FROBENIUS METHOD FOR MAHLER EQUATIONS

JULIEN ROQUES

ABSTRACT. Using Hahn series, one can attach to any linear Mahler equation a basis of solutions at 0 reminiscent of the solutions of linear differential equations at a regular singularity. We show that such a basis of solutions can be produced by using a variant of Frobenius method.

CONTENTS

Date: November 10, 2022.
2010 *Mathematics Subject Classification.* 39A06,12H10.
Key words and phrases. Linear difference equations.

1. Introduction

A linear Mahler equation with coefficients in $\mathbb{C}(z)$ is a functional equation of the form

$$a_n(z)y(z^{p^n}) + a_{n-1}(z)y(z^{p^{n-1}}) + \cdots + a_0(z)y(z) = 0 \tag{1}$$

for some $p \in \mathbb{Z}_{\geq 2}$, some $n \in \mathbb{Z}_{\geq 0}$ and some $a_0(z), \ldots, a_n(z) \in \mathbb{C}(z)$ such that $a_0(z)a_n(z) \neq 0$. These equations appear for instance in the theory of automatic sequences : the generating series of any automatic sequence satisfies a nontrivial linear Mahler equation with coefficients in $\mathbb{C}(z)$. The following list of references gives an idea of the many facets of the theory of Mahler equations [Mah29, Mah30, Kub77, LvdP78, Mas82, Ran92, Dum93, Bec94, Nis96, DF96, Zan98, CZ02, AS03, Pel09, Ngu11, Ngu12, Phi15, SS16, BCZ16, AB17, AF17, AF18, DHR18, CDDM18, BCCD18, Fer18, Roq20, Pou20].

In [Roq20], we have shown the relevance of Hahn series for the study of Mahler equations. It follows from [Roq20, Theorem 2] that (1) has n $\mathbb{C}$-linearly independent solutions $y_1(z), \ldots, y_n(z)$ of the form

$$y_i(z) = \sum_{(c,j) \in \mathbb{C}^\times \times \mathbb{Z}_{\geq 0}} f_{i,c,j}(z)e_c(z)\ell(z)^j \tag{2}$$

where the terms involved in this finite sum have the following properties :

— the $f_{i,c,j}(z)$ belong to the field $\mathscr{H}$ of Hahn series with coefficients in $\mathbb{C}$ and value group $\mathbb{Q}$;
— $e_c(z)$ satisfies $e_c(z^p) = ce_c(z)$;
— $\ell(z)$ satisfies $\ell(z^p) = \ell(z) + 1$.

Remark 1. *If $c \in \mathbb{C}^\times \setminus \{1\}$, then the equation $y(z^p) = cy(z)$ has no nonzero solution in $\mathscr{H}$. Similarly, the equation $y(z^p) = y(z)+1$ has no solution in $\mathscr{H}$. In this paper, the $e_c(z)$ and $\ell(z)$ will be "symbols", i.e., they will be constructed algebraically as elements of a certain difference field extension of $\mathscr{H}$; see Section 3. However, note that one can find "true functions" solutions of these equations; for instance, $(\log z)^{\frac{\log c}{\log p}}$ is a solution of $y(z^p) = cy(z)$ and $\frac{\log \log z}{\log p}$ is a solution of $y(z^p) = y(z)+1$.*

The shape of the $y_i(z)$ is reminiscent of the shape of the solutions of a linear differential equation at a *regular singularity*. Indeed, we recall that any linear differential equation of order n with coefficients in $\mathbb{C}(z)$ having at worst a regular singularity at 0 has n $\mathbb{C}$-linearly independent solutions $\widetilde{y}_1(z),\ldots,\widetilde{y}_n(z)$ of the form

$$\widetilde{y}_i(z)=\sum_{(\alpha,j)\in\mathbb{C}\times\mathbb{Z}_{\geq 0}}\widetilde{f}_{i,\alpha,j}(z)z^{\alpha}\log(z)^j \tag{3}$$

where the $\widetilde{f}_{i,\alpha,j}(z)$ involved in this finite sum belong to the field of formal Laurent series $\mathbb{C}((z))$. It is thus natural to wonder wether a basis of solutions of the form (2) can be derived by using a variant of the celebrated Frobenius method [Fro73]. The aim of the present paper is to bring a positive answer to this question.

Remark 2. *The original Frobenius method produces solutions of linear differential equations at a regular singular point. We remind that the fact that a given point is regular singular can be read from a certain Newton polygon: it corresponds to the case where this polygon has only one (finite) slope and that this slope is equal to* 0*. See Section 2 for references. It turns out that such restrictions are not necessary in the Mahler case: the method introduced in the present paper works for any Mahler equation with coefficients in the field $\mathscr{H}$ of Hahn series.*

1.1. **Organization of the paper.** Section 2 provides an overview of the classical Frobenius method for linear differential equations. An extension of Frobenius method to Mahler equations is presented in Section 5. Its statement requires notions and notations introduced in Section 3 and Section 4 (Section 3 introduces rings where the calculations of the method will take place, Section 4 introduces the notions of Newton polygons, slopes and exponents for Mahler equations). A simple but non trivial example of application of the method is given in Section 6. The proofs of the unjustified statements of Section 5 occupy Sections 7 and 8.

Acknowledgement. I would like to thank the anonymous referee for its useful comments. This work was supported by the ANR De rerum natura project, grant ANR-19-CE40-0018 of the French Agence Nationale de la Recherche.

2. Frobenius method for regular singular linear differential equations

In the theory of differential equations, Frobenius left his name to an important method for computing a basis of solutions of any linear differential equation at a regular singularity. Here is an outline of this method; we refer to [CL55, IKSY91, oM, Fro73] for more details.

Consider a (formal) linear differential operator

$$\mathcal{L}=\delta^n+b_{n-1}(z)\delta^{n-1}+\cdots+b_0(z) \tag{4}$$

with coefficients $b_0(z),\ldots,b_{n-1}(z)$ in the field $\mathbb{C}((z))$ of formal Laurent series written in terms of Euler operator $\delta=z\frac{\mathrm{d}}{\mathrm{d}z}$. As mentioned above, we will focus our attention on the regular singular case, *i.e.*, from now on, we assume that $b_0(z), \dots, b_{n-1}(z)$ belong to the ring $\mathbb{C}[[z]]$ of formal power series (see [vdPS03, Definition 3.14]).

Remark 3. *One classically attaches to $\mathcal{L}$ a Newton polygon $\mathcal{N}(\mathcal{L}) \subset \mathbb{R}^2$ and a (finite) set of slopes $\mathcal{S}(\mathcal{L}) \subset \mathbb{Q}$; see* [Per11, HS99] *and* [vdPS03, §3.3]. *The following properties are equivalent:*
— *$\mathcal{L}$ is regular singular;*
— *$b_0(z), \dots, b_{n-1}(z)$ belong to $\mathbb{C}[[z]]$;*
— *$\mathcal{S}(\mathcal{L}) = \{0\}$.*
See [vdPS03, Definition 3.14] *and* [vdPS03, Exercise 3.47].

