ON THE ALGEBRAIC AND ANALYTIC q-DE RHAM COMPLEXES ATTACHED TO q-DIFFERENCE EQUATIONS

by

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Abstract. — This paper is concerned with the algebraic and analytic q-de Rham complexes attached to linear q-difference operators with Laurent polynomial coefficients over the field of complex numbers. There is a natural morphism from the former to the latter complex. Whether or not it is a quasi-isomorphism, *i.e.*, whether or not the induced morphisms on the corresponding cohomology spaces are isomorphisms is the basic question considered in the present paper. We study this question following three distinct approches. The first one is based on duality, and leads to a direct connection between the problem considered in the present paper and the convergence of formal series solutions of q-difference equations. The second approach is sheaf theoretic, based on growth considerations. The third one relies on the local analytic theory of q-difference equations. The paper ends with an extension of our results to variants of the above q-de Rham complexes when certain q-spiral of poles are allowed. Our study includes the case $|q| = 1$.

Contents

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1. Introduction - Content of the paper

Let q be a nonzero complex number which is not a root of the unity and let σ_q be the q-dilatation operator acting on a function $f(x)$ of the complex variable x by

$$(\sigma_q f)(x) := f(qx).$$

x We consider a linear q-difference operator

$$L = a_n(x)\sigma_q^n + \cdots + a_1(x)\sigma_q + a_0(x)$$

with coefficients $a_0(x), a_1(x), \ldots, a_n(x) \in \mathbf{C}[x, x^{-1}]$ such that $a_0(x)a_n(x) \neq 0$. We let L act on $f(x)$ as follows :

$$L(f)(x) = a_n(x)f(q^n x) + \cdots + a_1(x)f(qx) + a_0(x)f(x).$$

We attach to L its algebraic and analytic q-De Rham complexes respectively given by

$$DR^{alg}(L) = \underset{\bullet}{\mathbf{C}[x, x^{-1}]} \xrightarrow{L} \mathbf{C}[x, x^{-1}] \quad \text{and} \quad DR^{an}(L) = \underset{\bullet}{\mathbb{O}} \xrightarrow{L} \mathbb{O}$$

where $\mathbb{O}$ is the ring of analytic functions over $\mathbf{C}^{\times}$ (the symbol $\bullet$ denotes the term of degree 0). This paper is concerned with the cohomology spaces

$$H^i_{DR^{alg}}(L) := H^i(DR^{alg}(L)) \quad \text{and} \quad H^i_{DR^{an}}(L) := H^i(DR^{an}(L)).$$

The inclusion $\mathbf{C}[x, x^{-1}] \hookrightarrow \mathbb{O}$ induces a morphism of complexes

$$DR^{alg}(L) \to DR^{an}(L). \tag{1.0.1}$$

Whether or not this is a quasi-isomorphism, *i.e.*, whether or not the induced morphisms of cohomology spaces

$$H^i_{DR^{alg}}(L) \to H^i_{DR^{an}}(L) \tag{1.0.2}$$

are isomorphisms is the basic question considered in the present paper. This is a quantization of questions addressed by Deligne in [**4**, II.6] about connections on algebraic varieties (see also Grothendieck's [**7**]).

Along the way, we will study the dimensions of the above cohomology spaces. We emphasize that the algebraic q-de Rham complex $DR^{alg}(L)$ has finite dimensional cohomology spaces and that its Euler characteristic can be computed explicitly (see Proposition 2.1 in Section 2). This finiteness property is not necessarily true for $DR^{an}(L)$ when $|q| = 1$; we will come back to this bellow.

A classical approach to study this kind of questions is to use perturbative methods in Banach algebras theory. This has been done by Bézivin in [**1**] when $|q| \neq 1$, inspired by anterior works of Malgrange and Ramis notably about linear differential equations (see [**10, 11**] and the references therein). We will not say much about this approach; instead, the aim of this paper is to present alternative approaches, that we shall now describe.

1.1. An approach via duality. — Whatever the method used to study the morphism (1.0.1), it turns out that the conditions ensuring that (1.0.1) is a quasi-isomorphism is closely related to the convergence of the formal solutions of the dual q-difference operator

$$L^\vee := \sum_{i=0}^{n} q^{-i}\sigma_q^{-i} a_i(x).$$

The first part of this paper gives an a priori reason for this, inspired by Chiarellotto's work in [3] on p-adic differential equations. More precisely, let us introduce the local analytic and formal q-De Rham complexes of $L^\vee$ at 0 respectively given by

$$DR_0^{an}(L^\vee) = \underset{\bullet}{\mathbf{C}(\{x\})} \xrightarrow{L^\vee} \mathbf{C}(\{x\}) \quad \text{and} \quad DR_0^{form}(L^\vee) = \underset{\bullet}{\mathbf{C}((x))} \xrightarrow{L^\vee} \mathbf{C}((x))$$

where $\mathbf{C}(\{x\})$ is the field of germs of analytic functions at $0 \in \mathbf{C}$ and $\mathbf{C}((z))$ is the field of formal Laurent series. The inclusion $\mathbf{C}(\{x\}) \hookrightarrow \mathbf{C}((x))$ induces a morphism of complexes $DR_0^{an}(L^\vee) \to DR_0^{form}(L^\vee)$ and, hence, morphisms of cohomology spaces

$$H^i(DR_0^{an}(L^\vee)) \to H^i(DR_0^{form}(L^\vee)). \tag{1.0.3}$$

We introduce similar complexes at ∞ :

$$DR_\infty^{an}(L^\vee) = \underset{\bullet}{\mathbf{C}(\{x^{-1}\})} \xrightarrow{L^\vee} \mathbf{C}(\{x^{-1}\})$$

and

$$DR_\infty^{form}(L) = \underset{\bullet}{\mathbf{C}((x^{-1}))} \xrightarrow{L^\vee} \mathbf{C}((x^{-1})).$$

The inclusion $\mathbf{C}(\{x^{-1}\}) \hookrightarrow \mathbf{C}((x^{-1}))$ induces a morphism of complexes $DR_\infty^{an}(L^\vee) \to DR_\infty^{form}(L^\vee)$ and, hence, morphisms of cohomology spaces

$$H^i(DR_\infty^{an}(L^\vee)) \to H^i(DR_\infty^{form}(L^\vee)). \tag{1.0.4}$$

Our first result is :

Theorem 1.1. — *The following properties are equivalent :*

- *the complex $DR^{an}(L)$ has finite dimensional cohomology spaces;*
- *the complexes $DR_0^{an}(L^\vee)$ and $DR_\infty^{an}(L^\vee)$ have finite dimensional cohomology spaces.*

If these spaces are finite dimensional, then their Euler characteristics are related by the formula

$$\begin{aligned} &\chi(DR^{an}(L)) - \chi(DR^{alg}(L)) = \\ &\quad \chi(DR_0^{form}(L^\vee)) - \chi(DR_0^{an}(L^\vee)) + \chi(DR_\infty^{form}(L^\vee)) - \chi(DR_\infty^{an}(L^\vee)) \end{aligned} \tag{1.1.1}$$

and the following properties are equivalent

– *the morphisms* (1.0.2) *are isomorphisms;*
– *the morphisms* (1.0.3) *and* (1.0.4) *are isomorphisms.*

In the above theorem, we have implicitly used the fact that $DR_0^{form}(L^\vee)$ and $DR_\infty^{form}(L^\vee)$ have finite dimensional cohomology spaces; this follows from Bézivin's [1, Proposition 2.7], which also gives an explicit formula for the Euler characteristics of these complexes.

Here is a first consequence of Theorem 1.1 in the case $|q| > 1$ (the case $|q| < 1$ can be deduced from the case $|q| > 1$ by viewing L as a q^{-1}-difference operator).

Corollary 1.2. — *Assume that $|q| > 1$. The following properties are equivalent :*

(i) the morphisms (1.0.2) *are isomorphisms;*
(ii) L has no positive slope at 0 and ∞;
(iii) the irregularity numbers $\mathrm{irr}_0(L)$ and $\mathrm{irr}_\infty(L)$ of L at 0 and ∞ are equal to 0.

The slopes mentioned in Corollary 1.2 are the slopes of the Newton polygons of L in the sense of Sauloy's [14] for instance. The Newton polygon $\mathcal{N}_0(L)$ of L at 0 is the convex hull in $\mathbf{R}^2$ of

$$\{(i,j) \mid i \in \mathbf{Z} \text{ and } j \geq v_0(a_{n-i})\},$$

where v_0 denotes the x-adic valuation. This polygon is made of two vertical half lines and of k vectors $(r_1,d_1),\ldots,(r_k,d_k) \in \mathbf{Z}_{>0} \times \mathbf{Z}$ having pairwise distinct slopes $\lambda_1 = \frac{d_1}{r_1},\ldots,\lambda_k = \frac{d_k}{r_k}$, called the slopes L at 0. The integer r_i is called the multiplicity of λ_i. We define the irregularity number $\mathrm{irr}_0(L)$ of L at 0 as the sum of the positive slopes of the Newton polygon of L at 0 counted with multiplicity, *i.e.*,

$$\mathrm{irr}_0(L) = \sum_{i=0}^{k} r_i \max\{\lambda_i, 0\}.$$

Notice that

$$\mathrm{irr}_0(L) = v_0(a_0) - \min\{v_0(a_i) \mid i \in \{0,\ldots,n\}\}.$$

We have similar notions and notations at ∞.

Here is a second consequence of Theorem 1.1 which is now concerned with the case $|q| = 1$. It relies on a technical assumption introduced by Di Vizio in [5].

Corollary 1.3. — *Under the Assumption 3.3 stated in Section 3.3, the morphisms* (1.0.2) *are isomorphisms.*

We emphasize that Assumption 3.3 is generically satisfied and has nothing to do with the signs of the slopes of L. This is in strong contrast with the case $|q| \neq 1$. This is very similar to the differences between complex differential equations and their p-adic counterparts.

We also have the following consequences of Theorem 1.1 concerning the Euler characteristics of $DR^{an}(L)$ and $DR^{alg}(L)$.

Corollary 1.4. — *If $|q| > 1$, then the complexes $DR^{an}(L)$ and $DR^{alg}(L)$ have finite dimensional cohomology and their Euler characteristics are given by*

$$\chi(DR^{an}(L)) = v_0(a_0(x)) - \deg(a_n(x))$$

and

$$\chi(DR^{alg}(L)) = v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L).$$

So,

$$\chi(DR^{an}(L)) - \chi(DR^{alg}(L)) = \mathrm{irr}_0(L) + \mathrm{irr}_\infty(L).$$

Corollary 1.5. — *Assume that $|q| = 1$ and that the Assumption 3.3 stated in Section 3.3 is satisfied. Then, the complexes $DR^{an}(L)$ and $DR^{alg}(L)$ have finite dimensional cohomology and their Euler characteristics are given by*

$$\chi(DR^{an}(L)) = \chi(DR^{alg}(L)) = v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L).$$

Note that if $|q| = 1$ but that Assumption 3.3 is not satisfied, then $H^1(DR^{an}(L))$ may be infinite dimensional; see Section 3.4 for an example.

1.2. A sheaf theoretic approach based on growth considerations. — When $|q| \neq 1$, the slopes of L involved in Corollary 1.2 are intimately related to the growth of the solutions of L. The following question is thus natural : is there a direct proof of the fact that (1.0.2) is an isomorphism if L has no positive slopes relying on growth considerations ? A positive answer is given in Section 4. Our starting point is the interpretation the q-de Rham cohomology spaces $H^i_{DR^{an}}(L)$ and $H^i_{DR^{alg}}(L)$ as the hypercohomology of complexes of sheaves on the curve $\mathbf{E}_q = \mathbf{C}^\times/q^{\mathbf{Z}}$:

$$H^i_{DR^{an}}(L) = \mathbb{H}^i(\mathbf{E}_q, \mathscr{DR}^{an}(L)) \text{ and } H^i_{DR^{alg}}(L) = \mathbb{H}^i(\mathbf{E}_q, \mathscr{DR}^{an,mod}(L))$$

where $\mathscr{DR}^{an}(L)$ is the "analytic q-de Rham complex of sheaves" and where $\mathscr{DR}^{an,mod}(L)$ is the "moderate analytic q-de Rham complex of sheaves" of L introduced in Sections 4.1 and 4.2 respectively. More precisely,

$$\mathscr{DR}^{an}(L) = \underset{\bullet}{\mathcal{A}^{an}} \xrightarrow{L} \mathcal{A}^{an}.$$

where $\mathcal{A}^{an} = \pi_*(\mathcal{O}_{\mathbf{C}^\times})$ is the direct image by the projection $\pi : \mathbf{C}^\times \to \mathbf{E}_q$ of the sheaf $\mathcal{O}_{\mathbf{C}^\times}$ of analytic functions on $\mathbf{C}^\times$. Moreover,

$$\mathscr{D}\mathscr{R}^{an,mod}(L) = \underset{\bullet}{\mathcal{A}^{an,mod}} \xrightarrow{L} \mathcal{A}^{an,mod}$$

where $\mathcal{A}^{an,mod}$ is the subsheaf of $\mathcal{A}^{an}$ whose sections have moderate growth at 0 and ∞; see Section 4.2 for the precise definition. The inclusion $\mathcal{A}^{an,mod} \hookrightarrow \mathcal{A}^{an}$ induces a morphism

$$\mathscr{D}\mathscr{R}^{an,mod}(L) \to \mathscr{D}\mathscr{R}^{an}(L) \tag{1.5.1}$$

and the fact that the morphisms (1.0.2) are isomorphisms is equivalent to the fact that (1.5.1) is a quasi-isomorphism. It turns out that it is possible to prove that the latter morphism is indeed a quasi-isomorphism if L has no positive slope by using simple growth considerations.

Unfortunately, we have not been able to find a similar proof in the case $|q| = 1$.

1.3. An approach relying on the local structure of q-difference equations. — In Sections 6 and 7, we give alternative proofs of Corollaries 1.2, 1.3, 1.4 and 1.5 based on the fact that any q-difference module is, analytically at 0, the successive extension of "simple" q-difference modules, namely of q-difference modules attached to q-difference operators of the form

$$\sigma_q^r - cx^s$$

for some $c \in \mathbf{C}^\times$, $r \in \mathbf{Z}_{\geq 1}$ and $s \in \mathbf{Z}$ (when $|q| = 1$, we assume that Assumption 3.3 stated in Section 3.3 is satisfied). The spirit of this proof is closed to Deligne's [4, II.6].

