On classical irregular q-difference equations

Julien Roques

Abstract

The primary aim of this paper is to (provide tools in order to) compute Galois groups of classical irregular q-difference equations. We are particularly interested in quantizations of omnipresent differential equations in the mathematical and physical literature, namely confluent generalized q-hypergeometric equations and q-Kloosterman equations.

1. Introduction - Organization

In the whole paper, q is a nonzero complex number such that $|q| < 1$. For all $\alpha \in \mathbb{C}$, we set $q^\alpha = e^{\alpha \log(q)}$ where $\log(q)$ is a fixed logarithm of q. We denote by $\mathbb{C}(z)\langle \sigma_q, \sigma_q^{-1} \rangle$ the noncommutative algebra of noncommutative Laurent polynomials with coefficients in $\mathbb{C}(z)$ such that $\sigma_q z = qz\sigma_q$.

1.1 Introduction

Here are examples of computations of q-difference Galois groups derived from the main results of this paper.

The generalized q-hypergeometric operator $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$ with parameters $\underline{a} = (a_1, ..., a_r) \in (q^{\mathbb{R}})^r$ ($r \in \mathbb{N}$), $\underline{b} = (b_1, ..., b_s) \in (q^{\mathbb{R}})^s$ ($s \in \mathbb{N}$) and $\lambda \in \mathbb{C}^*$ is given by

$$\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) = \prod_{j=1}^{s} (\frac{b_j}{q}\sigma_q - 1) - z\lambda \prod_{i=1}^{r} (a_i \sigma_q - 1).$$

We assume that $r \neq s$ (cf. [Roq11] for the case $r = s$). Up to replacing z by $1/z$, we can assume that $r > s$. For all $(i, j) \in \{1, ..., r\} \times \{1, ..., s\}$, we let $\alpha_i, \beta_j \in \mathbb{R}$ be such that $a_i = q^{\alpha_i}$ and $b_j = q^{\beta_j}$.

Theorem. *Assume that, for all $(i, j) \in \{1, ..., r\} \times \{1, ..., s\}$, $\beta_j - \alpha_i \notin \mathbb{Z}$ (this condition is empty if $s = 0$) and that the algebraic group generated by $\mathrm{diag}(e^{2\pi i \alpha_1}, ..., e^{2\pi i \alpha_r})$ is connected. Then the Galois group of $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$ is $GL(\mathbb{C}^r)$.*

Example. *The Galois group of $(q^{1/2}\sigma_q - 1)^s - z(\sigma_q - 1)^r$ is $GL(\mathbb{C}^r)$.*

The q-Kloosterman operator $Kl_q(U, V)$ associated to a pair (U, V) of elements of $\mathbb{C}[X]$ such that $U(0) = 0$ and $V(0) \neq 0$ is given by

$$Kl_q(U, V) = U(\sigma_q) + V(z^{-1}).$$

We let $c_1, ..., c_{\deg U}$ be the complex roots of $X^{\deg U}(U(X^{-1}) + V(0)) \in \mathbb{C}[X]$ and, for all $i \in \{1, ..., \deg U\}$, we denote by (u_i, α_i) the unique element of $\mathbb{U} \times \mathbb{R}$ such that $c_i = u_i q^{\alpha_i}$ ($\mathbb{U} \subset \mathbb{C}$ denotes the unit circle).

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Theorem. *Assume that* $\deg U$ *and* $\deg V$ *are relatively prime, that the algebraic group generated by* $\mathrm{diag}(u_1, ..., u_{\deg U})$ *and* $\mathrm{diag}(e^{2\pi i\alpha_1}, ..., e^{2\pi i\alpha_{\deg U}})$ *is connected and that there exists* $z_0 \in \mathbb{C}^*$ *such that* $V(z_0) = 0$ *and, for all* $k \in \mathbb{Z}^*$, $V(q^k z_0) \neq 0$. *Then the Galois group of* $Kl_q(U, V)$ *is* $GL(\mathbb{C}^{\deg U})$.

Example. *For relatively prime integers* m, n, *the Galois group of* $(1-\sigma_q)^n + (1-z^{-1})^m - 1$ *is* $GL(\mathbb{C}^n)$.

Proposition. *Let us consider* $V \in q + X\mathbb{C}[X]$. *Then, for any odd integer* $n \geqslant 2$ *prime to* $\deg V$, *the Galois group of* $Kl_q((q^{1/2} - X)^2(1-X)^{n-2} - q, V)$ *is* $GL(\mathbb{C}^n)$.

In order to achieve these goals, we present our results in two parts.

Part I is devoted to the following problem : *find simple and relevant characterizations of the classical linear algebraic groups.*

Part II is a Galoisian study of q-difference operators $L \in \mathbb{C}(z)\langle \sigma_q, \sigma_q^{-1}\rangle$ of rank n satisfying one of the following properties (cf. §4.2 for the notion of slope) :

($\mathscr{H}1$) At 0, L has a unique slope, this slope being of the form m/n with $m \in \mathbb{Z}^*$ prime to n.

($\mathscr{H}2$) At 0, L has two slopes, namely 0 and μ. Denoting by r the multiplicity of μ, we have $\mu = m/r$ for some $m \in \mathbb{Z}^*$ prime to r. The exponents attached to the slope 0 belong to $q^{\mathbb{R}}$.

For instance, the generalized q-hypergeometric operators with $s > 0$ considered above satisfy ($\mathscr{H}2$) whereas the generalized q-hypergeometric operators with $s = 0$ as well as the q-Kloosterman operators $Kl_q(U, V)$ with $\deg U$ prime to $\deg V$ satisfy ($\mathscr{H}1$).

Our starting point originates from the work of N. Katz [Kat87] : we exploit the structure of the local formal Galois groups. However, the q-difference and the differential cases are rather different; in particular, the "theta torus" is "poorer" than its differential analogue, Ramis' exponential torus. We make an essential use of works by M. van der Put and M. Reversat [PR07], M. van der Put and M. Singer [PS97] and J. Sauloy [Sau04]. In the theory of (irregular) linear differential equations, another way was explored in order to compute Galois groups : the use of Ramis' "wild fundamental group"(see [DM89, Mit96]). It would be interesting to compute q-difference Galois groups using the q-analogue of the wild fundamental group introduced by J.-P. Ramis and J. Sauloy in [RS07, RS09]. The essential difference lies in the presence of a unipotent Stokes component (and hence in the *analytic* properties of the slopes filtration).

In some cases, the classical equations studied in this paper can be seen as q-deformations of classical differential equations (this is exploited by Y. André in [And01]; see also §3, §4 and §5 of J. Sauloy's paper [Sau00]), namely the confluent generalized hypergeometric equations and the Kloosterman equations. These differential equations were studied by N. Katz, with contributions by O. Gabber, in [Kat87, Kat90], N. Katz and R. Pink in [KP87], F. Beukers, W. D. Brownawell and G. Heckman in [BBH88], A. Duval and C. Mitschi in [DM89] and C. Mitschi in [Mit96].

The original interest of the author in the classical equations studied in the present paper comes from the discrete Morales-Ramis theory developed in [CR08, CR11] in order to derive the nonintegrability of classical nonlinear q-difference equations, e.g. discrete Painlevé equations.

1.2 Organization

Part I essentially provides "easily checkable" characterizations of the classical linear algebraic groups. In §2 we give a new such characterization relying on pairs of semisimple elements with special spectra. In §3 we give consequences of results of N. Katz and of B. Kostant. Part II gives applications of these purely representation theoretic results to the Galois theory of irregular q-difference equations. In §4 we give elements of slopes theory and useful Galoisian results. In §5 and §6 we show that the connected algebraic groups occurring as Galois groups of irreducible equations satisfying either ($\mathscr{H}1$) or ($\mathscr{H}2$) belong to a very short list of linear algebraic groups. In §7 we compute Galois groups of q-Kloosterman equations and of generalized q-hypergeometric equations. In §8 we give a $\otimes$-indecomposability criterion that we apply to the calculation of q-difference Galois groups. In §9, combining several results of this paper, we give additional computations of Galois groups.

Part I. Characterizations of the classical linear algebraic groups

2. A characterization of the classical linear algebraic groups

Let E be a $\mathbb{C}$-vector space of finite dimension $n \geqslant 3$. Let us consider α, β in $\mathbb{N}$ such that $\alpha \geqslant 1$, $\beta \geqslant 2$ and $n = \alpha + \beta$.

Definition 1 (Property ($\mathcal{P}$)). *A pair f, g of semisimple elements of $GL(E)$ satisfies property ($\mathcal{P}$) if :*

- *the list of eigenvalues of f is of the form (a repeated α times, b repeated β times) where $a, b \in \mathbb{C}^*$ are such that $a \neq \pm b$;*
- *the list of eigenvalues of g is of the form (c repeated $\alpha + 1$ times, d_1,..., $d_{\beta-1}$) where $c, d_1, ..., d_{\beta-1}$ are pairwise distinct nonzero complex numbers.*

This § is devoted to the proof of the following result :

Theorem 2. *Let G be a connected algebraic subgroup of $GL(E)$ which acts irreducibly on E. If G contains a pair of semisimple elements f, g satisfying ($\mathcal{P}$) then the derived subgroup G' of G is either $SL(E)$ or $SO(E)$ or (if $n = \dim(E)$ is even) $Sp(E)$. Furthermore, $G' \subset G \subset \mathbb{C}^* G'$.*

Proposition 3. *Let G be a connected semisimple algebraic subgroup of $GL(E)$ which acts irreducibly on E. If G contains a semisimple element f whose list of eigenvalues is of the form (a repeated α times, b repeated β times) for some $a, b \in \mathbb{C}^*$ such that $a \neq \pm b$ then its Lie algebra $\mathfrak{g}$ contains a semisimple element whose list of eigenvalues is (β repeated α times, $-\alpha$ repeated β times).*

Proof. O. Gabber's Theorem 1.0 in [Kat90] applied to the Lie subalgebra $\mathfrak{g}$ of $End(E)$ and to the subgroup H of G generated by f ensures that, for any x, y in $\mathbb{C}$ such that $\alpha x + \beta y = 0$, $\mathfrak{g}$ contains a semisimple element whose list of eigenvalues is (x repeated α times, y repeated β times). □

Proposition 4. *Let G be a connected semisimple algebraic subgroup of $SL(E)$ which acts irreducibly on E. If G contains a pair of semisimple elements f, g satisfying ($\mathcal{P}$) then G is simple (in the sense that its Lie algebra is simple).*

Proof. Let $\rho : G \hookrightarrow GL(E)$ be the standard representation of G, which is irreducible by hypothesis. It comes from an irreducible representation $\widetilde{\rho} : \widetilde{G} \twoheadrightarrow G \hookrightarrow GL(E)$ of the universal covering $\widetilde{G}$ of G. We want to prove that G is simple, i.e. that its Lie algebra $Lie(G) = Lie(\widetilde{G}) = \mathfrak{g}$ is simple.

Assume at the contrary that $\mathfrak{g}$ is not simple. Then it splits into a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of nontrivial semisimple Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in such a way that the irreducible representation $Lie(\widetilde{\rho}) : \mathfrak{g} \hookrightarrow End(E)$ is (irreducible representation $\mathfrak{g}_1 \to End(E_1)$)$\otimes$(irreducible representation $\mathfrak{g}_2 \to End(E_2)$) with $n_1 = \dim(E_1) \geqslant 2$ and $n_2 = \dim(E_2) \geqslant 2$. Denoting by $\widetilde{G}_1$ and $\widetilde{G}_2$ the connected and simply connected semisimple Lie groups with respective Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and integrating the above representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ into representations $\widetilde{\rho}_1 : \widetilde{G}_1 \to GL(E_1)$ and $\widetilde{\rho}_2 : \widetilde{G}_2 \to GL(E_2)$, we get that $\widetilde{G}$ is $\widetilde{G_1} \times \widetilde{G_2}$ and that $\widetilde{\rho}$ is $\widetilde{\rho}_1 \otimes \widetilde{\rho}_2$. So the list of eigenvalues of any element of $G = Im(\widetilde{\rho})$ is of the form $(\lambda_i \mu_j ; 1 \leqslant i \leqslant n_1, 1 \leqslant j \leqslant n_2)$.

