Euler characteristics and q-difference equations

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Abstract

This paper is concerned with linear q-difference equations. Our main result is an explicit formula for the Euler characteristic of the sheaf of analytic solutions attached to any linear algebraic q-difference equation. This formula involves certain invariants attached to the so-called intermediate singularities. As an application, we interpret the index of rigidity recently introduced by Sakai and Yamaguchi in cohomological terms.

1 Introduction

This work grew out of an attempt by the first author to find a cohomological interpretation of the index of rigidity for q-difference equations defined by Sakai and Yamaguchi in [11, §3]; and of an attempt by the second author to understand the role of the so-called "intermediate singularities" (those other than $0, \infty$, see further below) in the global behaviour of rational q-difference equations. Only the former problem will be tackled here, we intend to pursue the latter one in a future work.

The approach developed in the present paper relies on a sheaf $\mathcal{F}_A$ of analytic solutions attached to any q-difference system

$$Y(qz) = A(z)Y(z) \tag{1.0.1}$$

with $q \in \mathbf{C}^\times$, $|q| \neq 1$, and $A(z) \in \mathrm{GL}_n(\mathbf{C}(x))$. This is a sheaf over the Riemann surface $\mathbf{E}_q^{an} = \mathbf{C}^\times / q^{\mathbf{Z}}$. It turns out that $\mathcal{F}_A$ is a locally free $\mathcal{O}_{\mathbf{E}_q^{an}}$-module of rank n and, hence, defines a vector bundle over $\mathbf{E}_q^{an}$. One of our main results is an explicit formula for the Euler characteristic $\chi(\mathcal{F}_A)$ of $\mathcal{F}_A$: we prove that it is the sum of *local* invariants of (1.0.1) attached to the intermediate singularities; see theorem 3.18. By intermediate singularities, we mean the poles of A or A^{-1} on $\mathbb{P}^1(\mathbf{C}) \setminus \{0, \infty\}$.

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When it is applied to the "internal End" of (1.0.1), this formula essentially gives Sakai and Yamaguchi's index of rigidity attached to (1.0.1); see paragraph 3.4. This is parallel to Katz's [7, Theorem 1.1.2].

This paper also includes a cohomological study of natural extensions of $\mathcal{F}_A$ to "completions" of $\mathbf{E}_q^{an}$. We refer to paragraphs 4 and 5 for the details.

Let us now explain the origin of our approach. The celebrated formula of Grothendieck-Ogg-Shafarevitch [10] was transposed by Deligne to the case of differential equations (see "théorème de comparaison" in [3]), then used by Daniel Bertrand in [1, 2]. This is the same formula that Katz uses in his study of rigidity. A long time ago, Daniel Bertrand asked one of us if it could be transposed to q-difference equations. Our answer, which roughly follows the lines of [3, chap. 6], is contained in the present paper.

2 General notations

- $\mathcal{O}_X$, resp. $\mathcal{M}_X$: sheaf of holomorphic, resp. meromorphic, functions over a Riemann surface X.
- $\mathcal{O}_x := \mathcal{O}_{X,x}$, $\mathcal{M}_x := \mathcal{M}_{X,x}$: shorthand for the stalks of those sheaves at $x \in X$ (usually unambiguous).
- $\mathbf{C}\{x\} := \mathcal{O}_{\mathbf{C},0}$, $\mathbf{C}(\{x\}) := \mathcal{M}_{\mathbf{C},0}$.
- u_x: a local coordinate centered at $x \in X$ (thus a uniformizer of the discrete valuation ring $\mathcal{O}_{X,x}$); $v_x(f)$: u_x-adic valuation of $f \in \mathcal{O}_{X,x}$.
- $q \in \mathbf{C}^\times$ such that $|q| > 1$; $\sigma_q f(x) := f(qx)$.
- $\mathbf{E}_q^{an} := \mathbf{C}^\times / q^{\mathbf{Z}}$ seen as a Riemann surface (complex torus); $\pi : \mathbf{C}^\times \to \mathbf{E}_q^{an}$ is the canonical covering.
- $\overline{a} := \pi(a)$; $[a;q] := aq^{\mathbf{Z}}$.
- We will freely use the identification : Field of σ_q-invariant meromorphic functions over $\mathbf{C}^\times$ = Field of meromorphic functions over $\mathbf{E}_q^{an}$ *i.e.*

$$\mathcal{M}(\mathbf{C}^\times)^{\sigma_q} = \mathcal{M}(\mathbf{E}_q^{an}).$$

3 Euler characteristics of some sheaves of modules over $\mathbf{E}_q^{an}$

3.1 Some basic constructs

3.1.1 q-difference systems

Let $A \in \mathrm{GL}_n(\mathbf{C}(x))$. The associated q-difference system is:

$$\sigma_q X = AX. \tag{3.0.1}$$

The singular locus of equation (3.0.1) is defined as:

$$\mathrm{Sing}(A) := \{\text{poles of } A\} \cup \{\text{poles of } A^{-1}\} = \{\text{poles of } A\} \cup \{\text{zeroes of } \det A\}.$$

Thus, if $U \subset \mathbf{C}^\times$ is an open subset which does not meet $\mathrm{Sing}(A)$, then A is *regular* on U, meaning that it is holomorphic over U as well as its inverse A^{-1}.

Let R a difference algebra over the difference field $(\mathbf{C}(x), \sigma_q)$. The solutions of (3.0.1) in R form the $\mathbf{C}$-vector space:

$$\mathrm{Sol}(A,R) := \{X \in R^n \mid \sigma_q X = AX\},$$

where we consistently identify R^n with the space of column vectors $\mathrm{Mat}_{n,1}(R)$. A fundamental matricial solution of (3.0.1) in R is a matrix $\mathcal{X} \in \mathrm{GL}_n(R)$ such that $\sigma_q \mathcal{X} = A\mathcal{X}$.

Let K be a difference field extension of $(\mathbf{C}(x), \sigma_q)$. The gauge transform of A by $F \in \mathrm{GL}_n(K)$ is:

$$F[A] := (\sigma_q F) A F^{-1} \in \mathrm{GL}_n(K).$$

We say that $A, B \in \mathrm{GL}_n(\mathbf{C}(x))$ are rationally equivalent if $B = F[A]$ for some $F \in \mathrm{GL}_n(\mathbf{C}(x))$. The same relation with $F \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$ (resp. $F \in \mathrm{GL}_n(\mathbf{C}(\{x^{-1}\}))$) defines local analytic equivalence at 0 (resp. at ∞). Note that in this case one has automatically $F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$ (resp. $F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^\times \cup \{\infty\}))$).

If $B = F[A]$ and if $\sigma_q X = AX$, then, setting $Y := FX$, one has $\sigma_q Y = BY$. Thus, if $F \in \mathrm{GL}_n(\mathbf{C}(x))$ and if R is a difference algebra over $(\mathbf{C}(x), \sigma_q)$, we have:

$$F\,\mathrm{Sol}(A,R) = \mathrm{Sol}(B,R).$$

We have similar statements for local analytic equivalence.

3.1.2 Some sheaves of functions

Note that σ_q operates on the direct image sheaves $\pi_* \mathcal{O}_{\mathbf{C}^\times}$ and $\pi_* \mathcal{M}_{\mathbf{C}^\times}$ and that we have the following obvious identifications for the fixed subsheaves:

$$\mathcal{O}_{\mathbf{E}_q^{an}} = (\pi_* \mathcal{O}_{\mathbf{C}^\times})^{\sigma_q}, \quad \mathcal{M}_{\mathbf{E}_q^{an}} = (\pi_* \mathcal{M}_{\mathbf{C}^\times})^{\sigma_q}.$$

We shall now introduce various subsheaves of $\pi_* \mathcal{M}_{\mathbf{C}^\times}$. Let V an open subset of $\mathbf{E}_q^{an}$ and let $U := \pi^{-1}(V)$, so that $\pi_* \mathcal{M}_{\mathbf{C}^\times}(V) = \mathcal{M}_{\mathbf{C}^\times}(U)$. We define the subsheaves $\mathcal{A}^{(0)}, \mathcal{A}^{(\infty)}, \mathcal{A}, \mathcal{A}'$ of $\pi_* \mathcal{M}_{\mathbf{C}^\times}$

by:

(3.0.2)
$\mathcal{A}^{(0)}(V) := \{f \in \mathcal{M}_{\mathbf{C}^\times}(U) \mid f \text{ is holomorphic over } U \cap D(0,r) \text{ for some } r > 0\}$,
(3.0.3)
$\mathcal{A}^{(\infty)}(V) := \{f \in \mathcal{M}_{\mathbf{C}^\times}(U) \mid f \text{ is holomorphic over } U \cap D(0,R)^c \text{ for some } R > 0\}$,
(3.0.4)
$\mathcal{A}(V) := \{f \in \mathcal{M}_{\mathbf{C}^\times}(U) \mid f \text{ is holomorphic over } U\} \quad (\text{ thus } \mathcal{A} = \pi_* \mathcal{O}_{\mathbf{C}^\times})$,
(3.0.5)
$\mathcal{A}'(V) := \{f \in \mathcal{M}_{\mathbf{C}^\times}(U) \mid f \text{ has at worst a finite number of poles over any } q\text{-spiral } [a;q] \subset U\}$.