1.4. The q-de Rham complexes with q-spirals of poles. — In the last section of the paper, we extend our results to more general q-de Rham complexes, when rational functions more general than Laurent polynomials are allowed.

1.4.1. The algebraic case. — A finite subset $\mathcal{S}$ of $\mathbf{C}^\times$ being given, we denote by $\mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}}$ the localization of $\mathbf{C}[x, x^{-1}]$ at $q^{\mathbf{Z}}\mathcal{S}$, *i.e.*,

$$\mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}} = \mathbf{C}[x, x^{-1}][\{(x - s)^{-1} \mid s \in q^{\mathbf{Z}}\mathcal{S}\}]. \tag{1.5.2}$$

It is natural to wonder whether the complex

$$\underset{\bullet}{\mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}}} \xrightarrow{L} \mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}} \tag{1.5.3}$$

has finite dimensional cohomology. If $\mathcal{S} = \emptyset$, the answer is positive since (1.5.3) coincides with $DR^{alg}(L)$ in this case; unfortunately, if $\mathcal{S} \neq \emptyset$, then the H^1 of (1.5.3) may be infinite dimensional (see Example 8.1 in Section 8.1).

However, a natural filtration of $\mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}}$ is given by the sequence of sub-$\mathbf{C}[x,x^{-1}]$-algebras $(\mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S},d})_{d\geq 0}$ where

$$\mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S},d} = \{f(x) \in \mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}} \mid f(x) \text{ has at most poles of order } d \text{ on } q^{\mathbf{Z}}\mathcal{S}\}.$$

For any $d \in \mathbf{Z}_{\geq 0}$, we attach to L the complex

$$DR^{alg}(L, q^{\mathbf{Z}}\mathcal{S}, d) = \underset{\bullet}{\mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S},d}} \xrightarrow{L} \mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S},d}$$

and denote the corresponding cohomology spaces by

$$H^i_{DR^{alg}}(L, q^{\mathbf{Z}}\mathcal{S}, d) := H^i(DR^{alg}(L, q^{\mathbf{Z}}\mathcal{S}, d)).$$

We have :

Theorem 1.6. — *The complex $DR^{alg}(L, q^{\mathbf{Z}}\mathcal{S}, d)$ has finite dimensional cohomology spaces and its Euler characteristic is given by*

$$\begin{aligned}\chi(DR^{alg}(L, q^{\mathbf{Z}}\mathcal{S}, d)) &= \chi(DR^{alg}(L)) - ndm \\ &= v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L) - ndm\end{aligned}$$

where $m = \sharp(\mathcal{S} \mod q^{\mathbf{Z}})$.

1.4.2. The analytic case and a comparison theorem. — We consider the $\mathbf{C}[x,x^{-1}]$-algebra $\mathbb{O}_{[q^{\mathbf{Z}}\mathcal{S}],d}$ of meromorphic functions on $\mathbf{C}^\times$ having finitely many poles, all in $q^{\mathbf{Z}}\mathcal{S}$ and of order $\leq d$. For any $d \in \mathbf{Z}_{\geq 0}$, we attach to L the complex

$$DR^{an}(L, [q^{\mathbf{Z}}\mathcal{S}], d) = \underset{\bullet}{\mathbb{O}_{[q^{\mathbf{Z}}\mathcal{S}],d}} \xrightarrow{L} \mathbb{O}_{[q^{\mathbf{Z}}\mathcal{S}],d}$$

and denote the corresponding cohomology spaces by

$$H^i_{DR^{an}}(L, [q^{\mathbf{Z}}\mathcal{S}], d) := H^i(DR^{an}(L, [q^{\mathbf{Z}}\mathcal{S}], d)).$$

The inclusion

$$\mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S},d} \hookrightarrow \mathbb{O}_{[q^{\mathbf{Z}}\mathcal{S}],d}$$

induces a morphism of complexes

$$DR^{alg}(L, q^{\mathbf{Z}}\mathcal{S}, d) \to DR^{an}(L, [q^{\mathbf{Z}}\mathcal{S}], d)$$

and, hence, morphisms of cohomology spaces

$$H^i_{DR^{alg}}(L, q^{\mathbf{Z}}\mathcal{S}, d) \to H^i_{DR^{an}}(L, [q^{\mathbf{Z}}\mathcal{S}], d). \tag{1.6.1}$$

Theorem 1.7. — *The morphisms (1.6.1) are isomorphisms if and only if the morphisms (1.0.2) are isomorphisms.*

Theorem 1.8. — *The following properties are equivalent :*

– *the complex $DR^{an}(L, [q^{\mathbf{Z}}\mathcal{S}], d)$ has finite dimensional cohomology spaces;*

– *the complex* $DR^{an}(L)$ *has finite dimensional cohomology spaces.*

If these spaces are finite dimensional, then their Euler characteristics are related by the formula

$$\chi(DR^{an}(L,[q^{\mathbf{Z}}\mathcal{S}],d)) - \chi(DR^{alg}(L,q^{\mathbf{Z}}\mathcal{S},d)) = \chi(DR^{an}(L)) - \chi(DR^{alg}(L)),$$

so

$$\chi(DR^{an}(L,[q^{\mathbf{Z}}\mathcal{S}],d)) = \chi(DR^{an}(L)) - ndm$$

1.5. Organization of the paper. — In Section 2, we show that $DR^{alg}(L)$ has finite dimensional cohomology spaces and we compute its Euler characteristic. Section 3.1 is devoted to the proof of Theorem 1.1. This proof uses some general results about the effect of (algebraic or topological) duality on the cohomology of complexes of vector spaces; these results are recalled in Appendix A. Corollaries 1.2 and 1.4 (resp. Corollaries 1.3 and 1.5) are then proved in Section 3.2 (resp. Section 3.3). In Section 3.4, we give an example of L and q (necessarily of norm 1) such that $H^1(DR^{an}(L))$ is infinite dimensional. Section 4 contains a sheaf theoretic proof of the implication $(ii) \Rightarrow (i)$ of Corollaries 1.2 relying on simple growth considerations. Section 5 is an interlude on q-difference systems : we introduce the q-de Rham complexes attached to q-difference systems and we make a link beteween the properties of the q-de Rham complex of a q-difference operator and the properties of the q-de Rham complex of the associated system. In Sections 6 and 7, we give alternative proofs of Corollaries 1.2, 1.3, 1.4 and 1.5 relying on the local analytic classification of q-difference modules. In Section 8, we study the complexes $DR^{alg}(L,q^{\mathbf{Z}}\mathcal{S},d)$ and $DR^{an}(L,[q^{\mathbf{Z}}\mathcal{S}],d)$; we prove Theorems 1.6, 1.7 and 1.8.

2. Study of the complex $DR^{alg}(L)$

Proposition 2.1. — *The complex* $DR^{alg}(L)$ *has finite dimensional cohomology and its Euler characteristic is given by*

$$\chi(DR^{alg}(L)) = v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L).$$

Proof. - For any $\ell_-,\ell_+ \in \mathbf{Z}$, let V_{ℓ_-,ℓ_+} be the finite dimensional sub-$\mathbf{C}$-vector space of $\mathbf{C}[x,x^{-1}]$ given by

$$V_{\ell_-,\ell_+} = \{P \in \mathbf{C}[x,x^{-1}] \mid \ell_- \leq v_0(P) \leq \deg(P) \leq \ell_+\}.$$

The operator L induces a $\mathbf{C}$-linear morphism

$$L : V_{\ell_-,\ell_+} \to V_{\ell_-+d_-,\ell_++d_+}$$

where

$$d_- = \min\{v_0(a_0),\ldots,v_0(a_n)\}$$

and

$$d_+ = \max\{\deg(a_0), \ldots, \deg(a_n)\}.$$

We claim that, for ℓ_- small enough and ℓ_+ large enough, the following morphism of complexes is a quasi-isomorphism :

$$\begin{array}{ccc} \underset{\bullet}{V_{\ell_-+1,\ell_+-1}} & \xrightarrow{L} & V_{\ell_-+d_-+1,\ell_++d_+-1} \ , \\ \downarrow & & \downarrow \\ \underset{\bullet}{V_{\ell_-,\ell_+}} & \xrightarrow{L} & V_{\ell_-+d_-,\ell_++d_+} \end{array} \tag{2.1.1}$$

where the vertical arrows are the inclusions. Indeed, (2.1.1) is a quasi-isomorphism if and only if the $\mathbf{C}$-linear morphism

$$V_{\ell_-,\ell_+}/V_{\ell_-+1,\ell_+-1} \to V_{\ell_-+d_-,\ell_++d_+}/V_{\ell_-+d_-+1,\ell_++d_+-1} \tag{2.1.2}$$

induced by L is an isomorphism. Setting, for any $i \in \{0, \ldots, n\}$,

$$a_i(x) = \sum_{j=d_-}^{d_+} a_{i,j} x^j,$$

we have, for any $f(x) = \sum_{j=\ell_-}^{\ell^+} f_j x^j \in V_{\ell_-,\ell_+}$,

$$L(f)(x) = \sum_{i=0}^{n} (\sum_{j=d_-}^{d_+} a_{i,j} x^j)(\sum_{j=\ell_-}^{\ell_+} f_j q^{ij} x^j) = \sum_{m=d_-+\ell_-}^{d_++\ell_+} b_m x^m$$

where

$$b_m = \sum_{i=0}^{n} \sum_{\alpha+\beta=m} a_{i,\alpha} f_\beta q^{i\beta}.$$

In particular,

$$b_{d_-+\ell_-} = p_{\ell_-} \sum_{i=0}^{n} a_{i,d_-} q^{i\ell_-}$$

and

$$b_{d_++\ell_+} = p_{\ell_+} \sum_{i=0}^{n} a_{i,d_+} q^{i\ell_+}.$$

This clearly implies that (2.1.2) is an isomorphism if and only if $\sum_{i=0}^{n} a_{i,d_-} q^{i\ell_-}$ and $\sum_{i=0}^{n} a_{i,d_+} q^{i\ell_+}$ are nonzero; this holds true if ℓ_- is small enough and if ℓ_+ is large enough. This justifies our claim.

Since $DR^{alg}(L)$ is the inductive limit of the complexes

$$\underset{\bullet}{V_{\ell_-,\ell_+}} \xrightarrow{L} V_{\ell_-+d_-,\ell_++d_+} \tag{2.1.3}$$

as $\ell_- \to -\infty$ and $\ell_+ \to +\infty$, we deduce that $DR^{alg}(L)$ is quasi-isomorphic to (2.1.3) for ℓ_- small enough and ℓ_+ large enough. It follows that $DR^{alg}(L)$ has finite dimensional cohomology and that

$$\chi(DR^{alg}(L)) = \chi(2.1.3) = \dim_{\mathbf{C}} V_{\ell_-,\ell_+} - \dim_{\mathbf{C}} V_{\ell_-+d_-,\ell_++d_+} = d_- - d_+.$$

We conclude the proof by noticing that

$$\begin{aligned} d_- - d_+ &= \min\{v_0(a_0),\dots,v_0(a_n)\} - \max\{\deg(a_0),\dots,\deg(a_n)\} \\ &= v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L). \end{aligned}$$

because

$$\mathrm{irr}_0(L) = v_0(a_0) - \min\{v_0(a_i) \mid i \in \{0,\dots,n\}\}$$

and

$$\mathrm{irr}_\infty(L) = -\deg(a_n) + \max\{\deg(a_i) \mid i \in \{0,\dots,n\}\}.$$

□

We refer to [13, Proposition 2.5.4] for a similar result for q-difference modules.

3. Proofs of Theorem 1.1, Corollaries 1.2, 1.3, 1.4 and 1.5 via duality

3.1. Proof of Theorem 1.1. — We let A_0 (resp. A_∞) be the ring of germs at 0 (resp. ∞) of analytic functions on a punctured neighborhood of 0 (resp. ∞). We let $K_0 = \mathbf{C}(\{x\})$ (resp. $K_\infty = \mathbf{C}(\{x^{-1}\})$) be the field of germs of meromorphic functions at 0 (resp. ∞). We have the $\mathbf{C}$-linear isomorphism

$$\begin{aligned} \phi : \mathbb{O}/\mathbf{C}[x,x^{-1}] &\xrightarrow{\sim} A_0/K_0 \oplus A_\infty/K_\infty \\ f + \mathbf{C}[x,x^{-1}] &\mapsto (f+K_0, f+K_\infty) \end{aligned} \tag{3.0.1}$$

and the commutative diagram

$$\begin{array}{ccc} \mathbb{O}/\mathbf{C}[x,x^{-1}] & \xrightarrow{L} & \mathbb{O}/\mathbf{C}[x,x^{-1}] \\ \phi \downarrow \cong & & \cong \downarrow \phi \\ A_0/K_0 \oplus A_\infty/K_\infty & \xrightarrow{L\oplus L} & A_0/K_0 \oplus A_\infty/K_\infty \end{array},$$

whence an isomorphism

$$\begin{aligned} &H^k(DR^{an}(L)/DR^{alg}(L)) \\ &\qquad \cong H^k(A_0/K_0 \xrightarrow{L} A_0/K_0) \oplus H^k(A_\infty/K_\infty \xrightarrow{L} A_\infty/K_\infty). \end{aligned} \tag{3.0.2}$$

$\bullet$ $\bullet$

In order to prove the first assertion of Theorem 1.1, it is thus sufficient to prove that

$$\underset{\bullet}{A_0/K_0} \xrightarrow{L} A_0/K_0 \text{ (resp. } \underset{\bullet}{A_\infty/K_\infty} \xrightarrow{L} A_\infty/K_\infty)$$

has finite dimensional cohomology if and only if $DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)$ (resp. $DR_\infty^{form}(L^\vee)/DR_\infty^{an}(L^\vee)$) has finite dimensional cohomology. Let us prove this equivalence at 0, the proof at ∞ being similar. In this respect, we can and will assume that the coefficients of L are in $\mathbf{C}[x]$. We have a $\mathbf{C}$-linear isomorphism

$$A_0/K_0 \cong H_0/H_{[0]} \text{ with } H_0 = A_0/O_0 \text{ and } H_{[0]} = K_0/O_0$$

where $O_0 = \mathbf{C}\{x\}$ is the ring of germs of analytic functions at 0 (resp. ∞). Since L has coefficients in $\mathbf{C}[x]$, L acts on $H_{[0]}$ and H_0, and these actions induce the action of L on A_0/K_0 above.