Since f belongs to G, its list of eigenvalues (a repeated α times, b repeated β times) is of the form $(\lambda_i \mu_j ; 1 \leqslant i \leqslant n_1, 1 \leqslant j \leqslant n_2)$.

Note that either $\mathrm{card}\{\lambda_i \mid 1 \leqslant i \leqslant n_1\} = 1$ or $\mathrm{card}\{\mu_j \mid 1 \leqslant j \leqslant n_2\} = 1$. Otherwise, there would exist $t, u \in \{\lambda_i \mid 1 \leqslant i \leqslant n_1\}$ and $v, w \in \{\mu_j \mid 1 \leqslant j \leqslant n_2\}$ such that $t \neq u$ and $v \neq w$. The sublist (tv, tw, uv, uw) of $(\lambda_i \mu_j ; 1 \leqslant i \leqslant n_1, 1 \leqslant j \leqslant n_2)$ would be made of at least 3 distinct numbers (otherwise, since $\{tv, uw\} \cap \{tw, uv\} = \emptyset$, we would have $tv = uw$ and $tw = uv$ so $v/w = (tv)/(tw) = (uw)/(uv) = w/v$ hence $v = -w$ and $t = -u$ therefore the inclusion $\{tv, -tv\} = \{tv, tw, uv, uw\} \subset \{\lambda_i \mu_j \mid 1 \leqslant i \leqslant n_1, 1 \leqslant j \leqslant n_2\} = \{a, b\}$ would be an equality, so $a = -b$: contradiction). This is a contradiction.

Up to relabeling, we can assume that $\mathrm{card}\{\lambda_i \mid 1 \leqslant i \leqslant n_1\} = 1$ and $\mathrm{card}\{\mu_j \mid 1 \leqslant j \leqslant n_2\} = 2$. Hence α and β are nonzero integral multiples of n_1; in particular $n_1 \leqslant \alpha$ and $n_1 \leqslant \beta$.

Since g belongs to G, its list of eigenvalues (c repeated $\alpha + 1$ times, d_1, ..., $d_{\beta-1}$) is of the form $(\lambda'_i \mu'_j ; 1 \leqslant i \leqslant n_1, 1 \leqslant j \leqslant n_2)$. So there exist $\alpha + 1$ distinct indices $(i_1, j_1), ..., (i_{\alpha+1}, j_{\alpha+1})$ in $\{1, .., n_1\} \times \{1, .., n_2\}$ such that $c = \lambda'_{i_1} \mu'_{j_1} = \cdots = \lambda'_{i_{\alpha+1}} \mu'_{j_{\alpha+1}}$. Since $n_1 < \alpha + 1$, we get that there exist $1 \leqslant k \neq k' \leqslant \alpha + 1$ such that $i_k = i_{k'}$. Hence $j_k \neq j_{k'}$ and $\lambda'_{i_k} \mu'_{j_k} = \lambda'_{i_{k'}} \mu'_{j_{k'}}$ so $\mu'_{j_k} = \mu'_{j_{k'}}$. Therefore, for all $1 \leqslant i \leqslant n_1$, $\lambda'_i \mu'_{j_k} = \lambda'_i \mu'_{j_{k'}}$, so $\lambda'_i \mu'_{j_k} = c$ (because c is the unique eigenvalue of g with multiplicity > 1). Hence, $\lambda'_1 = \cdots = \lambda'_{n_1}$. So, any element of $(\lambda'_i \mu'_j ; 1 \leqslant i \leqslant n_1, 1 \leqslant j \leqslant n_2)$ occurs at least $n_1 > 1$ times : this is a contradiction (g has at least one eigenvalue with multiplicity 1). So $\mathfrak{g}$ is simple. $\square$

We have proved that any connected semisimpe algebraic subgroup of $GL(E)$ which acts irreducibly on E and which contains a pair of semisimple elements f, g satisfying $(\mathcal{P})$ is simple and that its Lie algebra contains a morphism with exactly two eigenvalues. This restricts the possibilities for G in virtue of the following result of J.-P. Serre. For the notion of minuscule representations, we refer to N. Bourbaki [Bou75].

Theorem 5 (J.-P. Serre, §3 of [Ser79]). *If a simple Lie subalgebra $\mathfrak{g}$ of $End(E)$ which acts irreducibly on E contains a morphism with exactly two eigenvalues then $\mathfrak{g}$ is a classical Lie algebra (A_m, B_m, C_m or D_m) and its weights in E are minuscule.*

It is proved in §7.3 of chapter 8 of [Bou75] that the minuscule representations of classical Lie algebras are :

$$A_m, m \geqslant 1; \omega_1, ..., \omega_m$$
$$B_m, m \geqslant 3; \omega_m$$
$$C_m, m \geqslant 2; \omega_1$$
$$D_m, m \geqslant 4; \omega_1, \omega_{m-1}, \omega_m.$$

Remark 1. *This list is slightly different from the one given in loc. cit. because (we are only interested in classical Lie algebras and) we have taken into consideration accidental isomorphisms.*

The corresponding representations of connected Lie groups are conjugated to a factor of one of the following representations :

$$\begin{aligned} SL_{m+1}(\mathbb{C}), m \geqslant 1; & \, std, \Lambda^2(std)..., \Lambda^m(std) \\ Spin_{2m+1}(\mathbb{C}), m \geqslant 3; & \, \text{spin representation} \\ Sp_{2m}(\mathbb{C}), m \geqslant 2; & \, std \\ Spin_{2m}(\mathbb{C}), m \geqslant 4; & \, \text{half-spin representations or ``std representation of } SO_{2m}(\mathbb{C})\text{''}. \end{aligned}$$

For any subgroup G of $GL(E)$, we have denoted and we will denote by std the standard representation of G i.e. the inclusion $G \hookrightarrow GL(E)$.

In what follows, we shall prove that among the above representations, the only ones whose image contains a pair of semisimple elements satisfying $(\mathcal{P})$ are $SL_{m+1}(\mathbb{C})$ in std or in $\Lambda^m(std)$, $Sp_{2m}(\mathbb{C})$ in std and $Spin_{2m}(\mathbb{C})$ in the standard representation of $SO_{2m}(\mathbb{C})$.

Proposition 6. *For $1 < k < m$ (so $m \geqslant 3$), the image of $SL_{m+1}(\mathbb{C})$ in $\Lambda^k(std)$ does not contain a pair of semisimple elements satisfying $(\mathcal{P})$.*

Proof. By duality ($\Lambda^k(std) \cong (\Lambda^{m+1-k}(std))^*$), it is sufficient to consider the case that $1 < k \leqslant \frac{m+1}{2}$.

Assume at the contrary that the image of $SL_{m+1}(\mathbb{C})$ in $\Lambda^k(std)$ contains a pair of semisimple elements f, g satisfying $(\mathcal{P})$.

So, the list of eigenvalues (a repeated α times, b repeated β times) of f is of the form

$$(u_{i_1,\dots,i_k} = u_{i_1} \cdots u_{i_k};\ 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m+1).$$

We have $\mathrm{card}\{u_i \mid 1 \leqslant i \leqslant m+1\} \geqslant 2$ because $a \neq b$. We claim that $\mathrm{card}\{u_i \mid 1 \leqslant i \leqslant m+1\} = 2$. Assume at the contrary that $\mathrm{card}\{u_i \mid 1 \leqslant i \leqslant m+1\} > 2$. Up to renumbering, we can assume that u_1, u_2 and u_3 are pairwise distinct. Then $u_{3,\dots,k+2}$, $u_{2,4,\dots,k+2}$ and $u_{1,4,\dots,k+2}$ (note that $k+2 \leqslant \frac{m+1}{2} + 2 \leqslant m+1$ because $m \geqslant 3$) would be pairwise distinct, therefore $\mathrm{card}\{u_{i_1,\dots,i_k} \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m+1\} > 3$: this is a contradiction.

So, up to renumbering, we can assume that there exists $i \in \{1, \dots, m\}$ such that $u := u_1 = \cdots = u_i \neq u_{i+1} = \cdots = u_{m+1} =: v$.

We claim that $i = 1$ or $i = m$. Indeed, assume at the contrary that $2 \leqslant i \leqslant m-1$ (recall that $m \geqslant 3$) and denote by l the least nonnegative integer such that $i \leqslant l + k$ (so $l = 0$ if $i \leqslant k$ and $l = i - k$ if $i > k$). Then $u_{l+1,\dots,l+k}$, $u_{l+2,\dots,l+k+1}$ and $u_{l+3,\dots,l+k+2}$ would be pairwise distinct (indeed, there exists $t \in \mathbb{C}^*$ such that $u_{l+1,\dots,l+k} = u^2 t$, $u_{l+2,\dots,l+k+1} = uvt$ and $u_{l+3,\dots,l+k+2} = v^2 t$; these 3 numbers are pairwise distinct because $u \neq \pm v$), so $\mathrm{card}\{u_{i_1,\dots,i_k} \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m+1\} > 3$: this is a contradiction.

Consequently, we have either $u_1 \neq u_2 = \cdots = u_{m+1}$ or $u_1 = \cdots = u_m \neq u_{m+1}$ so we have either $(\alpha, \beta) = (\binom{m}{k-1}, \binom{m}{k})$ or $(\alpha, \beta) = (\binom{m}{k}, \binom{m}{k-1})$. In any case, we have $\alpha \geqslant \min\{\binom{m}{k-1}, \binom{m}{k}\} = \binom{m}{k-1}$ (the last equality holds because $k \leqslant \frac{m+1}{2}$).

On the other hand, the list of eigenvalues (c repeated $\alpha + 1$ times, d_1, ..., $d_{\beta-1}$) of g is of the form

$$(v_{i_1,\dots,i_k} = v_{i_1} \cdots v_{i_k};\ 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m+1).$$

This list is the concatenation of the $\binom{m}{k-1}$ lists of the form

$$(v_{i_1,\dots,i_{k-1},j} = v_{i_1} \cdots v_{i_{k-1}} v_j;\ i_{k-1} < j \leqslant m+1)$$

indexed by $1 \leqslant i_1 < i_2 < \cdots < i_{k-1} \leqslant m$.

Since $\alpha+1>\binom{m}{k-1}$, we get that there exist $1\leqslant i_1<i_2<\cdots<i_{k-1}\leqslant m$ and $i_{k-1}<j,j'\leqslant m+1$ with $j\neq j'$ such that $c=v_{i_1,\dots,i_{k-1},j}=v_{i_1,\dots,i_{k-1},j'}$. So $v_j=v_{j'}$. Up to renumbering, we can assume that $v_1=v_2$.

For all $3\leqslant i_2<\cdots<i_k\leqslant m+1$, we obviously have $v_1v_{i_2}\cdots v_{i_k}=v_2v_{i_2}\cdots v_{i_k}$. Since c is the only eigenvalue of g with multiplicity >1, we necessary have, for all $3\leqslant i_2<\cdots<i_k\leqslant m+1$, $c=v_1v_{i_2}\cdots v_{i_k}$. Therefore, $v_3=\cdots=v_{m+1}$.

If $k>2$ then it is clear that any element of the list $(v_{i_1,\dots,i_k};\ 1\leqslant i_1<i_2<\cdots<i_k\leqslant m+1)$ occurs with multiplicity at least 2 : this is a contradiction.

If $k=2$ then any element of the list $(v_{i_1,i_2};\ 1\leqslant i_1<i_2\leqslant m+1)$ occurs with multiplicity at least 2 except, maybe, the term corresponding to $i_1=1$ and $i_2=2$; in particular, $c=v_1v_3=v_3v_4=v_3^2$ so $v_1=v_3$ so $v_1=\cdots=v_{m+1}$ hence $\operatorname{card}\{v_{i_1,\dots,i_k}\mid 1\leqslant i_1<i_2<\cdots<i_k\leqslant m+1\}=1$: this is a contradiction. □

Proposition 7. *The image of $Spin_{2m}(\mathbb{C})$ with $m\geqslant 4$ in its 1/2-spin representations does not contain a pair of semisimple elements satisfying $(\mathcal{P})$.*

Proof. Assume at the contrary that the image G of $Spin_{2m}(\mathbb{C})$ in one of its 1/2-spin representations contains a pair of semisimple elements f,g satisfying $(\mathcal{P})$.

Let us first treat the case of the 1/2-spin representation ρ_- whose weights have an odd number of minus signs.