It is easily seen that $\mathcal{A} \subset \mathcal{A}^{(0)} \cap \mathcal{A}^{(\infty)} \subset \mathcal{A}'$.

3.1.3 Some sheaves of solutions

To any subsheaf $\mathcal{B}$ of $\pi_* \mathcal{M}_{\mathbf{C}^\times}$, we shall associate a sheaf of solutions on $\mathbf{E}_q^{an}$ for which the sections over an open subset $V \subset \mathbf{E}_q^{an}$ are the solutions $X \in \mathcal{B}(V)^n$ of (3.0.1). Taking successively for $\mathcal{B}$ the sheaves $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}^{(0)}$, $\mathcal{A}^{(\infty)}$, we obtain sheaves of solutions on $\mathbf{E}_q^{an}$ respectively denoted by $\mathcal{F}_A$, $\mathcal{F}'_A$, $\mathcal{F}_A^{(0)}$, $\mathcal{F}_A^{(\infty)}$. We check easily that $\mathcal{F}_A \subset \mathcal{F}_A^{(0)} \cap \mathcal{F}_A^{(\infty)} = \mathcal{F}'_A$. In the course of what follows, we shall find out that all these sheaves are locally free of rank n over $\mathcal{O}_{\mathbf{E}_q^{an}}$, whence define holomorphic vector bundles over $\mathbf{E}_q^{an}$.

Remark 3.1 *Taking $\mathcal{B} := \pi_* \mathcal{M}_{\mathbf{C}^\times}$ yields the sheaf of all meromorphic solutions, plainly a $(\pi_* \mathcal{M}_{\mathbf{C}^\times})^{\sigma_q} = \mathcal{M}_{\mathbf{E}_q^{an}}$-module. It was proved by Praagman in [8] that this is a free $\mathcal{M}_{\mathbf{E}_q^{an}}$-module of rank n. Said otherwise, there exists a fundamental matricial solution $X \in GL_n(\mathcal{M}(\mathbf{C}^\times))$ of (3.0.1). The proof relies on the fact that it is a meromorphic vector bundle on the compact Riemann surface $\mathbf{E}_q^{an}$ and that such a bundle is free.*

The proof of the following is immediate:

Lemma 3.2 *Let $F \in GL_n(\mathbf{C}(x))$ and $B := F[A]$. Then the automorphism $X \mapsto FX$ of $(\pi_* \mathcal{M}_{\mathbf{C}^\times})^n$ induces identifications:*

$$F \mathcal{F}'_A = \mathcal{F}'_B,$$
$$F \mathcal{F}_A^{(0)} = \mathcal{F}_B^{(0)},$$
$$F \mathcal{F}_A^{(\infty)} = \mathcal{F}_B^{(\infty)}.$$

Actually the equalities on the second line, resp. on the third line, only require F to be local analytic at 0, resp. at ∞.

Note however that it is not generally true that $F \mathcal{F}_A = \mathcal{F}_B$. For instance, take $A := 1$ and $F := 1/(z-1)$ so that $B := F[A] = \dfrac{z-1}{qz-1}$. Then, $\mathcal{F}_B$ is isomorphic to $\mathcal{O}_{\mathbf{E}_q^{an}}(-[\overline{1}])$, whereas $\mathcal{F}_A = \mathcal{O}_{\mathbf{E}_q^{an}}$.

3.2 Sheaves of solutions local at 0 and ∞

Lemma 3.3 *The sheaves $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$ are locally free $O_{\mathbf{E}_q^{an}}$-modules of rank n.*

Proof. - Let $\overset{\circ}{\mathrm{D}}(0,r)$, $r > 0$, any punctured disk on which A is regular. Let V be a trivializing open subset of $\mathbf{E}_q^{an}$ for the covering π so that $\pi^{-1}(V)$ is the disjoint union of the $q^k U$, for $k \in \mathbf{Z}$, where $\pi_{|U}$ is a homeomorphism onto V. We can assume that $U \subset \overset{\circ}{\mathrm{D}}(0,r)$. Let $V' \subset V$ be any open subset and set $U' = \pi^{-1}(V') \cap U$ so that $\pi^{-1}(V')$ is the disjoint union of the $q^k U'$ for $k \in \mathbf{Z}$. Then any $X \in O_{\mathbf{C}^\times}(U')^n$ extends successively holomorphically to $q^{-1}U', q^{-2}U', \ldots$ through (3.0.1) used as a recursive definition $X(z) := A(z)^{-1}X(qz)$; and meromorphically to $qU', q^2U', \ldots$ through (3.0.1) used as a recursive definition $X(qz) := A(z)X(z)$. Thus, X extends uniquely to a solution $X \in \mathcal{F}_A^{(0)}(V')$. This implies that the restriction $(\mathcal{F}_A^{(0)})_{|V}$ is isomorphic to $(O_{\mathbf{C}^\times})^n_{|U} = (O_{\mathbf{E}_q^{an}})^n_{|V}$. The proof at ∞ is similar. $\Box$

The goal of this section is to prove the following:

Proposition 3.4 *The Euler characteristics of the sheaves $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$ are given by the following formulas:*

(3.4.1)
$$\chi(\mathcal{F}_A^{(0)}) = v_0(\det A) \text{ and } \chi(\mathcal{F}_A^{(\infty)}) = v_\infty(\det A).$$

The proofs at 0 and ∞ are entirely similar, so we shall concentrate on the first case, which occupies this whole section. The proof proceeds in three steps: first, reduction to the case of a “pure isoclinic system”; then, reduction to the case of “integral slopes”; last, proof in this case.

Recall from [9] that one associates to A a Newton polygon, *i.e.* in essence slopes $\mu_1 < \cdots < \mu_k$ in $\mathbf{Q}$, with multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$ such that $r_1 + \cdots + r_k = n$ and $r_i\mu_i \in \mathbf{Z}$ for $i \in \{1, \ldots, k\}$. The set of these data is a local analytic invariant (actually, a formal invariant). Moreover, A is locally isomorphic to (*i.e.* there exists $F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$ such that $F[A]$ is equal to) a matrix of the form:

(3.4.2)
$$\begin{pmatrix} B_1 & \ldots & \ldots & \ldots \\ 0 & \ddots & U_{i,j} & \vdots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & B_k \end{pmatrix}$$

where each B_i has unique slope μ_i and has rank r_i; and where $U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}[z,z^{-1}])$ for $1 \leq i < j \leq k$. Up to some supplementary conditions on the B_i and the $U_{i,j}$ (which we do not recall here because they will not be used in what follows), this is the so-called Birkhoff-Guenther normal form. If $k = 1$, the system is said to be *pure isoclinic* of (unique) slope μ_1.

3.2.1 Reduction to the case of a pure isoclinic system

From the fact just recalled and from lemma 3.2, we see that we can as well take A in the form (3.4.2). The following fact is stated without proof in [9] (and other places).

Lemma 3.5 *The functor $A \rightsquigarrow \mathcal{F}_A^{(0)}$ is exact.*

Proof. - In the abelian category of q-difference systems over a fixed difference field K with difference operator σ_q, an exact sequence $0 \to A' \to A \to A'' \to 0$ takes (up to isomorphism) the form:

$$A = \begin{pmatrix} A' & U \\ 0 & A'' \end{pmatrix}.$$

In the associated sequence $0 \to \mathcal{F}_{A'}^{(0)} \to \mathcal{F}_A^{(0)} \to \mathcal{F}_{A''}^{(0)} \to 0$ the significant morphisms take (over any fixed $V \subset \mathbf{E}_q^{an}$) the form $X' \mapsto (X', 0)$ and $(X', X'') \mapsto X''$. Then exactness is obvious, except from right exactness $\mathcal{F}_A^{(0)} \to \mathcal{F}_{A''}^{(0)} \to 0$ which we now proceed to prove.

So let $x \in \mathbf{E}_q^{an}$ and let $X'' \in (\mathcal{F}_{A''}^{(0)})_x$, which we may represent by some $X'' \in \mathcal{F}_{A''}^{(0)}(V)$ for some open neighborhood $V \subset \mathbf{E}_q^{an}$ of x; and we can as well assume that V is a trivializing neighborhood for the covering π so that $\pi^{-1}(V)$ is the disjoint union of the $q^k U$, for $k \in \mathbf{Z}$, where $\pi_{|U}$ is a homeomorphism onto V. Let $\overset{\circ}{\mathrm{D}}(0,r)$, $r > 0$, a punctured disk centered at 0 on which A is regular, so the same holds for A', A''. We choose the above U such that $U \subset \overset{\circ}{\mathrm{D}}(0,r)$.