On the one hand, we have the $\mathbf{C}$-linear isomorphism

$$\begin{array}{rcl} \varphi : H_{[0]}^* & \xrightarrow{\sim} & \mathbf{C}[[t]] \\ u & \mapsto & \displaystyle\sum_{k\geq 0} u(x^{-k-1})t^k, \end{array}$$

where $H_{[0]}^*$ denotes the dual of $H_{[0]}$, and the isomorphism of complexes

$$\begin{array}{ccc} \underset{\bullet}{H_{[0]}^*} & \xrightarrow{L^*} & H_{[0]}^* \\ \varphi \downarrow \cong & & \cong \downarrow \varphi \\ \underset{\bullet}{\mathbf{C}[[t]]} & \xrightarrow{L^\vee} & \mathbf{C}[[t]] \end{array}, \tag{3.0.3}$$

where L^* denotes the dual of the $\mathbf{C}$-linear map $L : H_{[0]} \to H_{[0]}$. Since $\underset{\bullet}{\mathbf{C}[[t]]} \xrightarrow{L^\vee} \mathbf{C}[[t]]$ has finite dimensional cohomology (see [1, Proposition 2.3]), the isomorphism (A.0.1) ensures that $\underset{\bullet}{H_{[0]}} \xrightarrow{L} H_{[0]}$ has finite dimensional cohomology as well and that

$$\chi(\underset{\bullet}{\mathbf{C}[[t]]} \xrightarrow{L^\vee} \mathbf{C}[[t]]) = -\chi(\underset{\bullet}{H_{[0]}} \xrightarrow{L} H_{[0]}).$$

On the other hand, we can and will identify H_0 (as a $\mathbf{C}$-vector space) to the $\mathbf{C}$-vector space of holomorphic functions on $\mathbf{C}^\times \cup \{\infty\}$ vanishing at ∞. We endow H_0 with its usual structure of Fréchet space (see [8, Chapter 4, Part 1, 2, Example f)] for instance) and denote by H_0' its topological dual. Note that

L acts continuously on H_0. We have the $\mathbf{C}$-linear isomorphism

$$\begin{array}{rcl} \psi : H_0' & \xrightarrow{\sim} & \mathbf{C}\{t\} \\ u & \mapsto & \sum_{k\geq 0} u(z^{-k-1})t^k, \end{array}$$

and the isomorphism of complexes

$$\begin{array}{ccc} \underset{\bullet}{H_0'} & \xrightarrow{L^*} & H_0' \\ \psi \downarrow \cong & & \cong \downarrow \psi \\ \underset{\bullet}{\mathbf{C}\{t\}} & \xrightarrow{L^\vee} & \mathbf{C}\{t\} \end{array} . \tag{3.0.4}$$

The first part of Lemma A.3 applied to

$$\mathcal{N} = \underset{\bullet}{H_0} \xrightarrow{L} H_0$$

implies that $\underset{\bullet}{H_0} \xrightarrow{L} H_0$ has finite dimensional cohomology if and only if $\underset{\bullet}{\mathbf{C}\{t\}} \xrightarrow{L^\vee} \mathbf{C}\{t\}$ has finite dimensional cohomology and that, in this case, we have

$$\chi(\underset{\bullet}{\mathbf{C}\{t\}} \xrightarrow{L^\vee} \mathbf{C}\{t\}) = -\chi(\underset{\bullet}{H_0} \xrightarrow{L} H_0).$$

That we can apply Lemma A.3 necessitates some explanations. Firstly, H_0 is a Fréchet space, so any closed subspace of H_0 is Fréchet as well, and the first hypothesis of Lemma A.3 is satisfied. Secondly, we have to prove that, if $\underset{\bullet}{\mathbf{C}\{t\}} \xrightarrow{L^\vee} \mathbf{C}\{t\}$ has finite dimensional cohomology, then $\underset{\bullet}{H_0} \xrightarrow{L} H_0$ has finite dimensional cohomology as well. To prove this, we recall that we can identify H_0 (as a $\mathbf{C}$-vector space) to the $\mathbf{C}$-vector space of holomorphic functions on $\mathbf{C}^\times \cup \{\infty\}$ vanishing at ∞. So, we can identify H_0 with the topological dual $\mathbf{C}\{t\}'$ of the locally convex topological $\mathbf{C}$-vector space $\mathbf{C}\{t\}$, an isomorphism being given by

$$\begin{array}{rcl} \phi : H_0 & \xrightarrow{\sim} & \mathbf{C}\{t\}' \\ \sum_{k\geq 1} a_k x^{-k} \mod O_0 & \mapsto & \left(\sum_{k\geq 0} b_k t^k \mapsto \sum_{k\geq 0} a_{k+1} b_k \right). \end{array}$$

We have the following isomorphism of complexes

$$\begin{array}{ccc} \underset{\bullet}{H_0} & \xrightarrow{L} & H_0 \\ \phi\Big\downarrow\cong & & \cong\Big\downarrow\phi \\ \underset{\bullet}{\mathbf{C}\{t\}'} & \xrightarrow{L^{\vee *}} & \mathbf{C}\{t\}' \end{array} .$$

The first part of Lemma A.3 ensures that, if the complex $\underset{\bullet}{\mathbf{C}\{t\}} \xrightarrow{L^\vee} \mathbf{C}\{t\}$ has finite dimensional cohomology, then $\underset{\bullet}{H_0} \xrightarrow{L} H_0$ has finite dimensional cohomology as well, as expected.

Using what precedes, we see that the following properties are equivalent:

- $DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)$ has finite dimensional cohomology;
- $DR_0^{an}(L^\vee)$ has finite dimensional cohomology;
- $\underset{\bullet}{H_0} \xrightarrow{L} H_0$ has finite dimensional cohomology;
- $\underset{\bullet}{H_0/H_{[0]}} \xrightarrow{L} H_0/H_{[0]}$ has finite dimensional cohomology;
- $\underset{\bullet}{A_0/K_0} \xrightarrow{L} A_0/K_0$ has finite dimensional cohomology.

(For the first equivalence, we have used the fact that $DR_0^{form}(L^\vee)$ has finite dimensional cohomology according to [1, Proposition 2.3].) We have a similar statement at ∞. Therefore, $DR^{an}(L)/DR^{alg}(L)$ has finite dimensional cohomology if and only if $DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)$ and $DR_\infty^{form}(L^\vee)/DR_\infty^{an}(L^\vee)$ have finite dimensional cohomology and, in this case, we have

$$\chi(DR^{an}(L)/DR^{alg}(L)) = \chi(DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)) + \chi(DR_\infty^{form}(L^\vee)/DR_\infty^{an}(L^\vee)).$$

The first part of Theorem 1.1 follows from this and from the fact that the complexes $DR^{alg}(L)$, $DR_0^{form}(L^\vee)$ and $DR_\infty^{form}(L^\vee)$ have finite dimensional cohomology spaces (see Proposition 2.1 for the first complex, and [1, Proposition 2.3] for the latter two complexes).

Let us now prove the second part of Theorem 1.1. Putting (3.0.3) and (3.0.4) together, we obtain the following morphisms of complexes

$$
\begin{array}{ccc}
\mathbf{C}[[t]]_\bullet & \xrightarrow{L^\vee} & \mathbf{C}[[t]] \\
{\scriptstyle \varphi}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \varphi} \\
H^*_{[0]\bullet} & \xrightarrow{L^*} & H^*_{[0]} \\
{\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\
H'_{0\bullet} & \xrightarrow{L^*} & H'_0 \\
{\scriptstyle \psi^{-1}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \psi^{-1}} \\
\mathbf{C}\{t\}_\bullet & \xrightarrow{L^\vee} & \mathbf{C}\{t\}
\end{array}\quad ,
$$

where ι is given by the restriction. The map $\varphi\circ\iota\circ\psi^{-1}$ is simply the inclusion $i : \mathbf{C}\{t\} \hookrightarrow \mathbf{C}[[t]]$. So

$$
\begin{array}{ccc}
\mathbf{C}[[t]]_\bullet & \xrightarrow{L^\vee} & \mathbf{C}[[t]] \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\
\mathbf{C}\{t\}_\bullet & \xrightarrow{L^\vee} & \mathbf{C}\{t\}
\end{array}\ ,
$$

is a quasi-isomorphism if and only if

$$
\begin{array}{ccc}
H^*_{[0]\bullet} & \xrightarrow{L^*} & H^*_{[0]} \\
{\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\
H'_{0\bullet} & \xrightarrow{L^*} & H'_0
\end{array}
\tag{3.0.5}
$$

is a quasi-isomorphism. But, if $DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)$ has finite dimensional cohomology, we have already seen that the hypotheses of Lemma A.3 are satisfied by

$$
\mathcal{M} = H_{[0]\bullet} \xrightarrow{L} H_{[0]}, \quad \mathcal{N} = H_{0\bullet} \xrightarrow{L} H_0
$$

and $\varphi = j : H_{[0]} \hookrightarrow H_0$ the inclusion. Therefore, (3.0.5) is a quasi-isomorphism if and only if

$$\begin{array}{ccc} H_{[0]}^{\bullet} & \xrightarrow{L} & H_{[0]}^{\bullet} \\ \downarrow & & \downarrow \\ H_0^{\bullet} & \xrightarrow{L} & H_0^{\bullet} \end{array} \tag{3.0.6}$$

is a quasi-isomorphism and this concludes the proof of the theorem.

3.2. Proof of Corollaries 1.2 and 1.4. — We assume that $|q| > 1$. According to [1, Proposition 2.7], $DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)$ has finite dimensional cohomology,

$$H^1(DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)) = 0 \tag{3.0.7}$$

$$\dim H^0(DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)) = \max\{-v_0(a_i) \mid i \in \{1, \cdots, n\}\} + v_0(a_0) = \mathrm{irr}_0(L). \tag{3.0.8}$$

We have a similar result at ∞.

Using Theorem 1.1, we obtain that $DR^{an}(L)/DR^{alg}(L)$ has finite dimensional cohomology and that

$$\chi(DR^{an}(L)/DR^{alg}(L)) = \mathrm{irr}_0(L) + \mathrm{irr}_\infty(L).$$

But, according to Proposition 2.1 proven in Section 2, $DR^{alg}(L)$ has finite dimensional cohomology and its Euler characteristic is given by

$$\chi(DR^{alg}(L)) = v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L),$$

so $DR^{an}(L)$ has finite dimensional cohomology as well and the equality $\chi(DR^{an}(L)/DR^{alg}(L)) = \chi(DR^{an}(L)) - \chi(DR^{alg}(L))$ concludes the proof of Corollary 1.4.

Moreover, Theorem 1.1 ensures that the morphisms (1.0.2) are isomorphisms if and only if the morphisms (1.0.3) and (1.0.4) are isomorphisms if and only if $H^i(DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)) = 0$ and $H^i(DR_\infty^{form}(L^\vee)/DR_\infty^{an}(L^\vee)) = 0$ for $i \in \{0, 1\}$ if and only if (in virtue of (3.0.7) and (3.0.8)) $\mathrm{irr}_0(L) = \mathrm{irr}_\infty(L) = 0$. This concludes the proof of Corollary 1.2.

3.3. Proof of Corollary 1.3. — We assume that q has norm 1 but is not a root of the unity.

Consider a q-difference operator

$$P = b_n(x)\sigma_q^n + \cdots + b_1(x)\sigma_q + b_0(x)$$

with coefficients $b_0(x), b_1(x), \dots, b_n(x) \in K_0 = \mathbf{C}(\{x\})$ such that $b_0(x)b_n(x) \neq 0$. Let $d \geq 1$ be the least common denominator of the slopes of P at 0 and let q_d be a d-th root of q. We can see P as the q_d-difference operator P_d with integral slopes at 0 given by

$$P_d = b_n(x_d^d)\sigma_{q_d}^{dn} + \cdots + b_1(x_d^d)\sigma_{q_d}^d + b_0(x_d^d)$$

where $x_d^d = x$. Following [5, Section 2.2], to any slope λ of P_d at 0, we associate a characteristic polynomial $\mathrm{char}(P_d, \lambda; X) \in \mathbf{C}[X]$ whose complex roots are called the exponents of the slope λ of P_d. The set of these exponents is denoted by $\mathrm{Exp}_0(P_d, \lambda) \subset \mathbf{C}^\times$.

Definition 3.1 ([5, Definition 2.5]). — *We say that P is admissible at 0 if, for any slope λ of P_d at 0, for any two $a, b \in \mathrm{Exp}_0(P_d, \lambda)$ such that $ab^{-1} \notin q_d^{\mathbf{Z}_{\leq 0}}$, the series*

$$\sum_{k \geq 0} \frac{x^k}{1 - q_d^k ab^{-1}}$$

has a nonzero radius of convergence.

We say that P is very admissible at 0 if $(\sigma_q - 1)P$ is admissible at 0.