Proposition 3 ensures that $Lie(G)=\mathfrak{g}$ contains an element u whose list of eigenvalues is $E_u=(\beta$ repeated α times, $-\alpha$ repeated β times$)$. There exist $\lambda_1,\dots,\lambda_m$ in $\mathbb{C}$ such that

$$E_u=(\epsilon_1\lambda_1+\cdots+\epsilon_m\lambda_m;\ (\epsilon_1,\dots,\epsilon_m)\in\{-1,1\}^m \text{ such that } \epsilon_1\cdots\epsilon_m=-1).$$

Since $(\lambda_1+\cdots+\lambda_m-2\lambda_1,\lambda_1+\cdots+\lambda_m-2\lambda_2,\dots,\lambda_1+\cdots+\lambda_m-2\lambda_m)$ is a sublist of E_u, we get that $\operatorname{card}\{\lambda_i\mid 1\leqslant i\leqslant m\}\leqslant 2$.

Assume that $\operatorname{card}\{\lambda_i\mid 1\leqslant i\leqslant m\}=1$ i.e. that $\lambda:=\lambda_1=\cdots=\lambda_m$. Note that $\lambda\neq 0$. If $m\geqslant 5$ then

$$\begin{aligned}(\lambda_1+\cdots+\lambda_m-2\lambda_1,\lambda_1+\cdots+\lambda_m-2\lambda_1-2\lambda_2-2\lambda_3,\\ \lambda_1+\cdots+\lambda_m-2\lambda_1-2\lambda_2-2\lambda_3-2\lambda_4-2\lambda_5)\\ =((m-2)\lambda,(m-6)\lambda,(m-10)\lambda)\end{aligned}$$

is a sublist of E_u made of three mutually distinct numbers : contradiction. If $m=4$ then E_u is $(2\lambda$ repeated 4 times, -2λ repeated 4 times$)$. In particular $\alpha=\beta=2^{m-2}$.

Assume that $\operatorname{card}\{\lambda_i\mid 1\leqslant i\leqslant m\}=2$ i.e. that $\lambda:=\lambda_1=\cdots=\lambda_i$ and $\lambda_{i+1}=\cdots=\lambda_m=:\mu$ for some $1\leqslant i<m$ and some distinct complex numbers λ, μ. Since $m\geqslant 4$, up to relabeling, we can assume that $i\geqslant 2$. Then

$$\begin{aligned}(\lambda_1+\cdots+\lambda_m-2\lambda_1,\lambda_1+\cdots+\lambda_m-2\lambda_m,\lambda_1+\cdots+\lambda_m-2\lambda_1-2\lambda_2-2\lambda_m)\\ =(\lambda_1+\cdots+\lambda_m-2\lambda,\lambda_1+\cdots+\lambda_m-2\mu,\lambda_1+\cdots+\lambda_m-2(2\lambda+\mu))\end{aligned}$$

is a sublist of E_u. Since $\lambda\neq\mu$, we have $\lambda_1+\cdots+\lambda_m-2\lambda\neq\lambda_1+\cdots+\lambda_m-2\mu$ so, since E_u is made of 2 elements, $\lambda_1+\cdots+\lambda_m-2(2\lambda+\mu)$ is either equal to $\lambda_1+\cdots+\lambda_m-2\lambda$ or to $\lambda_1+\cdots+\lambda_m-2\mu$ that is $\lambda=0$ or $\mu=-\lambda$. If $\lambda=0$ and $i<m-1$ then

$$\begin{aligned}(\lambda_1+\cdots+\lambda_m-2\lambda_1,\lambda_1+\cdots+\lambda_m-2\lambda_1-2\lambda_2-2\lambda_m,\lambda_1+\cdots+\lambda_m-2\lambda_1-2\lambda_{m-1}-2\lambda_m)\\ =((m-i)\mu,(m-i-2)\mu,(m-i-4)\mu)\end{aligned}$$

is a sublist of E_u made of three pairwise distinct complex numbers (because $\mu \neq \lambda = 0$) : this is impossible. If $\lambda = 0$ and $i = m-1$ then E_u has the form (μ repeated 2^{m-2} times, $-\mu$ repeated 2^{m-2} times) and hence $\alpha = \beta = 2^{m-2}$. If $\mu = -\lambda$ and if $i \geqslant 3$ then

$$\begin{aligned}(\lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_1 - 2\lambda_2 - 2\lambda_3, \lambda_1 + \cdots + \lambda_m - 2\lambda_m)\\ = (\lambda_1 + \cdots + \lambda_m - 2\lambda, \lambda_1 + \cdots + \lambda_m - 6\lambda, \lambda_1 + \cdots + \lambda_m + 2\lambda)\end{aligned}$$

is a sublist of E_u made of three pairwise distinct complex numbers : this is impossible. Similarly, the case $\lambda = -\mu$ and $m - i \geqslant 3$ is impossible. So, since $m \geqslant 4$, the only possibility compatible with $\lambda = -\mu$ is $m = 4$ and $i = 2$, in which case E_u is of the form (2λ repeated 4 times, -2λ repeated 4 times) so, in particular, $\alpha = \beta = 2^{m-2}$.

Hence, in any possible case, we have $\alpha = \beta = 2^{m-2}$.

On the other hand, since g belongs to G, its list of eigenvalues $E_g = (c$ repeated $\alpha+1$ times, $d_1, ..., d_{\beta-1})$ has the form

$$E_g = (\mu_1^{\epsilon_1} \cdots \mu_m^{\epsilon_m};\ (\epsilon_1, ..., \epsilon_n) \in \{-1,1\}^m \text{ such that } \epsilon_1 \cdots \epsilon_m = -1).$$

This list is the concatenation of the 2^{m-2} lists of the form

$$\left(\prod_{i \in \{1,\dots,m\} \setminus \{i_1,\dots,i_{p-1},i_p\}} \mu_i \cdot \prod_{i \in \{i_1,\dots,i_{p-1},i_p\}} \mu_i^{-1};\ i_{p-1} < i_p \leqslant m \right)$$

indexed by $1 \leqslant i_1 < \cdots < i_{p-1} \leqslant m-1$ with $1 \leqslant p \leqslant m$ odd number. Since $\alpha + 1 > 2^{m-2}$, we see that there exist $1 \leqslant i_1 < \cdots < i_{p-1} \leqslant m-1$ and $i_{p-1} < j, j' \leqslant m$ with $j \neq j'$ such that

$$c = \prod_{i \in \{1,\dots,m\} \setminus \{i_1,\dots,i_{p-1},j\}} \mu_i \cdot \prod_{i \in \{i_1,\dots,i_{p-1},j\}} \mu_i^{-1} = \prod_{i \in \{1,\dots,m\} \setminus \{i_1,\dots,i_{p-1},j'\}} \mu_i \cdot \prod_{i \in \{i_1,\dots,i_{p-1},j'\}} \mu_i^{-1}$$

so $\mu_j^2 = \mu_{j'}^2$ i.e. $\mu_j = \pm \mu_{j'}$. Up to renumbering, we can assume that $\mu_1 = \pm\mu_2$. So, for all $3 \leqslant k, l \leqslant m$ with $k \neq l$ (recall that $m \geqslant 4$), we have

$$\mu_1 \mu_2^{-1} \mu_k^{-1} \mu_l^{-1} \prod_{i \in \{1,\dots,m\} \setminus \{1,2,k,l\}} \mu_i = \mu_1^{-1} \mu_2 \mu_k^{-1} \mu_l^{-1} \prod_{i \in \{1,\dots,m\} \setminus \{1,2,k,l\}} \mu_i$$

so $\mu_1 \mu_2^{-1} \mu_k^{-1} \mu_l^{-1} \prod_{i \in \{1,\dots,m\} \setminus \{1,2,k,l\}} \mu_i$ occurs with multiplicity > 1 in E_g hence

$$c = \mu_1 \mu_2^{-1} \mu_k^{-1} \mu_l^{-1} \prod_{i \in \{1,\dots,m\} \setminus \{1,2,k,l\}} \mu_i.$$

Similarly, for all $3 \leqslant k, l \leqslant m$ with $k \neq l$,

$$c = \mu_1 \mu_2^{-1} \mu_k \mu_l \prod_{i \in \{1,\dots,m\} \setminus \{1,2,k,l\}} \mu_i.$$

So, for all $3 \leqslant k, l \leqslant m$ with $k \neq l$, $\mu_k^2 \mu_l^2 = 1$. If $m \geqslant 5$ then, for all $3 \leqslant k, l \leqslant m$, there exists $3 \leqslant k' \leqslant m$ such that $k' \neq k, l$, so $\mu_k^2/\mu_l^2 = (\mu_k^2 \mu_{k'}^2)/(\mu_l^2 \mu_{k'}^2) = 1/1 = 1$ i.e. $\mu_k^2 = \mu_l^2$. Therefore, we get $\mu_3^2 = \cdots = \mu_m^2 = \pm 1$. This implies that any element of

$$E_g = (\mu_1^{\epsilon_1} \cdots \mu_m^{\epsilon_m};\ (\epsilon_1, ..., \epsilon_n) \in \{-1,1\}^m \text{ such that } \epsilon_1 \cdots \epsilon_m = -1)$$

has multiplicity at least 2 because $\mu_1^{\epsilon_1} \cdots \mu_m^{\epsilon_m} = \mu_1^{\epsilon_1} \cdots \mu_{m-2}^{\epsilon_{m-2}} \mu_{m-1}^{-\epsilon_{m-1}} \mu_m^{-\epsilon_m}$; this is a contradicion. If $m = 4$ then it is easily seen that E_g is of the form $(\nu_1, \nu_1^{-1}, ..., \nu_{2^{m-2}}, \nu_{2^{m-2}}^{-1})$ (this is more generaly true if m is even). If $m = 4$ and if $c^{-1} = c$ then $\alpha + 1$ would be an even number (because if $c \in \{\nu_i, \nu_i^{-1}\}$ then $\{\nu_i, \nu_i^{-1}\} = \{c\}$ so the number $\alpha+1$ of occurences of c in $E_g =$

$(\nu_1, \nu_1^{-1}, ..., \nu_{2^{m-2}}, \nu_{2^{m-2}}^{-1})$ is even) so α would be an odd number and hence would not be an integral power of 2 : contradiction. If $m = 4$ and if $c^{-1} \neq c$ then, the fact that c occurs with multiplicity $\alpha + 1$ in $E_g = (\nu_1, \nu_1^{-1}, ..., \nu_{2^{m-2}}, \nu_{2^{m-2}}^{-1})$ implies that c^{-1} occurs with multiplicity $\alpha + 1 > 1$ in E_g, so $c = c^{-1}$ (because c is the unique eigenvalue of g with multiplicity > 1) : contradiction.

Let us now treat the case of the 1/2-spin representation ρ_+ whose weights have an even number of minus signs.

Since ρ_+ is dual to ρ_- when m is odd, it is sufficient to consider the case that m is even. As mentionned above, the fact that m is even implies that the list $E_f =$ (a repeated α times, b repeated β times) of eigenvalues of f is of the form $E_f = (\nu_1, \nu_1^{-1}, ..., \nu_{2^{m-2}}, \nu_{2^{m-2}}^{-1})$. We claim that $\alpha = \beta = 2^{m-2}$. Indeed, assume first that $a = a^{-1}$ i.e. that $a = \pm 1$. This implies that $b^{-1} \neq b$ and $b^{-1} \neq a$ because $b \neq \pm a = \pm 1$. So b^{-1} does not belong to $E_f =$(a repeated α times, b repeated β times) and hence b itself does not belong to $E_f = (\nu_1, \nu_1^{-1}, ..., \nu_{2^{m-2}}, \nu_{2^{m-2}}^{-1})$: contradiction. A similar argument shows that $b \neq b^{-1}$. Therefore $a \neq a^{-1}$ and $b \neq b^{-1}$. Since b belongs to $E_f = (\nu_1, \nu_1^{-1}, ..., \nu_{2^{m-2}}, \nu_{2^{m-2}}^{-1})$, b^{-1} belongs to E_f. Since $b^{-1} \neq b$, the only possibility is $a = b^{-1}$ and hence the number of occurences of a and of b in $E_f = (\nu_1, \nu_1^{-1}, ..., \nu_{2^{m-2}}, \nu_{2^{m-2}}^{-1})$ are the same. Hence $\alpha = \beta = 2^{m-2}$. Now, the same argument as for the case $m = 4$ treated above allows us to conclude. □

Proposition 8. *The image of $Spin_{2m+1}(\mathbb{C})$ in its spin representation does not contain a pair of semisimple elements satisfying $(\mathcal{P})$.*

Proof. Assume that the image G of $Spin_{2m+1}(\mathbb{C})$ in its spin representation contains a pair of semisimple elements f, g satisfying $(\mathcal{P})$.

