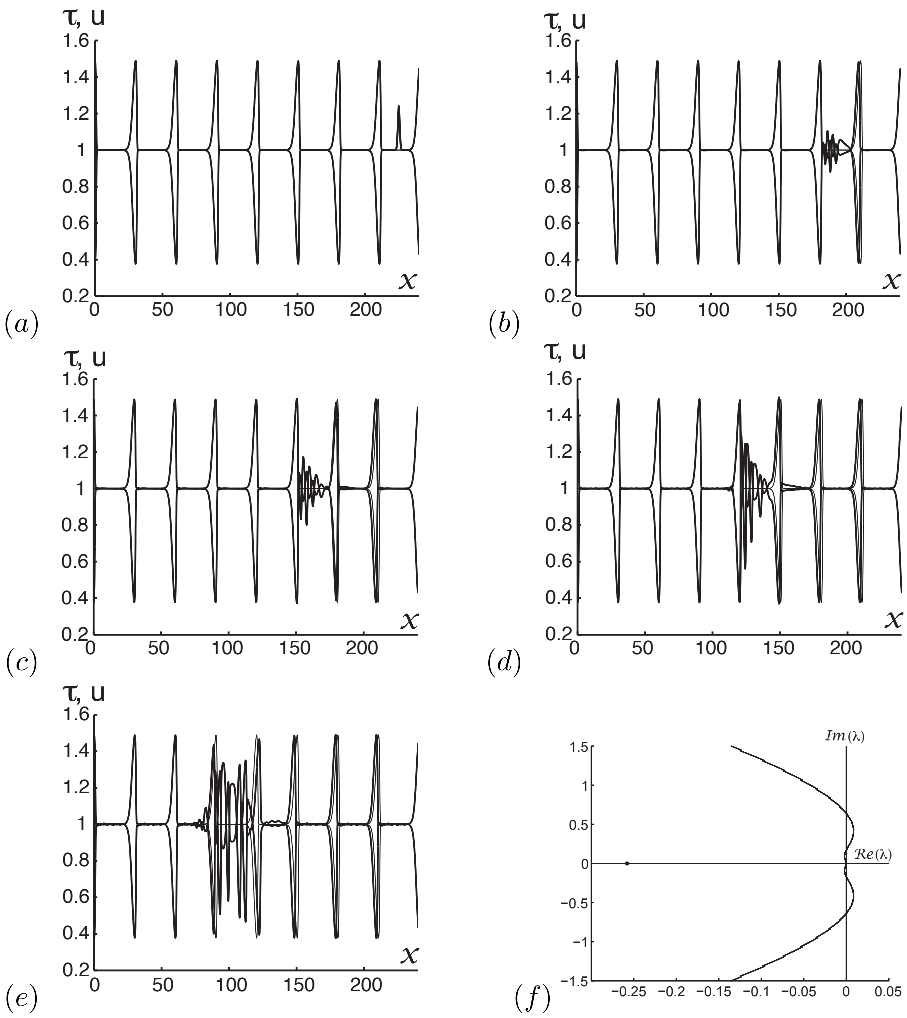



Figure 3.7.: Unstable periodic array.

Obviously the same mechanisms could stabilize less regular arrays of pulses. See discussion and numerical experiments for a regularly-spaced array semi-infinite on the left in [14]. The reader interested by dynamics of array of pulses or multiple pulses in dissipative systems is encouraged to read the reaction-diffusion literature [80, 256] as a gateway. We also point [198] to the attention of the reader. Although not directly comparable to any material described here, this piece of work is adherent to many topics of the present memoir. In particular, it may be thought as both a justification of a KdV approximation for the KdV-KS equation (1.1.18) when $\delta \to 0$ and a study of a form of metastability for multipulses. Besides providing various detailed numerical simulations of time-evolutions, the authors prove that solutions starting within $\mathcal{O}(1)$ distance of $N$-pulse solutions ($N \in \mathbf{N}$) of the KdV equation remain $\mathcal{O}(1)$-close up to time $\mathcal{O}(1/\delta)$. While the proof in [198] builds upon the proof of orbital stability of solitary waves of the KdV equation [18, 24] thus relies on conserved quantities of the KdV equation and not on spectral methods, it echoes the spectral bound

$$\sigma(L) \ \subset \ \{ \ \lambda \mid \mathrm{Re}(\lambda) \leq C\,\delta \ \}$$

(with $C > 0$) for near-KdV periodic waves that is proved en route for Theorem 2.2.1.

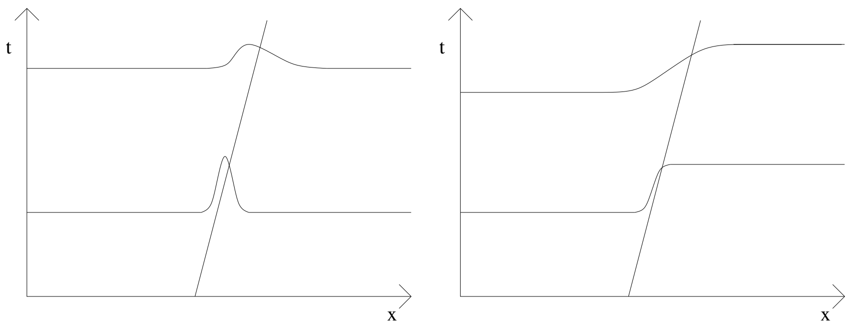
# 4. Nonlinear dynamics

Having thoroughly discussed various means to establish spectral stability, we now turn to proofs of nonlinear stability of diffusively spectrally stable waves and inspection of asymptotic behavior near them. Once again, our goal is not to trace back the development of the theory but to give a detailed account of its last stage offered by work of the author of the present memoir and his collaborators in [127, 128, 126]. Yet we shall also emphasize common points and differences with another strategy that yielded pioneering results[1] [48, 217, 218, 219] and whose recent instance in [212] provides results comparable to [127, 128] on the simplest case where the Whitham system reduces to a scalar equation. Details of the theory are fully given only for reaction-diffusion systems (1.1.13) [212, 127, 128] and systems of conservation laws (1.1.14) [126] but some indications about how to obtain similar results for general systems (1.1.1) under suitable assumptions are also given in [126, Appendix D]. Details of similar extensions of earlier partial results [129] are fully expanded for the KdV-KS equation (1.1.18) in [13] and for the St. Venant system (1.1.17) in [135].

We recall that elementary blocks of the asymptotic behavior are diffusion-waves in local parameters, that is diffusion-waves of system (1.4.6) lifted back by (1.4.5). Mark that actually we provide a good asymptotic description by *weakly-interacting* diffusion-waves and that diffusion-waves are in general of Burgers-type, so that asymptotic behavior follows a nonlinear evolution. Main contributions of these perturbation waves is through creation of phase shifts that are antiderivatives of wavenumber components. To make it slightly more concrete we draw a suggestive sketch of a wavenumber component of a diffusion-wave and its phase counterpart in Figure 4.1. Recall that a single wave with a nonzero wavenumber necessary emerges from a nonlocalized perturbation since limits at $\pm\infty$ of its phase-shift outcome differ by the mass of the wavenumber component, an invariant of the evolution. Yet multiple diffusion-waves in local parameters may arise from localized perturbations but with a massless wavenumber component, that is, with masses of wavenumber component of each diffusion-wave summing to zero. This is sketched in Figure 4.2.

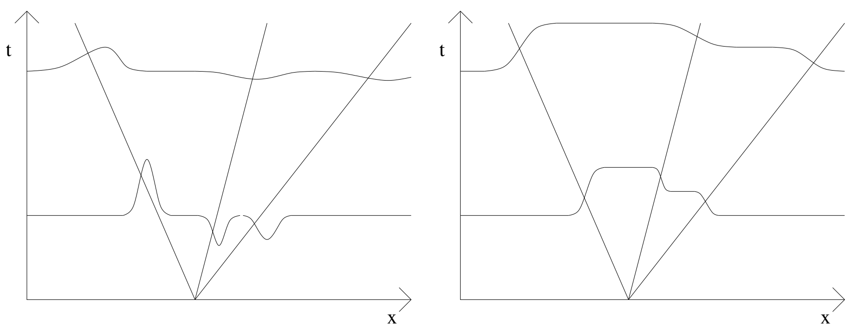
It should be clear that for systems supporting multiple diffusion-waves modifying local wavenumbers there is no fundamental distinction between behavior under localized or nonlocalized perturbations. However when only one of the diffusion-waves of (1.4.6) contributes directly to wavenumber, then the zero-mass condition, characterizing local wavenumbers of localized perturbations, implies that the coefficient of the distinguished diffusion-wave is zero and that no diffusion-wave affects directly local phase shift. Hence

[1] See the review paper [178] that among other things discusses breaking-through contributions of Guido Schneider to nonlinear stability, various extensions by Hannes Uecker and founding work of Alexander Mielke on spectral stability, already mentioned in Section 2.2.




(*a*) Local wavenumber. (*b*) Local phase.

Figure 4.1.: A single wave: nonlocalized perturbation.




(*a*) Local wavenumber. (*b*) Local phase.

Figure 4.2.: Three waves: localized perturbation.

no phase shift plateau is created and the situation does not require the introduction of the notion of space-modulated stability but indeed leads to usual asymptotic stability under localized perturbation. This is of course the case when the Whitham system (1.4.6) is scalar and there is only one diffusion-wave. The former occurs when periodic traveling wave profiles are parametrized by wavenumber alone as it holds generally for waves of system (1.1.13). But as is read on system (1.4.6) the condition is equivalent to the vanishing of $\partial_{\mathbf{M}}\omega(\underline{\mathbf{M}},\underline{\mathbf{k}})$, that is, to the vanishing of $\partial_{\mathbf{M}}c(\underline{\mathbf{M}},\underline{\mathbf{k}})$.

By discarding nonlinear interactions, in previous paragraph we have slightly oversimplified dynamics description since, under simplifying conditions of the previous paragraph — at the linear order phase velocity does not depend on average values and initial perturbation is localized —, dominant contributions are due to the influence on local phase shifts of quadratic forcing by other diffusion-waves. This does not preclude (usual) asymptotic stability but still main contributions reveal the presence of diffusion-waves and asymptotic behavior remains a nonlinear multiscale phenomenon.

To obtain asymptotically linear behavior under localized perturbation we need further quadratic decoupling, that is, vanishing of both $\partial_{\mathbf{M}}\omega(\underline{\mathbf{M}},\underline{\mathbf{k}})$ and $\partial_{\mathbf{M}}^2\omega(\underline{\mathbf{M}},\underline{\mathbf{k}})$ or equivalently of both $\partial_{\mathbf{M}}c(\underline{\mathbf{M}},\underline{\mathbf{k}})$ and $\partial_{\mathbf{M}}^2c(\underline{\mathbf{M}},\underline{\mathbf{k}})$.

In short, while under nonlocalized perturbations asymptotic behavior is the same without distinction, for behavior under localized perturbations there are at least three categories:

- when $\partial_{\mathbf{M}}c(\underline{\mathbf{M}},\underline{\mathbf{k}})$ is nonzero, behavior is the same than under nonlocalized perturbation, hence requires modulation in space and is resolved in weakly-interacting diffusion-waves;
- when both $\partial_{\mathbf{M}}c(\underline{\mathbf{M}},\underline{\mathbf{k}})$ and $\partial_{\mathbf{M}}^2c(\underline{\mathbf{M}},\underline{\mathbf{k}})$ vanish, then there is asymptotic stability in the usual sense with asymptotically linear behavior and decay rates for localized perturbative part and phase shift of heat-like type as in the scalar Whitham case originally treated by Guido Schneider;
- when only $\partial_{\mathbf{M}}c(\underline{\mathbf{M}},\underline{\mathbf{k}})$ vanishes, then there is asymptotic stability in the usual sense but with still asymptotically nonlinear and multiscale behavior and with intermediate decay rates between non-decaying case and heat-like decay for perturbative part and local phase shift.

We call the first case, generic or linearly phase-coupled, the second quadratically phase-decoupled and the union of second and third cases linearly phase-decoupled.