Proposition 3.2. — *We make the following two assumptions :*

- *the series $\sum_{k\geq 0} \frac{x^k}{(q;q)_k}$ has a nonzero radius of convergence, where $(q;q)_k = (1-q)(1-q^2)\cdots(1-q^k)$;*
- *P is very admissible at 0.*

Then, for any $f(x) \in \mathbf{C}((x))$:

$$P(f(x)) = g(x) \in K_0 \Rightarrow K_0.$$

Proof. - If $g = 0$, this follows from [5, Corollary 2.11]. If $g \neq 0$, then we have $M(f(x)) = 0$ where M is the q-difference operator given by

$$M = (\sigma_q - g(qx)/g(x))P.$$

The fact that P is very admissible at 0 implies that M is admissible at 0 and the result follows from [5, Corollary 2.11] again. □

Of course, we have similar notions and facts at ∞. We now make the following assumption :

Assumption 3.3. — *We assume that*

- *q has norm 1 but is not a root of the unity;*
- *the series $\sum_{k\geq 0} \frac{x^k}{(q;q)_k}$ has a nonzero radius of convergence, where $(q;q)_k = (1-q)(1-q^2)\cdots(1-q^k)$;*
- *$L^\vee$ (or, equivalently, L) is very admissible at 0;*
- *$L^\vee$ (or, equivalently, L) is very admissible at ∞.*

Proposition 3.2 (and its variant at ∞) ensures that

$$H^0(DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)) = H^0(DR_\infty^{form}(L^\vee)/DR_\infty^{an}(L^\vee)) = 0.$$

Moreover, as in the case $|q| > 1$, we have

$$H^1(DR_0^{form}(L^\vee)/DR_0^{an}(L^\vee)) = H^1(DR_\infty^{form}(L^\vee)/DR_\infty^{an}(L^\vee)) = 0.$$

(The proof of [1, Proposition 2.3] works as soon as q is not a root of the unity.) Now, Corollary 1.3 follows immediately from Theorem 1.1.

Moreover, according to Proposition 2.1 proven in Section 2, $DR^{alg}(L)$ has finite dimensional cohomology and its Euler characteristic is given by

$$\chi(DR^{alg}(L)) = v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L),$$

so $DR^{an}(L)$ has finite dimensional cohomology as well and

$$\chi(DR^{an}(L)) = \chi(DR^{alg}(L)) = v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L).$$

3.4. An example with infinite dimensional $H^1(DR^{an}(L))$. — Consider $q \in \mathbf{C}^\times$ such that the series $\sum_{k\geq 0} \frac{x^k}{1-q^{-k}}$ is divergent. Consider the operator $L = q\sigma_q - 1$; so $L^\vee = \sigma_q^{-1} - 1$. We claim that $H^1(DR^{an}(L))$ is infinite dimensional. According to Theorem 1.1, it is sufficient to prove that $H^1(DR_0^{an}(L^\vee))$ is infinite dimensional. It is thus sufficient to find an infinite dimensional sub-$\mathbf{C}$-vector space of K_0 which is in direct sum with $L^\vee(K_0)$. We claim that

$$V = \mathrm{Span}_{\mathbf{C}}\{(1-sz)^{-1} \mid s \in \mathbf{Z}_{\geq 2}\}$$

is such a sub-$\mathbf{C}$-vector space of K_0. Indeed, consider $g(x) = \sum_{k\geq 0} g_k x^k \in V \cap L^\vee(K_0)$ and $f(x) = \sum_{k\geq 0} f_k x^k \in K_0$ such that

$$g(x) = L^\vee(f(x)) = f(q^{-1}x) - f(x).$$

We have $f_k = \frac{g_k}{q^{-k}-1}$. If $g(x)$ is nonzero then $|b_k|$ tends to $+\infty$ as k tends to $+\infty$; since $\sum_{k\geq 0} \frac{x^k}{1-q^{-k}}$ is divergent, this implies that $\sum_{k\geq 0} f_k x^k$ is divergent, whence a contradiction. So $g(x) = 0$, as expected.

4. A moderate sheaf theoretic proof of $(ii) \Rightarrow (i)$ in Corollary 1.1

In this section, we assume that $|q| > 1$ and we give another proof of the fact that, if L has no positive slopes at 0 and ∞, then the morphisms (1.0.2) are isomorphisms. This new proof is more analytic in nature than the one given in the previous Section and based on growth considerations.

4.1. First step : the analytic q-De Rham cohomology as hypercohomology of sheaves. — We consider the quotient $\mathbf{E}_q = \mathbf{C}^\times/q^{\mathbf{Z}}$ and we denote by $\pi : \mathbf{C}^\times \to \mathbf{E}_q$ the corresponding quotient map. We endow $\mathbf{E}_q$ with its structure of Riemann surface.

We denote by $\mathcal{O}_{\mathbf{C}^\times}$ the sheaf of analytic functions on $\mathbf{C}^\times$ and we let

$$\mathcal{A}^{an} = \pi_*(\mathcal{O}_{\mathbf{C}^\times})$$

be the sheaf on $\mathbf{E}_q$ whose sections over $U \subset \mathbf{E}_q$ are the analytic functions on $\pi^{-1}(U)$.

Proposition 4.1. — *The sheaf $\mathcal{A}^{an}$ is acyclic.*

Proof. - First note that, for $i \geq 1$, $R^i\pi_*(\mathcal{A}^{an}) = 0$. Indeed, this is the sheaf associated to the presheaf $U \mapsto H^i(\pi^{-1}(U), \mathcal{O}_{\mathbf{C}^\times})$. If $\zeta = \pi(a) \in \mathbf{E}_q$ and if $U = \pi(D(a,\epsilon))$ is a small neighborhood of ζ, then $\pi^{-1}(U)$ is the disjoint union of small discs and these small discs are Stein, whence $H^i(\pi^{-1}(U), \mathcal{O}_{\mathbf{C}^\times}) = 0$ and, hence, $R^i\pi_*(\mathcal{A}^{an}) = 0$ as claimed. It follows that $H^i(\mathbf{C}^\times, \mathcal{O}_{\mathbf{C}^\times}) = H^i(\mathbf{E}_q, \mathcal{A}^{an})$ (see [9, Chapter III, Exercise 8.1]) and, $\mathbf{C}^\times$ being Stein, we have $H^i(\mathbf{C}^\times, \mathcal{O}_{\mathbf{C}^\times}) = 0$. □

We consider the complex of sheaves on $\mathbf{E}_q$ given by

$$\mathscr{DR}^{an}(L) = \underset{\bullet}{\mathcal{A}^{an}} \xrightarrow{L} \mathcal{A}^{an}.$$

Since $\mathcal{A}^{an}$ is acyclic with global sections $\mathbb{O}$, we get :

Corollary 4.2. — *We have*

$$\mathbb{H}^i(\mathbf{E}_q, \mathscr{DR}^{an}(L)) = H^i_{DR^{an}}(L).$$

4.2. Second step : the algebraic q-De Rham cohomology as hypercohomology of sheaves (moderate analytic q-de Rham cohomology). — We let $\mathcal{A}^{an,mod}$ be the subsheaf of $\mathcal{A}^{an}$ whose sections have moderate growth at 0 and ∞. A section of $\mathcal{A}^{an,mod}$ on an open subset U of $\mathbf{E}_q$ is an $f \in \mathcal{A}^{an}(U)$ such that, for any relatively compact subset K of U, there exist $C_{K,0}, N_{K,0}$ such that, for all $x \in \pi^{-1}(K) \cap D(0,1)$,

$$|f(x)| \leq C_{K,0}|x|^{N_{K,0}}$$

and there exist $C_{K,\infty}, N_{K,\infty}$ such that, for all $x \in \pi^{-1}(K) \cap \mathbf{C} \setminus D(0,1)$,

$$|f(x)| \leq C_{K,\infty}|x|^{N_{K,\infty}}.$$

Proposition 4.3. — *The sheaf $\mathcal{A}^{an,mod}$ is acyclic with global sections* $\mathbf{C}[x, x^{-1}]$.

Let us first prove some preliminary results. We denote by $\mathcal{A}^{diff,mod}$ the sheaf on $\mathbf{E}_q$ defined as follows : a section of $\mathcal{A}^{diff,mod}$ on an open subset U of $\mathbf{E}_q$ is a $\mathcal{C}^\infty$ function[1] $f : \pi^{-1}(U) \to \mathbf{C}$ with moderate growth at 0 and ∞, *i.e.*, such that, for any relatively compact subset K of U, there exist $C_{K,0}, N_{K,0}$ such that, for all $x \in \pi^{-1}(K) \cap D(0,1)$,

$$|f(x)| \leq C_{K,0}|x|^{N_{K,0}}$$

and there exist $C_{K,\infty}, N_{K,\infty}$ such that, for all $x \in \pi^{-1}(K) \cap \mathbf{C} \setminus D(0,1)$,

$$|f(x)| \leq C_{K,\infty}|x|^{N_{K,\infty}}.$$

Lemma 4.4. — *For any $g \in \mathcal{A}^{diff,mod}(\mathbf{E}_q)$, there exists $f \in \mathcal{A}^{diff,mod}(\mathbf{E}_q)$ such that $\overline{\partial}(f) = g$.*

Proof. - For any $g \in \mathcal{A}^{diff,mod}(\mathbf{E}_q)$, there exist $g_0 \in \mathcal{A}^{diff,mod}(\mathbf{E}_q)$ with support in $D(0,1)$ and $g_\infty \in \mathcal{A}^{diff,mod}(\mathbf{E}_q)$ with support in $\mathbf{C} \setminus D(0,1/2)$ such that $g = g_0 + g_\infty$. So, by linearity of the $\overline{\partial}$-equation, it is sufficient to treat the case when g has support in $D(0,1)$ or $\mathbf{C} \setminus D(0,1/2)$. Using the the change of variable $x \to 1/(2x)$, we see that it is sufficient to treat the case when g has support in $D(0,1)$. Since, for any $n \in \mathbf{Z}$, we have $x^n\overline{\partial}(f) = \overline{\partial}(x^n f)$, we can moreover assume that g is bounded at 0. We consider

$$f(x) = \int_{\mathbf{C}} \frac{g(\zeta)}{x-\zeta} d\zeta \wedge d\overline{\zeta} = \int_{D(0,1)} \frac{g(\zeta)}{x-\zeta} d\zeta \wedge d\overline{\zeta},$$

which is well-defined for any $x \in \mathbf{C}$. We claim that this f has the expected properties. Since g is bounded and since, for $x \in D(0,2)$,

$$\int_{D(0,1)} \frac{1}{|x-\zeta|} |d\zeta \wedge d\overline{\zeta}| \leq \int_{D(x,3)} \frac{1}{|x-\zeta|} |d\zeta \wedge d\overline{\zeta}| = \int_{D(0,3)} \frac{1}{|w|} |dw \wedge d\overline{w}| < \infty$$

and, for $x \notin D(0,2)$,

$$\int_{D(0,1)} \frac{1}{|x-\zeta|} |d\zeta \wedge d\overline{\zeta}| \leq \int_{D(0,1)} |d\zeta \wedge d\overline{\zeta}| < \infty,$$

we see that $f(x)$ is bounded as well. It remains to prove that f is $\mathcal{C}^\infty$ on $\mathbf{C}^\times$ and satisfies $\overline{\partial}(f) = g$. Consider $x_0 \in \mathbf{C}^\times$ and choose $\epsilon > 0$ small enough so that $0 \notin \overline{D}(x_0, 2\epsilon)$. We consider a decomposition $g = g_1 + g_2$ where g_2 is a $\mathcal{C}^\infty$ function on $\mathbf{C}^\times$ which is 0 on $D(x_0,\epsilon)$ and g_1 is a $\mathcal{C}^\infty$ function on $\mathbf{C}$ which is 0 on $\mathbf{C} \setminus D(x_0, 2\epsilon)$. So $f = f_1 + f_2$ where

$$f_i(x) = \int_{D(0,\eta)} \frac{g_i(\zeta)}{x-\zeta} d\zeta \wedge d\overline{\zeta}$$

[1] Here and in what follows, by $\mathcal{C}^\infty$ function we mean a $\mathcal{C}^\infty$ function of the real variables u, v where $x = u + iv$.

for an arbitrary $\eta > 1$ such that $D(x_0,\epsilon) \subset D(0,\eta)$. We have

$$f_2(x) = \int_{D(0,\eta)\setminus D(x_0,\epsilon)} \frac{g_2(\zeta)}{x-\zeta} d\zeta \wedge d\overline{\zeta}$$

and, for $x \in D(x_0,\epsilon/2)$ and $\zeta \in D(0,\eta)\setminus D(x_0,\epsilon)$, we have $|\frac{g_2(\zeta)}{x-\zeta}| \leq ||g_2||_\infty 2\epsilon^{-1}$. It follows that f_2 is holomorphic on $D(x_0,\epsilon/2)$, *i.e.*, f_2 is $\mathcal{C}^\infty$ and satisfies $\overline{\partial}(f_2) = 0$ on $D(x_0,\epsilon/2)$. Moreover, since g_1 is $\mathcal{C}^\infty$ on $\mathbf{C}$ with compact support, it is classical that f_1 is $\mathcal{C}^\infty$ and satisfies $\overline{\partial}(f_1) = g_1$ on $D(x_0,\epsilon/2)$. Hence, f is $\mathcal{C}^\infty$ on $D(x_0,\epsilon/2)$ and $\overline{\partial}(f) = g$ on $D(x_0,\epsilon/2)$. □

Lemma 4.5. — *We have the exact sequence*

$$0 \to \mathcal{A}^{an,mod} \to \mathcal{A}^{diff,mod} \xrightarrow{\overline{\partial}} \mathcal{A}^{diff,mod} \to 0. \tag{4.5.1}$$

Proof. - The only non trivial point is the surjectivity of $\mathcal{A}^{diff,mod} \xrightarrow{\overline{\partial}} \mathcal{A}^{diff,mod}$. Since, for any $\zeta \in \mathbf{E}_q$, the natural map $\mathcal{A}^{diff,mod}(\mathbf{E}_q) \to (\mathcal{A}^{diff,mod})_\zeta$ is surjective, the result is a direct consequence of Lemma 4.4. □

Lemma 4.6. — *The sheaf $\mathcal{A}^{diff,mod}$ is fine, and, hence, acyclic.*

Proof. - This is a direct consequence of the existence of $\mathcal{C}^\infty$ partitions of the unity subordinated to any open covering of $\mathbf{C}^\times$. These partitions of the unity are bounded and, hence, are global sections of $\mathcal{A}^{diff,mod}$. □

Proof of Proposition 4.3. - Combining Lemma 4.5 and Lemma 4.6, we get that (4.5.1) is an acyclic resolution of $\mathcal{A}^{an,mod}$, so

$$\mathbb{H}^i(\mathbf{E}_q, \mathcal{A}^{an,mod}) = H^i(\underbrace{\mathcal{A}^{diff,mod}(\mathbf{E}_q) \xrightarrow{\overline{\partial}} \mathcal{A}^{diff,mod}(\mathbf{E}_q)}_{\bullet})$$

and the result follows from Lemma 4.4. □

We will use the following immediate consequence of Proposition 4.3. Consider the complex of sheaves on $\mathbf{E}_q$ given by

$$\mathscr{DR}^{an,mod}(L) = \underbrace{\mathcal{A}^{an,mod} \xrightarrow{L} \mathcal{A}^{an,mod}}_{\bullet}.$$

Corollary 4.7. — *We have*

$$\mathbb{H}^i(\mathbf{E}_q, \mathscr{DR}^{an,mod}(L)) = H^i_{DR^{alg}}(L).$$

4.3. Third and last step. — In the following proofs, we use the notations

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n}\end{pmatrix}\in \mathrm{GL}_n(\mathbf{C}(x))$$

and $p=a_n$. Moreover, we consider the matricial q-difference operator

$$\nabla = p(\sigma_q - A)$$

acting on

$$F(x)=(f_1(x),f_2(x),\ldots,f_n(x))^\top$$

by

$$\nabla(F)(x)=p(x)(F(qx)-A(x)F(x)).$$

Lemma 4.8. — *Let $\zeta\in\mathbf{E}_q$ and $g\in(\mathcal{A}^{an,mod})_\zeta$. If L has no positive slope, then any $f\in(\mathcal{A}^{an})_\zeta$ such that $L(f)=g$ actually belongs to $(\mathcal{A}^{an,mod})_\zeta$.*

Proof. - We will only prove that f has moderate growth at 0, the proof at ∞ being similar. The functions

$$F(x)=(f(x),f(qx),\ldots,f(q^{n-1}x))^\top \text{ and } G(x)=(0,\ldots,0,g(x))^\top$$

satisfy $\nabla(F)=G$. Of course, G has moderate growth at 0. We have to prove that F has moderate growth at 0.