Proposition 3 ensures that $Lie(G) = \mathfrak{g}$ contains an element u whose list of eigenvalues is $E_u =$ (β repeated α times, $-\alpha$ repeated β times). So there exist $\lambda_1, ..., \lambda_m$ in $\mathbb{C}$ such that

$$E_u = (\epsilon_1\lambda_1 + \cdots + \epsilon_m\lambda_m;\ (\epsilon_1, ..., \epsilon_m) \in \{-1, 1\}^m).$$

Since $(\lambda_1 + \cdots + \lambda_m - 2\lambda_1, \lambda_1 + \cdots + \lambda_m - 2\lambda_2, ..., \lambda_1 + \cdots + \lambda_m - 2\lambda_m)$ is a sublist of E_u, we get that $\mathrm{card}\{\lambda_i \mid 1 \leqslant i \leqslant m\} \leqslant 2$.

Assume that $\mathrm{card}\{\lambda_i \mid 1 \leqslant i \leqslant m\} = 1$ i.e. that $\lambda := \lambda_1 = \cdots = \lambda_m$. We have $\lambda \neq 0$. Then

$$\begin{aligned}(\lambda_1 + \cdots + \lambda_m, \lambda_1 + \cdots + \lambda_m - 2\lambda_1,& \\ \lambda_1 + \cdots + \lambda_m - 2\lambda_1 - 2\lambda_2, ..., &\lambda_1 + \cdots + \lambda_m - 2\lambda_1 - \cdots - 2\lambda_m) \\ &= ((m - 2j)\lambda;\ 0 \leqslant j \leqslant m)\end{aligned}$$

is a sublist of E_u made of $m + 1 > 2$ mutually distinct numbers : contradiction.

Assume that $\mathrm{card}\{\lambda_i \mid 1 \leqslant i \leqslant m\} = 2$ i.e. that $\lambda := \lambda_1 = \cdots = \lambda_i$ and $\lambda_{i+1} = \cdots = \lambda_m =: \mu$ for some $1 \leqslant i < m$ and some distinct complex numbers λ, μ. Up to renumbering, we can assume that $i \geqslant 2$. Using the fact that $(\pm\lambda \pm \lambda + \lambda_3 + \cdots + \lambda_m)$ is a sublist of E_u, we see that $\lambda = 0$. Moreover, $i = m - 1$ because, otherwise, $(\lambda_1 + \cdots + \lambda_{m-2} \pm \mu \pm \mu)$ would be a sublist of E_u made of 4 distinct elements (because $\mu \neq \lambda = 0$), which is impossible. So, E_u has the form (μ repeated 2^{m-1} times, $-\mu$ repeated 2^{m-1} times), hence $\alpha = \beta = 2^{m-1}$.

On the other hand, since g belongs to G, its list of eigenvalues $E_g =$ (c repeated $\alpha + 1$ times, d_1, ..., $d_{\beta-1}$) is of the form $E_g = (\mu_1^{\epsilon_1} \cdots \mu_m^{\epsilon_m};\ (\epsilon_1, ..., \epsilon_n) \in \{-1, 1\}^m)$. This list is the concatenation of the 2^{m-1} lists

$$\left(\prod_{i\in\{1,\ldots,m\}\setminus\{i_1,\ldots,i_{p-1},i_p\}}\mu_i\cdot\prod_{i\in\{i_1,\ldots,i_{p-1},i_p\}}\mu_i^{-1};\ i_{p-1}<i_p\leqslant m\right)$$

indexed by $1\leqslant i_1<\cdots<i_{p-1}\leqslant m-1$ with $0\leqslant p\leqslant m$. Since $\alpha+1>2^{m-1}$, we see that there exist $1\leqslant i_1<\cdots<i_{p-1}\leqslant m-1$ and $i_{p-1}<j,j'\leqslant m$ with $j\neq j'$ such that

$$\prod_{i\in\{1,\ldots,m\}\setminus\{i_1,\ldots,i_{p-1},j\}}\mu_i\cdot\prod_{i\in\{i_1,\ldots,i_{p-1},j\}}\mu_i^{-1}=\prod_{i\in\{1,\ldots,m\}\setminus\{i_1,\ldots,i_{p-1},j'\}}\mu_i\cdot\prod_{i\in\{i_1,\ldots,i_{p-1},j'\}}\mu_i^{-1}$$

so $\mu_j^2=\mu_{j'}^2$. Up to renumbering, we can assume that $\mu_1^2=\mu_2^2$. So, for all $3\leqslant k\leqslant m$, we have

$$\mu_1\mu_2^{-1}\mu_k^{-1}\prod_{i\in\{1,\ldots,m\}\setminus\{1,2,k\}}\mu_i=\mu_1^{-1}\mu_2\mu_k^{-1}\prod_{i\in\{1,\ldots,m\}\setminus\{1,2,k\}}\mu_i$$

so $\mu_1\mu_2^{-1}\mu_k^{-1}\prod_{i\in\{1,\ldots,m\}\setminus\{1,2,k\}}\mu_i$ occurs with multiplicity >1 in E_g hence

$$c=\mu_1\mu_2^{-1}\mu_k^{-1}\prod_{i\in\{1,\ldots,m\}\setminus\{1,2,k\}}\mu_i.$$

Similarly, we have, for all $3\leqslant k\leqslant m$,

$$c=\mu_1\mu_2^{-1}\prod_{i\in\{1,\ldots,m\}\setminus\{1,2\}}\mu_i.$$

Therefore, for all $3\leqslant k\leqslant m$, $\mu_k^2=1$ i.e. $\mu_k=\pm1$. This clearly implies that any element of $E_g=(\mu_1^{\epsilon_1}\cdots\mu_m^{\epsilon_m};(\epsilon_1,\ldots,\epsilon_n)\in\{-1,1\}^m)$ occurs with multiplicity at least 2 : this is a contradiction. □

Proof of Theorem 2. Since G acts irreducibly on E, we have $G=Z(G)^\circ G'$ where $Z(G)^\circ$ denotes the connected center of G and where G' denotes the derived subgroup of G. Moreover $Z(G)^\circ$ is included in the scalars, so $G'\subset G\subset\mathbb{C}^*G'$ and G' is a connected semisimple algebraic subgroup of $SL(E)$ which acts irreducibly on E. Let f,g be a pair of semisimple elements of G satisfying $(\mathcal{P})$. Then there exists $t_f,t_g\in\mathbb{C}^*$ such that $f'=t_ff$ and $g'=t_gg$ belong to G'. It is clear that f',g' is a pair of semisimple elements of G' satisfying $(\mathcal{P})$. Proposition 4 ensures that G' is simple. Proposition 3 and Theorem 5 ensure that G' is classical and that, as a representation of G', E is minuscule. In view of the classification of minuscule representations given after Theorem 5, the result follows from Propositions 6, 7 and 8. □

3. Additional results

We let E be a $\mathbb{C}$-vector space of finite dimension $n\geqslant 2$.

Theorem 9. *Let G be a connected algebraic subgroup of $GL(E)$. Assume that G contains a semisimple element u having n distinct eigenvalues and an element v permuting cyclically the n eigenspaces of u. Then the derived subgroup G' of G is either the image of $\prod_{i=1}^{l}SL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^{l}std$ for some $l\in\mathbb{N}^*$ and some pairwise prime numbers $n_1,n_2,\ldots,n_l>1$ or the image of $Sp(\mathbb{C}^{n_1})\times\prod_{i=2}^{l}SL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^{l}std$ for some $l\in\mathbb{N}^*$ and some pairwise prime numbers n_1 even $\geqslant 4$ and $n_2,\ldots,n_l>1$. Moreover, $G'\subset G\subset\mathbb{C}^*G'$.*

Proof. The fact that G contains a semisimple element u having n distinct eigenvalues and an element v permuting cyclically the corresponding eigenspaces implies that G acts irreducibly on

E. So $G' \subset G \subset \mathbb{C}^* G'$ and G' is a connected semisimple algebraic subgroup of $SL(E)$ acting irreducibly on E (see the beginning of the proof of Theorem 2 for details) and containing an element u' ($= \xi u$ for some $\xi \in \mathbb{C}^*$) having n distinct eigenvalues and an element v' ($= \zeta v$ for some $\zeta \in \mathbb{C}^*$) permuting cyclically the corresponding eigenspaces.

In virtue of Corollary 3.2.8 in [Kat87], in order to conclude the proof, it is sufficient to find a maximal torus $\mathcal{T}$ in G' and an element w in the normalizer $N(\mathcal{T})$ of $\mathcal{T}$ such that, as a representation of $\mathcal{T}$, E is the direct sum of n distinct characters which are cyclically permuted by the conjugation action of w. But, since u' is a semisimple element of G', it is contained in a maximal torus $\mathcal{T}$ of G'. By commutativity, this maximal torus leaves invariant the n eigenspaces of u'. It is now clear that $\mathcal{T}$ and $w = v' \in N(\mathcal{T})$ have the required properties. □

Theorem 10. *Let G be a connected algebraic subgroup of $GL(E)$ which acts irreducibly on E. If G contains a semisimple element f whose list of eigenvalues is of the form (a, b repeated $n-1$ times) for some $a, b \in \mathbb{C}^*$ such that $a \neq \pm b$ then the derived subgroup G' of G is $SL(E)$. Furthermore, $G' \subset G \subset \mathbb{C}^* G'$.*

Proof. Since G acts irreducibly on E, $G' \subset G \subset \mathbb{C}^* G'$ and G' is a connected semisimple algebraic subgroup of $SL(E)$ acting irreducibly on E (see the beginning of the proof of Theorem 2 for details) and containing $f' = tf$ for some $t \in \mathbb{C}^*$. Proposition 3 ensures that the semisimple Lie algebra $\mathfrak{g}'$ of G' contains a semisimple morphism whose list of eigenvalues is ($n-1$, -1 repeated $n-1$ times). Since G' acts irreducibly on E, so do $\mathfrak{g}'$. B. Kostant's characterization of $\mathfrak{sl}(E)$ given in [Kos58] ensures that $\mathfrak{g}' = \mathfrak{sl}(E)$ and hence $G' = SL(E)$. □

Part II. Applications to q-difference Galois theory

4. Reminder and complements

4.1 q-Difference modules and systems

Let (K, σ) be a difference field and let $\mathcal{D}_{(K,\sigma)}$ be the noncommutative algebra $K\langle \sigma, \sigma^{-1} \rangle$ of noncommutative Laurent polynomials with coefficients in K satisfying the relation $\sigma a = \sigma(a)\sigma$ for any $a \in K$. The full subcategory of the category of $\mathcal{D}_{(K,\sigma)}$-modules whose objects are the $\mathcal{D}_{(K,\sigma)}$-modules of finite length is denoted by $\mathcal{E}_{(K,\sigma)}$. It is a K^σ-linear abelian tensor category, where $K^\sigma = \{a \in K \mid \sigma(a) = a\}$ is the subfield of constants of (K, σ).

It is sometimes convenient to choose specific bases. We introduce the category $\mathcal{E}'_{(K,\sigma)}$, which is tensor equivalent to $\mathcal{E}_{(K,\sigma)}$, described as follows : its objects are difference systems $(\sigma Y = AY)$ where $A \in GL_n(K)$ and its morphisms from $(\sigma Y = AY)$, $A \in GL_n(K)$, to $(\sigma Y = BY)$, $B \in GL_m(K)$, are the matrices $F \in M_{m,n}(K)$ such that $BF = \sigma(F)A$.

We refer to Chapter 1 (especially to §1.4) of [PS97] or to §1.1 of [Sau04] for details; in particular the tensor product, denoted by $\otimes$, and the dual, denoted by $\cdot^\vee$, are defined there.