The indicial polynomial of $\mathcal{L}$ at 0 is the polynomial of degree n given by

$$\chi(\mathcal{L}; X) = X^n + b_{n-1}(0)X^{n-1} + \cdots + b_0(0).$$

The complex roots of $\chi(\mathcal{L}; X)$ are called the exponents of $\mathcal{L}$ at 0. The multiplicity of such an exponent α is by definition its multiplicity as a root of $\chi(\mathcal{L}; X)$ and will be denoted by m_α. If $\alpha \in \mathbb{C}$ is not an exponent of $\mathcal{L}$ at 0 then we set $m_\alpha = 0$.

The Frobenius method attaches m_α solutions of $\mathcal{L}$ to any exponent α as follows. The fundamental idea is to introduce a parameter λ. One can prove that:

— there exists a unique $g_\alpha(\lambda, z) \in 1 + z\mathbb{C}(\lambda)[[z]]$ such that

$$\mathcal{L}(g_\alpha(\lambda, z)z^\lambda) = (\lambda - \alpha)^{s_\alpha + m_\alpha} z^\lambda$$

where

$$s_\alpha = \sum_{\beta \in \alpha + \mathbb{Z}_{\geq 1}} m_\beta;$$

— the coefficients of $g_\alpha(z, \lambda)$ have no pole at $\lambda = \alpha$;

— the derivatives

$$\partial_\lambda^{s_\alpha}(g_\alpha(\lambda, z)z^\lambda)_{|\lambda=\alpha},$$
$$\partial_\lambda^{s_\alpha+1}(g_\alpha(\lambda, z)z^\lambda)_{|\lambda=\alpha},$$
$$\dots$$
$$\partial_\lambda^{s_\alpha+m_\alpha-1}(g_\alpha(\lambda, z)z^\lambda)_{|\lambda=\alpha}$$

are m_α $\mathbb{C}$-linearly independent solutions of $\mathcal{L}$, where we have used the notation $\partial_\lambda = \frac{\partial}{\partial \lambda}$.

We then have :

Theorem 4 (Frobenius [Fro73])**.** *We have attached to any exponent α and to any $m \in \{0, \dots, m_\alpha - 1\}$ a solution*

$$y_{\alpha,m} = \partial_\lambda^{s_\alpha+m}(g_\alpha(\lambda, z)z^\lambda)_{|\lambda=\alpha}$$

of $\mathcal{L}$. We obtain in this way a family of n $\mathbb{C}$-linearly independent solutions of $\mathcal{L}$.

Using the Leibniz rule, we see that

$$\begin{aligned} y_{\alpha,m} &\in \operatorname{Span}_{\mathbb{C}((z))}(\partial_\lambda^0(z^\lambda)_{|\lambda=\alpha}, \partial_\lambda^1(z^\lambda)_{|\lambda=\alpha}, \ldots, \partial_\lambda^{s_\alpha+m}(z^\lambda)_{|\lambda=\alpha}) \\ &= \operatorname{Span}_{\mathbb{C}((z))}(z^\alpha, \log(z)z^\alpha, \ldots, (\log(z))^{s_\alpha+m}z^\alpha). \end{aligned}$$

Remark 5. *The solutions attached to α involve $\log(z)$ if α is an exponent of $\mathcal{L}$ of multiplicity ≥ 2 (i.e., $m_\alpha \geq 2$). But, this is not the only case : logarithms may also appear in the solutions attached to α if $\alpha + \mathbb{Z}_{\geq 1}$ contains at least one exponent of $\mathcal{L}$ (i.e., $s_\alpha \geq 1$).*

3. Hahn series and other useful rings

The calculations involved in the classical Frobenius method outlined in Section 2 take place in the differential ring

$$\mathbb{C}(\lambda)((z))[z^\lambda, \partial_\lambda(z^\lambda), \partial_\lambda^2(z^\lambda), \ldots]$$

equipped with the derivatives $\partial_z = \frac{\partial}{\partial z}$ and $\partial_\lambda = \frac{\partial}{\partial \lambda}$. This section aims at introducing an avatar of this differential ring in which the calculations of our variant of Frobenius method for Mahler equations will take place.

3.1. The ring of Hahn series.

The first fundamental fact is that the classical power series will not be sufficient for our purpose; we will need Hahn series [Hah07].

Let R be a ring. We denote by $\mathscr{H}_R$ the ring of Hahn series with coefficients in R and with value group $\mathbb{Q}$. An element of $\mathscr{H}_R$ is a sequence $(f_\gamma)_{\gamma\in\mathbb{Q}} \in R^{\mathbb{Q}}$ whose support

$$\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) = \{\gamma \in \mathbb{Q} \mid f_\gamma \neq 0\}$$

is well-ordered, *i.e.*, any nonempty subset of $\operatorname{supp}(f)$ has a least element. An element $(f_\gamma)_{\gamma\in\mathbb{Q}}$ of $\mathscr{H}_R$ is usually (and will be) denoted by

$$f(z) = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma.$$

The sum and product of two elements $f(z) = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma$ and $g(z) = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma$ of $\mathscr{H}_R$ are respectively defined by

$$f(z) + g(z) = \sum_{\gamma\in\mathbb{Q}} (f_\gamma + g_\gamma) z^\gamma$$

and

$$f(z)g(z) = \sum_{\gamma\in\mathbb{Q}} \left(\sum_{\gamma'+\gamma''=\gamma} f_{\gamma'} g_{\gamma''} \right) z^\gamma.$$

(Note that there are only finitely many $(\gamma', \gamma'') \in \mathbb{Q} \times \mathbb{Q}$ such that $\gamma' + \gamma'' = \gamma$ and $f_{\gamma'} g_{\gamma''} \neq 0$.)

If R is an integral domain (resp. a field), then $\mathscr{H}_R$ is an integral domain (resp. a field) as well.

For $R=\mathbb{C}$, we will use the shorthand notation

$$\mathscr{H}=\mathscr{H}_{\mathbb{C}}.$$

The z-adic valuation on $\mathscr{H}_R$ will be denoted by

$$\mathrm{val}_z : \mathscr{H}_R \to \mathbb{Q}\cup\{+\infty\}.$$

It is given, for any $f(z)\in\mathscr{H}_R$, by

$$\mathrm{val}_z(f(z)) = \min \mathrm{supp}(f(z))$$

with the convention $\min\emptyset = +\infty$.

For $f(z)=\sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}_R\setminus\{0\}$, we will denote by $\mathrm{cld}_{\mathrm{z}}(f(z))$ the coefficient of $z^{\mathrm{val}_z(f(z))}$ in $f(z)$:

$$\mathrm{cld}_{\mathrm{z}}(f(z)) = f_{\mathrm{val}_z(f(z))} \in R\setminus\{0\}.$$

We will say that $f(z)\in\mathscr{H}_R\setminus\{0\}$ is tangent to the identity if $\mathrm{val}_z(f(z))=0$ and $\mathrm{cld}_{\mathrm{z}}(f(z))=1$.