The above description is validated by our nonlinear results but it does not underly our proofs. Indeed our strategy is to validate formal prescription (1.4.5)-(1.4.6)-(1.4.7) and the weakly-interacting diffusion-waves scenario follows as an *a posteriori* outcome. More, detailed description stems from validation of the modulation theory but nonlinear stability is proved without this validation. Until [126] no asymptotic behavior result was available, not even under localized perturbations, save in the scalar Whitham case exhibiting trivial quadratic phase-decoupling, but proofs of nonlinear stability under localized perturbations were already available in work of two collaborators of the author of the present memoir, Mathew Johnson and Kevin Zumbrun [131, 129]. With the strategy developed there were recovered with a different proof results of Guido Schneider and was proved also stability under localized perturbations for waves of reaction-diffusion systems [132], of the St. Venant system [135] and of two fourth-order parabolic equations, the Swift–Hohenberg equation and the Korteweg–de Vries/Kuramoto–Sivashinsky equation [13]. The full modulation theory and clarifying distinction according to degree of phase-coupling were left unnoticed in [131, 129]. More, unnecessarily separated proofs were given for the linearly phase-decoupled [131] and phase-coupled [129] cases, with non sharp decay rates for the former. Even with modulation theory in hands sharp asymptotic description of the linearly phase-decoupled case is a subtle point resulting from strict hyperbolicity assumption that yields eventual separation in space of diffusion-waves and a fine analysis of their weak interactions. Although a decisive break-through is achieved in [127, 128, 126], some of the main features of proofs are clearly inherited from long-standing work of Kevin Zumbrun and its collaborators [188, 189, 190, 191, 193, 192, 131, 129, 132, 135, 13] that focused on nonlinear stability under localized perturbations and brought in tools originally tailored for stability of other patterns (fronts, shocks...) in systems of conservation laws.

In contrast, obtention of asymptotic behavior is built in in the strategy of proof of nonlinear stability culminating in [212]. Indeed it aims at the unraveling of a large-time asymptotic profile, a wavenumber Burgers-wave or heat-wave, moving with linear group velocity, and, in order to reveal asymptotic behavior hidden in the full system, it strongly relies on the expected asymptotic self-similarity of the scalar-Whitham thus single-wave case treated in [212]. To do so, Guido Schneider, his collaborators and successors accommodate renormalization techniques as introduced in [104, 105, 28, 29, 179] so that, at each step of the discrete renormalization process[2], terms that do not scale properly are damped, making concrete their asymptotic irrelevance. A clever use is needed to capture asymptotic self-similarity that holds only for modulation equation and it is somehow surprising that the strategy could handle the case where the linear group velocity does not coincide with phase velocity. Indeed strictly speaking self-similarity of a relevant expansion of modulation equation holds only in the linear group velocity frame, but we also need to keep original wave in its own co-moving frame — the phase velocity frame — to make it stationary. To accommodate these apparent incompatibilities is used the ability of the Bloch transform to perform a two-scale separation of slow-modulation evolutions (recall (1.3.7)). Explicitly if $g(t,\cdot)$ is slow and $h$ is co-periodic, then, for $f = g\,h$,

$$\mathrm{e}^{\mathrm{i}a\,\xi t}\;[f(t,\cdot)]\check{}(\xi,x) \tag{4.0.1}$$

[2]In short, once a scaling factor $\lambda$ chosen sufficiently large, time interval $\mathbf{R}_+$ is split as

$$\mathbf{R}_+ \;=\; \bigcup_{n\in\mathbf{N}^*}\left[\lambda^{2(n-1)}-1,\lambda^{2n}-1\right]$$

and on each $[\lambda^{2(n-1)}-1,\lambda^{2n}-1]$ ($n\in\mathbf{N}^*$) independent variables $(t,x)$ are scaled to $(\lambda^{2n}\,t,\lambda^{n}\,x)$ so that instead of studying directly a time evolution on an infinite interval, one examine a sequence of evolutions on finite interval $[\lambda^{-2},1]$.

is the Bloch transform of $x \mapsto g(t, x - at)\, h(x)$, and

$$\lambda\, [f(\lambda^2 t, \cdot)]\check{}(\lambda^{-1}\xi, x) \tag{4.0.2}$$

is the Bloch transform of $x \mapsto g(\lambda^2 t, \lambda x)\, h(x)$. With this motivation, the authors of [212] use transformations (4.0.1) and (4.0.2) as natural substitutes for transforms on modulation equations leaving original periodic scale unchanged. As elegant as this strategy may be it is unlikely[3] that it could receive an analog for the analysis of multiscale problems arising when multiple diffusion-waves travel with different linear group velocities.

Reflecting the above, results of [126] are detailed in two separate forthcoming sections, nonlinear stability in Section 4.1, asymptotic behavior and its implications in Section 4.2.

## 4.1. Nonlinear stability

### Statement

To state our nonlinear stability result, we make more precise spectral stability assumptions. We pick a periodic traveling wave $(t,x) \mapsto \underline{\mathbf{U}}(\underline{k}\,(x - ct))$ of (1.1.14) with $\underline{\mathbf{U}}$ one-periodic. In adequate co-moving frame, system (1.1.14) reads

$$\mathbf{U}_t \;+\; \underline{k}\;(\mathbf{f}(\mathbf{U})_x - c\,\mathbf{U}_x) \;=\; \underline{k}^2\;\mathbf{U}_{xx}. \tag{4.1.1}$$

and $\underline{\mathbf{U}}$ is a one-periodic standing wave of (4.1.1). Linearizing (4.1.1) about $\underline{\mathbf{U}}$ yields the one-periodic coefficient equation $(\partial_t - L)\mathbf{V} = 0$ with operator $L$ defined by

$$L\mathbf{V} \;=\; (\;\underline{k}^2\;\partial_x^2 + \underline{k}\;\partial_x(c - \mathrm{d}\mathbf{f}(\underline{\mathbf{U}}))\;)\mathbf{V}, \tag{4.1.2}$$

that comes with a family of operator-valued symbols

$$L_\xi \;:=\; \mathrm{e}^{-\mathrm{i}\xi\,\cdot}L\mathrm{e}^{\mathrm{i}\xi\,\cdot} \;=\; \underline{k}^2(\partial_x + \mathrm{i}\xi)^2 + \underline{k}(\partial_x + \mathrm{i}\xi)(c - \mathrm{d}\mathbf{f}(\underline{\mathbf{U}}))), \quad \xi \in [-\pi, \pi),$$

operating on one-periodic functions.

We assume that we are at a nondegenerate point of the one-periodic profile manifold, that is, we assume that

(H) up to translation nearby one-periodic traveling-wave profiles $(\underline{\mathbf{U}}^{(\mathbf{M},k)}, c(\mathbf{M},k))$ form a $(d+1)$-manifold parametrized by wavenumber $k$ and averages $\mathbf{M} = \langle \underline{\mathbf{U}}^{(\mathbf{M},k)} \rangle$.

This assumption is sufficient to prove that 0 is an eigenvalue of $L_0$ of algebraic multiplicity $(d+1)$ and that the $(d+1)$ continuous spectral curves $(\lambda_j)_{j=1}^{d+1}$ are differentiable at $\xi = 0$,

$$\lambda_j(\xi) = -\mathrm{i}\underline{k}\xi a_j + o(\xi), \quad j = 1, \ldots, d+1. \tag{4.1.3}$$

[3] This seems to be a common fact that strategies developed to analyze self-similar situations scarcely extend to prove natural multiscale analogs. To proceed with comparison started in Footnote 32, p.27, we encourage the reader to compare [92, 91, 207] with [119, 120, 208]. Yet self-similar techniques of [92, 91, 207] are not based on renormalization but on self-similar variables that convert algebraic decay rates in exponential decay rates accessible by spectral gap estimates. A direct analog in the nonlinear stability theory of periodic traveling waves may be found in [79].

To ensure higher regularity of the former spectral expansions, we add the following non-degeneracy assumption

(H')  $a_j$s are distinct.

Recall that we know from [222] that $c + \underline{k}a_j$ are characteristic velocities of the corresponding first-order system (1.4.2) linearized at $(\underline{\mathbf{M}}, \underline{k})$ therefore we are assuming that linear group velocities are distinct.

Now we recall and accommodate to the context diffusive spectral stability assumptions,

(D1)  $$\sigma(L) \subset \{\lambda \mid \mathrm{Re}\lambda < 0\} \cup \{0\},$$

there exists $\theta > 0$ such that

(D2)  $$\sigma_{per}(L_\xi) \subset \left\{\lambda \;\middle|\; \mathrm{Re}\lambda \leq -\theta|\xi|^2\right\} \quad \text{for any } \xi \in [-\pi, \pi],$$

and

(D3)  0 is, as an eigenvalue, of algebraic multiplicity $(d+1)$ for $L_0$.

Recall also that albeit introduced separately for exposition convenience conditions (H) and (D3) are equivalent.

Using space-modulated distances (recall (1.2.1)), we may state in a concise way a version of our nonlinear stability result.

**Theorem 4.1.1** ([126])**.** *Assume (H)-(H') and (D1)-(D2)-(D3). Then, for any $K \geq 3$,*

- $\underline{\mathbf{U}}$ *is* $\delta_{L^1(\mathbf{R})\cap H^K(\mathbf{R})}$*-to-*$\delta_{H^K(\mathbf{R})}$ *asymptotically stable;*
- $\underline{\mathbf{U}}$ *is* $\delta_{L^1(\mathbf{R})\cap H^K(\mathbf{R})}$*-to-*$\|\cdot\|_{L^\infty(\mathbf{R})}$ *(boundedly) orbitally stable.*

We may also state a longer version[4] with sharp decay rates in $L^p(\mathbf{R}^2)$, $2 \leq p \leq \infty$.

**Theorem 4.1.2** ([126])**.** *Assume (H)-(H') and (D1)-(D2)-(D3).*
*Then, for any $K \geq 3$, there exist positive $\varepsilon$ and $C$ such that if*

$$E_0 := \|\widetilde{\mathbf{U}}_0(\cdot - h_0(\cdot)) - \underline{\mathbf{U}}(\cdot)\|_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} + \|\partial_x h_0\|_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} \;\leq\; \varepsilon$$

*for some $h_0$, then, there exist $(\widetilde{\mathbf{U}}, \psi)$ with initial data $(\widetilde{\mathbf{U}}_0, h_0)$ such that*
*$\widetilde{\mathbf{U}}$ solves (4.1.1) and for $t > 0$ and $2 \leq p \leq \infty$,*

$$\begin{aligned} \|\widetilde{\mathbf{U}}(t, \cdot - \psi(t,\cdot)) - \underline{\mathbf{U}}(\cdot)\|_{L^p(\mathbf{R})} &\leq C\,E_0\ (1+t)^{-\frac{1}{2}(1-1/p)} \\ \|\nabla_{x,t}\psi(t,\cdot)\|_{L^p(\mathbf{R})} &\leq C\,E_0\ (1+t)^{-\frac{1}{2}(1-1/p)}, \end{aligned}$$

*and, with global phase shift $\psi_\infty = (h_0(-\infty) + h_0(\infty))/2$,*

$$\|\widetilde{\mathbf{U}}(t, \cdot - \psi_\infty) - \underline{\mathbf{U}}(\cdot)\|_{L^\infty(\mathbf{R})}, \quad \|\psi(t,\cdot) - \psi_\infty\|_{L^\infty(\mathbf{R})} \leq\; C\,E_0.$$

[4] Theorem 4.1.2 does not imply Theorem 4.1.1 but they are proved together. Decay rates provided by the proof are not expected to be sharp in $H^K(\mathbf{R})$.

### Outline of the proof

Our proof essentially follows the structure introduced in [127] to analyze systems (1.1.13), but requires considerably more involved linear estimates.

First, as hinted at by the statement of Theorem 4.1.2, we set

$$\mathbf{V}(t,x) \;=\; \widetilde{\mathbf{U}}(t,x-\psi(t,x)) - \underline{\mathbf{U}}(x).$$

This turns system (4.1.1) in

$$\begin{aligned}(1-\psi_x)\mathbf{V}_t \quad &+ \quad (\omega+\psi_t)\,(\underline{\mathbf{U}}+\mathbf{V})_x \\ &= \quad \underline{k}^2\left(\tfrac{1}{1-\psi_x}(\underline{\mathbf{U}}+\mathbf{V})_x\right)_x \;-\; \underline{k}\,(\mathbf{f}(\underline{\mathbf{U}}+\mathbf{V}))_x.\end{aligned} \tag{4.1.4}$$

Observe that since $\psi$ is introduced in the equation through commutators between differentiation and composition with $\mathrm{Id}-\psi$ it appears only through its derivatives. This is a well-known point that is all the more important so as derivatives of $\psi$ decay in time while $\psi$ is at best bounded in $L^\infty(\mathbf{R})$. By undoing the commutators on the linear part of the equation, one transforms system (4.1.4) in

$$(\partial_t - L)\;(\mathbf{V}(t)+\underline{\mathbf{U}}_x\psi(t)) \;=\; \mathcal{N}(t)$$

with $\mathcal{N} := \mathcal{Q}_x + \mathcal{R}_x + \mathcal{S}_t$, where

$$\mathcal{Q} := -\underline{k}\,(\mathbf{f}(\underline{\mathbf{U}}+\mathbf{V}) - \mathbf{f}(\underline{\mathbf{U}}) - \mathrm{d}\mathbf{f}(\underline{\mathbf{U}})(\mathbf{V}))\,, \qquad \mathcal{S} := \mathbf{V}\psi_x,$$

$$\mathcal{R} := -\mathbf{V}\psi_t + \underline{k}^2\mathbf{V}_x\psi_x + \underline{k}^2(\underline{\mathbf{U}}_x+\mathbf{V}_x)\tfrac{\psi_x^2}{1-\psi_x}.$$

The first turning point of the proof is the choice of some additional requirement leading to two coupled evolution equations for the couple $(\mathbf{V},\psi)$. In [212], such a choice stems from mode filtering of the analog of (4.1.4) for (1.1.13), that is, from going to Bloch variables and separating single critical mode from the rest with appropriate spectral projectors. This choice forces exchange of the original initial phase shift $h_0$ for a similar one satisfying some spectral constraint, hence it precludes $\psi(0,\cdot) = h_0$.