We denote by $\mathbb{C}\{z\}$ the local ring of germs of analytic functions at 0 and by $\mathbb{C}(\{z\})$ its field of fractions, by $\mathbb{C}[[z]]$ the local ring of formal series in z and by $\mathbb{C}((z))$ its field of fractions.

For $K = \mathbb{C}(z)$, $\mathbb{C}(\{z\})$ or $\mathbb{C}((z))$, we denote by σ_q the automorphism of K defined by $\sigma_q(a(z)) = a(qz)$. Then (K, σ_q) is a difference field with field of constants $\mathbb{C}$.

For any $N \in \mathbb{N}^*$, we set $q_N = q^{1/N}$, we denote by $[N] : \mathbb{C}^* \to \mathbb{C}^*$ the étale morphism $z \mapsto$

z^N and by $[N]^* : \mathcal{E}_{(\mathbb{C}((z)),\sigma_q)} \to \mathcal{E}_{(\mathbb{C}((z_N)),\sigma_{q_N})}$ the corresponding ramification functor (explicitly defined in §1.4 of [DV02] for instance).

4.2 Slopes

Our main reference for slopes theory is [Sau04] where it is assumed that $|q| > 1$ (in opposition with our hypothesis $|q| < 1$). The slopes defined in the present paper are the opposite of those defined in [Sau04]; but since we use only the formal part of *loc. cit.*, this has no impact on what follows.

The Newton polygon $\mathcal{N}(L)$ of $L = \sum_i a_i \sigma_q^i \in \mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}$ is the convex hull in $\mathbb{R}^2$ of $\{(i,j) \mid i \in \mathbb{Z}$ and $j \geqslant v_z(a_i)\}$ where v_z denotes the z-adic valuation on $\mathbb{C}((z))$. This polygon is made of two vertical half lines and of k vectors (r_1, d_1), ..., $(r_k, d_k) \in \mathbb{N}^* \times \mathbb{Z}$ having pairwise distinct slopes, called the slopes L. For any $i \in \{1, ..., k\}$, r_i is called the multiplicity of the slope $\frac{d_i}{r_i}$.

Let M be an object of $\mathcal{E}_{(\mathbb{C}((z)),\sigma_q)}$. The cyclic vector lemma (Lemma 1.3.1 in [DV02]) ensures that there exists $L \in \mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}$ such that $M \cong \mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}/\mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}L$. One can define the slopes of M as the slopes of L and the multiplicity of a slope λ of M as the multiplicity of λ as a slope of L. This definition is independent of the chosen L (Théorème et définition 2.2.5 in [Sau04]). An object M of $\mathcal{E}_{(\mathbb{C}((z)),\sigma_q)}$ is pure isoclinic if it has a unique slope.

For instance, for $a \in \mathbb{C}((z))^\times$, the Newton polygon of $M = \mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}/\mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}(\sigma_q - a)$ is the convex subset of $\mathbb{R}^2$ delimited by the vertical half lines $\{0\} \times \mathbb{R}^+$ and $\{1\} \times [v_z(a), +\infty[$ and by the segment from $(0,0)$ to $(1, v_z(a))$. So M is pure isoclinic with slope $v_z(a)$. Another example : $M = \mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}/\mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}(qz\sigma_q^2 - (1+z)\sigma_q + 1)$ has two slopes, namely 0 and 1, both with multiplicity 1.

4.3 Galois groups

Let $\mathcal{E}$ be a tannakian category over $\mathbb{C}$ and let ω be a $\mathbb{C}$-fiber functor on $\mathcal{E}$. For any object M of $\mathcal{E}$, we will denote by $\langle M \rangle$ the tannakian category generated by M in $\mathcal{E}$ and by $Gal(M, \omega)$ the complex linear algebraic group $Aut^{\otimes}(\omega_{|\langle M \rangle})$. The choice of a specific fiber functor has no consequence : since $\mathbb{C}$ is algebraically closed, any two $\mathbb{C}$-fiber functors on $\mathcal{E}$ are isomorphic. For the theory of tannakian categories, we refer to P. Deligne and J. S. Milne's paper [DM81].

4.3.1 *Connectedness* Let M be an object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$.

The categories $\mathcal{E}_{(\mathbb{C}((z)),\sigma_q)}$ and $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ are neutral tannakian over $\mathbb{C}$ (§1.4 of [PS97]). Let $\widehat{\omega}$ be a $\mathbb{C}$-fiber functor on $\mathcal{E}_{(\mathbb{C}((z)),\sigma_q)}$. The formalization functor $\widehat{\cdot} : \mathcal{E}_{(\mathbb{C}(z),\sigma_q)} \to \mathcal{E}_{(\mathbb{C}((z)),\sigma_q)}$ being an exact and faithful $\otimes$-functor, $\omega = \widehat{\omega} \circ \widehat{\cdot}$ is a $\mathbb{C}$-fiber functor on $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$.

The following result is Proposition 12.2 of [PS97] (compare with O. Gabber's Proposition 1.2.5 in [Kat87]).

PROPOSITION 11. *The natural closed immersion $Gal(\widehat{M}, \widehat{\omega}) \hookrightarrow Gal(M, \omega)$ of the local formal Galois group $Gal(\widehat{M}, \widehat{\omega})$ of M at 0 into the Galois group $Gal(M, \omega)$ of M induces a surjective morphism $Gal(\widehat{M}, \widehat{\omega})/Gal(\widehat{M}, \widehat{\omega})^\circ \twoheadrightarrow Gal(M, \omega)/Gal(M, \omega)^\circ$.*

COROLLARY 12. *If $Gal(\widehat{M}, \widehat{\omega})$ is connected then $Gal(M, \omega)$ is connected.*

We give an additional corollary for later use.

COROLLARY 13. *Assume that M satisfies $(\mathscr{H}1)$ and that it is regular singular at ∞ with exponents in $\{c \in \mathbb{C}^* \mid c^{n'} \in q^{\mathbb{Z}}\}$ for some $n' \in \mathbb{Z}^*$ prime to the rank n of M. Then $Gal(M, \omega)$ is* *connected.*

Proof. We set $G = Gal(M,\omega)$ and we denote by G_0 and G_∞ the local formal Galois groups of M at 0 and ∞ respectively. Proposition 16 below and Example 5.6 in §5.2 of [PR07] ensure that $G_0/G_0^\circ \cong (\mathbb{Z}/n^2\mathbb{Z})$. Proposition 11 implies that the order of any element of G/G° divides n^2. Moreover, using Chapter 12 of [PS97] or §2.2 of [Sau03] we see that the order of any element of G_∞/G_∞° divides n'. Proposition 11 ensures that the same property holds for the elements of G/G°. Therefore, G/G° is trivial. □

4.3.2 *Lie-irreducibility*

Definition 14. *We say that a list $c_1,...,c_n$ of nonzero complex numbers is q-Kummer induced if there exist a divisor $d \geqslant 2$ of n and a permutation ν of $\{1,...,n\}$ such that, for all $i \in \{1,...,n\}$, $c_i = q^{\frac{1}{d}} c_{\nu(i)}$ mod. $q^{\mathbb{Z}}$.*

Proposition 15. *If M is an irreducible object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$, of rank n, regular singular at ∞ with non q-Kummer induced exponents $c_1,...,c_n \in q^{\mathbb{R}}$ then M is Lie-irreducible i.e. the action of $Gal(M,\omega)^\circ$ on $\omega(M)$ is irreducible.*

Proof. For all $i \in \{1,...,n\}$, let $\gamma_i \in \mathbb{R}$ be such that $c_i = q^{\gamma_i}$. It follows from Chapter 12 of [PS97] or §2.2 of [Sau03] that the local formal Galois group of M at ∞ is generated, as an algebraic group, by its neutral component and by a semisimple morphism f with list of eigenvalues $e^{2\pi i\gamma_1},...,e^{2\pi i\gamma_n}$. Proposition 11 implies that $G = Gal(M,\omega)$ is generated, as an algebraic group, by G° and f. So, since the action of G on $\omega(M)$ is irreducible, its restriction to the abstract group H generated by G° and f is still irreducible. Assume that M is not Lie-irreducible and let $V \neq \{0\}, \omega(M)$ be a minimal invariant subspace of $\omega(M)$ for the action of G°. For all $k \in \mathbb{Z}$, f^kV is an invariant subspace of $\omega(M)$ for the action of G° because G° is a normal subgroup of G. Therefore $\sum_{k\in\mathbb{Z}} f^kV$ is an invariant subspace of $\omega(M)$ for the action of H and hence $\omega(M) = \sum_{k\in\mathbb{Z}} f^kV$. Let d be the least integer $\geqslant 2$ such that $\omega(M) = \sum_{k=0}^{d-1} f^kV$. It is easily seen that $\omega(M) = \bigoplus_{k=0}^{d-1} f^kV$. This implies that f and $e^{\frac{2\pi i}{d}} f$ are conjugate. Considering the eigenvalues of f, we see that there exists a permutation ν of $\{1,...,n\}$ such that, for all $i \in \{1,...,n\}$, $e^{2\pi i\gamma_i} = e^{\frac{2\pi i}{d}} e^{2\pi i\gamma_{\nu(i)}}$ i.e. $c_i = q^{\frac{1}{d}} c_{\nu(i)}$ mod. $q^{\mathbb{Z}}$. Since $n = d \dim V$, d divides n. □

5. Main theorem in the one slope case

Proposition 16 (Reformulation of $(\mathscr{H}1)$). *Let $\widehat{M}$ be an object of $\mathcal{E}_{(\mathbb{C}((z)),\sigma_q)}$ of rank $n \geqslant 2$. The following properties are equivalent :*

- *(a) $\widehat{M}$ is irreducible (i.e. simple);*
- *(b) $\widehat{M} \cong \widehat{M}_q(n,m,a) := \mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}/\mathcal{D}_{(\mathbb{C}((z)),\sigma_q)}(\sigma_q^n - q_n^{m\frac{n(n-1)}{2}} az^m)$ for some $m \in \mathbb{Z}^*$ prime to n and for some $a \in \mathbb{C}^*$;*
- *(c) $\widehat{M}$ satisfies $(\mathscr{H}1)$.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ is Proposition 1.3 of [PR07] and $(b) \Rightarrow (c)$ is obvious. It remains to prove $(c) \Rightarrow (a)$. Assume that $\widehat{M}$ satisfies $(\mathscr{H}1)$. Let $\widehat{M}'$ be a nonzero subobject of $\widehat{M}$. Then $\widehat{M}'$ is pure isoclinic with slope μ (Théorème 2.3.1 of [Sau04]). In order to prove that $\widehat{M} = \widehat{M}'$, it is sufficient to prove that the rank n' of $\widehat{M}'$ is $\geqslant n$. This is indeed the case because $n'\mu$ has to be a relative integer (immediate from the definition of the slopes of $\widehat{M}'$). □

Lemma 17. *If $M_1,...,M_l$ are objects of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of ranks >1 such that $M=M_1\otimes\cdots\otimes M_l$ satisfies $(\mathscr{H}1)$ then $M_1,...,M_l$ satisfy $(\mathscr{H}1)$.*

Proof. Let $n,n_1,...,n_l$ be the respective ranks of $M,M_1,...,M_l$. Note that $n=n_1\cdots n_l$. Since $M=M_1\otimes\cdots\otimes M_l$ is pure isoclinic at 0 with slope $\mu=m/n$, $M_1,...,M_l$ are pure isoclinic at 0 with respective slopes $\mu_1,...,\mu_l$ such that $\mu=\mu_1+\cdots+\mu_l$ (Théorème 2.3.1. of [Sau04]). For any $i\in\{1,...,l\}$, μ_i has the form m_i/n_i for some $m_i\in\mathbb{Z}$. The equalities $m/n=\mu=\mu_1+\cdots+\mu_l=m_1/n_1+\cdots+m_l/n_l$ and $n=n_1\cdots n_l$ together with the fact that m is prime to n imply that, for any $i\in\{1,...,l\}$, m_i is prime to n_i. □

Lemma 18. *Let M be an object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of rank n satisfying $(\mathscr{H}1)$. Assume that $M\cong M_1\otimes M_2$ for some objects M_1,M_2 of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ with respective ranks $n_1>1$ and n_2. If $M_1^\vee\cong U_1\otimes M_1$ for some rank one object U_1 of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ then $n_1=2$.*

Proof. We have $M^\vee\cong M_1^\vee\otimes M_2^\vee\cong U_1\otimes M_1\otimes M_2^\vee$. Lemma 17 ensures that both M_1 and M_2 satisfy $(\mathscr{H}1)$. Denoting by μ_1,μ_2 and ν the respective slopes of M_1,M_2 and U_1 at 0, we get that the unique slope $-\mu_1-\mu_2$ of $M^\vee$ at 0 is equal to the unique slope $\nu+\mu_1-\mu_2$ of $U_1\otimes M_1\otimes M_2^\vee$ at 0. So $2\mu_1=-\nu\in\mathbb{Z}$ (because U_1 has rank 1). Since M_1 satisfies $(\mathscr{H}1)$, we get $n_1=2$. □

This following result is (essentially) proved by M. van der Put and M. Singer in §1.2 of [PS97]. Following the referees' suggestions, we shall give a sketch of proof.