3.2. Structure of difference ring on the ring of Hahn series $\mathscr{H}_R$.

One can endow $\mathscr{H}_R$ with the ring automorphism

$$\begin{array}{cccc} \phi_p : & \mathscr{H}_R & \to & \mathscr{H}_R \\ & f(z)=\displaystyle\sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma & \mapsto & \displaystyle\sum_{\gamma\in\mathbb{Q}} f_\gamma z^{p\gamma}. \end{array} \tag{5}$$

We obtain in this way a structure of difference field on $\mathscr{H}_R$.

3.3. The differential-difference ring $\mathscr{H}_{\mathbb{C}(\lambda)}$.

We will make an essential use of the field $\mathscr{H}_{\mathbb{C}(\lambda)}$ of Hahn series with coefficients in the field of rational functions $\mathbb{C}(\lambda)$ and with value group $\mathbb{Q}$.

One can endow $\mathscr{H}_{\mathbb{C}(\lambda)}$ with the field automorphism

$$\phi_p : \mathscr{H}_{\mathbb{C}(\lambda)} \to \mathscr{H}_{\mathbb{C}(\lambda)} \tag{6}$$

defined by (5) and with the derivation

$$\begin{array}{cccc} \partial_\lambda = \dfrac{\partial}{\partial\lambda} : & \mathscr{H}_{\mathbb{C}(\lambda)} & \to & \mathscr{H}_{\mathbb{C}(\lambda)} \\ & f(\lambda,z)=\displaystyle\sum_{\gamma\in\mathbb{Q}} f_\gamma(\lambda) z^\gamma & \mapsto & \displaystyle\sum_{\gamma\in\mathbb{Q}} \frac{\mathrm{d}f_\gamma}{\mathrm{d}\lambda}(\lambda) z^{\gamma}. \end{array} \tag{7}$$

We obtain in this way a structure of differential-difference field on $\mathscr{H}_{\mathbb{C}(\lambda)}$. We emphasize that ∂_λ and ϕ_p commute on $\mathscr{H}_{\mathbb{C}(\lambda)}$:

$$\partial_\lambda\circ\phi_p = \phi_p\circ\partial_\lambda.$$

The role played by the differential-difference field $\mathscr{H}_{\mathbb{C}(\lambda)}$ in the present paper is similar to the role played by the differential field $\mathbb{C}(\lambda)((z))$ in the classical Frobenius method outlined in Section 2.

3.4. **The differential-difference ring $\mathscr{R}_\lambda$.** We shall now introduce a differential-difference ring $\mathscr{R}_\lambda$ that will play the same role as the differential ring

$$\mathbb{C}(\lambda)((z))[z^\lambda, \partial_\lambda(z^\lambda), \partial_\lambda^2(z^\lambda), \ldots]$$

(equipped with the derivatives ∂_z and ∂_λ) in the classical Frobenius method.

We consider the differential ring of differential polynomials

$$\mathscr{R}_\lambda = \mathscr{H}_{\mathbb{C}(\lambda)}[X, \partial_\lambda X, \partial_\lambda^2 X, \ldots].$$

As a ring, this is simply the ring of polynomials with coefficients in $\mathscr{H}_{\mathbb{C}(\lambda)}$ and in the indeterminates $X, \partial_\lambda X, \partial_\lambda^2 X, \ldots$ This ring is equipped with the unique derivation

$$\partial_\lambda : \mathscr{R}_\lambda \to \mathscr{R}_\lambda$$

extending (7) and such that $\partial_\lambda(\partial_\lambda^i X) = \partial_\lambda^{i+1} X$. We also equip $\mathscr{R}_\lambda$ with the unique ring automorphism

$$\phi_p : \mathscr{R}_\lambda \to \mathscr{R}_\lambda$$

extending (6) and such that, for all $i \geq 0$, $\phi_p(\partial_\lambda^i X) = \partial_\lambda^i(\lambda X)$. This turns $\mathscr{R}_\lambda$ into a differential-difference ring extension of $\mathscr{H}_{\mathbb{C}(\lambda)}$. We emphasize that ∂_λ and ϕ_p commute on $\mathscr{R}_\lambda$:

$$\partial_\lambda \circ \phi_p = \phi_p \circ \partial_\lambda.$$

It will be convenient to introduce the following notations :

— $e_\lambda = X \in \mathscr{R}_\lambda$;

— for all $i \geq 0$, $\ell_{\lambda,i} = \frac{\partial_\lambda^i e_\lambda}{i!} = \frac{\partial_\lambda^i X}{i!} \in \mathscr{R}_\lambda$;

so that $\mathscr{R}_\lambda$ can be rewritten as follows :

$$\begin{aligned} \mathscr{R}_\lambda &= \mathscr{H}_{\mathbb{C}(\lambda)}[e_\lambda, \partial_\lambda(e_\lambda), \partial_\lambda^2(e_\lambda), \ldots] \\ &= \mathscr{H}_{\mathbb{C}(\lambda)}[e_\lambda, \ell_{\lambda,1}, \ell_{\lambda,2}, \ldots]. \end{aligned}$$

It is of fundamental importance in what follows to keep in mind that e_λ satisfies the following equation

$$\phi_p(e_\lambda) = \lambda e_\lambda. \tag{8}$$

More generally, for all $i \geq 0$, we have

$$\phi_p(\ell_{\lambda,i}) = \lambda \ell_{\lambda,i} + \ell_{\lambda,i-1}$$

where $\ell_{\lambda,i-1} = \ell_{\lambda,-1} = 0$ for $i = 0$.

Remark 6. *The difference equation (8) satisfied by e_λ has to be compared with the differential equation*

$$z\partial_z(z^\lambda) = \lambda z^\lambda$$

satisfied by z^λ; this e_λ will play in the present paper a role similar to that of z^λ in Section 2. More generally, the $\ell_{\lambda,i}$ will play a role similar to that of the $\partial_\lambda^i(z^\lambda)$ in Section 2.

3.5. **The difference ring $\mathscr{R}$.** Throughout this paper, we denote by $\mathscr{R}$ a difference ring extension of $\mathscr{H}$ such that, for any $c \in \mathbb{C}^\times$ and any integer $i \geq 0$, there exists a nonzero $\ell_{c,i} \in \mathscr{R}$ such that

$$\phi_p(\ell_{c,i}) = c\ell_{c,i} + \ell_{c,i-1}, \tag{9}$$

where $\ell_{c,i-1} = \ell_{c,-1} = 0$ for $i = 0$. For $i = 0$, we use the notation

$$e_c = \ell_{c,0}.$$

Of course, equation (9) gives

$$\phi_p(e_c) = ce_c.$$

The solutions produced by the Frobenius method presented in this paper will belong to $\mathscr{R}$.

Remark 7. *It is sufficient to require that $\mathscr{R}$ satisfies the following properties:*
- *for any $c \in \mathbb{C}^\times$, there exists $e_c \in \mathscr{R}$ which is not a zero divisor such that $\phi_p(e_c) = ce_c$;*
- *there exists $\ell \in \mathscr{R}$ such that $\phi_p(\ell) = \ell + 1$.*

Indeed, a straightforward calculation shows that

$$\ell_{c,i} = c^{-i} e_c \binom{\ell}{i}$$

is nonzero and satisfies (9).