Here alternatively, as in [127], we modify appropriately the strategy of [131, 129]. We first write an integral form of the system

$$\mathbf{V}(t) \;+\; \underline{\mathbf{U}}_x\psi(t) \;=\; S(t)\;(\mathbf{V}(0)+\underline{\mathbf{U}}_x h_0) \;+\; \int_0^t S(t-s)\mathcal{N}(s)\mathrm{d}s \tag{4.1.5}$$

where $(S(t))_{t\geq 0} = (\mathrm{e}^{t\,L})_{t\geq 0}$ then we decompose $(S(t))_{t\geq 0}$ as the principal part of the critical contribution that is aligned with $\underline{\mathbf{U}}_x$ plus a faster-decaying localized part $(\widetilde{S}(t))_{t\geq 0}$. To make it precise, we state an extension of the relevant analog of Theorem 2.1.1. Recall that $\mathbf{F}(\mathbf{M},k) = \langle \mathbf{f}(\underline{\mathbf{U}}^{(\mathbf{M},k)})\rangle$.

**Proposition 4.1.3** ([126]). *Assume (H)-(H') and (D1)-(D2)-(D3). There exist $\epsilon_0 > 0$ and $0 < \xi_0 < \pi$ such that, for $|\xi| \leq \xi_0$,*

$$\sigma(L_\xi)\cap B(0,\epsilon_0) \;=\; \left\{\; \lambda_j(\xi) \;\middle|\; j\in\{1,\dots,d+1\} \;\right\}$$

*with analytic* $\lambda_j$ *such that*

$$\lambda_j(\xi) \stackrel{0}{=} -i\underline{k}\xi\ a_j + (i\underline{k}\xi)^2\ b_j + \mathcal{O}(|\xi|^3), \qquad a_j, b_j \quad \textit{real}, \quad \underline{k}^2 b_j \geq \theta > 0,$$

*and associated left and right eigenfunctions* $\phi_j(\xi)$ *and* $\tilde{\phi}_j(\xi)$ *such that*

$$\langle \tilde{\phi}_j(\xi), \phi_k(\xi) \rangle = i\underline{k}\xi\ \delta_k^j, \qquad 1 \leq j, k \leq d+1,$$

*obtained as*

$$\begin{aligned} \phi_j(\xi,\cdot) &= (i\underline{k}\xi) \sum_{l=1}^{d} \beta_l^{(j)}(\xi)\ q_l(\xi,\cdot) \ + \ \beta_{d+1}^{(j)}(\xi) \quad q_{d+1}(\xi,\cdot) \\ \tilde{\phi}_j(\xi,\cdot) &= \sum_{l=1}^{d} \tilde{\beta}_l^{(j)}(\xi)\ \tilde{q}_l(\xi,\cdot) \ + \ (i\underline{k}\xi)\, \tilde{\beta}_{d+1}^{(j)}(\xi)\ \tilde{q}_{d+1}(\xi,\cdot) \end{aligned}$$

*where*

- $(q_1(\xi), \ldots, q_{d+1}(\xi))$ *and* $(\tilde{q}_1(\xi), \ldots, \tilde{q}_{d+1}(\xi))$ *are dual families arising from* $(\partial_{\mathbf{M}_1}\underline{\mathbf{U}}, \ldots, \partial_{\mathbf{M}_d}\underline{\mathbf{U}}, \underline{\mathbf{U}}_x)$ *and* $(e_1, \ldots, e_d, *)$ *at* $\xi = 0$*;*
- $(\beta^{(1)}(\xi), \ldots, \beta^{(d+1)}(\xi))$ *and* $(\tilde{\beta}^{(1)}(\xi), \ldots, \tilde{\beta}^{(d+1)}(\xi))$ *are dual bases of right-left eigenvectors of some matrix* $\tilde{\Lambda}_\xi$ *departing from*

$$\tilde{\Lambda}_0 \ = \ -\left( d(\mathbf{F}, -\omega)(\underline{\mathbf{M}}, \underline{k}) - c\mathrm{Id} \right).$$

Proposition 4.1.3 holds regardless of the linear-coupling assumption. Yet in the linearly-decoupled case, we may also assume $\tilde{\beta}_j^{(d+1)}(0) = 0$ and $\beta_{d+1}^j(0) = 0$ for $j \neq d+1$. Then by replacing $\tilde{\phi}_{d+1}(\xi)$ with $(\mathrm{i}\underline{k}\xi)^{-1}\tilde{\phi}_{d+1}(\xi)$ and, for $j \neq d+1$, $\phi_j(\xi)$ with $(\mathrm{i}\underline{k}\xi)^{-1}\phi_j(\xi)$, we obtain *dual* critical bases, analytic in $\xi$. For localized data this difference translates directly *at the linear level* in different decay rates.

From the above, in Bloch variables critical evolution is then concentrate on small Floquet $|\xi| \leq \xi_0$ and is indeed given at first-order by phase modulation since, by $q_{d+1}(0,\cdot) = \underline{\mathbf{U}}_x$,[5]

$$\frac{1}{\mathrm{i}\underline{k}\xi} \sum_{j=1}^{d+1} \mathrm{e}^{\lambda_j(\xi)t}\ |\phi_j(\xi,\cdot)\rangle\ \left\langle \tilde{\phi}_j(\xi,\cdot) \right|$$

expands when $\xi \to 0$ as

$$|\underline{\mathbf{U}}_x\rangle \quad \left\langle \frac{1}{\mathrm{i}\underline{k}\xi} \sum_{j=1}^{d+1} \mathrm{e}^{\lambda_j(\xi)t}\ \beta_{d+1}^{(j)}(\xi)\ \tilde{\phi}_j(\xi,\cdot) \right| \ + \ \mathcal{O}(\mathrm{e}^{-\theta'\,|\xi|^2\,t}),$$

[5] We adopt here the Dirac bra-ket notation.

with $\theta' > 0$. Accordingly we pick a smooth high-Floquet cut-off function $\alpha$ and we decompose the solution operator $S(t) = \mathrm{e}^{tL}$ as $S(t) = S^{\mathrm{p}}(t) + \tilde{S}(t)$ where $S^{\mathrm{p}}(t) = \underline{\mathbf{U}}_x\, \mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t)$, with $s^{\mathrm{p}}(t) = \sum_{j=1}^{d+1} s_j^{\mathrm{p}}(t)$ and

$$(s_j^{\mathrm{p}}(t)g)(x) \;=\; \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\xi x}\alpha(\xi)\, \mathrm{e}^{\lambda_j(\xi)t}\frac{1}{\mathrm{i}\underline{k}\xi}\, \beta^{(j)}(\xi)\langle\tilde{\phi}_j(\xi,\cdot),\check{g}(\xi,\cdot)\rangle_{L^2([0,1])}\,\mathrm{d}\xi.$$

Mark that at this stage we disregard modulations in wavenumber and averages and retain only phase modulation as required for space-modulated stability.

Now, to simultaneously accommodate the initial datum constraint $\psi(0,\cdot) = h_0$ and absorb as much as possible $\mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t)$-contributions into the equation for $\psi$, we pick a large-time cut-off function $\chi$ and decompose (4.1.5) as

$$\begin{aligned}\psi(t) \;&=\; \mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t)\,(\mathbf{V}(0) + \underline{\mathbf{U}}_x h_0) + \int_0^t \mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)\mathrm{d}s\\ -\,\chi(t)\;&\left[\mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t)\,(\mathbf{V}(0) + \underline{\mathbf{U}}_x h_0) - h_0 + \int_0^t \mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)\mathrm{d}s\right]\end{aligned}$$

and

$$\begin{aligned}\mathbf{V}(t) \;&=\; \tilde{S}(t)\,(\mathbf{V}(0) + \underline{\mathbf{U}}_x h_0) + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)\mathrm{d}s\\ &+\; \chi(t)\left[S^{\mathrm{p}}(t)\,(\mathbf{V}(0) + \underline{\mathbf{U}}_x h_0) - \underline{\mathbf{U}}_x h_0 + \int_0^t S^{\mathrm{p}}(t-s)\mathcal{N}(s)\mathrm{d}s\right].\end{aligned}$$

In particular with a suitable cut-off we ensure

$$\psi(t) \;=\; \begin{cases} \mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t)(\mathbf{V}(0) + h_0\underline{\mathbf{U}}_x) + \displaystyle\int_0^t \mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t-s)\mathcal{N}(s)\,\mathrm{d}s & \text{for } t \geq 1\\ h_0 \qquad \text{for } t \leq 1/2 & \end{cases}.$$

We now discuss how to solve the evolution equations. Observe that by differentiating the $\psi$-equation we may obtain a set of three evolution equations in closed form for $(\mathbf{V}, \psi_x, \psi_t)$ and thus work with decaying quantities. Mark also that although system (4.1.1) looks like a quasilinear equation, since $\psi$ is by construction eventually slow, only regularity on $\mathbf{V}$ requires some care and with this respect $\mathbf{V}$ solves a semilinear parabolic equation with coefficients depending on derivatives of $\psi$. In particular a direct energy estimate yields the existence of positive $\epsilon$, $C$ and $\theta'$ such that, on any time-interval $[0,t]$, provided

$$\sup_{[0,t]}\,(\|\mathbf{V}\|_{H^K(\mathbb{R})}^2 + \|\psi_t\|_{H^{K-1}(\mathbb{R})}^2 + \|\psi_x\|_{H^K(\mathbb{R})}^2) \leq \epsilon$$

then

$$\begin{aligned}\|\mathbf{V}(t)\|_{H^K(\mathbb{R})}^2 \;&\leq\; C\,e^{-\theta' t}\|\mathbf{V}(0)\|_{H^K(\mathbb{R})}^2 + C\int_0^t \mathrm{e}^{-\theta'(t-s)}\|\mathbf{V}(s)\|_{L^2(\mathbb{R})}^2\mathrm{d}s\\ &+\; C\int_0^t \mathrm{e}^{-\theta'(t-s)}\left(\|\psi_t(s)\|_{H^{K-1}(\mathbb{R})}^2 + \|\psi_x(s)\|_{H^K(\mathbb{R})}^2\right)\mathrm{d}s\,.\end{aligned}$$

This is one of the few points where we rely on the structure of (4.1.1) and not on spectral stability assumptions. Yet we use only that at linear order parabolic systems, whatever their order, damp exponentially high-frequencies thus at the nonlinear order high-regularity norms of the solution are slaved to low-regularity ones. This also holds for partially parabolic systems satisfying a suitable genuine-coupling condition (see Appendix A) but requires in its implementation use of Friedrichs symmetrizers and Kawashima compensators instead of mere heat-like energy estimates[6].

Together with suitable linear estimates the above high-frequency slaving estimate allows for solving the system in $H^K(\mathbf{R})$ with decay rates of $\|\mathbf{V}(t)\|^2_{L^2(\mathbb{R})}$ by closing a nonlinear iteration with

$$\sup_{0\leq s\leq t}\ (1+s)^{1/4}\ \|\,(\mathbf{V},\psi_t,\psi_x)\,(s,\cdot)\,\|_{H^K(\mathbb{R})}.$$

Only *a posteriori* do we recover sharp decay rates but just in low-regularity norms[7].

We now come to the core of the proof, linear estimates. We try to get most of them by a direct application of Hausdorff–Young estimates. Action on localized functions of $s_j^{\mathrm{p}}(t)$ is bounded essentially as action of an operator of symbol $\xi\mapsto\alpha(\xi)\,\frac{1}{\mathrm{i}\xi}\,\mathrm{e}^{-\theta'|\xi|^2\,t}$, leading to

- when $l+m\geq 1$, for some $C$,

  $$\left\|\partial_x^l\partial_t^m s_j^{\mathrm{p}}(t)g\right\|_{L^p(\mathbb{R})}\leq C(1+t)^{-\frac12(1/q-1/p)+\frac12-\frac{l+m}{2}}\|g\|_{L^q(\mathbb{R})}$$

  for all $t\geq 0$ and $1\leq q\leq 2\leq p\leq\infty$;

- and, for some $C$,

  $$\left\|s_j^{\mathrm{p}}(t)g\right\|_{L^\infty(\mathbb{R})}\leq C\|g\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}$$

  for all $t\geq 0$.