The hypothesis relative to the slopes of L ensures that A^{-1} has analytic coefficients near 0.

Let $a\in\zeta$. It is easily seen that there exist $M\in\mathbf{R}_{>0}$, $n_0,N,N_G\in\mathbf{Z}_{>0}$, with $N\geq N_G$, $\eta\in]0,1[$, such that:

- for all $x\in\cup_{j\in\mathbf{Z}_{\leq -n_0}}D(aq^j,\eta q^j)$, $|A(x)^{-1}|\leq M$,
- for all $x\in\cup_{j\in\mathbf{Z}_{\leq -n_0}}D(aq^j,\eta q^j)$, $|A(x)^{-1}G(x)/p(x)|\leq |x|^{-N_G}$,
- for all $x\in\cup_{j\in\mathbf{Z}_{\leq -n_0}}D(aq^j,\eta q^j)$, $M|q|^{-N}+|x|^{N-N_G}\leq 1$, and,
- for all $x\in D(aq^{-n_0},\eta q^{-n_0})$, $|F(x)|\leq |x|^{-N}$.

We claim that $|F(x)|\leq |x|^{-N}$ for all $x\in\cup_{j\in\mathbf{Z}_{\leq -n_0}}D(aq^j,\eta q^j)$. In order to prove this, we set $F_j=F_{|D(aq^{-j},\eta q^{-j})}$ and we will prove by induction on $j\geq n_0$ that $|F_j(x)|\leq |x|^{-N}$ for all $x\in D(aq^{-j},\eta q^{-j})$. The result is true for $j=n_0$. Assume that it is true for $j=n_0,\ldots,n_0+k$ for some $k\geq 0$. Then, the equality

$$F_{n_0+k+1}(x)=A(x)^{-1}F_{n_0+k}(qx)-A(x)^{-1}G(x)/p(x)$$

implies

$$|F_{n_0+k+1}(x)|\leq M|qx|^{-N}+|x|^{-N_G}=|x|^{-N}(M|q|^{-N}+|x|^{N-N_G})\leq |x|^{-N}.$$

□

Lemma 4.9. — *Let $\zeta \in \mathbf{E}_q$. For any $g \in (\mathcal{A}^{an})_\zeta$, there exists $f \in (\mathcal{A}^{an})_\zeta$ such that $g - L(f)$ belongs to $(\mathcal{A}^{an,mod})_\zeta$.*

Proof. - Consider

$$G(x) = (0, \ldots, 0, g(x))^\top.$$

We have to prove that there exists $F \in (\mathcal{A}^{an})^n_\zeta$ such that $G - \nabla(F)$ belongs to $(\mathcal{A}^{an,mod})^n_\zeta$. Indeed, such an F has the form

$$F(x) = (f(x), f(qx), \ldots, f(q^{n-1}x))^\top$$

and $f(x)$ is such that $g - L(f)$ belongs to $(\mathcal{A}^{an,mod})_\zeta$.

Let $a \in \mathbf{C}^\times$ such that $\zeta = \pi(a)$. Let $\epsilon > 0$ be such that $G \in \mathcal{A}^{an}(\pi(D(a,\epsilon)))^n$, and such that the eventual zeroes of p and the eventual poles of A and A^{-1} in $\cup_{j\in\mathbf{Z}} D(aq^j, \epsilon q^j)$ belong to ζ.

Let $N > 0$ be such that $\{aq^j \mid j \in \mathbf{Z},\ |j| \geq N\}$ does not contain any zero of p or pole of A or A^{-1}.

We let G_0 be the function defined by $G_0 = G$ on $\cup_{j\leq -N} D(aq^j, \epsilon q^j)$ and $G_0 = 0$ on $\cup_{j\geq -N+1} D(aq^j, \epsilon q^j)$. We let G_∞ be the function defined by $G_\infty = G$ on $\cup_{j\geq N} D(aq^j, \epsilon q^j)$ and $G_\infty = 0$ on $\cup_{j\leq N-1} D(aq^j, \epsilon q^j)$. We consider the decomposition $G = G_0 + G_\infty + (G - G_0 - G_\infty)$. Since $G - G_0 - G_\infty$ belongs to $(\mathcal{A}^{an,mod})^n_\zeta$, it is sufficient to prove that there exists $F_0 \in (\mathcal{A}^{an})^n_\zeta$ and $F_\infty \in (\mathcal{A}^{an})^n_\zeta$ such that $\nabla(F_0) = G_0$ and $\nabla(F_\infty) = G_\infty$. But, it is easily seen that the unique function F_∞ such that $F_{\infty|\cup_{j\leq N} D(aq^j,\epsilon q^j)} = 0$ and $\nabla(F_\infty) = G_\infty$ has the required properties. We have a similar construction for F_0. □

Lemma 4.10. — *If L has no positive slope, then the morphism of complexes $\mathscr{D}\mathscr{R}^{an,mod}(L) \to \mathscr{D}\mathscr{R}^{an}(L)$ given by the inclusion of sheaves $\mathcal{A}^{an,mod} \hookrightarrow \mathcal{A}^{an}$ is a quasi-isomorphism.*

Proof. - The statement means that the morphism of complexes

$$\mathscr{D}\mathscr{R}^{an,mod}(L) \to \mathscr{D}\mathscr{R}^{an}(L)$$

induces isomorphisms

$$\mathcal{H}^k(\mathscr{D}\mathscr{R}^{an,mod}(L)) \to \mathcal{H}^k(\mathscr{D}\mathscr{R}^{an}(L)) \tag{4.10.1}$$

on the cohomology sheaves for $k = 0, 1$.

For $k = 0$, this means that, for any $\zeta \in \mathbf{E}_q$ and any $f \in (\mathcal{A}^{an})_\zeta$ such that $L(f) = 0$, we have $f \in (\mathcal{A}^{an,mod})_\zeta$. This is a direct consequence of Lemma 4.8.

The fact that (4.10.1) is injective for $k=1$ follows from Lemma 4.8 again. Moreover, it follows from Lemma 4.9 that, for any $\zeta \in \mathbf{E}_q$, any element of $\mathcal{H}^1(\mathscr{DR}^{an}(L))_\zeta$ is an equivalence class that can be represented by an element of $(\mathcal{A}^{an,mod})_\zeta$, whence the surjectivity of (4.10.1) for $k=1$. □

We are now ready to conclude the proof of the fact that $(ii) \Rightarrow (i)$ in Corollary 1.2. Indeed, Lemma 4.10 ensures that the morphism $\mathscr{DR}^{an,mod}(L) \to \mathscr{DR}^{an}(L)$ given by the inclusion of sheaves $\mathcal{A}^{an,mod} \hookrightarrow \mathcal{A}^{an}$ induces an isomorphism

$$\mathbb{H}^k(\mathbf{E}_q, \mathscr{DR}^{an,mod}(L)) \cong \mathbb{H}^k(\mathbf{E}_q, \mathscr{DR}^{an}(L)).$$

Corollary 4.2 and Corollary 4.7 conclude the proof.

5. Equations vs systems

So far, we have considered (scalar) q-difference operators. In the next section, we will consider matricial q-difference operators. The present section introduces some notations and contains basic results about matricial q-difference operators.

5.1. Matricial q-difference operators. — A matrix $A \in \mathrm{GL}_n(\mathbf{C}(x))$ and a Laurent polynomial $p \in \mathbf{C}[x,x^{-1}]$ such that $pA \in M_n(\mathbf{C}[x,x^{-1}])$ being given, we consider the matricial q-difference operator

$$\nabla_{A,p} = p(\sigma_q - A)$$

acting on

$$F(x) = (f_1(x), f_2(x), \dots, f_n(x))^\top$$

by

$$\nabla_{A,p}(F)(x) = p(x)(F(qx) - A(x)F(x)).$$

We attach to $\nabla_{A,p}$ its algebraic q-De Rham complex given by

$$DR^{alg}_\bullet(\nabla_{A,p}) = \mathbf{C}[x,x^{-1}]^n \xrightarrow{\nabla_{A,p}} \mathbf{C}[x,x^{-1}]^n$$

and its analytic q-De Rham complex given by

$$DR^{an}_\bullet(\nabla_{A,p}) = \mathbb{O}^n \xrightarrow{\nabla_{A,p}} \mathbb{O}^n.$$

This corresponding cohomology spaces are denoted by

$$H^i_{DR^{alg}}(\nabla_{A,p}) := H^i(DR^{alg}(\nabla_{A,p})) \quad \text{and} \quad H^i_{DR^{an}}(\nabla_{A,p}) := H^i(DR^{an}(\nabla_{A,p})).$$

Remark 5.1. — *We consider $p(\sigma_q - A)$ instead of $\sigma_q - A$ because we want an operator acting on $\mathbf{C}[x,x^{-1}]$ and $\mathbb{O}$.*

5.2. From q-difference operators to matricial q-difference operators. — To a given q-difference operator

$$L = a_n(x)\sigma_q^n + \cdots + a_1(x)\sigma_q + a_0(x)$$

with coefficients in $a_n(x), \dots, a_1(x), a_0(x) \in \mathbf{C}[x, x^{-1}]$ such that $a_0(x)a_n(x) \neq 0$, we attach the matrix

$$A_L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix} \in \mathrm{GL}_n(\mathbf{C}(x)) \tag{5.1.1}$$

and the operator

$$\nabla_L = \nabla_{A_L, a_n} = a_n(\sigma_q - A_L).$$

Note that a function $f(x)$ satisfies $L(f)(x) = 0$ if and only if

$$F(x) = (f(x), f(qx), \dots, f(q^{n-1}x))^\top$$

satisfies $\nabla_L(F)(x) = 0$. Conversely, a function

$$F(x) = (f_1(x), f_2(x), \dots, f_n(x))^\top$$

satisfies $\nabla_L(F)(x) = 0$ if and only if $f(x) = f_1(x)$ satisfies $L(f)(x) = 0$ and

$$F(x) = (f_1(x), f_2(x), \dots, f_n(x))^\top = (f(x), f(qx), \dots, f(q^{n-1}x))^\top.$$

Therefore, the map

$$f(x) \mapsto (f(x), f(qx), \dots, f(q^{n-1}x))^\top$$

induces an isomorphism

$$H^0_{DR^{alg}}(L) \to H^0_{DR^{alg}}(\nabla_L).$$

Actually, we have :

Lemma 5.2. — *The complex $DR^{alg}(L)$ has finite dimensional cohomology if and only if $DR^{alg}(\nabla_L)$ has finite dimensional cohomology. In this case, we have*

$$\dim H^0_{DR^{alg}}(\nabla_L) = \dim H^0_{DR^{alg}}(L)$$

and

$$\dim H^1_{DR^{alg}}(\nabla_L) = (n-1)\mathrm{length}(a_n) + \dim H^1_{DR^{alg}}(L)$$

where $\mathrm{length}(a_n) = \deg(a_n) - v_0(a_n)$.

Proof. - We consider the morphism of complexes given by

$$\begin{array}{ccc} \mathbf{C}[x,x^{-1}] & \xrightarrow{\quad L \quad} & \mathbf{C}[x,x^{-1}] \\ \varphi\downarrow & & \downarrow\psi \\ \mathbf{C}[x,x^{-1}]^n & \xrightarrow{\nabla_L} & a_n(x)\mathbf{C}[x,x^{-1}]^{n-1}\oplus\mathbf{C}[x,x^{-1}] \end{array}$$

where

$$\varphi : f \mapsto (f(x), f(qx), \dots, f(q^{n-1}x))^\top \text{ and } \psi : f \mapsto (0, \dots, 0, f)^\top.$$

We claim that this is a quasi-isomorphism. Indeed, the fact that φ induces an isomorphism on the H^0 is a consequence of the discussion preceding the Lemma. It remains to prove that ψ induces an isomorphism on the H^1. For

$$F(x) = (f_1(x), f_2(x), \dots, f_n(x))^\top \in \mathbf{C}[x,x^{-1}]^n,$$

we have

$$\nabla_L(F)(x) = (a_n(x)(f_1(qx) - f_2(x)), \dots, a_n(x)(f_{n-1}(qx) - f_n(x)), *)^\top.$$

It follows that, for any $G = (g_1, \dots, g_{n-1}, g_n)^\top \in \mathbf{C}[x,x^{-1}]^n$, there exists an $F(x) \in \mathbf{C}[x,x^{-1}]^n$ such that

$$\nabla_L(F)(x) = (a_n(x)g_1(x), \dots, a_n(x)g_{n-1}(x), *)^\top.$$

So, $G - \nabla_L(F)$ is in the image of ψ and, hence, ψ is surjective on the H^1. Moreover, assume that

$$(5.2.1) \quad \psi(f) = \nabla_L(F(x))$$
$$= (a_n(x)(f_1(qx)-f_2(x)), \dots, a_n(x)(f_{n-1}(qx)-f_n(x)), a_n f_n(x)+\cdots+a_0(x)f_0(x))^\top$$

for some $f \in \mathbf{C}[x,x^{-1}]$ and $F(x) \in \mathbf{C}[x,x^{-1}]^n$. Setting $g(x) = f_1(x)$, we have $f_i(x) = g(q^{i-1}x)$ and $f(x) = a_n f_n(x) + a_{n-1}f_{n-1}(x) + \cdots + a_0(x)f_0(x) = L(g)(x)$. So $\psi(f)$ is in the image of L and ψ is injective on the H^1.