A possible choice is the polynomial ring

$$\mathscr{R} = \mathscr{H}[(X_c)_{c\in\mathbb{C}^\times}, Y]$$

endowed with the unique automorphism

$$\phi_p : \mathscr{R} \to \mathscr{R}$$

extending (5) and such that $\phi_p(X_c) = cX_c$ and $\phi_p(Y) = Y + 1$. Of course, $e_c = X_c$ and $\ell = Y$ have the properties required in Remark 7.

However, we emphasize that the specific choices of e_c and $\ell_{c,i}$ is not important; the important thing are the functional equations (9) they satisfy.

3.6. **Evaluating elements of $\mathscr{R}_\lambda$ at $\lambda = c \in \mathbb{C}^\times$.** As in the classical Frobenius method (see Section 2), we will need to "specialize elements of $\mathscr{R}_\lambda$ at $\lambda = c \in \mathbb{C}^\times$". Here is what we mean by "specialize" in this context.

We let $\mathbb{C}[\lambda]_{(\lambda-c)}$ be the ring of rational fractions regular at $c \in \mathbb{C}^\times$. Then, $\mathscr{H}_{\mathbb{C}[\lambda]_{(\lambda-c)}}$ is a differential-difference subring of $\mathscr{H}_{\mathbb{C}(\lambda)}$ and

$$\mathscr{R}_{\lambda,c} = \mathscr{H}_{\mathbb{C}[\lambda]_{(\lambda-c)}}[e_\lambda, \ell_{\lambda,1}, \ell_{\lambda,2}, \ldots] \tag{10}$$

is a differential-difference subring of $\mathscr{R}_\lambda$. There is a unique morphism of rings

$$\mathrm{ev}_{\lambda=c} : \mathscr{R}_{\lambda,c} \to \mathscr{R}$$

such that, for all $f(\lambda, z) \in \mathscr{H}_{\mathbb{C}[\lambda]_{(\lambda-c)}}$,

$$\mathrm{ev}_{\lambda=c}(f(\lambda, z)) = f(c, z)$$

and, for all $i \geq 0$,

$$\mathrm{ev}_{\lambda=c}(\ell_{\lambda,i}) = \ell_{c,i}.$$

This is a morphism of difference rings. In the context of this paper, "specilizing at $\lambda = c$" means taking the image by $\mathrm{ev}_{\lambda=c}$.

4. Newton polygons, slopes and exponents

In this section, we consider a Mahler equation

$$a_n(z)y(z^{p^n}) + a_{n-1}(z)y(z^{p^{n-1}}) + \cdots + a_0(z)y(z) = 0 \tag{11}$$

with coefficients $a_0(z), \ldots, a_n(z) \in \mathscr{H}$. We can and will assume that $a_0(z)a_n(z) \neq 0$.

The aim of this Section is to attach to (11) certain slopes and to attach to each slope certain exponents.

4.1. **Mahler equations and Mahler operators.** The Mahler equation (11) can be written as follows:

$$L(y(z)) = 0$$

where

$$L = a_n(z)\phi_p^n + a_{n-1}(z)\phi_p^{n-1} + \cdots + a_0(z). \tag{12}$$

This is an element of the Ore algebra

$$\mathcal{D}_{\mathscr{H}} = \mathscr{H}\langle \phi_p, \phi_p^{-1} \rangle$$

of noncommutative Laurent polynomials with coefficients in $\mathscr{H}$ such that, for all $f \in \mathscr{H}$, $\phi_p f(z) = \phi_p(f(z))\phi_p$. An element of $\mathcal{D}_{\mathscr{H}}$ will be called a Mahler operator (with coefficients in $\mathscr{H}$). The Mahler operator L given by (12) will be called the Mahler operator associated with (11).

4.2. **Newton polygons.** We define the Newton polygon $\mathcal{N}(L)$ of L as the convex hull in $\mathbb{R}^2$ of

$$\{(p^i, j) \mid i, j \in \mathbb{Z},\ j \geq \mathrm{val}_z(a_i(z))\} \subset \mathbb{R}^2$$

where $\mathrm{val}_z : \mathscr{H} \to \mathbb{Q} \cup \{+\infty\}$ denotes the z-adic valuation (this is for instance the Newton polygon used in [CDDM18]).

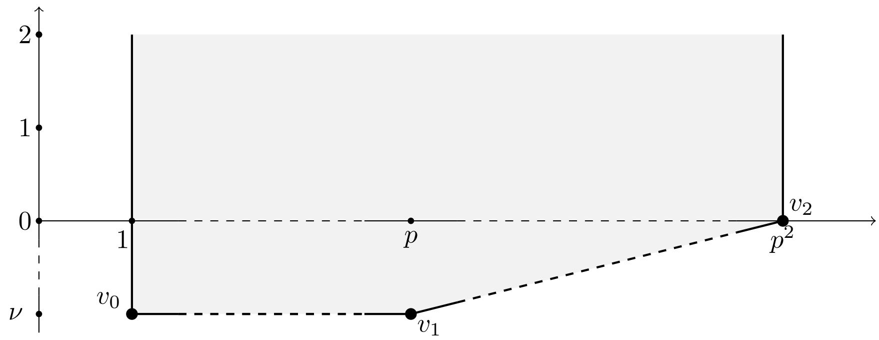

FIGURE 1. The Newton polygon of the Mahler operator (13). The • represent the points $(p^i, \mathrm{val}_z(a_i))$ for $i \in \{0,1,2\}$.

4.3. **Slopes.** Let

$$v_0 = (p^{\alpha_0}, \mathrm{val}_z(a_{\alpha_0}(z))) = (p^0, \mathrm{val}_z(a_0(z))), \ldots$$
$$\ldots, v_i = (p^{\alpha_i}, \mathrm{val}_z(a_{\alpha_i}(z))), \ldots$$
$$\ldots, v_k = (p^{\alpha_k}, \mathrm{val}_z(a_{\alpha_k}(z))) = (p^n, \mathrm{val}_z(a_n(z)))$$

be the vertices, ordered by increasing abscissa, of the polygon $\mathcal{N}(L)$. This polygon is delimited by two vertical half lines and by k vectors

$$w_1 = v_1 - v_0, \ldots, w_k = v_k - v_{k-1} \in \mathbb{Z}_{>0} \times \mathbb{Q}$$

having pairwise distinct slopes denoted by

$$\mu_1 < \cdots < \mu_k$$

and called the slopes of L. The set of slopes of L will be denoted by $\mathcal{S}(L)$. For any $i \in \{1, \ldots, k\}$, the integer

$$r(\mu_i, L) = \alpha_i - \alpha_{i-1}$$

is called the multiplicity of μ_i as a slope of L; in what follows, we will also use the shorthand notation

$$r_i = r(\mu_i, L).$$

Example 8. *The Newton polygon of the Mahler operator*

$$\begin{aligned} (13) \qquad L &= (\phi_p - z^\nu)h(z)^{-1}(\phi_p - 1) \\ &= \frac{1}{1+z^{-\frac{p\nu}{p-1}}}\phi_p^2 - \left(\frac{1}{1+z^{-\frac{p\nu}{p-1}}} + z^\nu\right)\phi_p + \frac{z^\nu}{1+z^{-\frac{\nu}{p-1}}} \end{aligned}$$

with $\nu \in \mathbb{Q}_{<0}$ *and* $h(z) = 1 + z^{-\frac{\nu}{p-1}}$ *is represented in Figure 1.*

This operator has two slopes, namely

$$\mu_1 = 0 < \mu_2 = -\frac{\nu}{(p-1)p},$$

with multiplicities $r_1 = r_2 = 1$.