To lift X'' to $\mathcal{F}_A^{(0)}(V)$, we have to find X' such that $X := (X', X'')$ is a solution of (3.0.1), which is equivalent (since we already know that $\sigma_q X'' = A'' X''$) to $\sigma_q X' = A'X' + B$, where $B := UX''$. Moreover, we want X to be holomorphic over $\overset{\circ}{\mathrm{D}}(0,r') \cap \pi^{-1}(V)$, for some $r' > 0$. So we choose $r' \leq r$ such that X'' is holomorphic over $\overset{\circ}{\mathrm{D}}(0,r') \cap \pi^{-1}(V)$ and proceed to solve the equation in X'. Let $U' := q^k U$ where $k \in \mathbf{Z}$ is chosen in such a way that $U' \subset \overset{\circ}{\mathrm{D}}(0,r')$. We set the value of X' on U' *arbitrarily* (we only require it to be holomorphic) and then use the functional equation $X' = A'^{-1}(\sigma_q X' - B)$ to extend it successively to $q^{-1}U'$, $q^{-2}U'$, $\ldots$: all these are holomorphic; and the equation $\sigma_q X' = A'X' + B$ to extend X' successively to qU', q^2U', $\ldots$: those ones are meromorphic. This X' and the corresponding X have the expected properties. □

We deduce from this lemma that (A having the form (3.4.2))

$$\chi(\mathcal{F}_A^{(0)}) = \chi(\mathcal{F}_{B_1}^{(0)}) + \cdots + \chi(\mathcal{F}_{B_k}^{(0)}).$$

Since obviously $v_0(\det A) = v_0(\det B_1) + \cdots + v_0(\det B_k)$, we see that we just have to prove formula (3.4.1) in the case of a pure isoclinic system.

3.2.2 Reduction to the case of integral slopes

We assume here that $B \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$ is pure isoclinic of slope $\mu = d/r$. From [9] we know that the change of variable (ramification) $z = z'^r$, $q = q'^r$ yields a q'-difference system with matrix $B'(z') := B(z)$ which is pure isoclinic with slope $\mu' = r\mu = d$. For this system, formula (3.4.1) must be interpreted with v_0 meaning the z'-adic valuation, *i.e.* $v_0(\det B') = r v_0(\det B)$.

Let $\rho : \mathbf{C}^\times \to \mathbf{C}^\times$, $z' \mapsto z := z'^r$. This induces a commutative diagram:

$$\begin{tikzcd} \mathbf{C}^\times \arrow[r, "\rho"] \arrow[d, "\pi'"] & \mathbf{C}^\times \arrow[d, "\pi"] \\ \mathbf{E}_{q'}^{an} \arrow[r, "\overline{\rho}"] & \mathbf{E}_q^{an}, \end{tikzcd}$$

where $\pi' : \mathbf{C}^\times \to \mathbf{E}_{q'}^{an}$ denotes the canonical projection.

Lemma 3.6 *With these notations,* $\overline{\rho}^* \mathcal{F}_B^{(0)} = \mathcal{F}_{B'}^{(0)}$.

Proof. - Let $V \subset \mathbf{E}_q^{an}$ an arbitrary open subset, $V' := \overline{\rho}^{-1}(V)$ its preimage in $\mathbf{E}_{q'}^{an}$, and $U := \pi^{-1}(V)$, $U' := \pi'^{-1}(V') = \rho^{-1}(U)$ their respective preimages in $\mathbf{C}^\times$. Any solution of $\sigma_q Y = BY$ analytic over U near 0 gives rise, by the changes of variables $B'(z') = B(z)$, $Y'(z') = Y(z)$, to a solution of $\sigma_{q'} Y' = B'Y'$ analytic over U' near 0. The maps $\mathcal{F}_B^{(0)}(V) \to \mathcal{F}_{B'}^{(0)}(V') = \overline{\rho}_* \mathcal{F}_{B'}^{(0)}(V)$ thus defined make up a morphism of sheaves of linear spaces $\mathcal{F}_B^{(0)} \to \overline{\rho}_* \mathcal{F}_{B'}^{(0)}$, whence, by adjunction, a morphism of sheaves of linear spaces $\overline{\rho}^{-1} \mathcal{F}_B^{(0)} \to \mathcal{F}_{B'}^{(0)}$ (the source here is the topological inverse image sheaf) and then a morphism of sheaves of modules $\overline{\rho}^* \mathcal{F}_B^{(0)} \to \mathcal{F}_{B'}^{(0)}$. We now show that this is an isomorphism. It is enough to do so by restriction to a basis of open subsets.

So let $V \subset \mathbf{E}_q$ be a trivializing open subset for the covering $\overline{\rho}$ and let $W' \subset V' := \overline{\rho}^{-1}(V)$ such that $W' \to V = \overline{\rho}(W')$ is a homeomorphism. Then a solution of $\sigma_q X = BX$ over $\pi^{-1}(V)$ gives rise to a solution $X'(z') := X(z)$ of $\sigma_{q'} X' = B'X'$ over $\pi'^{-1}(W')$. In this way, we get a $\mathbf{C}$-linear isomorphism from $\overline{\rho}^{-1} \mathcal{F}_B^{(0)}(W')$ to $\mathcal{F}_{B'}^{(0)}(W')$. The isomorphism of modules

$$\overline{\rho}^* \mathcal{F}_B^{(0)}(W') := \overline{\rho}^{-1} \mathcal{F}_B^{(0)}(W') \otimes_{\mathcal{O}_{\mathbf{E}_q^{an}}(V)} \mathcal{O}_{\mathbf{E}_{q'}^{an}}(W') \simeq \mathcal{F}_{B'}^{(0)}(W')$$

follows, because here $\mathcal{O}_{\mathbf{E}_{q'}^{an}}(W') = \mathcal{O}_{\mathbf{E}_q^{an}}(V)$. $\Box$

The following statement is obviously a particular case of much more general facts, but, for lack of a convenient reference, we give a direct proof.

Lemma 3.7 *Let $p : X' \to X$ an isogeny of degree r between two complex tori and let $\mathcal{F}$ a locally free sheaf on X. Then:*

$$\chi(p^* \mathcal{F}) = r\chi(\mathcal{F}).$$

Proof. - Since the inverse image functor is exact and since χ is additive for exact sequences, the triangularisation of holomorphic vector bundles over compact Riemann surfaces [5, corollary of theorem 10, p. 63] allow us to assume that $\mathcal{F}$ has rank 1, $\mathcal{F} = \mathcal{O}_X(D)$ for some divisor D. But then $p^* \mathcal{F} = \mathcal{O}_{X'}(D')$, where $D' := p^* D$. Writing $d := \deg D$, so that $\deg D' = rd$, we have, by Riemann-Roch theorem for line bundles (with here $g = 1$), $\chi(\mathcal{F}) = \deg D = d$ and $\chi(p^* \mathcal{F}) = \deg D' = rd$. $\Box$

Remark 3.8 For any finite morphism $p : X' \to X$ and any coherent sheaf $\mathcal{F}'$ on X', we have equality of the cohomology groups: $H^i(X', \mathcal{F}') = H^i(X, p_* \mathcal{F}')$ [4, p. 63], thus in the case of our lemma $\chi(p_* p^* \mathcal{F}) = \chi(p^* \mathcal{F}) = r\chi(\mathcal{F})$. However, even in the case of an etale covering, it is not true that $p_* p^* \mathcal{F} \simeq \mathcal{F}^r$. For instance, taking $\mathcal{F} := O_X$, we see that $p_* p^* O_X$ is locally free of rank r but its global sections has dimension 1: indeed, $p^* O_X = O_{X'}$.

Combining our two lemmas, we get the equality:

$$\chi(\mathcal{F}_{B'}^{(0)}) = r\chi(\mathcal{F}_B^{(0)}).$$

Since we found that $v_0(\det B') = r v_0(\det B)$, we see that it is enough to prove formula (3.4.1) for B', *i.e.* for a pure isoclinic system with integral slope.

3.2.3 Proof in the case of a pure system with integral slopes

From [9] any pure isoclinic system with slope $\mu \in \mathbf{Z}$ is equivalent to one with matrix $B = z^\mu C$, $C \in \mathrm{GL}_r(\mathbf{C})$. Then:

$$\mathcal{F}_B^{(0)} \simeq O_{\mathbf{E}_q^{an}}(\mu) \otimes \mathcal{F}_C,$$

and $\mathcal{F}_C$ is a flat vector bundle of rank r. Again from general facts, it follows that $\chi(\mathcal{F}_B^{(0)}) = r\mu$, but for the lack of convenient reference, we give a direct argument.

Lemma 3.9 *If $B = z^\mu C$, $C \in GL_r(\mathbf{C})$, then $\chi(\mathcal{F}_B^{(0)}) = r\mu$.*

Proof. - We can assume that C is triangular (conjugacy is a particular case of gauge equivalence), so that $\mathcal{F}_B^{(0)}$ is an iterated extension of r sheaves of the form $\mathcal{F}_{cz^\mu}^{(0)}$, $c \in \mathbf{C}^\times$. By additivity of χ, we are drawn to prove that $\chi(\mathcal{F}_{cz^\mu}^{(0)}) = \mu$. But a nontrivial meromorphic section of $\mathcal{F}_{cz^\mu}^{(0)}$ can be obtained as $s := \theta_q{}^{\mu-1}(z)\theta_q(cz)$, where the theta function $\theta_q \in O(\mathbf{C}^\times)$ satisfies $\sigma_q \theta_q = z\theta_q$ and $\mathrm{div}_{\mathbf{C}^\times}(\theta_q) = \sum\limits_{a \in [-1;q]} [a]$ (see [9]), so that the degree of the section s is μ, and we can apply Riemann-Roch theorem again. $\square$

Since $\det B = z^{r\mu} \det C$, the formula follows in this case. This terminates the proof of proposition 3.4.