The point is that a $t^{\frac12}$ factor of decay is lost because of the $\frac{1}{\mathrm{i}\xi}$ factor and that we may estimate in localization norms only after differentiating that brings down a $\mathcal{O}(\xi)$ compensating factor. Similarly a $\mathcal{O}(\xi)$ compensating factor is obtained from the fact that $\tilde\phi_j(0,\cdot)$ is constant and relation (1.3.6) that imply together

$$\langle\tilde\phi_j(\xi,\cdot),(\partial_x g)\check{}\,(\xi,\cdot)\rangle_{L^2([0,1])}\ =\ \mathcal{O}(\xi)$$

thus for some $C$

$$\left\|\partial_x^l\partial_t^m s_j^{\mathrm{p}}(t)\partial_x g\right\|_{L^p(\mathbb{R})}\leq C(1+t)^{-\frac12(1/q-1/p)-\frac{l+m}{2}}\|g\|_{L^q(\mathbb{R})}$$

[6] We point to the reader that there seems to be some room for improvement here since for the moment these estimates are carried out in a pointwise way sometimes adding pointwise requirements ensuring the existence of symmetrizers (see slope condition in [135]) whereas Floquet theory hints at the use of weighted symmetrizers leading to average constraints. In particular for (1.1.17) the natural candidate for symmetrizing averaged constraint would be automatically satisfied yielding no constraint at all.

[7] This is now a widely-spread standard strategy that is also illustrated by the already quoted [119, 208].

for all $t \geq 0$ and $1 \leq q \leq 2 \leq p \leq \infty$. The latter explains why we have insisted in writing nonlinear terms $\mathcal{N}$ in flux form[8] by in particular rewriting $\mathbf{V}_t\psi_x - \mathbf{V}_x\psi_t = \partial_t(\mathbf{V}\psi_x) - \partial_x(\mathbf{V}\psi_t)$.

The former estimates are not sufficient to handle transition time layer where is needed a bound on $\|\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t)g\|_{L^p(\mathbf{R})}$ for $1/2 \leq t \leq 1$. Mark that as illustrated on Figures 4.1 and 4.2 such an estimate is not available for phase contribution of single-wave evolution operator $\mathrm{e}_{d+1}\cdot s_j^{\mathrm{p}}(t)$ but holds for global phase contribution $\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t)$. By validating the scenario of Figure 4.2 through pointwise estimates[9] is proved for some positive $C$ and $\theta'$

$$\|\,\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t)g\,\|_{L^p(\mathbb{R})} \;\leq\; C\mathrm{e}^{-\theta' t}\;\|g\|_{L^2(\mathbb{R})} \;+\; t^{\frac{1}{p}}\;\|g\|_{L^1(\mathbb{R})}$$

for all $t \geq 0$ and $2 \leq p \leq \infty$. At the algebraic level, the zero-mass constraint on wavenumber is materialized by

$$\sum_{j=1}^{d+1}\beta^{(j)}(0)\;\langle\tilde{\phi}_j(0,\cdot)| \;\equiv\; \sum_{j=1}^{d+1}\beta^{(j)}(0)\tilde{\beta}^{(j)}(0)^T\;\begin{pmatrix}\mathrm{Id}_{d\times d}\\ 0\cdots 0\end{pmatrix} \;=\; \begin{pmatrix}\mathrm{Id}_{d\times d}\\ 0\cdots 0\end{pmatrix},$$

that implies

$$\mathrm{e}_{d+1}\cdot\sum_{j=1}^{d+1}\beta^{(j)}(0)\;\langle\tilde{\phi}_j(0,\cdot)| \;\equiv\; \begin{pmatrix}0 & \cdots & 0\end{pmatrix}.$$

As expected and already mentioned in Section 1.2 to motivate the introduction of space modulation, the latter bound grows in time, but we use it for moderate time only.

The remaining part $\widetilde{S}(t)$ involves both critical parts not taken in $\underline{\mathbf{U}}_x\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t)$ and parts that correspond to spectrum far from the imaginary axis. The former is bounded as for $\underline{\mathbf{U}}_x\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t)$ but with a gain stemming from the absence of the $\frac{1}{\mathrm{i}\xi}$ singularity. The latter is bounded by using a Theorem of Jan Prüss [203] converting spectral localization plus uniform bounds on resolvents in exponential decay of generated semigroup in the Hilbert-space framework. Indeed parabolic[10] energy estimates at the spectral level

[8] For the general case (1.1.1) with structure and notation of Section 1.4, analogous $\tilde{\phi}_j(0,\cdot)$ are constant with values in $\mathbf{R}^{d'}\times\{0_{(d-d')}\}$ and nonlinear terms come as

$$\mathcal{N} \;=\; \partial_t\mathcal{N}_0 + \partial_x\mathcal{N}_1 \;+\; \begin{pmatrix}0_{d'\times d'} & 0_{d'\times(d-d')}\\ 0_{(d-d')\times d'} & \mathrm{Id}_{(d-d')\times(d-d')}\end{pmatrix}\mathcal{N}_2$$

ensuring a similar compensation of the Jordan-block singularity by the special structure of nonlinear terms.

[9] Alternatively direct Hausdorff–Young estimates yield

$$\|\,\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t)g\,\|_{L^p(\mathbb{R})} \leq (1+t)\quad \|g\|_{L^1(\mathbb{R})}.$$

[10] This is second and last place where we crucially use the parabolic structure of system (1.1.14). Again we do it in a way that is not sensitive to order of parabolicity and also holds for hyperbolic-parabolic systems satisfying Kawashima condition. This simple robust strategy spares us a finer analysis in the spirit of [259, 260].

provide uniform resolvent bounds in[11] $H^s$, $s \geq 1$. This provides for instance, for any $l$, $m$, for some positive $C$ and $\theta'$,

$$\left\|\partial_x^l \partial_t^m \widetilde{S}(t) g\right\|_{L^p(\mathbb{R})} \leq C\mathrm{e}^{-\theta' t} \, \|g\|_{H^{l+2m+1}(\mathbb{R})} \; + \; C(1+t)^{-\frac{1}{2}(1/q-1/p)-\frac{l+m}{2}} \|g\|_{L^q(\mathbb{R})}$$

for all $t \geq 0$ and $1 \leq q \leq 2 \leq p \leq \infty$.

Up to now we have only discussed action of the evolution operators on localized functions but previous estimates have counterparts estimating contributions of nonlocalized perturbations of phase shift type $\underline{\mathbf{U}}_x h_0$ in terms of wavenumber perturbation $\partial_x h_0$. Assuming $\psi_\infty = 0$ by a suitable space translation, we obtain explicitly

- when $l + m \geq 1$, for some $C$,

  $$\left\|\partial_x^l \partial_t^m s_j^{\mathrm{p}}(t)(h_0 \underline{\mathbf{U}}_x)\right\|_{L^p(\mathbb{R})} \leq C(1+t)^{-\frac{1}{2}(1-1/p)+\frac{1}{2}-\frac{l+m}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R})}$$

  for all $t \geq 0$ and $2 \leq p \leq \infty$;

- for some $C$,

  $$\|\mathrm{e}_{d+1} \cdot s^{\mathrm{p}}(t)(h_0 \underline{\mathbf{U}}_x) - h_0\|_{L^p(\mathbb{R})} \leq C(1 + t^{\frac{1}{p}}) \; \|\partial_x h_0\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}$$

  for all $t \geq 0$ and $2 \leq p \leq \infty$;

- for any $l$, $m$, for some $C$,

  $$\left\|\partial_x^l \partial_t^m \widetilde{S}(t)(h_0 \underline{\mathbf{U}}_x)\right\|_{L^p(\mathbb{R})} \leq C(1+t)^{-\frac{1}{2}(1/q-1/p)-\frac{l+m}{2}} \|\partial_x h_0\|_{L^1(\mathbb{R}) \cap H^{l+2m+1}(\mathbb{R})}$$

  for all $t \geq 0$ and $2 \leq p \leq \infty$.

In all the proofs of the former inequalities we make use of the decomposition

$$(h_0 \underline{\mathbf{U}}_x)\check{}\,(\xi, x) \;=\; \underline{\mathbf{U}}_x(x) \; \widehat{h_0}(\xi) \;+\; \sum_{j' \neq 0} \underline{\mathbf{U}}_x(x) \; e^{2\mathrm{i}\pi j' x} \; \widehat{h_0}(\xi + 2j'\pi)\,. \tag{4.1.6}$$

Since it does not involve low frequencies of $h_0$, it is not surprising that the latter term on the right-hand side of the equality yields contributions estimated in terms of $\partial_x h_0$. In contrast estimation of the former term uses crucially the two-scale separation ability of the Bloch transform to take benefit of the special spectral role of $\underline{\mathbf{U}}_x$ unraveled by Proposition 4.1.3 and bring to $\widehat{h_0}(\xi)$ the needed $\xi$ factor.

[11] Actually we use the argument in $H^s([0,1])$ endowed with

$$\|g\|^2_{H^s_\xi([0,1])} := \sum_{j=0}^{s} \|(\partial_x + \mathrm{i}\xi)^j g\|^2_{L^2([0,1])}$$

when working with $L_\xi$ then lift it back by Parseval's identity.

## 4.2. Asymptotic behavior

### Statement

At last we now come to the validation of (1.4.5)-(1.4.6)-(1.4.7).

**Theorem 4.2.1** ([126]). *Assume (H)-(H') and (D1)-(D2)-(D3).*
*Then, for any $K \geq 4$ and $\eta > 0$, there exist positive $\varepsilon$ and $C$ such that if*

$$E_0 := \|\widetilde{\mathbf{U}}_0(\cdot - h_0(\cdot)) - \underline{\mathbf{U}}(\cdot)\|_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} + \|\partial_x h_0\|_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} \;\leq\; \varepsilon$$

*for some $h_0$, then, there exist $(\widetilde{\mathbf{U}}, \psi)$ with initial data $(\widetilde{\mathbf{U}}_0, h_0)$ and $\mathbf{M}$ such that $\widetilde{\mathbf{U}}$ solves (4.1.1) and, with $\psi_\infty = (h_0(-\infty) + h_0(\infty))/2$,*
*for $t > 0$ and $2 \leq p \leq \infty$*

$$\begin{aligned} &\|\widetilde{\mathbf{U}}(t, \cdot - \psi(t,\cdot)) - \;\underline{\mathbf{U}}^{\left(\underline{\mathbf{M}}+\mathbf{M}(t,\cdot),\frac{\underline{k}}{(1-\psi_x(t,\cdot))}\right)}(\,\cdot\,)\|_{L^p(\mathbf{R})} \\ &\qquad \leq\; C\,E_0\; \ln(2+t)\,(1+t)^{-\frac{3}{4}}\,, \end{aligned} \tag{4.2.1a}$$

$$\|(\mathbf{M}, \underline{k}\,\psi_x)(t,\cdot)\|_{L^p(\mathbf{R})} \leq\; C\,E_0\;(1+t)^{-\frac{1}{2}(1-1/p)}\,, \tag{4.2.1b}$$

$$\|\psi(t,\cdot) - \psi_\infty\|_{L^\infty(\mathbf{R})} \leq\; C\,E_0\,. \tag{4.2.1c}$$

*Moreover, setting $\Psi(t,\cdot) = (\mathrm{Id}_{\mathbf{R}} - \psi(t,\cdot))^{-1}$, $\kappa = \underline{k}\,\partial_x \Psi$,*

$$\boldsymbol{\mathcal{M}}(t,\cdot) = (\underline{\mathbf{M}} + \mathbf{M}(t,\cdot)) \circ \Psi(t,\cdot),$$

*and letting $(\boldsymbol{\mathcal{M}}_W, \kappa_W)$ and $\Psi_W$ solve (1.4.6) and (1.4.7) with initial data*

$$\begin{aligned} \kappa_W(0,\cdot) \;&=\; \underline{k}\,\partial_x\Psi(0,\cdot), \qquad \Psi_W(0,\cdot) \;=\; \Psi(0,\cdot), \\ \boldsymbol{\mathcal{M}}_W(0,\cdot) \;&=\; \underline{\mathbf{M}} + \tilde{\mathbf{U}}_0 - \underline{\mathbf{U}} \circ \Psi(0,\cdot) \\ &+\; \left(\frac{1}{\partial_x\Psi(0,\cdot)} - 1\right)(\underline{\mathbf{U}} \circ \Psi(0,\cdot) - \underline{\mathbf{M}})\,, \end{aligned}$$

*we have, for $t \geq 0$, $2 \leq p \leq \infty$,*

$$\begin{aligned} &\|(\mathcal{M}, \kappa)(t,\cdot) - (\mathcal{M}_W, \kappa_W)(t,\cdot)\|_{L^p(\mathbb{R})} \\ &\qquad \leq\; C\,E_0\;(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}, \end{aligned} \tag{4.2.2a}$$

$$\|\Psi(t,\cdot) - \Psi_W(t,\cdot)\|_{L^p(\mathbb{R})} \leq\; C\,E_0\;(1+t)^{-\frac{1}{2}(1-1/p)+\eta}. \tag{4.2.2b}$$

Since (4.2.1b) is sharp, estimate (4.2.1a) indeed proves a slow-modulation behavior as in (1.4.5). Then we have inverted the change of variable $\mathrm{Id}_{\mathbf{R}} - \psi(t,\cdot)$ to gather all the modulations — in phase, wavenumber and averages — on the same side as in *ansatz* of Section 1.4 to be able to show rigorously the role of (1.4.6)-(1.4.7). Once this done and suitable equivalent initial data are prescribed for the solutions of (1.4.6)-(1.4.7) to be compared with we validate modulation systems through (4.2.2a)-(4.2.2b), in view of sharp accuracy of (4.2.1b)-(4.2.1c).