Remark 9. *We have extracted two informations from $\mathcal{N}(L)$: its slopes and their multiplicities. Of course, $\mathcal{N}(L)$ is characterized by these data up to vertical translation.*

For any $\mu \in \mathbb{Q}$, we consider

$$\theta_\mu(z) = z^{\frac{\mu}{p-1}}.$$

It satisfies

$$\phi_p(\theta_\mu(z)) = z^\mu \theta_\mu(z).$$

We set

$$\begin{aligned} L^{[\theta_\mu(z)]} &= \theta_\mu(z)^{-1} L \theta_\mu(z) \\ &= \sum_{i=0}^{n} z^{(1+p+\cdots+p^{i-1})\mu} a_i(z) \phi_p^i = \sum_{i=0}^{n} z^{\frac{p^i-1}{p-1}\mu} a_i(z) \phi_p^i. \end{aligned}$$

Proposition 10. *The slopes of $L^{[\theta_\mu(z)]}$ are*

$$\mu_1 + \frac{\mu}{p-1} < \cdots < \mu_k + \frac{\mu}{p-1}$$

with respective multiplicities $r_1, \ldots, r_k$.

Proof. The Newton polygon $\mathcal{N}(L^{[\theta_\mu(z)]})$ is the convex hull of

$$\left\{ (p^i, j) \mid i, j \in \mathbb{Z},\ j \geq \mathrm{val}_z(a_i(z)) + \mu \frac{p^i - 1}{p-1} \right\}.$$

Its vertices are

$$\begin{aligned} &v'_0 = v_0 + (0, \mu \frac{p^{\alpha_0} - 1}{p-1}), \ldots \\ &\qquad \ldots, v'_i = v_i + (0, \mu \frac{p^{\alpha_i} - 1}{p-1}), \ldots \\ &\qquad\qquad \ldots, v'_k = v_k + (0, \mu \frac{p^{\alpha_k} - 1}{p-1}), \end{aligned}$$

and, hence, $\mathcal{N}(L^{[\theta_\mu]})$ is delimited by two vertical half lines and by the k vectors

$$\begin{aligned} &w'_1 = v'_1 - v'_0 = w_1 + (0, \mu \frac{p^{\alpha_1} - p^{\alpha_0}}{p-1}), \ldots \\ &\qquad \ldots, w'_i = v'_i - v'_{i-1} = w_i + (0, \mu \frac{p^{\alpha_i} - p^{\alpha_{i-1}}}{p-1}), \ldots \\ &\qquad\qquad \ldots, w'_k = v'_k - v'_{k-1} = w_k + (0, \mu \frac{p^{\alpha_k} - p^{\alpha_{k-1}}}{p-1}). \end{aligned}$$

□

In particular, the j-th slope of $L^{[\theta_{-(p-1)\mu_j}(z)]}$ is equal to 0. This will be used in the next section in order to attach to the slope μ_j a certain characteristic polynomial and certain exponents.

4.4. **Characteristic equations and exponents.** Consider a slope μ_j of L and set

$$L^{[\theta_{-(p-1)\mu_j}(z)]} = \sum_{i=0}^{n} b_i(z)\phi_p^i.$$

Let

$$\mathrm{val}_z(L^{[\theta_{-(p-1)\mu_j}]}) = \min\{\mathrm{val}_z(b_0(z)), \dots, \mathrm{val}_z(b_n(z))\}.$$

The characteristic polynomial $\chi(\mu_j, L; X)$ associated to the slope μ_j of L is

$$\begin{aligned}\chi(\mu_j, L; X) &= \left(z^{-\mathrm{val}_z(L^{[\theta_{-(p-1)\mu_j}(z)]})}\sum_{i=0}^{n} b_i(z)X^i\right)_{|z=0} \\ &= \left(z^{-\mathrm{val}_z(L^{[\theta_{-(p-1)\mu_j}(z)]})}\sum_{i=\alpha_{j-1}}^{\alpha_j} b_i(z)X^i\right)_{|z=0}.\end{aligned}$$

In what follows, we will see these characteristic polynomials as defined modulo a multiplicative factor in $\mathbb{C}^\times X^{\mathbb{Z}}$; in particular, the equalities involving the characteristic polynomials have to be interpreted modulo $\mathbb{C}^\times X^{\mathbb{Z}}$. The characteristic polynomial $\chi(\mu_j, L; X)$ has r_j roots (counted with multiplicities) in $\mathbb{C}^\times$ called the exponents of L attached to the slope μ_j. The multiplicity of such an exponent is its multiplicity as a root of $\chi(\mu_j, L; X)$.

Example 11. *We have already seen that the Mahler operator L given by (13) has two slopes*

$$\mu_1 = 0 < \mu_2 = -\frac{\nu}{(p-1)p},$$

with multiplicities $r_1 = r_2 = 1$.

Let us compute the characteristic polynomial of the Mahler operator (13) associated to the slope μ_1. We have

$$L^{[\theta_{-(p-1)\mu_1}(z)]} = L = \frac{1}{1+z^{-\frac{p\nu}{p-1}}}\phi_p^2 - \left(\frac{1}{1+z^{-\frac{p\nu}{p-1}}} + z^\nu\right)\phi_p + \frac{z^\nu}{1+z^{-\frac{\nu}{p-1}}}$$

and

$$\mathrm{val}_z(L^{[\theta_{-(p-1)\mu_1}(z)]}) = \nu.$$

Thus, the characteristic polynomial $\chi(\mu_1, L; X)$ associated to the slope μ_1 of L is

$$\begin{aligned}&\chi(\mu_1, L; X) = \\ &\left(z^{-\nu}\left(\frac{1}{1+z^{-\frac{p\nu}{p-1}}}X^2 - \left(\frac{1}{1+z^{-\frac{p\nu}{p-1}}} + z^\nu\right)X + \frac{z^\nu}{1+z^{-\frac{\nu}{p-1}}}\right)\right)_{|z=0} \\ &\qquad = -X + 1.\end{aligned}$$

So, 1 is the unique exponent of L attached to the slope μ_1 and it has multiplicity 1.