Remark 3.10 A somewhat different proof is possible along the following lines. Using the results of this section, one can prove (using the notations of proposition 3.4) that:

$$\det \mathcal{F}_A^{(0)} = \mathcal{F}_{\det A}^{(0)}. \tag{3.10.1}$$

Indeed, the equality is easy when A has integral slopes, using the existence of a triangular form with diagonal terms cz^μ and lemma 3.5; and one can reduce to this case by extension of the base just as in 3.2.2. Once equality (3.10.1) is proved, the theorem of Riemann-Roch for vector bundles over compact Riemann surfaces [5] allows one to conclude immediately.

3.3 Sheaves of solutions related to intermediate singularities

In this section, we intend to compute $\chi(\mathcal{F}'_A)$ as a sum of local terms defined at 0, ∞ and at the "intermediate singularities", *i.e.* points in $\mathrm{Sing}(A)$. The reason for using $\mathcal{F}'_A$ instead of $\mathcal{F}_A$ is the fact, mentioned at the end of subsection 3.1, that the former is in some sense intrinsic (up to rational isomorphisms) while the latter is not. This is related to so-called "resonancies" and we shall first show how to deal with them.

3.3.1 Resonancies

Definition 3.11 A singularity $a \in \mathrm{Sing}(A)$ is called *resonant*[1] if $q^k a \in \mathrm{Sing}(A)$ for some $k \in \mathbf{Z}$, $k \neq 0$. The system A is said to be *nonresonant* if it has no resonant singularities, *i.e.* if $\mathrm{Sing}(A) \cap q^{\mathbf{N}^*}\mathrm{Sing}(A) = \emptyset$.

Lemma 3.12 *If A is nonresonant, then $\mathcal{F}_A = \mathcal{F}'_A$.*

Proof. - Let $a \in \mathbf{C}^\times$ and let X a meromorphic solution of (3.0.1). In order to prove the lemma, it is sufficient to prove that X is either holomorphic over $[a;q]$ or has infinitely many poles over $[a;q]$. If $a \notin q^{\mathbf{Z}}\mathrm{Sing}(A)$, the relation $X(qz) = A(z)X(z)$ and the fact that A is regular over $[a;q]$ imply that X either has no poles over $[a;q]$ or has infinitely many poles over $[a;q]$. It remains to consider the case $a \in q^{\mathbf{Z}}\mathrm{Sing}(A)$. Up to replacing a by aq^j for some $j \in \mathbf{Z}$, we can assume that $a \in \mathrm{Sing}(A)$. Then, no $q^k a$ with $k \neq 0$ belongs to $\mathrm{Sing}(A)$, so we deduce that the same dichotomy as above holds separately over both half q-spirals $aq^{-\mathbf{N}}$ and $q^{\mathbf{N}^*}$. In any case, the conclusion follows. $\Box$

Lemma 3.13 *For every $A \in GL_n(\mathbf{C}(x))$, there exists a rational gauge transformation $F \in GL_n(\mathbf{C}(x))$ such that $F[A]$ has all its singularities within the fundamental annulus $\mathcal{C}(1, |q|)$:*

$$\forall a \in Sing(F[A]) \;,\; 1 \leq |a| < |q| \,.$$

In particular, $F[A]$ is nonresonant.

Proof. - Note that $A = uA_0$ where $u \in \mathbf{C}(x)^\times$ and $A_0 \in \mathrm{GL}_n(\mathbf{C}(x)) \cap \mathrm{Mat}_n(\mathbf{C}[x])$. We may write in the same way $F = fF_0$, and then clearly $F[A] = f[u]F_0[A_0]$. We shall deal separately with the scalar components f, u and with the polynomial components F_0, A_0.
Write $u = c\prod(z - a_i)^{r_i}$, $c \in \mathbf{C}^\times$. Then, if for instance $|a_j| \geq |q|$, the gauge transform $(z - a_j)^{r_j}[u]$ has singularity a_j replaced by a_j/q, so, iterating, we can move it to the fundamental annulus. The case where $|a_j| < 1$ is tackled similarly. In this way we get $\mathrm{Sing} f[u] \subset \mathcal{C}(1, |q|)$.
Since A_0 is polynomial, its singularities are the zeroes of $\det A_0$. So let a such that $\det(A_0(a)) = (\det A_0)(a) = 0$ and let $X_0 \in \mathbf{C}^n$ non trivial such that $A_0(a)X_0 = 0$. Complete X_0 to a basis of $\mathbf{C}^n$, thus yielding $P \in \mathrm{GL}_n(\mathbf{C})$ such that P has first column X_0. Then the first column of $P^{-1}AP$ vanishes at a, so it is a multiple of $z - a$ in $\mathbf{C}[x]^n$. Now assume for instance that $|a| \geq |q|$ and use the gauge transformation $S := \mathrm{Diag}(z - a, 1, \ldots, 1)$: we see that $S[P^{-1}AP] = (SP^{-1})[A]$ has the same singularities as A except that one zero a of the determinant has been replaced by a/q. Iterating,

[1] The notion of resonancy comes from the local study of q-different systems, where one has to get rid of resonant *exponents* (eigenvalues of the fuchsian components of the system), [9].

we may move it to the fundamental annulus. The case where $|a| < 1$ is tackled similarly. In this way we get the wanted property. □

Note that, writing $B := F[A]$ we then have:

$$\left(F\mathcal{F}_A^{(0)} = \mathcal{F}_B^{(0)} \text{ and } F\mathcal{F}_A' = \mathcal{F}_B'\right) \Longrightarrow \mathcal{F}_A^{(0)}/\mathcal{F}_A' \simeq \mathcal{F}_B^{(0)}/\mathcal{F}_B' = \mathcal{F}_B^{(0)}/\mathcal{F}_B.$$

3.3.2 Computation of $\chi(\mathcal{F}_A')$ for a nonresonant system

Let $A \in \mathrm{GL}_n(\mathbf{C}(x))$ nonresonant, so that $\mathcal{F}_A' = \mathcal{F}_A$. We know that $\mathcal{F}_A \subset \mathcal{F}_A^{(0)}$. From the argument in the proof of lemma 3.12, we know that at every $x = \pi(a)$, $a \not\in \mathrm{Sing}(A)$, we have $(\mathcal{F}_A)_x = (\mathcal{F}_A^{(0)})_x$, so that the coherent sheaf $\mathcal{F}_A^{(0)}/\mathcal{F}_A$ is concentrated at $\pi(\mathrm{Sing}(A))$. However, if $a \in \mathrm{Sing}(A)$, the relation $X(qz) = A(z)X(z)$ shows that a solution X maybe holomorphic at a but not at qa and conversely, *i.e.* the inclusion $\mathcal{F}_A \subset \mathcal{F}_A^{(0)}$ will generally be strict. Now, since $\mathrm{Sing}(A) \cap [a;q] = \{a\}$, again arguing as in the proof of lemma 3.12, we know at least that the polarity of X is "the same" all along $q^{-\mathbf{N}}a$ (*i.e.*, if some $q^{-k}a$ with $k \in \mathbf{N}$ is a pole of X then any element of $q^{-\mathbf{N}}a$ is a pole of X), and also all along $q^{\mathbf{N}^*}a$,.

Lemma 3.14 *Let $R := \mathcal{O}_{\mathbf{C}^\times,a} = \mathcal{O}_{\mathbf{E}_q^{an},x}$ (thus a discrete valuation ring). Then we have isomorphisms of R-modules:*

$$\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x \simeq \frac{R^n}{R^n \cap A^{-1}R^n}, \quad \left(\mathcal{F}_A^{(\infty)}/\mathcal{F}_A\right)_x \simeq \frac{R^n}{R^n \cap AR^n}.$$

Proof. - If $a \not\in \mathrm{Sing}(A)$, the stalks are trivial and, since $A \in \mathrm{GL}_n(R)$, so are the modules $R^n/(R^n \cap A^{-1}R^n)$ and $R^n/(R^n \cap AR^n)$, so the isomorphisms are valid. We assume that $a \in \mathrm{Sing}(A)$.
Let U a connected neighborhood of a small enough that π induces a homeomorphism $U \to V := \pi(U)$ and that $U \cap \mathrm{Sing}(A) = \{a\}$. Then the restriction maps $\mathcal{F}_A(V) \to (\mathcal{F}_A)_x$ and $\mathcal{F}_A^{(0)}(V) \to (\mathcal{F}_A^{(0)})_x$ are bijective.
As we already saw, a solution $X \in \mathcal{F}_A^{(0)}(V)$ can be taken arbitrarily in $\mathcal{O}_{\mathbf{C}^\times}(U)^n$ and then extended using the functional equation $X(z) = A(z)^{-1}X(qz)$. This yields an identification of $(\mathcal{F}_A^{(0)})_x$ with R^n. Under this identification, the condition that X belongs to $\mathcal{F}_A(V)$ is that all $X(q^kz)$, $k \geq 1$ be holomorphic; since the polarity of a solution is the same along $q^{\mathbf{N}^*}a$, it is enough to check it for $X(qz) = A(z)X(z)$, *i.e.* it is enough to require that $AX \in R^n$, whence the identification of $(\mathcal{F}_A)_x$ with $R^n \cap A^{-1}R^n$ and, in the end, of $\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x$ with $R^n/(R^n \cap A^{-1}R^n)$.
The isomorphism at ∞ is proven in the same way. □