Proposition 19. *If $(\sigma_qY=AY)$ is an object of $\mathcal{E}'_{(\mathbb{C}(z),\sigma_q)}$ of rank n having a connected Galois group G then there exists an object $(\sigma_qY=BY)$ of $\mathcal{E}'_{(\mathbb{C}(z),\sigma_q)}$ isomorphic to $(\sigma_qY=AY)$ such that B belongs to $G(\mathbb{C}(z))$.*

Proof. We keep and specialize to our situation the notations of §1.2 of [PS97] : $k=\mathbb{C}(z)$, $\phi=\sigma_q$, $C=\mathbb{C}$. The Galois group G can be seen as the group of k-automorphisms which commute with ϕ of some Picard-Vessiot ring R over k of $(\sigma_qY=AY)$. We consider the algebraic group $G_k=G\otimes_{\mathbb{C}}k$ in $GL_{n;k}$. Also, we consider the reduced algebraic subset Z of $GL_{n;k}$ corresponding to R. Theorem 1.13 in [PS97] ensures that Z/k has a natural structure of G-torsor : the morphism $Z\times_k G_k\rightarrow G_k\times_k G_k$ given by $(z,g)\mapsto(zg,g)$ is an isomorphism. But $k=\mathbb{C}(z)$ is a $\mathcal{C}^1$-field and G is connected so Corollary 1.18 in *loc. cit.* and the discussion following this result ensure that Z/k is a trivial G-torsor. Therefore $Z(k)$ is non empty and, for $U\in Z(k)$, we have $Z(\overline{k})=UG(\overline{k})$. We now use the τ-invariance of Z (the map τ is defined at the beginning of §1.2 of *loc. cit.* and the τ-invariance property is Lemma 1.10) : since $\tau Z(\overline{k})=Z(\overline{k})$, we have $\tau(UG(\overline{k}))=UG(\overline{k})$ i.e. $A^{-1}\phi(U)G(\overline{k})=UG(\overline{k})$ (we have used the fact that $\tau(UG(\overline{k}))=A^{-1}\phi(U)\phi G(\overline{k})=A^{-1}\phi(U)G(\overline{k})$). Hence $\phi(U)^{-1}AU\in G(k)$. □

Theorem 20 (Main Theorem in the one slope case). *Let M be an object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of rank n having a connected Galois group and satisfying $(\mathscr{H}1)$. Then $Gal(M,\omega)$ is the image of $\prod_{i=1}^l GL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^l std$ for some $l\in\mathbb{N}^*$ and some pairwise prime numbers $n_1,...,n_l>1$ such that $n=n_1\cdots n_l$.*

Proof. We set $G=Gal(M,\omega)$. Proposition 16 and Example 5.6 in §5.2 of [PR07] show that the hypotheses of Theorem 9 are satisfied by G and hence that the derived subgroup G' of G is either the image of $\prod_{i=1}^l SL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^l std$ for some $l\in\mathbb{N}^*$ and some pairwise prime numbers $n_1,n_2,...,n_l>1$ or the image of $Sp(\mathbb{C}^{n_1})\times\prod_{i=2}^l SL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^l std$ for some $l\in\mathbb{N}^*$ and some pairwise prime numbers n_1 even $\geqslant 4$ and $n_2,...,n_l>1$ and that $G'\subset G\subset\mathbb{C}^*G'$. Since $\det(M)$ is a rank one irregular object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$, its Galois group is $\mathbb{C}^*$, so $G=\mathbb{C}^*G'$. Therefore, G is either the image of $\prod_{i=1}^{l} GL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^{l} std$ or the image of $\mathbb{C}^* Sp(\mathbb{C}^{n_1}) \times \prod_{i=2}^{l} GL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^{l} std$. It remains to exclude the second case. Assume at the contrary that G is $\mathbb{C}^* Sp(\mathbb{C}^{n_1}) \times \prod_{i=2}^{l} GL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^{l} std$. Using Proposition 19, we would get $M \cong M_1 \otimes \cdots \otimes M_l$ for some objects $M_1, ..., M_l$ of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ where M_1 is such that $M_1^\vee \cong U_1 \otimes M_1$ for some rank one object U_1 of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$. Lemma 18 would imply that $n_1 = 2$. This is a contradiction. $\square$

Definition 21. *An object M of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ is $\otimes$-decomposable if there exist two objects M_1, M_2 of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of ranks $\geqslant 2$ such that $M \cong M_1 \otimes M_2$.*

Corollary 22. *Let M be an object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of rank n having a connected Galois group and satisfying $(\mathscr{H}1)$. If M is $\otimes$-indecomposable then $Gal(M,\omega)$ is $GL(\omega(M))$.*

Proof. Direct consequence of Theorem 20 and Proposition 19. $\square$

6. Main theorem in the two slopes case

Lemma 23. *Let M be an object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of rank $n \geqslant 3$ satisfying $(\mathscr{H}2)$. Then $Gal(M,\omega)$ is neither a subgroup of $\mathbb{C}^* SO(\omega(M))$ nor a subgroup of $\mathbb{C}^* Sp(\omega(M))$ (for some bilinear forms).*

Proof. Let H be either $SO(\omega(M))$ or $Sp(\omega(M))$ and set $G = \mathbb{C}^* H$. Assume that $Gal(M,\omega)$ is a subgroup of G. Let ρ be the representation of $Gal(M,\omega)$ corresponding to M by tannakian duality. Let χ be the character of G defined, for any $t \in \mathbb{C}^*$ and for any $A \in H$, by $\chi(tA) = t^2$. The dual $\rho^\vee$ of ρ is conjugated to $\rho \otimes (\chi^{-1} \circ \rho)$. Therefore, there exists a rank one object U of $\langle M \rangle$ such that $M^\vee \cong U \otimes M$. But, at 0, (see Théorème 2.3.1 of [Sau04]) $M^\vee$ has two slopes, namely 0 with multiplicity $n-r$ and $-\mu$ with multiplicity r, and $U \otimes M$ has two slopes, namely ν with multiplicity $n-r$ and $\mu+\nu$ with multiplicity r where $\nu \in \mathbb{Z}$ denotes the unique slope of U. The only possibility is $\mu = 0$: contradiction. $\square$

Theorem 24 (Main Theorem in the 2 slopes case). *Let M be an irreducible object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of rank n having a connected Galois group and satisfying $(\mathscr{H}2)$. Then $Gal(M,\omega) = GL(\omega(M))$.*

Proof. The formal slopes decomposition (Théorème 3.1.7 in [Sau04]) ensures that $\widehat{M} \cong \widehat{M_0} \oplus \widehat{M_\mu}$ where $\widehat{M_0}$ is a regular singular object of $\mathcal{E}_{(\mathbb{C}((z)),\sigma_q)}$ with exponents in $q^{\mathbb{R}}$ and $\widehat{M_\mu}$ is a pure isoclinic object of $\mathcal{E}_{(\mathbb{C}((z)),\sigma_q)}$ of slope μ and of rank r. Proposition 16 ensures that $\widehat{M_\mu} \cong \widehat{M_q}(r,m,a)$ for some $a \in \mathbb{C}^*$ so $\widehat{M} \cong \widehat{M_0} \oplus \widehat{M_q}(r,m,a)$. So $Gal(M,\omega)$ contains, with respect to a suitable basis, $I_{n-r} \oplus \mathbb{C}^* I_r$ and $I_{n-r} \oplus \mathrm{diag}(1,\zeta,...,\zeta^{r-1})$ where ζ is a primitive rth root of 1 (consequence of §5 of [PR07] or of §3.2 of [RS07] applied to $[r]^*\widehat{M} \cong [r]^*\widehat{M_0} \bigoplus_{c^r=a} \widehat{M}_{q_r}(1,0,c) \otimes \widehat{M}_{q_r}(1,m,1)$). If $r \geqslant 2$, Theorem 2 implies that $G \subset Gal(M,\omega) \subset \mathbb{C}^* G$ with $G = SL(\omega(M))$, $SO(\omega(M))$ or $Sp(\omega(M))$. Note that the Galois group of $\det(M)$ is $\mathbb{C}^*$ because $\det(M)$ is irregular of rank one, so $Gal(M,\omega)$ is $\mathbb{C}^* G$. Lemma 23 leads to the conclusion. If $r = 1$, the result follows from Theorem 10. $\square$

7. Some computations of Galois groups

7.1 Generalized q-hypergeometric equations with two slopes

We keep the notations of §1 (and the hypothesis that $r > s$) for the generalized q-hypergeometric operator with parameters $\underline{a} = (a_1, ..., a_r) \in (q^{\mathbb{R}})^r$, $\underline{b} = (b_1, ..., b_s) \in (q^{\mathbb{R}})^s$ and $\lambda \in \mathbb{C}^*$ and we set

$$\mathcal{H}_q(\underline{a};\underline{b};\lambda) = \mathcal{D}_{(\mathbb{C}(z),\sigma_q)}/\mathcal{D}_{(\mathbb{C}(z),\sigma_q)}\mathcal{L}_q(\underline{a};\underline{b};\lambda).$$

If $s>0$ then $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ satisfies ($\mathscr{H}2$) (its slopes at 0 are 0 with multiplicity s and $\frac{1}{r-s}$ with multiplicity $r-s$). Theorem 24 leads to :

Theorem 25. *The general linear group $GL(\mathbb{C}^r)$ is the unique connected algebraic group occurring as Galois group of some irreducible generalized q-hypergeometric module $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ with parameters $\underline{a}=(a_1,...,a_r)\in(q^{\mathbb{R}})^r$, $\underline{b}=(b_1,...,b_s)\in(q^{\mathbb{R}})^s$ with $r>s>0$.*

We now turn to explicit computations of q-hypergeometric Galois groups. For all $i\in\{1,...,r\}$, we denote by α_i the unique element of $\mathbb{R}$ such that $a_i=q^{\alpha_i}$.

Theorem 26. *Assume that $s>0$, that for all $(i,j)\in\{1,...,r\}\times\{1,...,s\}$, $\beta_j-\alpha_i\notin\mathbb{Z}$ and that the algebraic group generated by $\mathrm{diag}(e^{2\pi i\alpha_1},...,e^{2\pi i\alpha_r})$ is connected then $Gal(\mathcal{H}_q(\underline{a};\underline{b};\lambda),\omega)=GL(\mathbb{C}^r)$.*

Proof. Since, for all $(i,j)\in\{1,...,r\}\times\{1,...,s\}$, $\beta_j-\alpha_i\notin\mathbb{Z}$, $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible (same arguments as in §5.1 of [Roq11]). Moreover, $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is regular singular at ∞ with exponents $a_1,...,a_r$. It follows easily from Chapter 12 of [PS97] or §2.2 of [Sau03] that if the algebraic group generated by $\mathrm{diag}(e^{2\pi i\alpha_1},...,e^{2\pi i\alpha_r})$ is connected then the local formal Galois group of $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ at ∞ is connected and hence, in virtue of (the variant at ∞ of) Corollary 12, $Gal(\mathcal{H}_q(\underline{a};\underline{b};\lambda),\omega)$ is connected. Theorem 25 leads to the desired result. □

For instance, the algebraic group generated by $\mathrm{diag}(e^{2\pi i\alpha_1},...,e^{2\pi i\alpha_r})$ is connected if $\underline{a}\in\left(q^{\mathbb{Z}}\right)^r$ or if $\alpha_1,...,\alpha_r$ are $\mathbb{Z}$-linearly independent.