It follows that

$$H^1_{DR^{alg}}(\nabla_L) \cong H^1(\underset{\bullet}{\mathbf{C}[x,x^{-1}]^n} \xrightarrow{\nabla_{A,p}} a_n(x)\mathbf{C}[x,x^{-1}]^{n-1} \oplus \mathbf{C}[x,x^{-1}]) \oplus V$$
$$\cong H^1_{DR^{alg}}(L) \oplus V$$

where V is a complement of $a_n(x)\mathbf{C}[x,x^{-1}]^{n-1}$ in $\mathbf{C}[x,x^{-1}]^{n-1}$, whence the desired result. □

With variants of the previous proof, we obtain the following results.

Lemma 5.3. — *Lemma 5.2 remains true when the algebraic q-de Rham cohomology is replaced by the analytic q-de Rham cohomology.*

Lemma 5.4. — *Let R be a $\mathbf{C}[x,x^{-1}]\langle\sigma_q,\sigma_q^{-1}\rangle$-module. Assume that the multiplication by $a_n(x)$ gives a bijection $R\to R$. Then, the complexes*

$$\underset{\bullet}{R}\xrightarrow{L}R \quad \text{and} \quad \underset{\bullet}{R^n}\xrightarrow{\nabla_L}R^n$$

are quasi-isomorphic.

6. Proofs of Corollaries 1.2 and 1.4 via the local structure of q-difference equations

In this section, we assume that $|q|>1$ and we give proofs of Corollaries 1.2 and 1.4 using basic results about the local analytic structure of q-difference systems at 0.

6.1. Proof of Corollary 1.2. — We are going to prove that $(ii)\Rightarrow(i)$ in Corollary 1.2; the rest of the proof of Corollary 1.2 is a consequence Corollary 1.4 that will be proved in the next Section. We assume that L has no positive slope. We have to prove that the complex

$$\underset{\bullet}{\mathbb{O}/\mathbf{C}[x,x^{-1}]}\xrightarrow{L}\mathbb{O}/\mathbf{C}[x,x^{-1}]$$

has trivial cohomology. Starting as in Section 3.1, we see that it is equivalent to prove that the complexes

$$\underset{\bullet}{A_0/K_0}\xrightarrow{L}A_0/K_0 \text{ and } \underset{\bullet}{A_\infty/K_\infty}\xrightarrow{L}A_\infty/K_\infty$$

have trivial cohomology. Let us prove this for the first complex, the proof for the second complex being similar. According to Lemma 5.4, it is equivalent to prove that the complex

$$\underset{\bullet}{(A_0/K_0)^n}\xrightarrow{\nabla_L}(A_0/K_0)^n \tag{6.0.1}$$

has trivial cohomology. According to [6, Section 6], there exists $F\in\mathrm{GL}_n(\mathbf{C}(\{x\}))$ such that

$$F[A]=F(qx)A(x)F(x)^{-1}=\begin{pmatrix}A_1&\cdots&\cdots&\cdots\\0&\ddots&U_{i,j}&\vdots\\0&0&\ddots&\vdots\\0&0&0&A_k\end{pmatrix} \tag{6.0.2}$$

where each $A_i\in\mathrm{GL}_{r_i}(\mathbf{C}(x))$ is the companion matrix (5.1.1) associated to the q-difference operator $\sigma_q^{r_i}-c_ix^{s_i}$ for some $c_i\in\mathbf{C}^\times$, $r_i\in\mathbf{Z}_{>0}$ and $s_i\in\mathbf{Z}$ and where each $U_{i,j}$ belongs to $\mathrm{Mat}_{r_i,r_j}(\mathcal{M}(\mathbf{C}))$. We can and will moreover assume that $c_i\notin q^{\mathbf{Z}_{\leq -1}}$. The s_i/r_i are the slopes of L, so the s_i are ≤ 0.

The fact that the complex (6.0.1) has trivial cohomology is equivalent to the fact that each

$$\underset{\bullet}{(A_0/K_0)^{r_i}} \xrightarrow{\sigma_q - A_i} (A_0/K_0)^{r_i}$$

has trivial cohomology, and, according to Lemma 5.4, this is equivalent to the fact that

$$\underset{\bullet}{A_0/K_0} \xrightarrow{\sigma_q^{r_i} - c_i x^{s_i}} A_0/K_0$$

has trivial cohomology. Thus, the following lemma concludes the proof of Corollary 1.2.

Lemma 6.1. — *For any $r \in \mathbf{Z}_{\geq 1}$, $\lambda \in \mathbf{Z}_{\geq 0}$ and $c \in \mathbf{C}^{\times} \setminus q^{\mathbf{Z}_{\leq -1}}$, the complex*

$$\underset{\bullet}{A_0/K_0} \xrightarrow{x^{\lambda}\sigma_q^{r} - c} A_0/K_0$$

has trivial cohomology.

Proof. — Replacing q by q^r, we can assume that $r = 1$. We set

$$P = x^{\lambda}\sigma_q - c.$$

We have to prove that the following morphism is a quasi-isomorphism

$$\begin{array}{ccc} \underset{\bullet}{K_0} & \xrightarrow{P} & K_0 \\ \downarrow & & \downarrow \\ \underset{\bullet}{A_0} & \xrightarrow{P} & A_0 \end{array} \qquad (6.1.1)$$

where the vertical arrows are the inclusion.

Let us first assume that $\lambda > 0$. Up to replacing x by $c^{1/s}x$ for some s-th root $c^{1/s}$ of c, we can and will assume that $c = 1$. We consider the morphisms of complexes

$$\begin{array}{ccc} \underset{\bullet}{\mathbf{C}\{x\}} & \xrightarrow{P} & \mathbf{C}\{x\} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \underset{\bullet}{A_0} & \xrightarrow{P} & A_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} \underset{\bullet}{\mathbf{C}\{x\}} & \xrightarrow{P} & \mathbf{C}\{x\} \\ \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle r} \\ \underset{\bullet}{A_0} & \xrightarrow{P} & A_0 \end{array} \qquad (6.1.2)$$

where $i : \mathbf{C}\{x\} \hookrightarrow A_0$ is the inclusion and $r : A_0 \to \mathbf{C}\{x\}$ is given, for any $f(x) = \sum_{k \in \mathbf{Z}} a_k x^k \in A_0$, by

$$r(f(x)) = \sum_{k \geq 0} a_k x^k.$$

We claim that i is an equivalence of homotopy, and that r is an inverse up to homotopy of i; in particular, this implies that i is a quasi-isomorphism. Indeed, we have $r \circ i = \mathrm{id}_{\mathbf{C}\{x\}}$. Moreover, we consider the $\mathbf{C}$-linear map $H : A_0 \to A_0$ defined, for any $f(x) = \sum_{k\in\mathbf{Z}} a_k x^k \in A_0$, by

$$H(f) = \sum_{k\leq -1} b_k x^k$$

where

$$-b_k = a_k + q^{k-\lambda} a_{k-\lambda} + q^{k-\lambda} q^{k-2\lambda} a_{k-2\lambda} + q^{k-\lambda} q^{k-2\lambda} q^{k-3\lambda} a_{k-3\lambda} + \cdots .$$

A straightforward calculation shows that

$$H \circ P = \mathrm{id}_{A_0} - i \circ r$$

This proves our claim.

The proof in the case $\lambda = 0$ is similar by considering

$$H(f) = \sum_{k\leq -1} \frac{a_k}{q^k - c} x^k.$$

Using similar arguments, one can prove that

$$\begin{array}{ccc} \mathbf{C}\{x\}_{\bullet} & \xrightarrow{P} & \mathbf{C}\{x\} \\ \downarrow j & & \downarrow j \\ \mathbf{C}(\{x\})_{\bullet} & \xrightarrow{P} & \mathbf{C}(\{x\}) \end{array} , \tag{6.1.3}$$

where $j : \mathbf{C}\{x\} \hookrightarrow \mathbf{C}(\{x\})$ is the inclusion, is a quasi-isomorphism.

The fact that (6.1.2) and (6.1.3) are quasi-isomorphisms imply that (6.1.1) is a quasi-isomorphism. □

6.2. Proof of Corollary 1.4. — Using the notations of Section 6.1, we see that it is sufficient to prove that the complexes

$$A_0/K_{0\,\bullet} \xrightarrow{\sigma_q^{r_i} - c_i x^{s_i}} A_0/K_0 \tag{6.1.4}$$

have finite dimensional cohomology with Euler characteristics $\max\{0, s_i\}$ (indeed, using (3.0.2), this implies that the Euler characteristic of $DR^{an}(L)/DR^{alg}(L)$ is equal to the sum of the Euler characteristics of the complexes (6.1.4), *i.e.*, to $\sum_{i=1}^{k} \max\{0, s_i\}$, plus a similar term at ∞; this gives $\mathrm{irr}_0(\nabla) + \mathrm{irr}_\infty(\nabla)$ as expected). Therefore, the following two lemmas conclude the proof.

Lemma 6.2. — *For $r \in \mathbf{Z}_{\geq 1}$, $s \in \mathbf{Z}$ and $c \in \mathbf{C}^{\times}$, consider the complex*

$$A_0 \xrightarrow{\sigma_q^r - cx^s} A_0. \tag{6.2.1}$$

We have :

- *the H^0 of (6.2.1) has dimension*
  - *s if $s > 0$;*
  - *1 if ($s = 0$ and $c \in q^{\mathbf{Z}}$);*
  - *0 if $s < 0$ or ($s = 0$ and $c \notin q^{\mathbf{Z}}$);*
- *the H^1 of (6.2.1) has dimension*
  - *0 if $s > 0$ or ($s = 0$ and $c \notin q^{\mathbf{Z}}$);*
  - *1 if ($s = 0$ and $c \in q^{\mathbf{Z}}$);*
  - *$-s$ si $s < 0$.*

In particular, the complex (6.2.1) has finite dimensional cohomology and its Euler characteristic is s.

Proof. - This is inspired by [**15**, Lemmas 4.7 and 4.8] and [**12**, Section 2.3.1 and 3.2]. Replacing q by q^r, we can and will assume that $r = 1$. We set

$$P = \sigma_q - cx^s.$$

Note that $f = \sum_{k \in \mathbf{Z}} f_k x^k \in A_0$ satisfies $P(f) = 0$ if and only if, for all $k \in \mathbf{Z}$,

$$q^k f_k = c f_{k-s},$$

if and only if, for all $k, j \in \mathbf{Z}$,

$$f_{k-js} = q^{kj - s\frac{j(j-1)}{2}} c^{-j} f_k.$$

On the one hand, this formula shows that, if $s < 0$, then $\sum_{k \geq 0} f_k x^k$ is divergent except if $f = 0$; so, $H^0((6.2.1)) = 0$ in this case. On the other hand, the same formula shows that, if $s > 0$, then $H^0((6.2.1))$ has dimension s, a basis being given by the series

$$\sum_{j \in \mathbf{Z}} q^{kj - s\frac{j(j-1)}{2}} c^{-j} x^{k-js}$$

for $k \in \{0, \ldots, s-1\}$.

The case $s = 0$ is easy and left to the reader.

It remains to study $H^1((6.2.1))$.

We first assume that $s > 0$. Note that the $\mathbf{C}$-linear automorphism $f(x) \mapsto f(c^{-1/s}x)$ of A_0 conjugates $\sigma_q - cx^s$ to $\sigma_q - x^s$. So, we can and will assume that $c = 1$.

We first consider the case $s = 1$. In order to prove that $H^1((6.2.1)) = 0$, it is sufficient to prove the following two properties :

1. for any $g = \sum_{k\geq 0} b_k x^k \in \mathbf{C}\{x\}$, there exists $f = \sum_{k\geq 0} a_k x^k \in \mathbf{C}\{x\}$ such that $P(f) = g$;
2. for any $g = \sum_{k\leq 0} b_k x^k$ holomorphic on $\mathbf{C}^\times \cup \{\infty\}$, there exists $f = \sum_{k\leq 0} a_k x^k$ holomorphic on $\mathbf{C}^\times \cup \{\infty\}$ such that $P(f) = g$.

Let us prove 1. Let us first note that the series

$$\gamma_n(x) = \sum_{j\geq 0} q^{-n(j+1)} q^{-\frac{j(j+1)}{2}} x^{n+j}$$

satisfies

$$P(\gamma_n) = x^n.$$

Therefore, for any $g = \sum_{k\geq 0} b_k x^k \in \mathbf{C}\{x\}$, the series given by

$$f(x) = \sum_{n\geq 0} b_n \gamma_n(x) = \sum_{k\geq 0} a_k x^k$$

with

$$a_k = \sum_{n\leq k} b_n q^{-n(k-n+1)} q^{-\frac{(k-n)(k-n+1)}{2}} = q^{-\frac{k(k+1)}{2}} \sum_{n\leq k} b_n q^{\frac{n(n-1)}{2}}$$

satisfies

$$P(f) = g.$$

Moreover, if $A, B > 0$ are such that $|b_n| \leq AB^n$ for all $n \geq 0$, then $|a_k| \leq \sum_{n=0}^{k} AB^n$. It follows that $f(x) \in \mathbf{C}\{x\}$. This concludes the proof of 1.