Let us compute the characteristic polynomial of the Mahler operator (13) *associated to the slope* μ_2*. We have*

$$L^{[\theta_{-(p-1)\mu_2}(z)]} = \frac{z^{(p+1)\frac{\nu}{p}}}{1+z^{-\frac{p\nu}{p-1}}}\phi_p^2 - z^{\frac{\nu}{p}}\left(\frac{1}{1+z^{-\frac{p\nu}{p-1}}}+z^{\nu}\right)\phi_p + \frac{z^{\nu}}{1+z^{-\frac{\nu}{p-1}}}$$

and

$$\mathrm{val}_z(L^{[\theta_{-(p-1)\mu_2}(z)]}) = (p+1)\frac{\nu}{p}.$$

Thus, the characteristic polynomial $\chi(\mu_2, L; X)$ *associated to the slope* μ_2 *of* L *is*

$$\chi(\mu_2, L; X) =$$
$$\left(z^{-(p+1)\frac{\nu}{p}}\left(\frac{z^{(p+1)\frac{\nu}{p}}}{1+z^{-\frac{p\nu}{p-1}}}X^2 - z^{\frac{\nu}{p}}\left(\frac{1}{1+z^{-\frac{p\nu}{p-1}}}+z^{\nu}\right)X + \frac{z^{\nu}}{1+z^{-\frac{\nu}{p-1}}}\right)\right)_{|z=0}$$
$$= X^2 - X.$$

So, 1 *is the unique exponent of* L *attached to the slope* μ_2 *and it has multiplicity* 1*.*

5. An extension of Frobenius method to Mahler equations

We consider a Mahler equation

$$a_n(z)y(z^{p^n}) + a_{n-1}(z)y(z^{p^{n-1}}) + \cdots + a_0(z)y(z) = 0$$

with $a_0(z), \ldots, a_n(z) \in \mathscr{H}$ and $a_0(z)a_n(z) \neq 0$. It can be rewritten as

$$L(y(z)) = 0 \tag{14}$$

where

$$L = a_n(z)\phi_p^n + a_{n-1}(z)\phi_p^{n-1} + \cdots + a_0(z) \in \mathcal{D}_{\mathscr{H}}.$$

We denote by

$$\mu_1 < \cdots < \mu_k$$

the slopes of L and by $r_1, \ldots, r_k$ their respective multiplicites. The multiplicity of an exponent c of L attached to the slope μ_j will be denoted by $m_{c,j} \in \mathbb{Z}_{\geq 1}$. If c is a nonzero complex number that is not an exponent attached to the slope μ_j, we set $m_{c,j} = 0$. We refer the reader to Section 4 for details about the notions of slopes and exponents.

The extension of Frobenius method to Mahler equations presented below attaches $m_{c,j}$ solutions of (14) to any exponent c associated to the slope μ_j of L as follows. We will prove that

— there exists a unique $g_{c,j}(\lambda, z) \in \mathscr{H}_{\mathbb{C}(\lambda)}$ such that

$$L(g_{c,j}(\lambda, z)e_\lambda) = z^{\mathrm{val}_z(a_0(z)) - \frac{\nu_j}{p-1}}(\lambda - c)^{s_{c,j}+m_{c,j}}e_\lambda \tag{15}$$

where

$$s_{c,j} = m_{c,1} + \cdots + m_{c,j-1}$$

and

$$\nu_j = (p-1)(p^{r_1+\cdots+r_{j-1}}(\mu_j - \mu_{j-1}) + \cdots + p^{r_1}(\mu_2 - \mu_1) + \mu_1);$$

— the z-adic valuation of $g_{c,j}(\lambda, z)$ is equal to $-\mu_j$;
— the coefficients of $g_{c,j}(\lambda, z)$ have no pole at $\lambda = c$;
— the specializations

$$\begin{gathered} \mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}}(g_{c,j}(\lambda, z)e_\lambda)), \\ \mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}+1}(g_{c,j}(\lambda, z)e_\lambda)), \\ \ldots \\ \mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}+m_{c,j}-1}(g_{c,j}(\lambda, z)e_\lambda)) \end{gathered}$$

are $m_{c,j}$ solutions of (14).

For the justifications of the first three properties above, see Proposition 22 in Section 8. For the last one, see Proposition 27 in Section 9.

We can now state the main result of this paper.

Theorem 12. *We have attached to any slope μ_j, to any exponent c attached to the slope μ_j and to any $m \in \{0, \ldots, m_{c,j} - 1\}$, a solution*

$$y_{c,j,m} = \mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}+m}(g_{c,j}(\lambda, z)e_\lambda))$$

of (14). We obtain in this way a family of n $\mathbb{C}$-linearly independent solutions of (14).

The proof of this result will be given in Section 9.

Note that

$$\begin{aligned} y_{c,j,m} &\in \mathrm{Span}_{\mathscr{H}}(\mathrm{ev}_{\lambda=c}(\partial_\lambda^0(e_\lambda)), \mathrm{ev}_{\lambda=c}(\partial_\lambda^1(e_\lambda)), \ldots \\ &\qquad\qquad \ldots, \mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}+m_{c,j}-1}(e_\lambda))) \\ &= \mathrm{Span}_{\mathscr{H}}(e_c, \ell_{c,1}, \ldots, \ell_{c,s_{c,j}+m_{c,j}-1}). \end{aligned}$$

Remark 13. *The solutions attached to c involve an $\ell_{c,k}$ with $k \geq 1$ if c has multiplicity ≥ 2 as an exponent of L attached to a slope μ_j (i.e., $m_{c,j} \geq 2$). But, this is not the only case : such an $\ell_{c,k}$ may also appear if c is an exponent of L attached to two distinct slopes μ_i and μ_j (i.e., $s_{c,j} \geq 1$ if $j > i$).*

Remark 14. *The relevant fact with the term $z^{-\frac{\nu_j}{p-1}}$ involved in the right-hand side of (15) is its z-adic valuation equals to $-\frac{\nu_j}{p-1}$: it guarantees that $g_{c,j}(\lambda, z)$ has z-adic valuation $-\mu_j$ and, hence, is "associated" with the slope μ_j.*

6. Frobenius method for Mahler equations : an example

In this section, we apply the method described in Section 5 to an operator allowing explicit computations, namely to the operator considered in Example 8 given by

$$\begin{aligned} L &= (\phi_p - z^{\nu})h(z)^{-1}(\phi_p - 1) \\ &= \frac{1}{1+z^{-\frac{p\nu}{p-1}}}\phi_p^2 - \left(\frac{1}{1+z^{-\frac{p\nu}{p-1}}} + z^{\nu}\right)\phi_p + \frac{z^{\nu}}{1+z^{-\frac{\nu}{p-1}}} \\ &= a_2(z)\phi_p^2 + a_1(z)\phi_p + a_0(z) \end{aligned}$$

where $\nu \in \mathbb{Q}_{<0}$ and $h(z) = 1 + z^{-\frac{\nu}{p-1}}$.

In what follows, we use the notations μ_i, r_i, ν_j, *etc*, of Section 5.