All estimates are known or expected to be sharp save estimate (4.2.1a). As the restriction to[12] $2 \leq p \leq \infty$ decay in (4.2.1a) is a technical drawback of our choice to prove most of linear estimates by resorting to Hausdorff–Young estimates. Our choice is commanded by simplicity but there are various means[13] to obviate both issues: finer multiplier lemmas, pointwise bounds, weighted-in-time energy estimates... We refer the reader to [138] for a first piece of work in this direction for systems (1.1.13). Likewise we have not aimed at optimizing regularity requirements but at a simple and robust proof.

Prescription of the initial data for solutions of the Whitham system, especially for $\boldsymbol{\mathcal{M}}_W(0,\cdot)$, is a subtle point[14] not evident from the viewpoint of formal approximation. In particular, the appearance of a term related to phase variations in prescription of $\boldsymbol{\mathcal{M}}_W(0,\cdot)$ arises in our analysis through a detailed study of the contribution of high frequencies of the local wavenumber to variations of the low-Floquet part of the solution hence by essence is not captured by a *slow* modulation *ansatz*. Nevertheless in [126, Remark 1.14] some light is shed on these prescriptions by some heuristic arguments leading unfortunately to slightly incorrect formulas.

With the full scenario in hands we may now track simplifications that occur for phase-decoupled systems when the reference wave experience a localized perturbation.

**Corollary 4.2.2** ([126]). *Assume (H)-(H'), (D1)-(D2)-(D3).*
*Assume moreover that $\underline{\mathbf{U}}$ is linearly phase-decoupled.*
*Then, for any $K \geq 4$ and any $\eta > 0$, there exist positive $\varepsilon$ and $C$ such that if*

$$E_1 := \|\widetilde{\mathbf{U}}_0 - \underline{\mathbf{U}}\|_{L^1(\mathbf{R};(1+|\cdot|))\cap H^K(\mathbf{R})} \ \leq \ \varepsilon$$

*then, there exist $\widetilde{\mathbf{U}}$ with initial data $\widetilde{\mathbf{U}}_0$ such that $\widetilde{\mathbf{U}}$ solves (4.1.1) and for $t > 0$ and $2 \leq p \leq \infty$,*

$$\|\widetilde{\mathbf{U}}(t,\cdot) - \underline{\mathbf{U}}\|_{L^p(\mathbf{R})} \ \leq \ C\,E_1\ (1+t)^{\frac{1}{2p}-\frac{1}{4}+\eta}\,.$$

---

[12] That it is indeed only a technical restriction contrasts with some similar multidimensional situations, for instance [119, 208], where $L^2$ is a critical space, scattering of diffusion-waves enhancing decay in $L^p$ with $p > 2$ but slackening decay in $L^p$ when $p < 2$.

[13] Yet some other natural candidates do not work. In particular the weighted-norm strategy that when it works, as in [212], allows the choice of the same functional space for initial data and further values of the solution leads to spurious growth in deducing estimation of $L^p$-norms when applied to a multiscale evolution. Incidentally we point to the reader that weighted norms may also play a deeper role in the stability analysis than just offering a convenient control of Lebesgue norms. This is illustrated by pioneering work of David Sattinger [213, 214] but also by elsewhere mentioned [92, 91, 207] and [200].

[14] This issue does not arise in the related analysis [128] of the reaction-diffusion case, as $\mathbf{M}$ does not appear. Besides, by some special feature of the scalar case, there it is not even needed to invert the change of variable !

**Corollary 4.2.3** ([126])**.** *Assume (H)-(H'), (D1)-(D2)-(D3).*
*Assume moreover that* $\underline{\mathbf{U}}$ *is quadratically phase-decoupled.*
*Then, for any* $K \geq 4$ *and any* $\eta > 0$*, there exist positive* $\varepsilon$ *and* $C$ *such that if*

$$E_0 := \|\widetilde{\mathbf{U}}_0 - \underline{\mathbf{U}}\|_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} \ \leq \ \varepsilon$$

*then, there exist* $\widetilde{\mathbf{U}}$ *with initial data* $\widetilde{\mathbf{U}}_0$ *such that* $\widetilde{\mathbf{U}}$ *solves (4.1.1) and for* $t > 0$ *and* $2 \leq p \leq \infty$*,*

$$\|\widetilde{\mathbf{U}}(t,\cdot) - \underline{\mathbf{U}}\|_{L^p(\mathbf{R})} \ \leq \ C\,E_0\ (1+t)^{\frac{1}{2p}-\frac{1}{2}+\eta}\,.$$

Decay rates provided by Corollaries 4.2.2 and (4.2.3) are sharp. Though no phase shift appears in their statements they are proved by applying estimates of Theorem 4.2.1 and in the merely linearly phase-decoupled case it seems unlikely that a similar result could be obtained without a precise description of local phase shift. Mark that our refined treatment of the linearly phase-decoupled case requires more initial localization than our other results because it involves a fine spatial validation of the weakly-interacting diffusion-wave scenario for near-constant solutions of hyperbolic-parabolic systems proved in [170].

One may wish to express localization as a mean-free condition on $\partial_x h_0$. Actually, in the above bounds, the condition $h_0 \equiv 0$ may indeed be relaxed to the condition that $\partial_x h_0$ is mean-free and either

$$E_1 := E_0 + \||\cdot|\,\partial_x h_0\|_{L^1(\mathbb{R})}$$

is small in the quadratically phase-decoupled case or

$$E_1 := E_0 + \||\cdot|\,\partial_x h_0\|_{L^1(\mathbb{R})} + \||\cdot|\,(\tilde{u}_0(\cdot - h_0(\cdot)) - \bar{U})\|_{L^1(\mathbb{R})}$$

is small in the linearly phase-decoupled case. In either case, the conclusion is asymptotic orbital stability with asymptotic phase $\psi_\infty = (h_0(-\infty)+h_0(\infty))/2$ (in the sense of [118]).

We recall now with some more details processes determining decay rates. For the quadratically phase-decoupled case the point is that if $\kappa(t,\cdot)$ writes as $\underline{k}+k(t,\cdot)$ with for some real $a$, $b$ and some positive $d$

$$k(0,\cdot) = 0\,, \qquad k_t + (a\,k + b\,k^2)_x - d\,k_{xx} \ = \ \partial_x r\,,$$

and the remainder $r$ satisfies

$$\|r(t,\cdot)\|_{L^p(\mathbf{R})} \leq C(1+t)^{-\frac{1}{2}(1-1/p)-1}\,, \qquad 1 \leq p \leq \infty\,,\ t \geq 0\,,$$

then for any arbitrary positive $\eta$ provided $C$ is small enough

$$\|\kappa(t,\cdot) - \underline{k}\|_{L^p(\mathbf{R})} \leq C'(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta}\,, \qquad 2 \leq p \leq \infty\,,\ t \geq 0\,.$$

Likewise for the linearly phase-decoupled case the point is that if $\kappa(t,\cdot)$ writes as $\underline{k}+k(t,\cdot)$ with for some real $a$, $b$ and some positive $d$

$$k(0,\cdot) = 0\,, \qquad k_t + (a\,k + b\,k^2)_x - d\,k_{xx} \ = \ \partial_x r\,,$$

and the remainder $r$ writes for some Burgers-wave or heat-wave profile $W$ and some speed $a' \neq a$

$$r(t,x) \;=\; [W(t,x-a't)]^2\,, \qquad t\geq 0\,,\ x\in\mathbf{R}\,,$$

then for any arbitrary positive $\eta$ provided $W$ is small enough

$$\|\kappa(t,\cdot)-\underline{k}\|_{L^p(\mathbf{R})}\leq C'(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}+\eta}\,, \qquad 2\leq p\leq\infty\,,\ t\geq 0\,.$$

In view of the latter mark that for the analysis of localized perturbations in the linearly phase-decoupled case, assumption (H') plays a role deeper than just providing regularity in a simple unified way for both the Whitham system and low-Floquet critical spectral expansions. Indeed, in this case, the extra damping $(1+t)^{-1/4}$ encodes the fact that quadratic interactions between diffusion waves traveling at different characteristic speeds are asymptotically irrelevant [170]. Thus, whereas elsewhere we expect that, by usual considerations, it could be replaced with symmetrizability of the Whitham system and a direct smoothness assumption on spectral expansions, here one should not expect to be able to replace (H') with something weaker than: the linear group velocity associated to the wavenumber mode is different from all other characteristic speeds.

## Outline of the proof

The first step of the proof requires the use of the full critical block $s^{\mathrm{p}}(t)$ to refine decomposition of $S(t)$ by involving all types of modulation. To achieve the appropriate splitting we need to unravel the role of $\partial_k\underline{\mathbf{U}}$ in the description of low-Floquet critical spectral expansions.

**Proposition 4.2.4** ([185])**.** *Assume (H)-(H'), (D1)-(D2)-(D3).*
*In Proposition 2.1, one may ensure for a suitable normalization*

$$\partial_\xi q_{d+1}(0) \;=\; i\underline{k}\ \partial_k\underline{\mathbf{U}}_{|(\underline{\mathbf{M}},\underline{k})}\,.$$

As an outcome

$$\phi_j(\xi) \;=\; (\mathrm{i}\underline{k}\xi)\sum_{l=1}^{d}\beta_l^{(j)}(\xi)q_l(\xi) \;+\; \beta_{d+1}^{(j)}(\xi)\quad q_{d+1}(\xi).$$

with

$$q_j(\xi) \;=\; \partial_{\mathbf{M}_j}\underline{\mathbf{U}} \;+\; \mathcal{O}(\xi), \qquad 1\leq j\leq d,$$

$$q_{d+1}(\xi) \;=\; \underline{\mathbf{U}}_x \;+\; (\mathrm{i}\underline{k}\xi)\,\partial_k\underline{\mathbf{U}} \;+\; \mathcal{O}(\xi^2).$$

This leads to refined decomposition

$$S(t) \;=\; R^{\mathrm{p}}(t)+R^M(t)+\tilde{R}(t)\,,$$

with

$$R^{\mathrm{p}}(t) \;:=\; (\underline{\mathbf{U}}_x+\partial_k\underline{\mathbf{U}}\ \underline{k}\partial_x)\,\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t),$$

$$R^M(t) \;:=\; \mathrm{d}_{\mathbf{M}}\underline{\mathbf{U}}\cdot s^M(t),$$

where

$$s^M(t) \;:=\; \begin{pmatrix}\mathrm{I}_d & 0_{d\times 1}\end{pmatrix} \underline{k}\partial_x s^{\mathrm{p}}(t)\,.$$

Observe the obtention of the expected relation between phase shift evolution operator $\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t)$ and wavenumber evolution operator $\underline{k}\partial_x\,\mathrm{e}_{d+1}\cdot s^{\mathrm{p}}(t)$.