Let us prove 2. Let us first note that the series

$$\delta_n(x) = -\sum_{j\leq -1} q^{-n(j+1)} q^{-\frac{j(j+1)}{2}} x^{n+j}$$

satisfies

$$P(\delta_n) = x^n.$$

Therefore, for any $g = \sum_{k\leq 0} b_k x^k$ holomorphic over $\mathbf{C}^\times \cup \{\infty\}$, the series

$$f(x) = \sum_{n\leq 0} b_n \delta_n(x) = \sum_{k\leq 0} a_k x^k$$

with

$$a_k = \sum_{n\geq k+1} b_n q^{-n(k-n+1)} q^{-\frac{(k-n)(k-n+1)}{2}} = q^{-\frac{k(k+1)}{2}} \sum_{n\geq k+1} b_n q^{\frac{n(n-1)}{2}}$$

satisfies

$$P(f) = g.$$

Moreover, if $A, B > 0$ are such that $|b_n| \leq AB^n$ for all $n \geq 0$, then $|a_k| \leq \sum_{n=0}^{k} AB^n$. It follows that $f(x) \in \mathbf{C}\{x^{-1}\}$. The functional equation $P(f) = g$

implies that f is actually holomorphic over $\mathbf{C}^{\times}\cup\{\infty\}$. This concludes the proof of 2.

We now consider the case $s \geq 1$. We have

$$A_0 = \bigoplus_{i=0}^{s-1} x^i A_0'$$

where A_0' is the subring of A_0 made of the functions of x^s. Note that each $x^i A_0'$ is stable by $\sigma_q - cx^s$, so

$$H^1(A_0 \xrightarrow{\sigma_q - cx^s} A_0) \cong \bigoplus_{i=0}^{s-1} H^1(x^i A_0' \xrightarrow{\sigma_q - x^s} x^i A_0').$$

Moreover, for any $i \in \{0, \ldots, s-1\}$, we have the isomorphism of complexes

$$\begin{array}{ccc} A_0 & \xrightarrow{q^i \sigma_{q^s} - x} & A_0 \\ \varphi \downarrow \cong & & \cong \downarrow \varphi \\ x^i A_0' & \xrightarrow{\sigma_q - x^s} & x^i A_0' \end{array},$$

where φ is the $\mathbf{C}$-linear isomorphism given by

$$\begin{array}{rcl} \varphi : A_0 & \xrightarrow{\sim} & x^i A_0' \\ f(x) & \mapsto & x^i f(x^s). \end{array}$$

It follows from the case $s = 1$ treated previously that

$$H^1(x^i A_0' \xrightarrow{\sigma_q - x^s} x^i A_0') = 0,$$

whence the desired result.

The case $s = 0$ is easy and left to the reader.

It remains to study the case $s < 0$. As in the case $s > 0$, we can and will assume that $c = 1$.

We first study the case $s = -1$. We consider the q-Borel-Ramis transformation defined by

$$(6.2.2) \qquad \begin{array}{rcl} \mathcal{B}_{q,1} : A_0 & \xrightarrow{\sim} & E \\ \displaystyle\sum_{n \in \mathbf{Z}} f_n x^n & \mapsto & \displaystyle\sum_{n \in \mathbf{Z}} \frac{f_n}{q^{n(n-1)/2}} x^n. \end{array}$$

where E is the set of $\phi(x) = \sum_{n \in \mathbf{Z}} \phi_n x^n \in A_0$ such that :

- for all $A > 0$, we have $\phi_n = O(A^n q^{-n(n-1)/2})$ as $n \to -\infty$;
- there exists $B > 0$, such that $\phi_n = O(B^n q^{-n(n-1)/2})$ as $n \to +\infty$.

It gives rise to the isomorphism of complexes

$$
(6.2.3) \qquad \begin{array}{ccc}
\underset{\bullet}{A_0} & \xrightarrow{1-x\sigma_q} & A_0 \\
{\scriptstyle \mathcal{B}_{q,1}} \downarrow \cong & & \cong \downarrow {\scriptstyle \mathcal{B}_{q,1}} \\
\underset{\bullet}{E} & \xrightarrow{\times(1-x)} & E
\end{array} .
$$

Proving that

$$h^1(\underset{\bullet}{A_0} \xrightarrow{1-x\sigma_q} A_0) = 1$$

is thus equivalent to proving that

$$h^1(\underset{\bullet}{E} \xrightarrow{\times(1-x)} E) = 1.$$

The latter equality follows directly from the fact that the image of the bottom arrow in (6.2.3) is the set of $\phi \in E$ such that $\phi(1) = 0$. It remains to justify this description of this image. Let $\phi \in E$ be such that $\phi(1) = 0$. We have to prove that

$$\gamma(x) = \frac{\phi(x)}{1-x}$$

belongs to E. Note that

$$\gamma(x) = \sum_{n \in \mathbf{Z}} \gamma_n x^n \text{ where } \gamma_n = \sum_{k \leq n} \phi_k.$$

On the one hand, for any $A > 0$, there exists $C > 0$ such that, for all $n \in \mathbf{Z}_{\leq 0}$, $|\phi_n| \leq CA^n |q^{-n(n-1)/2}|$. So, for $n \in \mathbf{Z}_{\leq 0}$,

$$
\begin{aligned}
|\gamma_n| &\leq \sum_{k \leq n} |\phi_k| \\
&\leq CA^n \, |q|^{-n(n-1)/2} \sum_{l \leq 0} A^l \, |q|^{-l(2n+l-1)/2} \\
&\leq C' A^n \, |q|^{-n(n-1)/2}
\end{aligned}
$$

where

$$C' = C \sum_{l \leq 0} A^l \, |q|^{-l(l-1)/2} < \infty.$$

On the other hand, the equality $\phi(1)=0$ implies $\gamma_n = -\sum_{k>n}\phi_k$. Let $B, C > 0$ be such that, for all $n \in \mathbf{Z}_{\geq 0}$, $|\phi_n| \leq CB^n |q^{-n(n-1)/2}|$. So, for $n \in \mathbf{Z}_{\geq 0}$,

$$\begin{aligned} |\gamma_n| &\leq \sum_{k>n} |\phi_k| \\ &\leq CB^n\,|q|^{-n(n-1)/2} \sum_{l>0} B^l\,|q|^{-l(2n+l-1)/2} \\ &\leq C'B^n\,|q|^{-n(n-1)/2} \end{aligned}$$

where

$$C' = C\sum_{l>0} B^l\,|q|^{-l(l-1)/2} < \infty.$$

Therefore, γ belongs to E as expected.

The case $s \leq -1$ can be deduced from the case $s=-1$ as we did above to deduce the case $s \geq 1$ from the case $s=1$. □

Lemma 6.3. — *For $r \in \mathbf{Z}_{\geq 1}$, $s \in \mathbf{Z}$ and $c \in \mathbf{C}^{\times}$, the complex*

$$K_{\underset{\bullet}{0}} \xrightarrow{\sigma_q^r - cx^s} K_0 \tag{6.3.1}$$

has finite dimensional cohomology and its Euler characteristic is $\min\{0,s\}$.

Proof. - See [12, Section 3.2]. □

7. Proofs of Corollaries 1.3 and 1.5 via the local structure of q-difference equations

Of course, Corollary 1.5 follows from Corollary 1.3. Section 6.2 can be easily modified in order to obtain a proof of Corollary 1.3 by using the following two facts. Firstly, that there exists an $F \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$ satisfying (6.0.2) is still true according to [6, Section 6] (provided that Assumption 3.3 is satisfied). Secondly, we need a variant of Lemma 6.1 under Assumption 3.3; such a variant is given by:

Lemma 7.1. — *For any $r \in \mathbf{Z}_{\geq 1}$, $\lambda \in \mathbf{Z}_{\geq 1}$ and $c \in \mathbf{C}^{\times} \setminus q^{\mathbf{Z}_{\leq -1}}$ such that $\sum_{k\geq 0} \frac{x^k}{1-q^k c}$ has a nonzero radius of convergence if $\lambda = 0$, the complex*

$$A_0/K_{\underset{\bullet}{0}} \xrightarrow{x^{\lambda}\sigma_q^r - c} A_0/K_0$$

has trivial cohomology.

The proof of this lemma is an obvious variant of the proof of Lemma 6.1 and is thus left to the reader.

8. The q-de Rham complex with q-spirals of poles. Proofs of Theorems 1.6, 1.7 and 1.8

In what follows, we use the notations of Section 1.4.

8.1. An example. — We have mentioned in Section 1.4 that

$$\mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}} \xrightarrow{L} \mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}}$$

may have infnite dimensional H^1 if $\mathcal{S} \neq \emptyset$; here is an example.

Example 8.1. — *Consider $L = \sigma_q - 1$ and $\mathcal{S} = \{1\}$ and assume that q is not a root of the unity. We claim that the H^1 of (1.5.2) is infinite dimensional in this case. In order to prove this, it is sufficient to prove that the infinite dimensional sub-$\mathbf{C}$-vector space of $\mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}}$ given by*

$$V = \mathrm{Span}_{\mathbf{C}}\{(x-1)^{-k} \mid k \in \mathbf{Z}_{\geq 1}\}$$

is in direct sum with $L(\mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}})$. Assume at the contrary that $V \cap L(\mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}})$ contains a nonzero element $g(x)$ and consider $f(x) \in \mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}}$ such that

$$g(x) = L(f(x)) = f(qx) - f(x).$$

This functional equation together with the fact that $g(x)$ has a pole at 1 imply that $f(x)$ has at least one pole on $q^{\mathbf{Z}}$. Let q^{i_+} be the greater power of q which is a pole of $f(x)$; then q^{i_+} is a pole of $f(qx) - f(x) = g(x)$, so $i_+ = 0$. Let q^{i_-} be the least power of q which is a pole of $f(x)$; then $q^{i_- - 1}$ is a pole of $f(qx) - f(x) = g(x)$, so $i_- = 1$. The inequality $i_- > i_+$ is absurd. The reader interested in equations of the form $f(qz) - f(z) = g(z)$ with $f(x), g(x) \in \mathbf{C}(x)$ is referred to [2] and to the references therein.

8.2. Proof of Theorem 1.6. — Without loss of generality, we can and will assume that, for any $s, s' \in \mathcal{S}$, we have $s'/s \notin q^{\mathbf{Z}\setminus\{0\}}$. For any $i, j \in \mathbf{Z}_{\geq 0}$ such that $i \leq j$, we consider the sub-$\mathbf{C}$-vector space of $\mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}}$ given by

$$E_{i,j} = \{f(x) \in \mathbf{C}[x, x^{-1}]_{q^{\mathbf{Z}}\mathcal{S}} \mid \text{the poles of } f(x) \text{ belong to } q^{\{i,\dots,j\}}\mathcal{S}\}.$$

It isn easily seen that L induces a $\mathbf{C}$-linear morphism

$$L : E_{i,j} \to E_{i-n,j}.$$

We claim that, if $a_n(x)$ does not vanish on $q^{i-n}\mathcal{S}$ and if $a_0(x)$ does not vanish on $q^j\mathcal{S}$, then the following morphism of complexes is a quasi-isomorphism :

(8.1.1)
$$\begin{array}{ccc} E_{i-1,j-1} & \xrightarrow{L} & E_{i-n-1,j-1} \\ \bullet & & \\ \downarrow & & \downarrow \\ E_{i,j} & \xrightarrow{L} & E_{i-n,j} \\ \bullet & & \end{array},$$

where the vertical arrows are the inclusions. Indeed, (8.1.1) is a quasi-isomorphism if and only if the $\mathbf{C}$-linear morphism

(8.1.2)
$$E_{i,j}/E_{i-1,j-1} \rightarrow E_{i-n,j}/E_{i-n-1,j-1}$$

induced by L is an isomorphism. In order to prove the injectivity of (8.1.2), we have to prove that any $f(x) \in E_{i,j}$ such that $L(f(x)) \in E_{i-n-1,j-1}$ actually belongs to $E_{i-1,j-1}$, *i.e.*, that such an $f(x)$ has no pole on $q^i\mathcal{S} \cup q^j\mathcal{S}$. Assume at the contrary that $f(x)$ has a pole at some $q^is \in q^{\mathbf{Z}}\mathcal{S}$ then $a_n(x)f(q^nx)$ has a pole at $q^{i-n}s$ but none of $a_0(x)f(x), \ldots, a_{n-1}(x)f(q^{n-1}x)$ has a pole at $q^{i-n}s$, therefore $L(f(x))$ has a pole at $q^{i-n}s$ and this contradicts the fact that $L(f(x)) \in E_{i-n-1,j-1}$. Similarly, assume at the contrary that $f(x)$ has a pole at some $q^js \in q^{\mathbf{Z}}\mathcal{S}$ then $a_0(x)f(x)$ has a pole at q^js but none of $a_1(x)f(qx), \ldots, a_n(x)f(q^nx)$ has a pole at q^js, therefore $L(f(x))$ has a pole at q^js and this contradicts the fact that $L(f(x)) \in E_{i-n-1,j-1}$. The fact that (8.1.2) is an isomorphism now follows from the fact that $E_{i,j}/E_{i-1,j-1}$ and $E_{i-n,j}/E_{i-n-1,j-1}$ are finite dimensional $\mathbf{C}$-vector spaces having the same dimensions.

Since $DR^{alg}(L, \mathcal{S}, d)$ is the inductive limit of the complexes

(8.1.3)
$$\underset{\bullet}{E_{i,j}} \xrightarrow{L} E_{i-n,j}$$

as $i \rightarrow -\infty$ and $j \rightarrow +\infty$, we deduce that $DR^{alg}(L)$ is quasi-isomorphic to (8.1.3) for i small enough and j large enough. It remains to prove that (8.1.3) has finite dimensional cohomology and to compute its Euler characteristic.