We have seen in Example 8 that L has two slopes, namely

$$\mu_1 = 0 < \mu_2 = -\frac{\nu}{(p-1)p},$$

with multiplicities

$$r_1 = r_2 = 1.$$

So,

$$\nu_1 = (p-1)\mu_1 = 0, \quad \nu_2 = (p-1)(p^{r_1}(\mu_2 - \mu_1) + \mu_1) = -\nu.$$

Moreover, we have seen in Example 11 that 1 is the unique exponent of L and that the multiplicites $m_{1,1}$ and $m_{1,2}$ of 1 as an exponent of L attached to the slopes μ_1 and μ_2 respectively are

$$m_{1,1} = m_{1,2} = 1.$$

Thus

$$s_{1,1} = 0, \quad s_{1,2} = 1.$$

The Frobenius method described in Section 5 relies on the equations

$$\begin{aligned} L(g_1(\lambda,z)e_\lambda) &= z^{\mathrm{val}_z(a_0(z))}\theta_{-\nu_1}(z)(\lambda-1)^{m_{1,1}}e_\lambda \\ &= z^{\nu}(\lambda-1)e_\lambda \end{aligned} \tag{16}$$

and

$$\begin{aligned} L(g_2(\lambda,z)e_\lambda) &= z^{\mathrm{val}_z(a_0(z))}\theta_{-\nu_2}(z)(\lambda-1)^{m_{1,1}+m_{1,2}}e_\lambda \\ &= z^{\nu}\theta_\nu(z)(\lambda-1)^2 e_\lambda. \end{aligned} \tag{17}$$

We shall now give explicit formulas for the unique $g_1(\lambda,z), g_2(\lambda,z) \in \mathscr{H}_{\mathbb{C}(\lambda)}$ satisfying the above equations. In this purpose, we will freely use Lemma 24 stated and proven in Section 8.1 below.

Let us start with equation (16). This equation can be written as follows

$$(\lambda\phi_p - z^{\nu})h(z)^{-1}(\lambda\phi_p - 1)(g_1(\lambda,z)) = z^{\nu}(\lambda-1).$$

Multiplying by $\theta_\nu(z)^{-1}z^{-\nu}$, we see that the latter equation is equivalent to

$$(\lambda\phi_p - 1)(\theta_\nu(z)^{-1}h(z)^{-1}(\lambda\phi_p - 1)(g_1(\lambda,z))) = \theta_\nu(z)^{-1}(\lambda-1).$$

Since the z-adic valuation of the right hand-side of the latter equality is positive, Lemma 24 ensures that the latter equation is equivalent to

$$\theta_{\nu}^{-1}h(z)^{-1}(\lambda\phi_p-1)(g_1(\lambda,z))=-(\lambda-1)\sum_{k\geq 0}\lambda^k\phi_p^k(\theta_\nu(z)^{-1}),$$

i.e., to

$$\begin{aligned}(\lambda\phi_p-1)(g_1(\lambda,z))&=-(\lambda-1)(1+z^{-\frac{\nu}{p-1}})\sum_{k\geq 0}\lambda^k z^{\frac{\nu(1-p^k)}{p-1}}\\ &=-(\lambda-1)-(\lambda-1)\sum_{k\geq 1}\lambda^k z^{\frac{\nu(1-p^k)}{p-1}}-(\lambda-1)\sum_{k\geq 0}\lambda^k z^{\frac{-\nu p^k}{p-1}}.\end{aligned}$$

The z-adic valuations of the last two terms of the right hand-side of the latter equality are positive. Applying Lemma 24 again, we find

$$\begin{aligned}g_1(\lambda,z)&=-1+(\lambda-1)\sum_{j\geq 0}\sum_{k\geq 1}\lambda^{j+k}z^{p^j\frac{\nu(1-p^k)}{p-1}}+(\lambda-1)\sum_{j\geq 0}\sum_{k\geq 0}\lambda^{j+k}z^{\frac{-\nu p^{j+k}}{p-1}}\\ &=-1+(\lambda-1)\sum_{j\geq 0}\sum_{k\geq 1}\lambda^{j+k}z^{p^j\frac{\nu(1-p^k)}{p-1}}+(\lambda-1)\sum_{l\geq 0}(l+1)\lambda^l z^{\frac{-\nu p^l}{p-1}}.\end{aligned}$$

We now solve the equation (17). This equation can be written as follows

$$(\lambda\phi_p-z^\nu)h(z)^{-1}(\lambda\phi_p-1)(g_2(\lambda,z))=z^\nu\theta_\nu(z)(\lambda-1)^2.$$

Multiplying by $\theta_\nu(z)^{-1}z^{-\nu}$, we see that the latter equation is equivalent to

$$(\lambda\phi_p-1)\theta_\nu(z)^{-1}h(z)^{-1}(\lambda\phi_p-1)(g_2(\lambda,z))=(\lambda-1)^2.$$

Applying Lemma 24, we find that the latter equation is equivalent to

$$\theta_\nu(z)^{-1}h(z)^{-1}(\lambda\phi_p-1)(g_2(\lambda,z))=\lambda-1,$$

i.e., to

$$(\lambda\phi_p-1)(g_2(\lambda,z))=(\lambda-1)\theta_\nu(z)h(z)=(\lambda-1)z^{\frac{\nu}{p-1}}+(\lambda-1).$$

Applying Lemma 24 again, we find

$$g_2(\lambda,z)=(\lambda-1)\sum_{k\leq -1}\lambda^k\phi_p^k(z^{\frac{\nu}{p-1}})+1=(\lambda-1)\sum_{k\leq -1}\lambda^k z^{\frac{p^k\nu}{p-1}}+1.$$

Theorem 12 guarantees that

$$y_1=\mathrm{ev}_{\lambda=1}(g_1(\lambda,z)e_\lambda)=-e_1$$

and

$$y_2=\mathrm{ev}_{\lambda=1}(\partial_\lambda(g_2(\lambda,z)e_\lambda))=(\sum_{k\leq -1}z^{\frac{p^k\nu}{p-1}})e_1+\ell_{1,1}$$

are $\mathbb{C}$-linearly independent solutions of L.

7. Factorization of Mahler operators

The aim of this section is to prove the following result relative to a Mahler operator

$$L = a_n(z)\phi_p^n + a_{n-1}(z)\phi_p^{n-1} + \cdots + a_0(z) \in \mathcal{D}_{\mathscr{H}}. \tag{18}$$

We let

— $\mu_1 < \cdots < \mu_k$ be the slopes of L with respective multiplicities $r_1, \ldots, r_k$;

— $c_{i,1}, \ldots, c_{i,r_i}$ be the exponents (repeated with multiplicities) of L attached to the slope μ_i.

Proposition 15. *The operator L has a factorization*

$$L = a(z)L_k \cdots L_1$$

where

— $a(z) \in \mathscr{H}^\times$ *is such that* $\mathrm{val}_z(a(z)) = \mathrm{val}_z(a_0(z))$*;*

— $\mathrm{cld}_z\, a(z) = \prod_{i=1}^k \prod_{j=1}^{r_i} (-c_{i,j})^{-1}\, \mathrm{cld}_z\, a_0(z)$*;*

— *the $L_i \in \mathcal{D}_{\mathscr{H}}$ are given by*

$$L_i = (z^{\nu_i}\phi_p - c_{i,r_i})h_{i,r_i}(z)^{-1} \cdots (z^{\nu_i}\phi_p - c_{i,1})h_{i,1}(z)^{-1}$$

for some $h_{i,j}(z) \in \mathscr{H}^\times$ tangent to the identity and with

$$\nu_i = (p-1)(p^{r_1+\cdots+r_{i-1}}(\mu_i - \mu_{i-1}) + \cdots + p^{r_1}(\mu_2 - \mu_1) + \mu_1). \tag{19}$$

Remark 16. *This is a refinement of* [Roq20, Theorem 15], *where a similar statement is proved but without the explicit values of the ν_i in terms of the μ_j.*

The proof of Proposition 15 given in Section 7.3 will use some preliminary results gathered in the following two sections.