Lemma 3.15 *Let R a discrete valuation ring, u a uniformizer, K the fraction field of R and $A \in GL_n(K)$. Write $A = PDQ$, $P,Q \in GL_n(R)$ and $D = Diag(u^{d_1},\ldots,u^{d_n})$, $d_1 \leq \cdots \leq d_n$. Then we have isomorphisms of R-modules:*

$$\frac{R^n}{R^n \cap AR^n} \simeq \prod_{d_i > 0} \frac{R}{u^{d_i}R}, \quad \frac{R^n}{R^n \cap A^{-1}R^n} \simeq \prod_{d_i < 0} \frac{R}{u^{-d_i}R}.$$

Proof. - The given decomposition $A = PDQ$ comes from the theorem of invariant factors for finitely generated modules over principal ideal rings. We shall only prove the second formula, the first one being similar (and simpler). Since $P, Q \in \mathrm{GL}_n(R)$, one has $QR^n = P^{-1}R^n = R^n$ and:

$$\frac{R^n}{R^n \cap A^{-1}R^n} = \frac{R^n}{R^n \cap Q^{-1}D^{-1}P^{-1}R^n} \simeq \frac{QR^n}{Q(R^n \cap Q^{-1}D^{-1}P^{-1}R^n)} = \frac{QR^n}{QR^n \cap D^{-1}P^{-1}R^n} = \frac{R^n}{R^n \cap D^{-1}R^n},$$

and $R^n \cap D^{-1}R^n = \prod_{i=1}^{n}(R \cap u^{-d_i}R)$. Last:

$$R \cap u^k R = \begin{cases} R \text{ if } k \leq 0, \\ u^k R \text{ if } k > 0. \end{cases}$$

□

We introduce the following notations. Let $a \in \mathbf{C}^\times$ and $x = \pi(a)$. We write $O_x := O_{\mathbf{E}_q^{an},x} = O_{\mathbf{C}^\times,a}$. If $a \in \mathrm{Sing}(A)$ and $R := O_x$, we write $\Delta_a(A) = \Delta_x(A)$ the multiset of all the d_i appearing as exponents of the diagonal part D of A in the two lemmas above, and Δ^+, resp. Δ^- the submultisets of positive, resp. negative exponents. For a nonsingular a, we can consider that $\Delta_a(A)$ consists in n times 0, and that Δ^+, Δ^- are empty. To summarize:

Proposition 3.16 *The coherent sheaves $\mathcal{F}_A^{(0)}/\mathcal{F}_A$ and $\mathcal{F}_A^{(\infty)}/\mathcal{F}_A$ are supported at $\pi(Sing(A))$. They are skyscraper sheaves with stalks:*

$$\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x \simeq \prod_{d \in \Delta_x^-(A)} \frac{O_x}{u_x^{-d} O_x}, \quad \left(\mathcal{F}_A^{(\infty)}/\mathcal{F}_A\right)_x \simeq \prod_{d \in \Delta_x^+(A)} \frac{O_x}{u_x^{d} O_x}.$$

□

Corollary 3.17 *For a nonresonant $A \in GL_n(\mathbf{C}(x))$, the Euler characteristic of $\mathcal{F}_A$ is given by*

$$\chi(\mathcal{F}_A) = v_0(\det A) - \sum_{x \in \pi(Sing(A))} \dim_{\mathbf{C}} \left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x = v_0(\det A) + \sum_{a \in Sing(A)} \sum_{d \in \Delta_a^-(A)} d,$$

and also by

$$\chi(\mathcal{F}_A) = v_\infty(\det A) - \sum_{x \in \pi(Sing(A))} \dim_{\mathbf{C}} \left(\mathcal{F}_A^{(\infty)}/\mathcal{F}_A\right)_x = v_\infty(\det A) - \sum_{a \in Sing(A)} \sum_{d \in \Delta_a^+(A)} d.$$

Proof. - Since the cohomology in degree ≥ 1 of skyscraper sheaves is trivial, we obtain:

$$\chi\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right) = \sum_{x \in \pi(\mathrm{Sing}(A))} \dim_{\mathbf{C}} \left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x.$$

Using the additivity of the Euler characteristic, we get:

$$\chi(\mathcal{F}_A) = \chi(\mathcal{F}_A^{(0)}) - \sum_{x \in \pi(\mathrm{Sing}(A))} \dim_{\mathbf{C}} \left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x.$$

But, we have already seen that $\chi(\mathcal{F}_A^{(0)}) = v_0(\det A)$. Moreover, proposition 3.16 ensures that, for all $x \in \pi(\mathrm{Sing}(A))$, $\dim_{\mathbf{C}}\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x = \sum_{d\in\Delta_a^-(A)}(-d)$. Whence the first formula. The proof of the second formula is similar. □

3.3.3 Computation of $\chi(\mathcal{F}'_A)$ in the general case

We now release all resonancy conditions: we consider an arbitrary $A \in \mathrm{GL}_n(\mathbf{C}(x))$. We introduce the following notations for every $x \in \mathbf{E}_q^{an}$:

$$\begin{aligned}
\ell_x^-(A) &:= \dim_{\mathbf{C}}\left(\mathcal{F}_A^{(0)}/\mathcal{F}'_A\right)_x,\\
\ell_x^+(A) &:= \dim_{\mathbf{C}}\left(\mathcal{F}_A^{(\infty)}/\mathcal{F}'_A\right)_x,\\
\ell_x(A) &:= \ell_x^-(A) + \ell_x^+(A).
\end{aligned}$$

Note that the dimensions over $\mathbf{C}$ are as well lengths of O_x-modules. Also, if $x \notin \pi(\mathrm{Sing}(A))$, we have $\ell_x^-(A) = \ell_x^+(A) = \ell_x(A) = 0$.

Theorem 3.18 *(i) For an arbitrary $A \in GL_n(\mathbf{C}(x))$:*

$$\begin{aligned}
\chi(\mathcal{F}'_A) &= v_0(\det A) - \sum_{x\in\mathbf{E}_q^{an}} \ell_x^-(A),\\
\chi(\mathcal{F}'_A) &= v_\infty(\det A) - \sum_{x\in\mathbf{E}_q^{an}} \ell_x^+(A),\\
2\chi(\mathcal{F}'_A) &= v_0(\det A) + v_\infty(\det A) - \sum_{x\in\mathbf{E}_q^{an}} \ell_x(A).
\end{aligned}$$

(ii) For a nonresonant $A \in GL_n(\mathbf{C}(x))$:

$$\begin{aligned}
\chi(\mathcal{F}_A) &= \chi(\mathcal{F}'_A),\\
\ell_x^+(A) &= \sum_{d\in\Delta_x^+(A)} d,\\
\ell_x^-(A) &= \sum_{d\in\Delta_x^-(A)} (-d).
\end{aligned}$$

Proof. - We know that there exists $F \in \mathrm{GL}_n(\mathbf{C}(x))$ such that $B := F[A]$ is nonresonant. Since $F\mathcal{F}'_A = \mathcal{F}_B$ and $F\mathcal{F}_A^{(0)} = \mathcal{F}_B^{(0)}$, we see that $\mathcal{F}'_A \cong \mathcal{F}_B$ and $\mathcal{F}_A^{(0)}/\mathcal{F}'_A \cong \mathcal{F}_B^{(0)}/\mathcal{F}'_B$. Therefore, using proposition 3.17, we obtain:

$$\begin{aligned}
\chi\left(\mathcal{F}'_A\right) = \chi(\mathcal{F}'_B) &= v_0(\det B) - \sum_{x\in\mathbf{E}_q^{an}} \dim_{\mathbf{C}}\left(\mathcal{F}_B^{(0)}/\mathcal{F}_B\right)_x\\
&= v_0(\det A) - \sum_{x\in\mathbf{E}_q^{an}} \dim_{\mathbf{C}}\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x = v_0(\det A) - \sum_{x\in\mathbf{E}_q^{an}} \ell_x^-(A).
\end{aligned}$$

The proof of the second formula is similar and the third formula is an obvious consequence of the first and second ones.

The last part of the result, about the nonresonant case, is a direct consequence of the paragraph 3.3.2. $\Box$

3.4 Application to a formula of Sakai and Yamaguchi

3.4.1 Euler characteristics and index of rigidity

Recall from [9] that the "internal End" of a q-difference system $A \in \mathrm{GL}_n(\mathbf{C}(x))$ is defined as:

$$\underline{\mathrm{End}}(A) := A^\vee \otimes A,$$

where the "dual" $A^\vee$ of A is the contragredient ${}^tA^{-1}$. Some identification of $\mathbf{C}^n \otimes \mathbf{C}^n$ with $\mathbf{C}^{n^2}$ must be fixed in order to be able to consider $\underline{\mathrm{End}}(A)$ as a matrix $B \in \mathrm{GL}_{n^2}(\mathbf{C}(x))$. We intend here to compute $\chi(\mathcal{F}_B')$ and to compare it to the *rigidity index* introduced by Sakai and Yamaguchi in [11, §3]. An important preliminary fact is that, for any scalar $f \in \mathbf{C}(x)^\times$:

$$\underline{\mathrm{End}}(fA) = \underline{\mathrm{End}}(A).$$

Therefore, we may and will assume that A is polynomial and that its coefficients have no common factor. From general linear algebra, $\det B = (\det {}^tA^{-1})^n (\det A)^n = 1$. Obviously, $\mathrm{Sing} B = \mathrm{Sing} A$ (the inclusion might have been strict if the coefficients of A had a common factor); this singular set is the set of zeroes of $\det A$ over $\mathbf{C}^\times$. We write N the number of these zeroes, counted with multiplicities. Also, from now on, we assume A (and therefore B) to be nonresonant.