7.2 q-Kloosterman equations

We maintain the notations of §1 for the q-Kloosterman operators and we set

$$\mathcal{K}l_q(U,V)=\mathcal{D}_{(\mathbb{C}(z),\sigma_q)}/\mathcal{D}_{(\mathbb{C}(z),\sigma_q)}Kl_q(U,V).$$

Note that $\mathcal{K}l_q(U,V)$ is pure isoclinic at 0 with slope $\deg V/\deg U$. In particular, if $\deg U$ is prime to $\deg V$ then $\mathcal{K}l_q(U,V)$ satisfies ($\mathscr{H}1$). Theorem 20 and Corollary 22 lead to :

Theorem 27. *Let G be a connected algebraic group occurring as Galois group of some q-Kloosterman module $\mathcal{K}l_q(U,V)$ such that $\deg U$ is prime to $\deg V$. Then G is the image of $\prod_{i=1}^l GL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^l std$ for some $l\in\mathbb{N}^*$ and some pairwise prime numbers $n_1,...,n_l>1$ such that $\deg U=n_1\cdots n_l$. If, moreover, $\mathcal{K}l_q(U,V)$ is $\otimes$-indecomposable then G is $GL(\mathbb{C}^{\deg U})$.*

We denote by $c_1,...,c_{\deg U}$ the roots of $X^u(U(X^{-1})+V(0))\in\mathbb{C}[X]$. For all $i\in\{1,...,\deg U\}$, we denote by (u_i,α_i) the unique element of $\mathbb{U}\times\mathbb{R}$ such that $c_i=u_iq^{\alpha_i}$.

Theorem 28. *If $\deg U$ is prime to $\deg V$ and if the algebraic group generated by $\mathrm{diag}(u_1,...,u_{\deg U})$ and $\mathrm{diag}(e^{2\pi i\alpha_1},...,e^{2\pi i\alpha_{\deg U}})$ is connected then $Gal(\mathcal{K}l_q(U,V),\omega)$ is the image of $\prod_{i=1}^l GL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^l std$ for some $l\in\mathbb{N}^*$ and some pairwise prime numbers $n_1,...,n_l>1$ such that $\deg U=n_1\cdots n_l$. If, moreover, $\mathcal{K}l_q(U,V)$ is $\otimes$-indecomposable then $Gal(\mathcal{K}l_q(U,V),\omega)$ is $GL(\mathbb{C}^{\deg U})$.*

Proof. Note that $\mathcal{K}l_q(U,V)$ is regular singular at ∞ with exponents $c_1,...,c_{\deg U}$. It follows easily from Chapter 12 of [PS97] or §2.2 of [Sau03] that if the algebraic group generated by $\mathrm{diag}(u_1,...,u_{\deg U})$ and $\mathrm{diag}(e^{2\pi i\alpha_1},...,e^{2\pi i\alpha_{\deg U}})$ is connected then the local formal Galois group of $\mathcal{K}l_q(U,V)$ at ∞ is connected and hence, in virtue of (the variant at ∞ of) Corollary 12, $Gal(\mathcal{K}l_q(U,V),\omega)$ is connected. Theorem 27 leads to the desired result. □

Note that a q-Kloosterman module $\mathcal{K}l_q(U,V)$ with $\deg U$ prime to $\deg V$ is not necessarily $\otimes$-indecomposable. For instance, $\mathcal{K}l_q(X^6, -(1+q^{-4}X)(1+q^{-3}X)(1+q^{-2}X)(1+X)^2) \cong \mathcal{K}l_q(X^2, -(1+X)) \otimes \mathcal{K}l_q(X^3, -(1+X))$.

8. A $\otimes$-indecomposability criterion and application to q-Kloosterman operators (including $\mathcal{H}_q(\underline{a};\emptyset;\lambda)$)

8.1 A $\otimes$-indecomposability criterion

Slopes theory leads to a simple proof of the $\otimes$-indecomposability of the Kloosterman differential modules with bidegree (u,v) such that u is prime to v; see [Kat87]. In contrast, we gave at the end of §7.2 an example of $\otimes$-decomposable q-Kloosterman module $\mathcal{K}l_q(U,V)$ with $\deg U$ prime to $\deg V$. In this §, we propose an obstruction to $\otimes$-decomposability (Theorem 31 below) coming from residues at points of $\mathbb{C}^*$ of intrinsic Birkhoff matrices. In [Roq11], we used related ideas in order to obtain an avatar of the usual monodromy for the generalized q-hypergeometric equations.

We first work with q-difference systems.

DEFINITION 29 (Property (H_q)). *We say that an object $(\sigma_q Y = AY)$ of $\mathcal{E}'_{(\mathbb{C}(z),\sigma_q)}$ of rank n satisfies the condition (H_q) if :*

1) *there exists $z_0 \in \mathbb{C}^*$ such that A is analytic at any point of $q^{\mathbb{Z}}z_0$, $A(z_0)$ has rank $n-1$ and, for all $k \in \mathbb{Z}^*$, $A(q^k z_0) \in GL_n(\mathbb{C})$;*
2) *$(\sigma_q Y = AY)$ is pure isoclinic at both 0 and ∞.*

LEMMA 30. *Let $(\sigma_q Y = AY)$ be an object of $\mathcal{E}'_{(\mathbb{C}(z),\sigma_q)}$ of rank n. If $(\sigma_q Y = AY)$ is pure isoclinic at 0 and ∞ with integral slopes, respectively denoted by μ_0 and μ_∞, then :*

i) *There exist $A^{(0)} \in GL_n(\mathbb{C})$ and $F^{(0)} \in GL_n(\mathbb{C}(\{z\}))$ such that $F^{(0)}$ is an isomorphism in $\mathcal{E}'_{(\mathbb{C}((z)),\sigma_q)}$ from $(\sigma_q Y = z^{\mu_0}A^{(0)}Y)$ to $(\sigma_q Y = AY)$. Similarly, there exist $A^{(\infty)} \in GL_n(\mathbb{C})$ and $F^{(\infty)} \in GL_n(\mathbb{C}(\{z^{-1}\}))$ such that $F^{(\infty)}$ is an isomorphism in $\mathcal{E}'_{(\mathbb{C}((z^{-1})),\sigma_{q_r})}$ from $(\sigma_q Y = z^{\mu_\infty}A^{(\infty)}Y)$ to $(\sigma_q Y = AY)$.*

If moreover $(\sigma_q Y = AY)$ satisfies (H_q) then :

ii) *For any $A^{(0)}$, $F^{(0)}$, $A^{(\infty)}$ and $F^{(\infty)}$ satisfying the conditions of i), we have, for z near z_0, $(F^{(0)})^{-1}F^{(\infty)}(z) = H$ mod. $(z-z_0)M_n(\mathbb{C}\{z-z_0\})$, for some $H \in M_n(\mathbb{C})$ with rank $n-1$.*

Proof. For i), we refer to §2.2 of [RS07] and to the references therein. We now prove that ii) holds. Since $F^{(0)}$ is an isomorphism from $(\sigma_q Y = z^{\mu_0}A^{(0)}Y)$ to $(\sigma_q Y = AY)$, we have, for z near 0, $F^{(0)}(qz)z^{\mu_0}A^{(0)} = A(z)F^{(0)}(z)$. Similarly, for z near ∞, $F^{(\infty)}(qz)z^{\mu_\infty}A^{(\infty)} = A(z)F^{(\infty)}(z)$. These equations together with the fact that $F^{(0)} \in GL_n(\mathbb{C}(\{z\}))$ and $F^{(\infty)} \in GL_n(\mathbb{C}(\{z^{-1}\}))$ show that $F^{(0)}$ and $F^{(\infty)}$ can be extended meromorphically to $\mathbb{C}$ and $\mathbb{C}^*$ respectively, and that, for all $m \in \mathbb{N}^*$, we have, over $\mathbb{C}^*$:

$$\begin{aligned}
&(F^{(0)})^{-1}F^{(\infty)}(z)\\
&= z^{-m\mu_0}q^{-\frac{m(m-1)}{2}\mu_0}(A^{(0)})^{-m}(F^{(0)})^{-1}(q^m z)A(q^{m-1}z)\cdots A(z)\cdot\\
&\qquad \cdot A(q^{-1}z)\cdots A(q^{-m}z)F^{(\infty)}(q^{-m}z)(A^{(\infty)})^{-m}z^{-m\mu_\infty}q^{\frac{m(m+1)}{2}\mu_\infty}.
\end{aligned}$$

Now the result follows easily from the facts that $(F^{(0)})^{-1} \in GL_n(\mathbb{C}(\{z\}))$, $F^{(\infty)} \in GL_n(\mathbb{C}(\{z^{-1}\}))$, $A(z) = A(z_0)$ mod. $(z-z_0)M_n(\mathbb{C}\{z-z_0\})$ and, for any $k \in \mathbb{Z}^*$, $A(q^k z) \in GL_n(\mathbb{C}) + (z-$

$z_0)M_n(\mathbb{C}\{z-z_0\})$. $\square$

Theorem 31 ($\otimes$-Indecomposability criterion for systems). *Let $(\sigma_q Y = AY)$ be an object of $\mathcal{E}'_{(\mathbb{C}(z),\sigma_q)}$ which satisfies (H_q). Then $(\sigma_q Y = AY)$ is $\otimes$-indecomposable.*

Proof. Assume at the contrary that $(\sigma_q Y = AY)$ is $\otimes$-decomposable. Then there exist $A_1 \in GL_{n_1}(\mathbb{C}(z))$ and $A_2 \in GL_{n_2}(\mathbb{C}(z))$ $(n_1, n_2 > 1)$ such that $(\sigma_q Y = AY) \cong (\sigma_q Y = A_1 Y) \otimes (\sigma_q Y = A_2 Y)$. For further use, we denote by $R \in GL_n(\mathbb{C}(z))$ an isomorphism from $(\sigma_q Y = A_1 Y) \otimes (\sigma_q Y = A_2 Y)$ to $(\sigma_q Y = AY)$. Since $(\sigma_q Y = AY) \cong (\sigma_q Y = A_1 Y) \otimes (\sigma_q Y = A_2 Y)$ is pure isoclinic, both $(\sigma_q Y = A_1 Y)$ and $(\sigma_q Y = A_2 Y)$ are pure isoclinic (see Théorème 2.3.1 of [Sau04]). Let $N \in \mathbb{N}^*$ be such that $[N]^*(\sigma_q Y = A_1 Y) \cong (\sigma_q Y = [N]^* A_1 Y)$, $[N]^*(\sigma_q Y = A_2 Y) \cong (\sigma_q Y = [N]^* A_1 Y)$ and $[N]^*(\sigma_q Y = AY) \cong (\sigma_q Y = [N]^* A_1 Y) \otimes (\sigma_q Y = [N]^* A_2 Y)$ are pure isoclinic with integral slopes. Lemma 30 ensures that there exist $\mu_{1;0}, \mu_{1;\infty}, \mu_{2;0}, \mu_{1;\infty} \in \mathbb{Z}$ such that there exist :

- $A_1^{(0)} \in GL_{n_1}(\mathbb{C})$ and $F_1^{(0)} \in GL_{n_1}(\mathbb{C}(\{z_N\}))$ such that $F_1^{(0)}$ is an isomorphism from $\sigma_{q_N} Y = z_N^{\mu_{1;0}} A_1^{(0)} Y$ to $\sigma_{q_N} Y = [N]^* A_1 Y$;
- $A_1^{(\infty)} \in GL_{n_1}(\mathbb{C})$ and $F_1^{(\infty)} \in GL_{n_1}(\mathbb{C}(\{z_N^{-1}\}))$ such that $F_1^{(\infty)}$ is an isomorphism from $\sigma_{q_N} Y = z_N^{\mu_{1;\infty}} A_1^{(\infty)} Y$ to $\sigma_{q_N} Y = [N]^* A_1 Y$;
- $A_2^{(0)} \in GL_{n_2}(\mathbb{C})$ and $F_2^{(0)} \in GL_{n_2}(\mathbb{C}(\{z_N\}))$ such that $F_2^{(0)}$ is an isomorphism from $\sigma_{q_N} Y = z_N^{\mu_{2;0}} A_2^{(0)} Y$ to $\sigma_{q_N} Y = [N]^* A_2 Y$;
- $A_2^{(\infty)} \in GL_{n_2}(\mathbb{C})$ and $F_2^{(\infty)} \in GL_{n_2}(\mathbb{C}(\{z_N^{-1}\}))$ such that $F_2^{(\infty)}$ is an isomorphism from $\sigma_{q_N} Y = z_N^{\mu_{2;\infty}} A_2^{(\infty)} Y$ to $\sigma_{q_N} Y = [N]^* A_2 Y$.