At the nonlinear level this naturally suggests the decomposition

$$\mathbf{V}(t,x) \;=\; \partial_k\underline{\mathbf{U}}(x)\;\underline{k}\psi_x(t,x) \;+\; \mathrm{d}_{\mathbf{M}}\underline{\mathbf{U}}(x)\cdot\mathbf{M}(t,x) \;+\; \mathbf{Z}(t,x)$$

with $(\mathbf{V},\psi)$ as in Theorem 4.1.2 and

$$\mathbf{M}(t) \;=\; s^M(t)(\mathbf{V}(0)+h_0\underline{\mathbf{U}}_x)+\int_0^t s^M(t-s)\mathcal{N}(s)\mathrm{d}s\,.$$

As a consequence

$$\begin{aligned}\mathbf{Z}(t) \;&=\tilde{R}(t)\,(\mathbf{V}(0)+h_0\underline{\mathbf{U}}_x) \;+\; \int_0^t \tilde{R}(t-s)\mathcal{N}(s)\mathrm{d}s\\ &+\;(1-\chi(t))\,\Big(R^{\mathrm{p}}(t)(\mathbf{V}(0)+h_0\underline{\mathbf{U}}_x)-(\underline{\mathbf{U}}_x+\partial_k\underline{\mathbf{U}}\,\underline{k}\partial_x)h_0+\int_0^t R^{\mathrm{p}}(t-s)\mathcal{N}(s)\mathrm{d}s\Big)\,.\end{aligned}$$

Now since the critical part of $\tilde{R}(t)$ contains an extra $\mathcal{O}(\xi)$ factor compared to the one of $\tilde{S}(t)$, we expect and obtain an extra $(1+t)^{-1/2}$ factor of decay for $\tilde{R}(t)$, that yields almost directly bound (4.2.1a). Besides, bounds on $\partial_x s^{\mathrm{p}}_j(t)$ readily provides bounds on $s^M(t)$ that leads to the missing part of (4.2.1b).

To complete the proof of Theorem 4.2.1 we now compare $(\mathbf{M},\underline{k}\psi_x)$ with the appropriate solution of the Whitham system composed with $\mathrm{Id}_{\mathbf{R}}-\psi$. We start from integral evolution system

$$\begin{pmatrix}\mathbf{M}(t)\\ \underline{k}\psi_x(t)\end{pmatrix} \;=\; \underline{k}\partial_x s^{\mathrm{p}}(t)(\mathbf{V}(0)+h_0\underline{\mathbf{U}}_x)+\int_0^t \underline{k}\partial_x s^{\mathrm{p}}(t-s)\mathcal{N}(s)\,\mathrm{d}s \qquad \text{for } t\geq 1.$$

Therefore we need both an analysis of $\mathcal{N}$ and a connection between $s^{\mathrm{p}}(t)$ and the linearized evolution of (1.4.6).

Once substituted time derivatives in $\mathcal{N}$ by appropriate combination of space derivatives, by using bounds (4.2.1a)-(4.2.1b), nonlinear terms are written as derivatives of quadratic functions of $(M,\underline{k}\psi_x)$ with periodic coefficients, plus a faster-decaying remainder. The relevant time-space substitution is obtained from analytical validation of the role of linear group velocities. Indeed one shows by the same techniques used to prove other linear estimates that application of total derivative

$$D \;=\; \partial_t+\left(\begin{pmatrix}\mathrm{d}\mathbf{F}_{|(\underline{\mathbf{M}},\underline{k})}\\ -\mathrm{d}\omega_{|(\underline{\mathbf{M}},\underline{k})}\end{pmatrix}-c\mathrm{I}_{d+1}\right)\underline{k}\partial_x$$

enhances decay of $s^{\mathrm{p}}(t)$ by factor $(1+t)^{-1}$ as would do a second-order differentiation. We stress that up to inessential terms $\mathcal{N}$ is in the end written as a combination of derivatives of slow-modulation terms $\partial_t(g\,h)$ or $\partial_x(g\,h)$ with $h$ slow and $g$ one-periodic.

In particular, we only need then to make the connection at the level of evolution operators when applied to initial data $\mathbf{V}(0)+h_0\underline{\mathbf{U}}_x$ and to such slow-modulation terms $\partial_x(g\,h)$ and $\partial_t(g\,h)$. The analytical connection between $s^{\mathrm{p}}(t)$ and the linearized evolution of (1.4.6), which again involves some intricate analysis, reveals in which way data for the original system are translated into data for the Whitham system. Formulas involve a large number of averages that are in the end explicitly evaluated by differentiating a sufficient number of times then averaging wave profile equation. Contributions of terms of type $\partial_t(g\,h)$ are handled by integration by parts in time in Duhamel's formula then use of linear group velocities to reduce it to a combination of terms $\partial_x s^{\mathrm{p}}(t)(g\,h)$. To make comparisons precise we introduce

$$\Sigma(t) \;=\; \sum_{j=1}^{d+1}\sigma_j(t)\;\beta^{(j)}(0)\;\tilde{\beta}^{(j)}(0)^T.$$

with $\sigma_j(t)$ the solution operator of

$$u_t \;=\; \underline{k}\,a_j u_x+\underline{k}^2\,b_j u_{xx}$$

and $\mathcal{I}$ the natural symmetric antiderivative operator. To give an idea of the nature of the needed average computations, we state the full intricate comparisons: for any $g$ one-periodic and any $l\geq 0$, for some $C$, for any $2\leq p\leq\infty$ and any $t\geq 0$,

$$\begin{aligned}
&\Big\|\partial_x^l s^{\mathrm{p}}(t)(h_0\underline{\mathbf{U}}_x+\mathbf{d})-\frac{1}{\underline{k}}\Sigma(t)\mathcal{I}\partial_x^l\begin{pmatrix}\mathbf{d}-(\underline{\mathbf{U}}-\langle\underline{\mathbf{U}}\rangle)\partial_x h_0\\ \underline{k}\partial_x h_0\end{pmatrix}\Big\|_{L^p(\mathbb{R})}\\
&\leq C\begin{cases}(1+t)^{-\frac12(1-1/p)-\frac12}t^{-\frac{l-1}{2}}(\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})}+\|\mathbf{d}\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})}), & l\geq 1\\ (1+t)^{-\frac12(1-1/p)}\qquad(\|\partial_x h_0\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}+\|\mathbf{d}\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}), & l=0\end{cases}
\end{aligned}$$

$$\begin{aligned}
&\Big\|\partial_x^l s^{\mathrm{p}}(t)\partial_x(g\,h)-\frac{1}{\underline{k}}\Sigma(t)\mathcal{I}\partial_x^l\begin{pmatrix}[\langle g\rangle-\mathrm{i}\sum_{j=1}^{d}\langle\partial_\xi\tilde{q}_j(0),\partial_x g\rangle\mathrm{e}_j]\partial_x h\\ \underline{k}\langle\tilde{q}_{d+1}(0),\partial_x g\rangle\partial_x h\end{pmatrix}\Big\|_{L^p(\mathbb{R})}\\
&\qquad\lesssim\begin{cases}(1+t)^{-\frac12(1-1/p)-\frac12}t^{-\frac{l-1}{2}}\|\partial_x h\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})}, & l\geq 1\\ (1+t)^{-\frac12(1-1/p)}\qquad\|\partial_x h\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}, & l=0\end{cases}\\
&\Big\|\partial_x^{l+1} s^{\mathrm{p}}(t)(g\,h)-\frac{1}{\underline{k}}\Sigma(t)\mathcal{I}\partial_x^l\begin{pmatrix}\langle g\rangle\partial_x h\\ 0\end{pmatrix}\Big\|_{L^p(\mathbb{R})}\\
&\qquad\lesssim\begin{cases}(1+t)^{-\frac12(1-1/p)-\frac12}t^{-\frac{l-1}{2}}\|\partial_x h\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})}, & l\geq 1\\ (1+t)^{-\frac12(1-1/p)}\qquad\|\partial_x h\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})}, & l=0\end{cases}.
\end{aligned}$$

In order to offer a glimpse at some of the key points of the proof of the former crucial inequalities we inspect briefly the contribution of $h_0\underline{\mathbf{U}}_x$. Recall (4.1.6). The form of the equivalent contribution of the first term in the right-hand side of (4.1.6) stems from

$$\langle\tilde{\phi}_j(0)+\xi\,\partial_\xi\tilde{\phi}_j(0),\underline{\mathbf{U}}_x\widehat{h}_0(\xi)]\rangle \;=\; \xi\langle\partial_\xi\tilde{\phi}_j(0),\underline{\mathbf{U}}_x\rangle\;\widehat{h}_0(\xi)\;=\;\tilde{\beta}_j(0)\cdot\begin{pmatrix}0\\ \underline{k}\partial_x h_0\end{pmatrix}$$

while the contribution of the second term is read on

$$\begin{aligned}
&\langle \tilde{\phi}_j(0), \underline{\mathbf{U}}_x[\check{h}_0(\xi) - \widehat{h}_0(\xi)]\rangle \\
&\quad = -\tilde{\beta}_j(0) \cdot \begin{pmatrix} [(\underline{\mathbf{U}} - \langle \underline{\mathbf{U}} \rangle)\partial_x h_0]^\frown(\xi) \\ 0 \end{pmatrix} + \mathrm{i}\xi\, \tilde{\beta}_j(0) \cdot \begin{pmatrix} \langle \underline{\mathbf{U}}, [\check{h}_0(\xi) - \widehat{h_0}(\xi)]\rangle \\ 0 \end{pmatrix} ,
\end{aligned}$$

the latter term being negligible essentially because it exhibits an extra $\xi$ factor.

Finally, having checked that all relevant terms match by computing needed averages, we prove the nonlinear connection to (1.4.6) by closing nonlinear estimates in a standard way.

# 5. Prospects

As a conclusion, we discuss some future directions of research. They naturally split in two blocks, one aiming at developing further nonlinear theory about periodic wavetrains in dissipative systems, another one having as ambition a similar trail-blazing piece of work for other related fields.

The former is organized around three tasks:

- to apply further the theory, that is, to check diffusive spectral stability. One guide for such further tasks is the will to incorporate capillary effects in thin films description and to elucidate what occurs in inviscid or large-Froude number limits.
- to deepen it. By this we mean to test its robustness by trying to extend it to challenging situations. Again part of the motivation comes from applications. Extension to free-surface incompressible equations clearly stands there.
- to complete it. Among missing pieces of the theory we point a direct validation of the averaged equations and a nonlinear instability theory.

Achievement of goals of the latter direction is probably a long-term task that should start by accumulating small steps in the good direction. Among important coherent structures still awaiting an advanced stability theory we list:

- periodic wavetrains of *conservative* systems. Our spectral studies of some Hamiltonian systems is clearly meant to be one of these first advances in the good direction.
- multiperiodic waves, even in dissipative systems. As a motivation, we recall that in hydrodynamic instablity scenarios bi-periodic waves are expected to emerge as secondary instabilities.
- compound patterns. Some important coherent structures are built from simpler elementary blocks, among which periodic wavetrains. It seems now possible to tackle their stability.

With this scheme in mind we review now some open questions directly related to some topics of the memoir. Most of them are already under investigation by the author and his collaborators, either with some hope for some complete or preliminary results with mid-term expectations, for instance for the inviscid limit or the multiperiodic problem, or as parts of long-term projects, concerning the consideration of free-surface flows, identification and obtention of dispersive stability or analysis of dispersive shocks.

## Applying further

Checking spectral stability requires a case-by-case treatment. Hence an endless task list for both numerical verification and inspection of asymptotic limits. We only point two of them here as related to thin film problems.

For the St. Venant system some natural asymptotic limits are still to investigate. One of them, though it could be formulated as a particular $\mathbf{F} \to \infty$ limit, is better apprehended as an inviscid limit. After scaling out the Froude number and leaving instead a viscosity coefficient to obtain

$$\begin{cases} \tau_t - u_x = 0, \\ u_t + (\frac{1}{2}\tau^{-2})_x = 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x. \end{cases}$$

in mass-Lagrangian variables, it ammounts to the analysis of the singular limit $\nu \to 0^+$. In this case limit profiles are periodic arrays of shocks built by Robert Dressler [68]. Among other difficulties of the problem, we point that the analysis of low-viscosity profiles performed by Jörg Härterich and Pascal Noble [117, 183] involves extensions of standard geometric singular perturbation initiated by Neil Fenichel [85, 137] to non-hyperbolic cases as developed by Freddy Dumortier and Robert Roussarie, and Martin Krupa and Peter Szmolyan [78, 72, 73, 70, 71, 74, 56, 57, 55, 58, 75, 196, 154, 153, 155, 241]. As for the KdV limits of Section 2.2 [125, 10] or for the Zeldovich–von Neumann–Doering limit for detonation profiles [258], natural strategies involve Evans' functions analysis that are better suited than direct spectral studies to deal with singular situations.

Among other directly related problems stands the inspection of waves of the St. Venant system incorporating both viscosity and capillary effects

$$\begin{cases} \tau_t - u_x = 0, \\ u_t + ((2\mathbf{F}^2)^{-1}\tau^{-2})_x = 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x - \sigma(\tau^{-5}\tau_{xx} - \frac{5}{2}\,\tau^{-6}(\tau_x)^2)_x, \end{cases}$$

when both viscosity coefficient $\nu$ and capillarity coefficient $\sigma$ are positive. In contrast with purely capillary cases as encoded by the Euler–Korteweg system, these equations can be reduced by Kotschote's method of auxiliary variables [152] — that is, by introducing $z := \tau_x$ — to a $3 \times 3$ second-order quasilinear parabolic system

$$\begin{cases} \tau_t - u_x + z_x = \tau_{xx}, \\ z_t = u_{xx}, \\ u_t + ((2\mathbf{F}^2)^{-1}\tau^{-2})_x = 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x - \sigma(\tau^{-5}z_x - \frac{5}{2}\,\tau^{-6}z^2)_x \end{cases}$$

to which a standard extension of our results could be applied so that it is natural to examine whether its periodic waves are diffusively spectrally stable, either by the numerical strategy exposed in Section 3.1 or in one of the many possible asymptotic limits.