Let $a \in \mathrm{Sing} A$ and $A = PDQ$ the corresponding decomposition as in 3.3.2. Then $A^\vee = P^\vee D^\vee Q^\vee$ whence $B = (P^\vee \otimes P)(D^\vee \otimes D)(Q^\vee \otimes Q)$. Since $P^\vee \otimes P$ and $Q^\vee \otimes Q$ are regular at a, we see that $\Delta_a(B) = \Delta_a(A) - \Delta_a(A)$, meaning that if $\Delta_a(A)$ is the multiset $d_1 \leq \cdots \leq d_n$, then $\Delta_a(B)$ is the multiset of all $d_i - d_j$, $i, j = 1, \ldots, n$. Thus, writing $x := \pi(a)$:

$$\ell_x^\pm(B) = \sum_{1 \leq i < j \leq n} (d_j - d_i) = \sum_{1 \leq i < j \leq n} d_j - \sum_{1 \leq i < j \leq n} d_i = \sum_{1 \leq j \leq n} (j-1) d_j - \sum_{1 \leq n} (n-i) d_i = \sum_{i=1}^n (2i - 1 - n) d_i.$$

We deduce:

$$nv_a(\det A) - \frac{1}{2}\ell_x(B) = \sum_{i=1}^n (2n - 2i + 1) d_i = \sum_{i=1}^n (2i-1) d_{n-i+1} = d_n + 3d_{n-1} + \cdots + (2n-1) d_1 =$$

$$d_1(1 + 3 + \cdots + (2n-1)) + (d_2 - d_1)(1 + 3 + \cdots + (2n-3)) + \cdots$$
$$= n^2 d_1 + (n-1)^2 (d_2 - d_1) + \cdots + 1^2 (d_n - d_{n-1}),$$

that is $e_1^2 + \cdots + e_p^2$, where $p := d_n$ and $e_1, \ldots, e_p$ is the *dual Young tableau* of the *Young tableau* $d_1, \ldots, d_n$. Summing these equalities for all $a \in \mathrm{Sing}(A)$, we get, with obvious notations:

$$\chi(\mathcal{F}_B') = \sum_{a \in \mathrm{Sing}(A)} \sum e_i^2(a) - nN.$$

This is the part of Sakai-Yamaguchi's index of rigidity [11, §3] that depends only on intermediate singularities $a \in \mathrm{Sing}(A) \subset \mathbf{C}^{\times}$ and not on 0 and ∞.

3.4.2 Taking into account 0 and ∞

We shall now introduce a topological space $\widetilde{\mathbf{E}_q^{an}}$ and a sheaf on it in order to take into account 0 and ∞. We consider the set

$$\widetilde{\mathbf{E}_q^{an}} = \{0\} \sqcup \mathbf{E}_q^{an} \sqcup \{\infty\}.$$

We endow this set with the following topology: a basis of open sets is given by the open sets of $\mathbf{E}_q^{an}$, and by the subsets $\{0\}$ and $\{\infty\}$, so that $\widetilde{\mathbf{E}_q^{an}}$ has three connected components $\mathbf{E}_q^{an}$, $\{0\}$ and $\{\infty\}$. For any $A \in \mathrm{GL}_n(\mathbf{C}(x))$, we let $\widetilde{\mathcal{F}'_A}$ be the sheaf on $\widetilde{\mathbf{E}_q^{an}}$ such that $(\widetilde{\mathcal{F}'_A})_{|\mathbf{E}_q^{an}} = \mathcal{F}'_A$ and with stalks at 0 and ∞ given by $(\widetilde{\mathcal{F}'_A})_0 = \mathrm{Sol}(A, \mathbf{C}(\{z\}))$ and $(\widetilde{\mathcal{F}'_A})_\infty = \mathrm{Sol}(A, \mathbf{C}(\{z^{-1}\}))$. Then, we obviously have

$$\chi(\widetilde{\mathcal{F}'_A}) = \chi(\mathcal{F}'_A) + \dim_{\mathbf{C}} \mathrm{Sol}(A, \mathbf{C}(\{z\})) + \dim_{\mathbf{C}} \mathrm{Sol}(A, \mathbf{C}(\{z^{-1}\})).$$

We shall now apply this formula to the above $B \in \mathrm{GL}_{n^2}(\mathbf{C}(x))$ when A is regular singular at 0 and ∞ *i.e.* we assume that there exist $A^{(0)}, A^{(\infty)} \in \mathrm{GL}_n(\mathbf{C})$, $F^{(0)}(z) \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ and $F^{(\infty)}(z) \in \mathrm{GL}_n(\mathbf{C}(\{z^{-1}\}))$ such that

$$F^{(0)}(qz)A^{(0)} = A(z)F^{(0)}(z) \text{ and } F^{(\infty)}(qz)A^{(\infty)} = A(z)F^{(\infty)}(z).$$

We can and will assume that both $A^{(0)}$ and $A^{(\infty)}$ are nonresonant *i.e.* that, for any eigenvalue λ, μ of $A^{(0)}$ (resp. $A^{(\infty)}$), we have $\lambda/\mu \notin q^{\mathbf{Z}^*}$. Then, we have

$$\dim_{\mathbf{C}} \mathrm{Sol}(B, \mathbf{C}(\{z\})) = \dim_{\mathbf{C}} Z(A^{(0)}) \text{ and } \dim_{\mathbf{C}} \mathrm{Sol}(B, \mathbf{C}(\{z^{-1}\})) = \dim_{\mathbf{C}} Z(A^{(\infty)})$$

where $Z(\cdot)$ denotes the centralizer in $\mathrm{Mat}_n(\mathbf{C})$. So,

$$\chi(\widetilde{\mathcal{F}'_B}) = \sum_{a \in \mathrm{Sing}(A)} \sum e_i^2(a) + \dim_{\mathbf{C}} Z(A^{(0)}) + \dim_{\mathbf{C}} Z(A^{(\infty)}) - nN.$$

This is Sakai-Yamaguchi's index of rigidity [11, §3].

It would have been more natural to look for a connected topological space X (instead of the non connected $\widetilde{\mathbf{E}_q^{an}}$) and for a sheaf $\mathcal{F}$ on X such that $\chi(\mathcal{F})$ is the index of rigidity of A. Unfortunately, we were not able to find such a topological space. However, this led us to compute the Euler characteristics of natural "extensions" of $\mathcal{F}_A$ of independent interest; this is the content of the rest of the paper.

4 A natural extension of $\mathcal{F}'_A$ and its Euler characteristic

We consider the set

$$\overline{\mathbf{E}_q^{an}} = \{0\} \sqcup \mathbf{E}_q^{an} \sqcup \{\infty\}.$$

We endow this set with the following topology: the open sets of $\overline{\mathbf{E}_q^{an}}$ are the open subsets of $\mathbf{E}_q^{an}$, and the subsets $\{0\} \sqcup \mathbf{E}_q$, $\mathbf{E}_q \sqcup \{\infty\}$ and $\overline{\mathbf{E}_q^{an}}$.

We denote by $\overline{\varpi} : \mathbb{P}^1(\mathbf{C})^{an} \to \overline{\mathbf{E}_q^{an}}$ the natural continuous map.

Let V an open subset of $\overline{\mathbf{E}_q^{an}}$ and let $U := \overline{\varpi}^{-1}(V)$. We consider the subsheaf $\overline{\mathcal{A}'}(V)$ of $\overline{\varpi}_* \mathcal{M}_{\mathbb{P}^1(\mathbf{C})^{an}}$ given by

$$\overline{\mathcal{A}'}(V) := \mathcal{A}'(V \cap \mathbf{E}_q^{an}) \cap \mathcal{M}_{\mathbb{P}^1(\mathbf{C})^{an}}(U). \tag{4.0.1}$$

Note that

$$\overline{\mathcal{A}'}_{|\mathbf{E}_q^{an}} = \mathcal{A}'.$$

As in paragraph 3.1.3, we associate to the sheaf $\overline{\mathcal{A}'}$ the sheaf of solutions on $\overline{\mathbf{E}_q^{an}}$ denoted by $\overline{\mathcal{F}'_A}$. The sections of this sheaf on an open subset V of $\overline{\mathbf{E}_q^{an}}$ are given by

$$\overline{\mathcal{F}'_A}(V) = \left\{ F \in (\overline{\mathcal{A}'}(V))^n \mid \forall z \in \overline{\varpi}^{-1}(V), \sigma_q(F)(z) = A(z)F(z) \right\}.$$

Note that

$$(\overline{\mathcal{F}'_A})_{|\mathbf{E}_q^{an}} = \mathcal{F}'_A.$$

In order to compute the Euler characteristic of this sheaf, we will need the following lemmas.