So $F^{(0)} = ([N]^* R)(F_1^{(0)} \otimes F_2^{(0)}) \in GL_n(\mathbb{C}(\{z_N\}))$ is an isomorphism from $(\sigma_{q_N} Y = z_N^{\mu_{1;0}} A_1^{(0)} Y) \otimes (\sigma_{q_N} Y = z_N^{\mu_{2;0}} A_2^{(0)} Y)$ to $(\sigma_{q_N} Y = [N]^* AY)$ and $F^{(\infty)} = ([N]^* R)(F_1^{(\infty)} \otimes F_2^{(\infty)}) \in GL_n(\mathbb{C}(\{z_N^{-1}\}))$ is an isomorphism from $(\sigma_{q_N} Y = z_N^{\mu_{1;\infty}} A_1^{(\infty)} Y) \otimes (\sigma_{q_N} Y = z_N^{\mu_{2;\infty}} A_2^{(\infty)} Y)$ to $(\sigma_{q_N} Y = [N]^* AY)$. It is easily seen that $(\sigma_{q_N} Y = [N]^* AY)$ satisfies (H_{q_N}). So Lemma 30 ensures that, near some $z_0 \in \mathbb{C}^*$, $(F^{(0)})^{-1} F^{(\infty)}(z_N) = H \bmod. (z_N - z_0) M_n(\mathbb{C}\{z_N - z_0\})$ for some $H \in M_n(\mathbb{C})$ with rank $n-1$. Since $(F^{(0)})^{-1} F^{(\infty)} = (F_1^{(0)})^{-1} F_1^{(\infty)} \otimes (F_2^{(0)})^{-1} F_2^{(\infty)}$, H has the form $H_1 \otimes H_2$ for some $H_1 \in M_{n_1}(\mathbb{C})$ and $H_2 \in M_{n_2}(\mathbb{C})$. Therefore the rank of H is the product of the ranks of H_1 and H_2. This implies that either $n_1 = 1$ or $n_2 = 1$. This is a contradiction. $\square$

Let us now switch to operators. Recall that the q-difference system $(\sigma_q Y = AY)$ associated to $L = \sum_{k=0}^{n} a_{n-k} \sigma_q^k \in \mathcal{D}_{(\mathbb{C}(z),\sigma_q)}$ with $a_0 a_n \neq 0$ is given by :

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & \cdots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{pmatrix} \in GL_n(\mathbb{C}(z)).$$

Theorem 32 ($\otimes$-Indecomposability criterion for operators). *Assume that $L = \sum_{k=0}^{n} a_{n-k} \sigma_q^k \in \mathcal{D}_{(\mathbb{C}(z),\sigma_q)}$ with $a_0 a_n \neq 0$ is such that :*

1) *there exists $z_0 \in \mathbb{C}^*$ such that $a_n/a_0, ..., a_1/a_0$ are analytic at any point of $q^{\mathbb{Z}} z_0$, $a_n/a_0(z_0) = 0$ and, for all $k \in \mathbb{Z}^*$, $a_n/a_0(q^k z_0) \neq 0$;*

2) L is pure isoclinic at both 0 and ∞.

Then L is $\otimes$-indecomposable.

Proof. Since L is $\otimes$-indecomposable if and only if the associated q-difference system $(\sigma_q Y = AY)$ is $\otimes$-indecomposable, the result is an immediate consequence of Theorem 31. □

8.2 Application to q-Kloosterman operators (including $\mathcal{H}_q(\underline{a};\emptyset;\lambda)$)

We keep the notations of §7.2.

Theorem 33. *The general linear group $GL(\mathbb{C}^{\deg U})$ is the unique connected algebraic group occurring as Galois group of some q-Kloosterman module $\mathcal{K}l_q(U,V)$ such that $\deg U$ is prime to $\deg V$ and such that there exists $z_0 \in \mathbb{C}^*$ satisfying $V(z_0)=0$ and, for all $k \in \mathbb{Z}^*$, $V(q^k z_0) \neq 0$.*

Proof. Immediate consequence of Theorems 32 and 27. □

Corollary 34. *The general linear group $GL(\mathbb{C}^r)$ is the unique connected algebraic group occurring as Galois group of some confluent generalized q-hypergeometric module $\mathcal{H}_q(\underline{a};\emptyset;\lambda)$.*

Proof. This is a special case of Theorem 33 since $\mathcal{L}_q(\underline{a};\emptyset;\lambda) = zKl_q(-\lambda\prod_{i=1}^r(a_iX-1)+(-1)^r\lambda, -(-1)^r\lambda+X)$. □

In the following result, $c_1,...,c_{\deg U}$ denote the complex roots of $X^{\deg U}(U(X^{-1})+V(0)) \in \mathbb{C}[X]$ and, for all $i \in \{1,...,\deg U\}$, (u_i,α_i) denotes the unique element of $\mathbb{U}\times\mathbb{R}$ such that $c_i = u_i q^{\alpha_i}$.

Theorem 35. *Assume that $\deg U$ is prime to $\deg V$, that the algebraic group generated by $\mathrm{diag}(u_1,...,u_{\deg U})$ and $\mathrm{diag}(e^{2\pi i\alpha_1},...,e^{2\pi i\alpha_{\deg U}})$ is connected and that there exists $z_0 \in \mathbb{C}^*$ such that $V(z_0)=0$ and, for all $k\in\mathbb{Z}^*$, $V(q^k z_0)\neq 0$. Then, $Gal(\mathcal{K}l_q(U,V),\omega)$ is $GL(\mathbb{C}^{\deg U})$.*

Proof. Immediate consequence of Theorems 32 and 28. □

In the following result, for all $i \in \{1,...,r\}$, (u_i,α_i) denotes the unique element of $\mathbb{U}\times\mathbb{R}$ such that $a_i = u_i q^{\alpha_i}$.

Theorem 36. *If the algebraic group generated by $\mathrm{diag}(u_1,...,u_n)$ and $\mathrm{diag}(e^{2\pi i\alpha_1},...,e^{2\pi i\alpha_r})$ is connected then $Gal(\mathcal{H}_q(\underline{a};\emptyset;\lambda),\omega)$ is $GL(\mathbb{C}^r)$.*

Proof. This is a special case of Theorem 35 since $\mathcal{L}_q(\underline{a};\emptyset;\lambda) = zKl_q(-\lambda\prod_{i=1}^r(a_iX-1)+(-1)^r\lambda, -(-1)^r\lambda+X)$. □

8.3 Equations satisfying $(\mathscr{H}1)$ with Galois group $\otimes_{i=1}^l GL(\mathbb{C}^{n_i})$

Theorem 37. *For any $l \in \mathbb{N}^*$, for any pairwise prime numbers $n_1,...,n_l > 1$, the image of $\prod_{i=1}^l GL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^l std$ occurs as Galois group of some object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of rank $n = n_1\cdots n_l$ satisfying $(\mathscr{H}1)$.*

Proof. Theorem 36 ensures that, for any $i\in\{1,...,l\}$, there exists an object M_i of rank n_i of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ satisfying $(\mathscr{H}1)$ whose Galois group is $GL(\mathbb{C}^{n_i})$. It is easily seen that $\otimes_{i=1}^l M_i$ satisfies $(\mathscr{H}1)$. For any $i \in\{1,...,l\}$, let ρ_i be the representation of $Gal(\oplus_{i=1}^l M_i,\omega)$ corresponding to M_i by tannakian duality. Then, for any $i\in\{1,...,l\}$, the image of ρ_i is $GL(\mathbb{C}^{n_i})$ and $\oplus_{i=1}^l\rho_i$ is a faithful representation (because it is the representation of $Gal(\oplus_{i=1}^l M_i,\omega)$ corresponding to $\oplus_{i=1}^l M_i$ itself). So the image of $\otimes_{i=1}^l\rho_i$ coincides with the image of $\prod_{i=1}^l GL(\mathbb{C}^{n_i})$ in $\otimes_{i=1}^l std$ in virtue of Goursat-Kolchin-Ribet Proposition 1.8.2 in [Kat90]. □

9. More computations

9.1 Non q-Kummer induced equations in the two slopes case

Theorem 38. *Let M be an irreducible object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of rank n satisfying $(\mathscr{H}2)$ with r prime to n. Assume that M is regular singular at ∞ with exponents $c_1, ..., c_n \in q^{\mathbb{R}}$. If $c_1, ..., c_n$ is not q-Kummer induced then $Gal(M, \omega) = GL(\omega(M))$.*

Proof. We let $G = Gal(M, \omega)$. Proposition 15 ensures that G°, and hence its Lie algebra $\mathfrak{g}$, acts irreducibly on $\omega(M)$. Moreover, the proof of Theorem 24 shows that G° contains, with respect to some basis, $I_{n-r} \oplus \mathbb{C}^* I_r$. So $\mathfrak{g}$ contains, with respect to some basis, $0_{n-r} \oplus \mathbb{C} I_r$ and hence contains an element with two eigenvalues with relatively prime multiplicities. According to J.-P. Serre (§4 of [Ser67]), this implies that $\mathfrak{g}$ is either $\mathfrak{sl}(\omega(M))$ of $\mathfrak{gl}(\omega(M))$. Since $\det(M)$ is irregular of rank one, its Galois group is $\mathbb{C}^*$. So $G = GL(\omega(M))$. □

A immediate application is (see § 7.1 for $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$) :

Theorem 39. *If $a_1, ..., a_r \in q^{\mathbb{R}}$ is not q-Kummer induced and if r is prime to $s > 0$ then $Gal(\mathcal{H}_q(\underline{a}; \underline{b}; \lambda), \omega) = GL(\mathbb{C}^r)$.*

9.2 Another example of q-Kloosterman equation

The proof of the following $\otimes$-indecomposability criterion is left to the reader.

Proposition 40. *Let M be an object of $\mathcal{E}_{(\mathbb{C}(z),\sigma_q)}$ of rank n. Assume that M is regular singular at ∞ with exponents $c_1, ..., c_n$ in $q^{\mathbb{R}}$. If M is $\otimes$-decomposable then there exists a divisor $1 < d < n$ of n such that $c_1, ..., c_n$ mod. $q^{\mathbb{Z}}$ is of the form $(c'_i c''_j; 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant n/d)$ mod. $q^{\mathbb{Z}}$ for some $c'_1, ..., c'_d \in \mathbb{C}^*$ and some $c''_1, ..., c''_{n/d} \in \mathbb{C}^*$.*

We now give an illustration of the previous result. Note that, we cannot apply Theorem 35 to $\mathcal{K}l_q((q^{1/2} - X)^2(1 - X)^{n-2} - q, V)$ where $V \in \mathbb{C}[X]$ is such that $V(0) = q$. However :

Proposition 41. *Let us consider $V \in q + X\mathbb{C}[X]$. Then, for any odd integer $n \geqslant 2$ prime to $\deg V$, the Galois group of $\mathcal{K}l_q((q^{1/2} - X)^2(1 - X)^{n-2} - q, V)$ is $GL(\mathbb{C}^n)$.*

Proof. Recall (see §7.2) that $M = \mathcal{K}l_q((q^{1/2} - X)^2(1 - X)^{n-2} - q, V)$ is pure isoclinic at 0 with slope $\deg V/n$ and regular singular at ∞ with exponents $q^{1/2}$ with multiplicity 2 and 1 with multiplicity $n - 2$. Since n is odd, Corollary 13 ensures that the Galois group of M is connected. It is easily seen that M is $\otimes$-indecomposable by using Proposition 40. Theorem 27 leads to the conclusion. □

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Julien Roques Julien.Roques@ujf-grenoble.fr
Institut Fourier, Université Grenoble 1, UMR CNRS 5582, 100 rue des Maths, BP 74, 38402 St Martin d'Hères.