## Deepening

We have already pointed in the main text two interesting directions of extension. The first is to relax the strict hyperbolicity assumption on the first-order averaged system, at least to some relevant symmetrizable applications. The second is to develop a better strategy to obtain energy estimates, tailored to the periodic-wave case by incorporating Floquet weights. This could remove the slope restriction that is needed to apply results of [135] to St. Venant waves. Indeed this condition seems unnecessary in view of spectral considerations. Going further with considerations on energy estimates, one may also hope to extend the theory to cases where dissipation of the original system is not of parabolic type, such as hyperbolic-relaxation systems satisfying a suitable Kawashima condition.

Coming back to applications to thin fluid films, we add to the former two problems motivated by the will to bring theory closer to real laboratory experiments. First, one may wish to relax the extended systems description of open flows. To match closely experiments it is indeed more appropriate to consider injected flows, an initial-boundary value problem. A direct analog of our nonlinear stability program would investigate stability of semi-infinite periodic traveling waves. It would be at least as interesting to examine how an injected perturbation on a constant steady state grows in space into an invading front leaving in its wake a periodic traveling wave.

Recalling that even the St. Venant system offers only an approximate description of a reduced dynamics, one may also wish to analyze the original free-surface incompressible evolution. Obviously it does not follow from a standard extension of our theory and the nonlinear problem seems technically very challenging. But the strategy followed by Yan Guo and Ian Tice to prove nonlinear asymptotic stability of trivial constant solutions [113] does not seem incompatible with our scheme of proof.

## Completing

The conviction of the author is that nonlinear results of Chapter 4 unifies, clarifies and generalizes a large number of previous results in nonlinear stability and modulation theory of periodic traveling waves of dissipative systems, resolving some long-standing questions. Yet they leave open a few fundamental questions and raise a lot of new problems. We provide now examples for the former and the latter.

By a suitable choice of initial data in Theorem 4.2.1, any small regular localized initial data of the second-order Whitham system (1.4.6) may be realized in a slow modulation process giving analysis of asymptotic behavior. Yet it is not a direct validation of slow modulation *ansatz* encoded at first-order by (1.4.5)-(1.4.2) and at second-order by (1.4.5)-(1.4.6). Indeed such a validation in the spirit of [66, 69] is still awaiting a proof. One asset of direct validations is that, since they often rely on Cauchy–Kowaleskaya-type arguments, they sometimes include unstable dynamics.

Concerning precisely unstable dynamics, a new question is raised by the introduction of the notion of space-modulated stability. We have proved — and tried to convince the reader — that the notion is flexible enough to capture the kind of stability occuring in dynamics near periodic waves. But to obtain that the notion is in some sense sharp an instability result is missing. Indeed the question of whether spectral instability implies nonlinear space-modulated instability remains open. Mark that a proof that nonlinear space-modulated stability fails should show that no near-identity modulation in space could compensate for the growing instability.

## Dispersive nonlinear stability

A field that is left widely-open by the present memoir is nonlinear stability of periodic waves of Hamiltonian systems (1.1.2). Yet to the opinion of the author time is come for a long-term effort in this direction. An encouraging sign is that for Hamiltonian dispersive equations some of the proofs of asymptotic stability for other *localized* patterns, as pioneered by Michael Weinstein and his collaborators [247, 224, 225, 226, 227, 200, 248, 201, 232, 234, 235, 233, 231], already exhibit some common features with recent proofs of dissipative stability of periodic waves.

Actually there are already some nonlinear stability results but under co-periodic or subharmonic perturbations and that are thus proved by variational arguments stemming from the Hamiltonian structure supplemented by other local conservation laws. We refer the reader to [89, 4, 5, 124, 182, 61] for a significant sample of the literature. At the spectral level, it is known from the Floquet-Bloch theory that stability with respect to localized perturbations implies co-periodic stability. Remarkably enough, as far as nonlinear stability is concerned, there is no known relationship between stability to localized perturbations and co-periodic stability. If the latter is much easier to show, the author is convinced that the former is much more relevant. Indeed, systems in the whole space are to be viewed as idealizations of bounded domains situations. Unfortunately, to the best knowledge of the author, none has been proved so far.

On the spectral level we have already mentioned that there are now a few situations where a full spectral stability has been proved, see for instance [90, 116]. Yet nonlinear stability is likely to require a refined notion of spectral stability, which would incorporate prescription of some constraints on the Floquet parametrization of the spectrum. This we would call dispersive spectral stability. Identifying this notion is still a wide-open problem.

## Multiperiodic wavetrains

As already stressed, in dissipative systems, analyzing the stability of spectrally-stable planar periodic traveling waves gets easier as the dimension increases. In higher dimensions, a more challenging task is to examine the stability of multiperiodic patterns. In particular, for those, the formally-obtained modulation averaged systems are also posed in higher dimensions and local wavenumbers satisfy a curl-free condition. Besides it may well be that the potential ability of considering wilder non-localized perturbations, offered by higher decay rates expected for linear evolution of localized perturbations, are needed to consider isolated simple elements of the asymptotic behavior.

We recall that a motivation for considering planar traveling waves is that they emerge as primary hydrodynamic instabilities. Since investigation of secondary instabilities naturally leads to bi-periodic problems, it is the next step to achieve to improve our understanding of hydrodynamic stability.

## Compound patterns

Lastly, a natural direction of extension is towards the analysis of more complicated, unsteady, patterns. In particular for dissipative systems we are now in position to investigate compound patterns that integrate periodic waves as elementary blocks. Our strategy to obtain asymptotic behavior was to prove a form of averaging in the large-time to reduce the dynamics near periodic waves to dynamics of an averaged system near constant states. One may naturally expect that this successful strategy could be transferred to achieve the stability analysis of patterns corresponding to more sophisticated solutions of the averaged systems — rather than constant states, we may think of fronts, shocks, rarefaction waves,... Some such extensions have already been obtained for the scalar Whitham case of reaction-diffusion systems, see [66] for example.

We point as a particularly interesting situation that in some regime dispersive shocks of Hamiltonian systems (1.1.2) may be thought of as rarefaction waves of the associated first-order averaged system (1.4.1). These dispersive shocks are formed by hyperbolic systems regularized by higher-order dispersive terms. Approximately at the time when the original hyperbolic system would form a shock an oscillating zone appears. In contrast with what occurs for viscous shocks the regularizing zone of dispersive shocks expands in space with time and the oscillation amplitude exceeds amplitude of the original hyperbolic shock. To the opinion of the author, the analysis, by robust general methods, of the stability of such compound patterns in conservative systems will probably await a good notion of dispersive stability for periodic wavetrains of Hamiltonian systems.

# A. A glimpse at the Kawashima condition

Although it is not strictly needed in the course of the text we believe that a terse introduction to the classical theory of hyperbolic-dissipative systems could help the reader not familiar with it to gain more insight on extensions of the periodic stability theory to such systems and more familiarity with diffusion-waves behavior near constant states. Of course these few lines do not intend to replace a thorough consultation of related litterature [249, 147, 223, 170, 251, 252, 253, 149, 254, 209, 114, 23, 173, 16, 52, 53].

To serve as a simple example we consider only a linear constant-coefficient symmetric hyperbolic-parabolic system

$$\mathbf{U}_t + \mathbf{A}\,\mathbf{U}_x = \mathbf{B}\,\mathbf{U}_{xx} \tag{A.0.1}$$

with $\mathbf{A} = \mathbf{A}^*$, $\mathbf{B} = \mathbf{B}^*$ and $\mathbf{B}$ positive semidefinite. We assume the genuine-coupling Kawashima condition: no eigenvector of $\mathbf{A}$ belongs to the kernel of $\mathbf{B}$. Were the condition not satisfied there would be a solution to (A.0.1) experiencing no dissipation effect.

A simple but fundamental example is obtained by choosing ($d = 2$ and)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

## Energy estimates

In the present section we show energy estimates for solutions to (A.0.1).

In any[1] $H^s$, $s \in \mathbf{N}$, a direct energy estimate yields a bound

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{U}\|_{H^s}^2 = -\|\mathbf{B}^{1/2}\mathbf{U}_x\|_{H^s}^2 .$$

Since $\mathbf{B}$ may not be positive definite, the former estimate is insufficient to prove, for instance, asymptotic decay.

The point now is to distort above Lyapunov functionals in equivalent norms that are strict Lyapunov functionals by adding a small seemingly negligible term, called in this context a Kawashima compensator. Behind these vague words the expert reader may have recognized in disguise an instance of hypocoercive estimates. As emphasized in [245, Remark 17], Kawashima-type estimates are indeed special instances of these that have preceded general abstract theory.

[1] We do not specify spacial domain, as it could be either the whole line $\mathbf{R}$ or a periodic cell $[0, 1]$.

To this purpose, we observe that for any skew-symmetric[2] matrix $\mathbf{K}$, for any $0<\alpha<\min(\{1,1/\|\mathbf{K}\|\})$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{U}\|_{H^1}^2 \quad &+\quad 2\alpha\,\mathrm{Re}(\langle \mathbf{U}_x,\mathbf{K}\mathbf{U}\rangle)\Big)\\ &\leq\ -\alpha\langle \mathbf{U}_x,((\mathbf{K}\mathbf{A}+(\mathbf{K}\mathbf{A})^*)+\mathbf{B})\mathbf{U}_x\rangle\ -\ \|\mathbf{B}^{1/2}\mathbf{U}_{xx}\|_{L^2}^2\\ &\quad +2\alpha\|\mathbf{K}\mathbf{B}^{1/2}\|\ \|\mathbf{U}_x\|_{L^2}\|\mathbf{B}^{1/2}\mathbf{U}_{xx}\|_{L^2}\end{aligned}$$

so that if $(\mathbf{K}\mathbf{A}+(\mathbf{K}\mathbf{A})^*)+\mathbf{B}$ is positive definite, up to restricting further the size of $\alpha$ one may ensure

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{U}\|_{H^1}^2 \quad &+\quad 2\alpha\,\mathrm{Re}(\langle \mathbf{U}_x,\mathbf{K}\mathbf{U}\rangle)\Big)\\ &\leq\ -\tfrac{\alpha}{2}\langle \mathbf{U}_x,((\mathbf{K}\mathbf{A}+(\mathbf{K}\mathbf{A})^*)+\mathbf{B})\mathbf{U}_x\rangle\ -\ \tfrac{1}{2}\|\mathbf{B}^{1/2}\mathbf{U}_{xx}\|_{L^2}^2\,.\end{aligned}$$

To build a suitable $\mathbf{K}$, we first diagonalize $\mathbf{A}$ as $\mathbf{A}=\sum_{j=1}^r a_j\,\mathbf{\Pi}_j$ where $a_1,\,\dots,\,a_r$ are real and distinct and $\mathbf{\Pi}_1,\,\dots,\,\mathbf{\Pi}_r$ are complementary spectral orthogonal projectors. By the Kawashima condition, $\sum_{j=1}^r \mathbf{\Pi}_j\,\mathbf{B}\,\mathbf{\Pi}_j$ is positive definite so that it is sufficient to find a skew-symmetric $\mathbf{K}$ such that

$$-(\mathbf{K}\mathbf{A}+(\mathbf{K}\mathbf{A})^*)\ =\ \mathbf{B}\ -\ \sum_{j=1}^r \mathbf{\Pi}_j\,\mathbf{B}\,\mathbf{\Pi}_j\,.$$

Now a classical lemma of linear algebra ensures that

$$\mathcal{M}_d(\mathbf{R})\to\mathcal{M}_d(\mathbf{R}),\qquad \mathbf{X}\longmapsto\mathbf{X}\ -\ \sum_{j=1}^r \mathbf{\Pi}_j\,\mathbf{X}\,\mathbf{\Pi}_j$$

is the orthogonal projector on the range of

$$\mathcal{M}_d(\mathbf{R})\to\mathcal{M}_d(\mathbf{R}),\qquad \mathbf{X}\longmapsto\mathbf{A}\,\mathbf{X}\ -\ \mathbf{X}\,\mathbf{A}$$

along its kernel. Hence a $\mathbf{X}$ such that $\mathbf{A}\,\mathbf{X}-\mathbf{X}\,\mathbf{A}=\mathbf{B}-\sum_{j=1}^r \mathbf{\Pi}_j\,\mathbf{B}\,\mathbf{\Pi}_j$ and the appropriate $\mathbf{K}=-\frac{1}{2}(\mathbf{X}-\mathbf{X}^*)$.