Lemma 4.1 *Any sheaf of abelian groups on the topological subspace $\{0\} \sqcup \mathbf{E}_q^{an}$ (resp. $\mathbf{E}_q^{an} \sqcup \{\infty\}$) of $\overline{\mathbf{E}_q^{an}}$ is acyclic.*

Proof. - Let $\mathcal{F}$ be a sheaf of abelian groups on $\{0\} \sqcup \mathbf{E}_q^{an}$. Let $0 \to \mathcal{F} \to I^{\cdot}$ be an injective resolution of $\mathcal{F}$. Taking the stalks at 0, we get the exact sequence $0 \to \mathcal{F}_0 \to I_0^{\cdot}$. Since $\{0\} \sqcup \mathbf{E}_q^{an}$ is the only open subset of $\{0\} \sqcup \mathbf{E}_q^{an}$ containing 0, we see that the stalks at 0 coincide with the global sections so that the sequence of global sections obtained from $0 \to \mathcal{F} \to I^{\cdot}$ is exact, whence the result. □

Lemma 4.2 *Let $\mathcal{F}$ be a sheaf of $\mathbf{C}$-vector spaces on $\overline{\mathbf{E}_q^{an}}$. Assume that $H^0(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{F})$, $H^0(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{F})$ and any $H^k(\mathbf{E}_q^{an}, \mathcal{F})$ are finite dimensional. Then, the $H^k(\overline{\mathbf{E}_q^{an}}, \mathcal{F})$ are finite dimensional and we have*

$$\chi(\overline{\mathbf{E}_q^{an}}, \mathcal{F}) = -\chi(\mathbf{E}_q^{an}, \mathcal{F}_{|\mathbf{E}_q^{an}}) + h^0(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{F}) + h^0(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{F}).$$

Proof. - The Mayer-Vietoris long exact sequence for $\mathcal{F}$ with respect the open covering $\{\{0\} \sqcup \mathbf{E}_q^{an}, \mathbf{E}_q^{an} \sqcup \{\infty\}\}$ of $\overline{\mathbf{E}_q^{an}}$ reads as follow:

$$\cdots \to H^{k-1}(\mathbf{E}_q^{an}, \mathcal{F}) \to H^k(\overline{\mathbf{E}_q^{an}}, \mathcal{F}) \to H^k(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{F}) \oplus H^k(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{F})$$
$$\to H^k(\mathbf{E}_q^{an}, \mathcal{F}) \to H^{k+1}(\overline{\mathbf{E}_q^{an}}, \mathcal{F}) \to \cdots$$

Using Lemma 4.1, we see that, for all $k \geq 1$, $H^k(\mathbf{E}_q^{an}, \mathcal{F})$ and $H^{k+1}(\overline{\mathbf{E}_q^{an}}, \mathcal{F})$ are isomorphic $\mathbf{C}$-vector spaces. Moreover, the first terms of the Mayer-Vietoris sequence give:

$$\cdots 0 \to H^0(\overline{\mathbf{E}_q^{an}}, \mathcal{F}) \to H^0(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{F}) \oplus H^0(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{F}) \\ \to H^0(\mathbf{E}_q^{an}, \mathcal{F}) \to H^1(\overline{\mathbf{E}_q^{an}}, \mathcal{F}) \to 0$$

So that,

$$\begin{aligned} h^1(\overline{\mathbf{E}_q^{an}}, \mathcal{F}) &= h^0(\mathbf{E}_q^{an}, \mathcal{F}) - (h^0(\{0\} \sqcup \mathbf{E}_q, \mathcal{F}) + h^0(\mathbf{E}_q \sqcup \{\infty\}, \mathcal{F})) + h^0(\overline{\mathbf{E}_q^{an}}, \mathcal{F}) \\ &= h^0(\mathbf{E}_q^{an}, \mathcal{F}) - (h^0(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{F}) + h^0(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{F})) + h^0(\overline{\mathbf{E}_q^{an}}, \mathcal{F}). \end{aligned}$$

□

Applying this lemma to $\overline{\mathcal{F}'_A}$ and using theorem **??**, we get the following result:

Theorem 4.3 *For an arbitrary $A \in GL_n(\mathbf{C}(x))$:*

$$\begin{aligned} \chi(\overline{\mathcal{F}'_A}) &= -v_0(\det A) + \sum_{x \in \mathbf{E}_q^{an}} \ell_x^-(A) + \dim_{\mathbf{C}} Sol(A, R_0) + \dim_{\mathbf{C}} Sol(A, R_\infty), \\ \chi(\overline{\mathcal{F}'_A}) &= -v_\infty(\det A) + \sum_{x \in \mathbf{E}_q^{an}} \ell_x^+(A) + \dim_{\mathbf{C}} Sol(A, R_0) + \dim_{\mathbf{C}} Sol(A, R_\infty), \\ 2\chi(\overline{\mathcal{F}'_A}) &= -v_0(\det A) - v_\infty(\det A) + \sum_{x \in \mathbf{E}_q^{an}} \ell_x(A) + 2\dim_{\mathbf{C}} Sol(A, R_0) + 2\dim_{\mathbf{C}} Sol(A, R_\infty) \end{aligned}$$

where R_0 (resp. R_∞) is the $\mathbf{C}$-vector space of meromorphic functions over $\mathbf{C}$ (resp. $\mathbf{C}^\times \sqcup \{\infty\}$) with at most finitely many poles on any q-spiral $[a;q] \subset \mathbf{C}^\times$.

5 Another natural extension of $\mathcal{F}'_A$

Let $\mathbf{E}_q^{an'}$ be a copy of $\mathbf{E}_q^{an}$. We consider the set

$$X = \{0\} \sqcup \mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}.$$

We endow X with the following topology: the open subsets of X are of the form

- $U \sqcup U'$ where U (resp. U') is a subset of $\mathbf{E}_q^{an}$ (resp. $\mathbf{E}_q^{an'}$) such that $U' \subset U$;
- $\{0\} \sqcup \mathbf{E}_q^{an} \sqcup U'$ where U' is an open subset of $\mathbf{E}_q^{an'}$;
- $\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup U'$ where U' is an open subset of $\mathbf{E}_q^{an'}$;
- $\{0\} \sqcup \mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup U'$ where U' is an open subset of $\mathbf{E}_q^{an'}$.

We let $\mathcal{B}$ be the sheaf on X whose sections on an open subset V of X are the meromorphic functions $f(z)$ on $\pi^{-1}(V \cap \mathbf{E}_q^{an})$ (where $\pi : \mathbf{C}^\times \to \mathbf{E}_q^{an}$ is the natural projection) such that

- $f(z)$ is meromorphic at 0 if $0 \in V$;
- $f(z)$ is meromorphic at ∞ if $\infty \in V$;
- $f(z)$ is meromorphic on $\pi'^{-1}(V \cap \mathbf{E}'_q)$ with at most finitely many poles on any q-spiral $[a;q] \subset \pi'^{-1}(V \cap \mathbf{E}'_q)$ (where $\pi' : \mathbf{C}^\times \to \mathbf{E}_q^{an'}$ is the natural projection).

The restriction maps are the natural ones (restriction of functions).

The corresponding sheaf of solutions on X is given, for any open subset V of X, by

$$\mathcal{H}'_A(V) = \left\{ F \in (\mathcal{B}(V))^n \mid \forall z \in \pi^{-1}(V \cap \mathbf{E}_q^{an}), \sigma_q(F)(z) = A(z)F(z) \right\}.$$

This section is devoted to the proof of the following result.

Theorem 5.1 *We have:*

- *for all* $k \geq 2$, $H^k(X, \mathcal{H}'_A) = 0$;
- $\dim_{\mathbf{C}} H^1(X, \mathcal{H}'_A) = \infty$;
- $H^0(X, \mathcal{H}'_A) = Sol(A, \mathbf{C}(x))$;

In order to prove this result, we state and prove some lemmas.

5.1 Lemmas

We let Y be the topological subspace of X given by

$$Y = \mathbf{E}_q^{an} \sqcup \mathbf{E}_q^{an'}.$$

We denote by

$$i : \mathbf{E}_q^{an} \to Y \text{ and } j : \mathbf{E}_q^{an'} \to Y$$

the natural (continuous) inclusions.

Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathbf{E}_q^{an}$. We set

$$\widetilde{\mathcal{F}} = i_* \mathcal{F}$$

Lemma 5.2 *Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathbf{E}_q^{an}$. We have, for all $k \geq 0$,*

$$H^k(Y, \widetilde{\mathcal{F}}) = H^k(\mathbf{E}_q^{an}, \mathcal{F}).$$

Proof. - This follows from the facts that the direct image functor i_* is exact (be careful, since i is the inclusion of an open subset, the exactness of i_* is not a general fact but is true in our special case) and sends flasque sheaves on flasque sheaves (this is a general fact for direct images). □

Lemma 5.3 *Let $\mathcal{F}$ be a sheaf of abelian groups on Y. Assume that $i^{-1}\mathcal{F}$ is acyclic. Then, for all $k \geq 0$, we have*

$$H^k(Y, \mathcal{F}) \cong H^k(\mathbf{E}_q^{an'}, j^{-1}\mathcal{F}).$$

Proof. - We start with the exact sequence ([6, II, Exercise 1.19])

$$0 \to i_! i^{-1} \widetilde{i^{-1}\mathcal{F}} = i_! i^{-1}\mathcal{F} \to \widetilde{i^{-1}\mathcal{F}} \to j_* j^{-1} \widetilde{i^{-1}\mathcal{F}} \to 0, \tag{5.3.1}$$

where $i_!$ is the extension by zero outside $\mathbf{E}_q^{an}$ functor. Note that

$$H^k(Y, j_* j^{-1} \widetilde{i^{-1}\mathcal{F}}) = H^k(\mathbf{E}_q^{an'}, j^{-1} \widetilde{i^{-1}\mathcal{F}}) = H^k(\mathbf{E}_q^{an}, i^{-1}\mathcal{F}).$$

(The first equality is general [6, III, Lemma 2.10] because $\mathbf{E}_q^{an'}$ is closed in Y; the second one follows from the fact that there is an obvious identification of the topological space $\mathbf{E}_q^{an'}$ with $\mathbf{E}_q^{an}$, and that $j^{-1}\widetilde{i^{-1}\mathcal{F}}$ corresponds to $i^{-1}\mathcal{F}$ under this identification.) Since $i^{-1}\mathcal{F}$ is acyclic, we get that $j_* j^{-1}\widetilde{i^{-1}\mathcal{F}}$ is acyclic. Considering the long exact sequence of cohomology group obtained from (5.3.1), we obtain that, for all $k \geq 2$,

$$H^k(Y, i_! i^{-1}\mathcal{F}) = H^k(Y, \widetilde{i^{-1}\mathcal{F}}) = 0$$

(the last equality follows from Lemma 5.2 and from the fact that $i^{-1}\mathcal{F}$ is acyclic by assumption) and we also obtain the exact sequence

$$0 \to H^0(Y, i_! i^{-1}\mathcal{F}) \to H^0(Y, \widetilde{i^{-1}\mathcal{F}}) \to H^0(Y, j_* j^{-1}\widetilde{i^{-1}\mathcal{F}}) \to H^1(Y, i_! i^{-1}\mathcal{F}) \to H^1(Y, \widetilde{i^{-1}\mathcal{F}}) \to \cdots$$

But, we have

$$H^0(Y, j_* j^{-1}\widetilde{i^{-1}\mathcal{F}}) = H^0(\mathbf{E}_q^{an'}, j^{-1}\widetilde{i^{-1}\mathcal{F}}) = H^0(Y, \widetilde{i^{-1}\mathcal{F}})$$

and the map $H^0(Y, \widetilde{i^{-1}\mathcal{F}}) \to H^0(Y, j_* j^{-1}\widetilde{i^{-1}\mathcal{F}})$ is actually the identity. Moreover, lemma 5.2 ensures that $H^1(Y, \widetilde{i^{-1}\mathcal{F}}) = H^1(\mathbf{E}_q^{an}, i^{-1}\mathcal{F}) = 0$. It follows that

$$H^1(Y, i_! i^{-1}\mathcal{F}) = 0.$$

Using the long exact sequence of cohomology groups obtained from the exact sequence

$$0 \to i_! i^{-1}\mathcal{F} \to \mathcal{F} \to j_* j^{-1}\mathcal{F} \to 0,$$

we get, for all $k \geq 1$,

$$H^k(Y, \mathcal{F}) = H^k(Y, j_* j^{-1}\mathcal{F}) = H^k(\mathbf{E}_q^{an'}, j^{-1}\mathcal{F}).$$

This equality is obviously true for $k = 0$. $\square$

5.2 Proof of theorem 5.1

In order to compute the cohomology of $\mathcal{H}'_{\mathcal{A}}$ on X, we first use the Mayer-Vietoris long exact sequence for the open covering $\{\{0\} \sqcup \mathbf{E}_q^{an}, \mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}\}$ of X:

$$\cdots \to H^{k-1}(\mathbf{E}_q^{an}, \mathcal{H}'_A) \to H^k(X, \mathcal{H}'_A) \to H^k(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{H}'_A) \oplus H^k(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A)$$
$$\to H^k(\mathbf{E}_q^{an}, \mathcal{H}'_A) \to H^{k+1}(X, \mathcal{H}'_A) \to \cdots$$

But, for $k \geq 2$, we have $H^{k-1}(\mathbf{E}_q^{an}, \mathcal{H}'_A) = H^k(\mathbf{E}_q^{an}, \mathcal{H}'_A) = 0$ (because the restriction of $\mathcal{H}'_A$ to $\mathbf{E}_q^{an}$ is a meromorphic fiber bundle) and $H^k(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{H}'_A) = 0$ (follows from lemma 4.1), so

$$H^k(X, \mathcal{H}'_A) = H^k(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A).$$

Now, we use the Mayer-Vietoris long exact sequence for the open covering $\{\mathbf{E}_q^{an} \sqcup \{\infty\}, Y\}$ of $\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}$:

$$\cdots \to H^{k-1}(\mathbf{E}_q^{an}, \mathcal{H}'_A) \to H^k(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A) \to H^k(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{H}'_A) \oplus H^k(Y, \mathcal{H}'_A)$$
$$\to H^k(\mathbf{E}_q^{an}, \mathcal{H}'_A) \to H^{k+1}(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A) \to \cdots$$

Arguing as above, we get that, for $k \geq 2$,

$$H^k(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A) = H^k(Y, \mathcal{H}'_A).$$

But, $i^{-1}\mathcal{H}'_A$ is acyclic (it is a meromorphic vector bundle on $\mathbf{E}_q^{an}$), so lemma 5.3 ensures that, for $i \geq 0$,

$$H^k(Y, \mathcal{H}'_A) = H^k(\mathbf{E}_q^{an'}, j^{-1}\mathcal{H}'_{\mathcal{A}}).$$

Therefore, we have proved that, for $k \geq 2$,

$$H^k(X, \mathcal{H}'_A) = H^k(Y, \mathcal{H}'_A) = H^k(\mathbf{E}_q^{an'}, j^{-1}\mathcal{H}'_A) = 0. \tag{5.3.2}$$

Moreover, the first terms of the first Mayer-Vietoris sequence above gives:

$$0 \to H^0(X, \mathcal{H}'_A) \to H^0(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{H}'_A) \oplus H^0(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A) \to H^0(\mathbf{E}_q^{an}, \mathcal{H}'_A)$$
$$\to H^1(X, \mathcal{H}'_A) \to \cdots$$

But $H^0(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{H}'_A)$ and $H^0(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A)$ are finite dimensional $\mathbf{C}$-vector spaces, whereas $H^0(\mathbf{E}_q^{an}, \mathcal{H}'_A)$ is infinite dimensional. Therefore, $H^1(X, \mathcal{H}'_A)$ is infinite dimensional.

The last assertion of the theorem is obvious. □

References

[1] D. Bertrand. Extensions de $\mathcal{D}$-modules et groupes de Galois differentiels. (Extensions of $\mathcal{D}$-modules and differential Galois groups). p-adic analysis, Proc. Int. Conf., Trento/Italy 1989, Lect. Notes Math. 1454, 125-141 (1990)., 1990.

[2] Daniel Bertrand. Un analogue différentiel de la théorie de Kummer. In *Approximations diophantiennes et nombres transcendants. Comptes-rendus du colloque tenu au C.I.R.M. de Luminy, France, 18-22 Juin, 1990*, pages 39–49. Berlin etc.: de Gruyter, 1992.

[3] Pierre Deligne. Equations différentielles à points singuliers réguliers. Lecture Notes in Mathematics. 163. Berlin-Heidelberg-New York: Springer-Verlag. 133 p. DM 12.00; $ 3.30 (1970)., 1970.

[4] Gerd Fischer. Complex analytic geometry. Lecture Notes in Mathematics. 538. Berlin-Heidelberg-New York: Springer-Verlag. VII, 201 p. (1976)., 1976.

[5] R.C. Gunning. Lectures on vector bundles over Riemann surfaces. Preliminary informal notes of university courses and seminars in mathematics. Mathematical Notes. Princeton: Princeton University Press and the University of Tokyo Press. V, 243 p. (1967)., 1967.

[6] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[7] N. M. Katz. *Rigid local systems*, volume 139 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[8] C. Praagman. Fundamental solutions for meromorphic linear difference equations in the complex plane, and related problems. *Journal für Mathematik*, (369):101–109, 1985.

[9] J.-P. Ramis, J. Sauloy, and C. Zhang. Local analytic classification of irregular q-difference equations. *Astérisque*, 355, 2013.

[10] M. Raynaud. Caractéristique d'Euler-Poincaré d'un faisceau et cohomologie des variétés abéliennes. Dix Exposés Cohomologie Schémas, Advanced Studies Pure Math. 3, 12-30 (1968) = Sém. Bourbaki 17 (1964/65), No.286, 19 p. (1966)., 1966.

[11] H. Sakai and M. Yamaguchi. Spectral types of linear q-difference equations and q-analog of middle convolution. arXiv:1410.3674.