To this purpose, consider the exact sequence of complexes :

$$
(8.1.4)\qquad \begin{array}{ccc}
\underset{\bullet}{\mathbf{C}[x,x^{-1}]} & \xrightarrow{\quad L \quad} & \mathbf{C}[x,x^{-1}] \\
\downarrow & & \downarrow \\
\underset{\bullet}{E_{i,j}} & \xrightarrow{\quad L \quad} & E_{i-n,j} \\
\downarrow & & \downarrow \\
\underset{\bullet}{E_{i,j}/\mathbf{C}[x,x^{-1}]} & \xrightarrow{L} & E_{i-n,j}/\mathbf{C}[x,x^{-1}]
\end{array}
$$

But, according to Proposition 2.1, the top complex in (8.1.4) has finite dimensional cohomology with Euler characteristic

$$v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L).$$

The bottom complex in (8.1.4) has finite dimensional cohomology with Euler characteristic

$$\dim_{\mathbf{C}} E_{i,j}/\mathbf{C}[x,x^{-1}] - \dim_{\mathbf{C}} E_{i-n,j}/\mathbf{C}[x,x^{-1}] = (j-i+1)dm - (j-i+n+1)dm = -ndm.$$

Therefore, the middle complex in (8.1.4) has finite dimensional cohomology and its Euler characteristic is equal to

$$v_0(a_0(x)) - \deg(a_n(x)) - \mathrm{irr}_0(L) - \mathrm{irr}_\infty(L) - ndm.$$

8.3. Proof of Theorem 1.7. — The morphisms (1.6.1) are isomorphisms if and only if the quotient complex

$$(8.1.5)\qquad DR^{an}(L,[q^{\mathbf{Z}}\mathcal{S}],d)/DR^{alg}(L,q^{\mathbf{Z}}\mathcal{S},d) = \underset{\bullet}{\mathbb{O}_{[q^{\mathbf{Z}}\mathcal{S}],d}/\mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S},d}} \xrightarrow{L} \mathbb{O}_{[q^{\mathbf{Z}}\mathcal{S}],d}/\mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S},d}$$

has trivial cohomology. But, the inclusion $\mathbb{O} \hookrightarrow \mathbb{O}_{[q^{\mathbf{Z}}\mathcal{S}],d}$ induces an isomorphism

$$\mathbb{O}/\mathbf{C}[x,x^{-1}] \cong \mathbb{O}_{[q^{\mathbf{Z}}\mathcal{S}],d}/\mathbf{C}[x,x^{-1}]_{q^{\mathbf{Z}}\mathcal{S},d}.$$

It follows that (8.1.5) has trivial cohomology if and only if the complex

$$DR^{an}(L)/DR^{alg}(L) = \underset{\bullet}{\mathbb{O}/\mathbf{C}[x,x^{-1}]} \xrightarrow{L} \mathbb{O}/\mathbf{C}[x,x^{-1}]$$

has trivial cohomology if and only if the morphisms (1.0.2) are isomorphisms.

8.4. Proof of Theorem 1.8. — Arguing as in the proof of Theorem 1.7, we see that the quotient complex $DR^{an}(L,[q^{\mathbf{Z}}\mathcal{S}],d)/DR^{alg}(L,q^{\mathbf{Z}}\mathcal{S},d)$ has finite dimensional cohomology if and only if $DR^{an}(L)/DR^{alg}(L)$ has finite dimensional cohomology and that, in this case,

$$\chi(DR^{an}(L,[q^{\mathbf{Z}}\mathcal{S}],d)/DR^{alg}(L,q^{\mathbf{Z}}\mathcal{S},d)) = \chi(DR^{an}(L)/DR^{alg}(L)).$$

Since $DR^{alg}(L,q^{\mathbf{Z}}\mathcal{S},d)$ and $DR^{alg}(L)$ have finite dimensional cohomology, we get that $DR^{an}(L,[q^{\mathbf{Z}}\mathcal{S}],d)$ has finite dimensional cohomology if and only $DR^{an}(L)$ has finite dimensional cohomology and that, in this case,

$$\chi(DR^{an}(L,[q^{\mathbf{Z}}\mathcal{S}],d)) - \chi(DR^{alg}(L,q^{\mathbf{Z}}\mathcal{S},d)) \\ = \chi(DR^{an}(L)) - \chi(DR^{alg}(L)).$$

A
Cohomology and duality

This appendix contains results about the effect of duality on cohomology. These results are well-known but we have not been able to find suitable references with complete proofs with the exact hypotheses we need. The results and proofs below are straightforward extensions of results and proofs in Serre's [16].

A.1. Algebraic duality. — Consider a complex

$$\mathcal{M} : \cdots \xrightarrow{f_{-2}} M_{-1} \xrightarrow{f_{-1}} \underset{\bullet}{M_0} \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots$$

of $\mathbf{C}$-vector spaces. Its dual is

$$\mathcal{M}^* : \cdots \xrightarrow{f_1^*} M_1^* \xrightarrow{f_0^*} \underset{\bullet}{M_0^*} \xrightarrow{f_{-1}^*} M_{-1}^* \xrightarrow{f_{-2}^*} \cdots$$

Then $H^i(\mathcal{M}^*)$ and $H^{-i}(\mathcal{M})^*$ are isomorphic, more precisely a $\mathbf{C}$-linear isomorphism is given by

$$\begin{array}{rcl} H^i(\mathcal{M}^*) = \ker(f^*_{-i-1})/\operatorname{im}(f^*_{-i}) & \xrightarrow{\sim} & (\ker(f_{-i})/\operatorname{im}(f_{-i-1}))^* = H^{-i}(\mathcal{M})^* \\ u \mod \operatorname{im}(f^*_{-i}) & \mapsto & \widetilde{u_{|\ker(f_{-i})}} \end{array} \tag{A.0.1}$$

where $\widetilde{u_{|\ker(f_{-i})}}$ is the map induced by u on $\ker(f_{-i})/\operatorname{im}(f_{-i-1})$.

Moreover, this isomorphism is natural in the sense that, for any morphism $\varphi : \mathcal{M} \to \mathcal{N}$, we have the commutative diagram :

$$\begin{array}{ccc} H^i(\mathcal{N}^*) & \longrightarrow & H^{-i}(\mathcal{N})^* \\ {\scriptstyle H^i(\varphi^*)}\downarrow & & \downarrow{\scriptstyle H^{-i}(\varphi)^*} \\ H^i(\mathcal{M}^*) & \longrightarrow & H^{-i}(\mathcal{M})^* \end{array}$$

where the horizontal arrows are given by (A.0.1). In particular, we see that φ is a quasi-isomorphism if and only if φ^* is a quasi-isomorphism.

A.2. Topological duality. — Consider a complex

$$\mathcal{M} : \cdots \xrightarrow{f_{-2}} M_{-1} \xrightarrow{f_{-1}} \underset{\bullet}{M_0} \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots$$

of topological **C**-vector spaces (the M_i are topological **C**-vector spaces and the f_i are linear continuous). Its topological dual is

$$\mathcal{M}' : \cdots \xrightarrow{f_1^*} M_1' \xrightarrow{f_0^*} \underset{\bullet}{M_0'} \xrightarrow{f_{-1}^*} M_{-1}' \xrightarrow{f_{-2}^*} \cdots$$

where M_i' is the topological dual of M_i.

We have a **C**-linear morphism (where $H^{-i}(\mathcal{M})$ is endowed with its quotient structure)

$$\begin{array}{rcl} H^i(\mathcal{M}') = \ker(f_{-i-1}^*)/\operatorname{im}(f_{-i}^*) & \to & (\ker(f_{-i})/\operatorname{im}(f_{-i-1}))' = H^{-i}(\mathcal{M})' \\ u \mod \operatorname{im}(f_{-i}^*) & \mapsto & \widetilde{u_{|\ker(f_{-i})}} \end{array} \qquad \text{(A.0.2)}$$

where $\widetilde{u_{|\ker(f_{-i})}}$ is the map induced by u on $\ker(f_{-i})/\operatorname{im}(f_{-i-1})$.

Moreover, this morphism is natural in the sense that, for any morphism $\varphi : \mathcal{M} \to \mathcal{N}$ of complex of topological **C**-vector spaces, we have the commutative diagram :

$$\begin{array}{ccc} H^i(\mathcal{N}') & \longrightarrow & H^{-i}(\mathcal{N})' \\ {\scriptstyle H^i(\varphi^*)}\downarrow & & \downarrow{\scriptstyle H^{-i}(\varphi)^*} \\ H^i(\mathcal{M}') & \longrightarrow & H^{-i}(\mathcal{M})' \end{array}$$

where the horizontal arrows are given by (A.0.2).

Lemma A.1. — *Assume that*

- *for any subspace V of M_{-i}, the restriction morphism $M'_{-i} \to V'$ is surjective (according to* [8, Corollary 1, Chapter 2, 6, p.55]*, this holds true if M_i is locally convex);*
- *f_{-i} is a homomorphism (in the sense of* [8, Definition 2, Chapter 1, 3, p.16]*).*

Then, (A.0.2) *is a* $\mathbf{C}$*-linear isomorphism.*

Proof. — Let us prove that (A.0.2) is injective. Let $u \mod \operatorname{im}(f^*_{-i})$ be in the kernel of (A.0.2). Then, $u \in \ker(f^*_{-i-1}) \subset M'_{-i}$ vanishes on $\ker(f_{-i})$. So u induces $\widetilde{u} \in (M_{-i}/\ker(f_{-i}))'$. Moreover, since f_{-i} is a homomorphism, it induces an isomorphism of topological $\mathbf{C}$-vector spaces $\widetilde{f_{-i}} : M_{-i}/\ker(f_{-i}) \to \operatorname{im}(f_{-i})$. Then, $w := \widetilde{u} \circ \widetilde{f_{-i}}^{-1} \in \operatorname{im}(f_{-i})'$ satisfies $u = w \circ f_{-i}$. Now any extension $v \in M'_{-i}$ of w satisfies $u = f^*_{-i}(v) \in \operatorname{im}(f^*_{-i})$ and this concludes the proof of the injectivity of (A.0.2).

The proof of the surjectivity is easy and left to the reader. □

In what follows, we use the terminology "$\mathcal{LF}$" from [8, Definition 4, Chap. 4, Part 1, 5, p. 146]

Lemma A.2. — *Let* E, F *be two* ($\mathcal{LF}$)*-topological* $\mathbf{C}$*-vector spaces and let* $f : E \to F$ *be continuous linear. If* $\operatorname{im}(f)$ *has finite codimension in* F*, then* f *is an homomorphism and* $\operatorname{im}(f)$ *is closed in* F*.*

Proof. — Since the quotient of a ($\mathcal{LF}$)-topological $\mathbf{C}$-vector spaces by a closed subspace is a ($\mathcal{LF}$)-topological $\mathbf{C}$-vector spaces, up to replacing E by $E/\ker(f)$ and f by the map $E/\ker(f) \to F$ induced by f, we can and will assume that f is injective. Let Z be a supplement of $\operatorname{im}(f)$ in F. Since Z is Hausdorff finite dimensional, $E \times Z$ endowed with the product topology is a ($\mathcal{LF}$)-topological $\mathbf{C}$-vector spaces. Consider the surjective continuous linear map $g : E \times Z \to F$, $(x, y) \mapsto f(x) + y$. According to [8, Theorem 2, 1), Chap. 4, Part 1, 5, p. 148], g is an homomorphism in the sense of [8, Definition 2, Chapter 1, 3, p.16] meaning that the map $E/\ker(g) \to F$ induced by g is an isomorphism of topological $\mathbf{C}$-vector spaces. But g is injective, so g is an isomorphism of topological $\mathbf{C}$-vector spaces, so $\operatorname{im}(f) = g(E \times \{0\})$ is closed in F and $g_{|E\times\{0\}} : E \times \{0\} \to \operatorname{im}(f)$, $(x, 0) \mapsto f(x)$ is an isomorphism of topological $\mathbf{C}$-vector spaces, whence the result. □

Lemma A.3. — *Consider a complex*

$$\mathcal{N} : \cdots \xrightarrow{g_{-2}} N_{-1} \xrightarrow{g_{-1}} \underset{\bullet}{N_0} \xrightarrow{g_0} N_1 \xrightarrow{g_1} \cdots$$

of ($\mathcal{LF}$)*-topological* $\mathbf{C}$*-vector spaces (the* N_i *are topological* $\mathbf{C}$*-vector spaces and the* g_i *are linear continuous). Assume that*

- *the* $\ker(g_i)$ *are* ($\mathcal{LF}$)*-topological* $\mathbf{C}$*-vector spaces;*
- $\mathcal{N}$ *has finite dimensional cohomology.*

Consider a complex

$$\mathcal{M} : \cdots \xrightarrow{f_{-2}} M_{-1} \xrightarrow{f_{-1}} \underset{\bullet}{M_0} \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots$$

of vector spaces and a morphism $\varphi : \mathcal{M} \to \mathcal{N}$. Then, $\varphi^* : \mathcal{N}' \to \mathcal{M}^*$ is a quasi-isomorphism if and only if φ is a quasi-isomorphism.

Proof. — We have the commutative diagram

$$\begin{array}{ccc} H^i(\mathcal{N}') & \xrightarrow{\cong} & H^{-i}(\mathcal{N})' \\ {\scriptstyle H^i(\varphi^*)}\downarrow & & \downarrow{\scriptstyle H^{-i}(\varphi)^*} \\ H^i(\mathcal{M}^*) & \xrightarrow{\cong} & H^{-i}(\mathcal{M})^* \end{array}$$

where the top (resp. bottom) horizontal arrow is given by (A.0.2) (resp. (A.0.1)). The fact that the top arrow is an isomorphism follows from Lemma A.1 and Lemma A.2. Moreover, $H^{-i}(\mathcal{N})$ is finite dimensional (by hypothesis) and Hausdorff (because $\mathrm{im}(g_{-i-1})$ is closed in $\ker(g_{-i})$ in virtue of Lemma A.2), so $H^i(\mathcal{N})' = H^i(\mathcal{N})^*$. Therefore, we get that φ^* is a quasi-isomorphism if and only if, for all $i \in \mathbf{Z}$, $H^{-i}(\varphi)^* : H^{-i}(\mathcal{N})^* \to H^{-i}(\mathcal{M})^*$ is an isomorphism if and only if, for all $i \in \mathbf{Z}$, if $H^{-i}(\varphi) : H^{-i}(\mathcal{M}) \to H^{-i}(\mathcal{N})$ is an isomorphism if and only if φ is a quasi-isomorphism. □

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