## Resolvent bounds

Now we prove that there exists a positive $\theta_0$ such that if $0<\theta\leq\theta_0$ and $\mathcal{P}_\theta:=\{\ \lambda\in\mathbf{C}\mid \mathrm{Re}(\lambda)\geq-\theta\ \}$ is included in the resolvent set of $L=\mathbf{B}\,\partial_x^2-\mathbf{A}\partial_x$ (acting on $H^1$) then $(\lambda-L)^{-1}$ is uniformly bounded in $\lambda\in\mathcal{P}_\theta$.

[2]There is no choice there since anyway

$$\mathrm{Re}(\langle \mathbf{U}_x,\mathbf{K}\mathbf{U}\rangle)\ =\ \mathrm{Re}(\langle \mathbf{U}_x,\tfrac{1}{2}(\mathbf{K}-\mathbf{K}^*)\mathbf{U}\rangle)\,.$$

Performing *mutatis mutandis* the same algebraic manipulations as above yields

$$(\mathrm{Re}(\lambda)+\alpha)\,\|\mathbf{U}\|_{H^1}^2 \;+\;\alpha\,\|\mathbf{B}^{1/2}\mathbf{U}_{xx}\|_{L^2}^2 \;\leq\; M\;[\,\|\mathbf{U}\|_{L^2}^2+\|(\lambda-L)\mathbf{U}\|_{H^1}^2\,]$$

for some positive constants $\alpha$ and $M$, whenever $\mathrm{Re}(\lambda)\geq-\alpha$. Hence if $\mathrm{Re}(\lambda)\geq 2M-\alpha$ then

$$\|\mathbf{U}\|_{H^1}\;\leq\;\|(\lambda-L)\mathbf{U}\|_{H^1}.$$

Besides, a direct bound gives

$$|\mathrm{Im}(\lambda)|\,\|\mathbf{U}\|_{L^2}^2\;\leq\;M'\;[\,\|\mathbf{U}\|_{H^1}^2+\|(\lambda-L)\mathbf{U}\|_{L^2}^2\,]$$

for some positive $M'$ so that if moreover $\mathrm{Re}(\lambda)\geq-\frac{1}{2}\alpha$ then

$$|\mathrm{Im}(\lambda)|\,\|\mathbf{U}\|_{L^2}^2\;\leq\;M''\;[\,\|\mathbf{U}\|_{L^2}^2+\|(\lambda-L)\mathbf{U}\|_{H^1}^2\,]$$

with another positive $M''$. Hence if $\mathrm{Re}(\lambda)\geq-\frac{1}{2}\alpha$ and $|\mathrm{Im}(\lambda)|\geq 2M''$ then

$$\|\mathbf{U}\|_{H^1}\;\leq\;\sqrt{\frac{4M}{\alpha}}\;\|(\lambda-L)\mathbf{U}\|_{H^1}.$$

We may then choose $\theta_0=\frac{1}{2}\alpha$ and complete the above bounds by a continuity-compactness argument.

## Diffusion-waves

At last we illustrate at the linear level the diffusion-wave resolution of the large-time behavior. First mark that energy estimates performed on the Fourier side yield bounds uniformly in $(t,\xi)\in\mathbf{R}_+\times\mathbf{R}$

$$|\widehat{\mathbf{U}}(t,\xi)|\;\leq\;M\;\mathrm{e}^{-\eta\,t\,\frac{\xi^2}{1+\xi^2}}\;|\widehat{\mathbf{U}_0}(\xi)|$$

for some positive $M$ and $\eta$, when $\mathbf{U}$ is a solution to (A.0.1) starting from initial data $\mathbf{U}_0$. Therefore frequencies bounded away from zero are exponentially damped.

For simplicity we assume now that we are in the strictly hyperbolic case ($r=d$) so that spectral modes of $-\mathrm{i}\xi\mathbf{A}+(\mathrm{i}\xi)^2\mathbf{B}$ expand analytically in $\xi$ near the origin. We deduce for $\xi$ sufficiently small

$$\mathrm{e}^{t\,(-\mathrm{i}\xi\mathbf{A}+(\mathrm{i}\xi)^2\mathbf{B})}\;=\;\sum_{j=1}^{d}\mathrm{e}^{t\,\lambda_j(\xi)}\mathbf{\Pi}_j(\xi)$$

with, for $j=1,\,\dots,\,d$,

$$\mathbf{\Pi}_j(\xi)\;\stackrel{\xi\to0}{=}\;\mathbf{\Pi}_j\;+\;\mathcal{O}(\xi)\,,\qquad\lambda_j(\xi)\;\stackrel{\xi\to0}{=}\;-\mathrm{i}\xi a_j\;-\;\xi^2 b_j\;+\;\mathcal{O}(\xi^2)\,,$$

and

$$b_j\;\mathbf{\Pi}_j\;=\;\mathbf{\Pi}_j\;B\;\mathbf{\Pi}_j\,.$$

Kawashima condition ensures positivity of all $b_j$. Therefore there exist positive $\xi_0$, $\theta$ and $C$ such that for any $t \geq 0$ and $|\xi| \leq \xi_0$

$$\left| \mathrm{e}^{t\,(-\mathrm{i}\xi\mathbf{A}+(\mathrm{i}\xi)^2\mathbf{B})} \;-\; \sum_{j=1}^{d} \mathrm{e}^{-\mathrm{i}\,a_j\,\xi\,t\,-b_j\xi^2\,t}\,\mathbf{\Pi}_j \right| \;\leq\; C\,|\xi|\,\mathrm{e}^{-\theta\xi^2\,t}\,.$$

Since $\xi$-factors enhance time decay, the former readily proves the expected diffusion-waves scenario at the linear level.

# B. Evans' function and the first Chern number

For the convenience of the reader we briefly recall from [95] how the topologic nature of winding numbers of Evans' functions is uncovered by building a suitable vector bundle. We keep using throughout notational conventions of p. 65 – 67.

Fix a Floquet exponent $\xi$ and a simple curve $\Gamma$ contained in the resolvent set and bounding a domain $\Omega$. Let us explain how is built a vector bundle whose first Chern number is the winding number of $D(\cdot,\xi)$ along $\Gamma$. Each fiber is a complex vector space of dimension $N$, where $N$ is the size of the differential system $\mathbf{X}' = \mathbf{A}(\lambda;\cdot)\mathbf{X}$, and the basis is a topological sphere of dimension two. Actually the topological sphere is naturally embedded in $\mathbf{C}\times\mathbf{R}$ as the cylinder $\mathcal{S} := (\Omega\times\{0\})\cup(\Gamma\times[0,\Xi])\cup(\Omega\times\{\Xi\})$. In vague words, essentially the bottom $\Omega\times\{0\}$ is fibered with a space encoding diagonal $\Delta_0$ and the top $\Omega\times\{\Xi\}$ is fibered with a space encoding twisted diagonal $\Delta_\xi$ while along the vertical boundary $\Gamma\times[0,\Xi]$ the bottom is evolved through dynamics of $\mathbf{X}' = \mathbf{A}(\lambda;\cdot)\mathbf{X}$.

The fact that $\lambda$ lies in the resolvent set of $L_\xi$ is already expressed as a transversality condition for the intersection of $\widetilde{R}(\lambda;\Xi)\Delta_0$ and $\Delta_\xi$. To build a topological object one needs to convert it in a continuity condition allowing to glue vertical boundary with top boundary. This is achieved by introducing the quotient space $\mathbf{C}^{2N}/\Delta_\xi$ and looking for a $N$-vector bundle embedded as a subbundle of the $2N$-vector bundle

$$\bigcup_{(\lambda,x)\in\mathcal{S}} \mathbf{C}^{2N}/\Delta_{\frac{x}{\Xi}\xi}\times\mathbf{C}^{2N}/(\Delta_{\frac{x}{\Xi}\xi})^{\perp}\,.$$

Then on the bottom we fiber with the constant vector space[1] $\{\bar{0}\}\times\mathbf{C}^{2N}/(\Delta_0)^{\perp}$ and the top is fibered with the constant vector space $\mathbf{C}^{2N}/\Delta_\xi\times\{\bar{0}\}$. The choice of the orthogonal complement as a particular complement is rather arbitrary so that some trick is needed to enable gluing. Explicitly one endows any point $(\lambda,x)$ of the vertical boundary with the fiber obtained as the image of $\Delta_0$ under the map

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} \;\mapsto\; \left(\overline{\widetilde{R}(\lambda;x)\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}}\,,\;\frac{\Xi-x}{\Xi}\,\overline{\widetilde{R}(\lambda;x)\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}}\right).$$

With this choice the fact that the curve $\Gamma$ lies in the resolvent set of $L_\xi$ indeed implies that the above construction yields a topological bundle.

Now comes the natural question: is the bundle trivial ? Since we are fibering over a 2-sphere, the bundle may be covered by two trivializing charts with a common equator

[1] Relevant equivalence class of an element $\mathbf{z}$ is denoted $\bar{\mathbf{z}}$. Which equivalence relation is involved is decided by the ambient $2N$-vector bundle.

and its trivialization depends on the compatibility of the two charts along this equator. To each point of such an equator the transition map associates an isomorphism of $\mathbf{C}^N$. Bundles are then classified by homotopy classes of the obtained $\mathrm{Isom}(\mathbf{C}^N)$-valued equatorial loop. As is well-known these classes are parametrized by winding numbers of the image of the loop by the determinant application. In this context this number is called the first Chern number of the bundle. Our construction already comes with two natural trivializing charts, one for $(\Omega \times \{0\}) \cup (\Gamma \times [0, \Xi])$, another for $\Omega \times \{\Xi\}$, with a junction on the equator $\Gamma \times \{\Xi\}$. It follows that the first Chern number of the obtained bundle coincides with the winding number of the Evans' function along the loop $\Gamma$.

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**Abstract.** The present memoir reports on recent investigations of the author and his collaborators on stability of periodic wavetrains. It includes answers to the last general questions concerning their asymptotic stability as solutions to dissipative systems. These general results both extend in many directions and unify many other contributions obtained over the last twenty years. In the end, despite a quite technical proof, even the most surprising parts of the description receive nice formal interpretations. It also describes various investigations of spectral stability assumptions, needed to apply the general theory, for *roll-waves*, a special kind of periodic waves that emerge as primary hydrodynamic instabilities in shallow fluid films flowing down an inclined ramp. At last, for conservative systems, it contains some spectral results that are expected to lay sound foundations for a nonlinear theory still to come.

**Keywords:** partial differential equations, periodic wavetrains, asymptotic stability, spectral analysis, slow modulation, roll-waves, thin films, hydrodynamic instability, Hamiltonian systems, hyperbolic-parabolic systems, balance and conservation laws, Bloch transform, Floquet theory, Evans' function, Euler–Korteweg system, St. Venant system, Korteweg–de Vries equation, Kuramoto–Sivashinsky equation.

**Mathematical classification (2010) :** 35B10, 35B35, 35B40, 35K55, 35P15, 35Q30, 35Q35, 35Q53, 37K05, 37K45, 37L15, 37L50, 47F05, 35B25, 35B27, 35B36, 35L65, 76E09, 76E30.

**Résumé.** Ce mémoire se propose de discuter les résultats de son auteur et des ses collaborateurs concernant la stabilité des ondes progressives périodiques. Il inclut des réponses aux dernières questions ouvertes générales sur leur stabilité asymptotique comme solutions de systèmes dissipatifs. Ces résultats unifient, étendent dans de nombreuses directions et réconcilient un certain nombres de travaux antérieures de ces 20 dernières années. Bien que la démonstration soit parfois extrêmement technique, mêmes les parties les plus surprenantes *a posteriori* s'interprètent formellement de manière assez élégante. Le mémoire décrit également l'examen des hypothèses de stabilité spectrale, nécessaires à l'application des résultats généraux, pour les *rouleaux*, des ondes périodiques qui émergent comme instabilités hydrodynamiques primaires dans les fluides en couche mince s'écoulant le long d'une pente. Enfin, à propos des systèmes conservatifs, le mémoire contient quelques résultats spectraux, obtenus avec l'espoir qu'ils puissent servir de premières bases à un théorie non linéaire, encore à venir.

**Mots-clés :** équations aux dérivées partielles, trains d'ondes périodiques, stabilité asymptotique, analyse spectrale, modulation lente, rouleaux, films minces, instabilité hydrodynamique, systèmes Hamiltoniens, systèmes hyperboliques-paraboliques, bilans et lois de conservation, transformée de Bloch, théorie de Floquet, fonction d'Evans, système d'Euler–Korteweg, système de St. Venant, équation de Korteweg–de Vries, équation de Kuramoto–Sivashinsky.