7.1. Preliminary results : basic properties of slopes and exponents.

In this Section, we collect basic results relative to the notions of slopes and exponents introduced in Sections 4.3 and 4.4.

7.1.1. *Slopes and exponents of the gauge transform $L^{[\theta_\mu(z)]}$.* We remind the notation

$$\begin{aligned} L^{[\theta_\mu(z)]} &= \theta_\mu(z)^{-1} L \theta_\mu(z) \\ &= \sum_{i=0}^n z^{(1+p+\cdots+p^{i-1})\mu} a_i(z)\phi_p^i = \sum_{i=0}^n z^{\frac{p^i-1}{p-1}\mu} a_i(z)\phi_p^i \end{aligned}$$

introduced in Section 4.3 where

$$\theta_\mu(z) = z^{\frac{\mu}{p-1}}$$

with $\mu \in \mathbb{Q}$. We have $L^{[\theta_\mu(z)]}(f(z)) = 0$ if and only if $L(f(z)\theta_\mu(z)) = 0$.

Lemma 17. *The slopes of* $L^{[\theta_\mu(z)]}$ *are*

$$\mu_1 + \frac{\mu}{p-1} < \cdots < \mu_k + \frac{\mu}{p-1}$$

with respective multiplicities $r_1, \ldots, r_k$. *Moreover,*

$$\chi(\mu_j + \frac{\mu}{p-1}, L^{[\theta_\mu(z)]}; X) = \chi(\mu_j, L; X)$$

and, hence, the exponents counted with multiplicities of $L^{[\theta_\mu(z)]}$ *attached to the slope* $\mu_j + \frac{\mu}{p-1}$ *coincide with those of* L *attached to the slope* μ_j.

Proof. For the first assertions concerning the slopes and their multiplicities, see Proposition 10. Moreover, the equality

$$\chi(\mu_j + \frac{\mu}{p-1}, L^{[\theta_\mu(z)]}; X) = \chi(\mu_j, L; X)$$

follows immediately from the definition of the characteristic polynomials by using the equality

$$(L^{[\theta_\mu(z)]})^{[\theta_{-(p-1)(\mu_j+\frac{\mu}{p-1})}(z)]} = L^{[\theta_{-(p-1)\mu_j}(z)]}.$$

The very last assertion of the lemma follows from this equality of characteristic polynomials and from the definition of the exponents and of their multiplicities. □

7.1.2. *The gauge transform* $L^{[e_c]}$. We will also use the notation

$$L^{[e_c]} = e_c^{-1} L e_c = \sum_{i=0}^{n} c^i a_i(z) \phi_p^i$$

where $c \in \mathbb{C}^\times$. We have $L^{[e_c]}(f(z)) = 0$ if and only if $L(f(z)e_c) = 0$.

Lemma 18. *The operators* $L^{[e_c]}$ *and* L *have the same slopes* $\mu_1 < \cdots < \mu_k$ *with the same multiplicities* $r_1, \ldots, r_k$. *Moreover,*

$$\chi(\mu_j, L^{[e_c]}; X) = \chi(\mu_j, L; cX)$$

and, hence, the exponents of $L^{[e_c]}$ *and of* L *attached to a given slope* μ_j *are related as follows :*

list of exponents of $L^{[e_c]}$ *counted with mult. attached to the slope* μ_j

$= c^{-1}\cdot$(*list of exponents of* L *counted with mult. attached to the slope* μ_j).

Proof. Since $\mathrm{val}_z(c^i a_i) = \mathrm{val}_z(a_i)$, we obviously have

$$\mathcal{N}(L^{[e_c]}) = \mathcal{N}(L) \tag{20}$$

and, hence, $L^{[e_c]}$ and L have the same slopes with the same multiplicities. Moreover, the equality

$$\chi(\mu_j, L^{[e_c]}; X) = \chi(\mu_j, L; cX)$$

follows immediately from the definition of the characteristic polynomials by using the equality

$$(L^{[e_c]})^{[\theta-(p-1)\mu_j(z)]} = (L^{[\theta-(p-1)\mu_j(z)]})^{[e_c]}.$$

The very last assertion of the lemma follows from the above equality of characteristic polynomials and from the definition of the exponents and of their multiplicities. □

7.1.3. *The gauge transform $L^{[g(z)]}$.* We will also use the notation

$$L^{[g(z)]} = g(z)^{-1}Lg(z) = \sum_{i=0}^{n} \frac{\phi_p^i(g(z))}{g(z)} a_i(z)\phi_p^i$$

for $g(z) \in \mathscr{H}^\times$. We have $L^{[g(z)]}(f(z)) = 0$ if and only if $L(g(z)f(z)) = 0$.

Lemma 19. *If $\mathrm{val}_z(g(z)) = 0$ then the operators $L^{[g(z)]}$ and L have the same slopes $\mu_1 < \cdots < \mu_k$ with the same multiplicities $r_1, \ldots, r_k$. Moreover,*

$$\chi(\mu_j, L^{[g(z)]}; X) = \chi(\mu_j, L; X)$$

and, hence, the exponents counted with multiplicities of $L^{[g(z)]}$ attached to the slope μ_j coincide with those of L attached to the slope μ_j.

Proof. Since $\mathrm{val}_z(\frac{\phi_p^i(g(z))}{g(z)} a_i(z)) = \mathrm{val}_z(a_i(z))$, we obviously have

$$\mathcal{N}(L^{[g(z)]}) = \mathcal{N}(L) \tag{21}$$

and, hence, L and $L^{[g(z)]}$ have the same slopes with the same multiplicities. Moreover, the equality

$$\chi(\mu_j, L^{[g(z)]}; X) = \chi(\mu_j, L; X)$$

follows immediately from the definition of the characteristic polynomials by using the equality

$$(L^{[g(z)]})^{[\theta-(p-1)\mu_j(z)]} = (L^{[\theta-(p-1)\mu_j(z)]})^{[g(z)]},$$

and the fact that the $\frac{\phi_p^i(g(z))}{g(z)} \in \mathscr{H}^\times$ are tangent to the identity. The very last assertion of the lemma follows from this equality of characteristic polynomials and from the definition of the exponents and of their multiplicities. □

7.2. **Preliminary results : on operators with smallest slope 0.**

Lemma 20. *Assume that 0 is the smallest slope of L (i.e., $\mu_1 = 0$) and let c be an exponent attached to the slope 0. Then, there exist $L' \in \mathcal{D}_{\mathscr{H}}$ and $h(z) \in \mathscr{H}^\times$ tangent to the identity such that*

$$L = L'(\phi_p - c)h(z)^{-1}.$$

Proof. Indeed, [Roq20, Lemma 13] ensures that there exists $h(z) \in \mathscr{H}^{\times}$ tangent to the identity such that

$$L(h(z)e_c) = 0.$$

Since $h(z)e_c$ is also a solution of $(\phi_p - c)h(z)^{-1}$, we get, by right euclidean division,

$$L = L'(\phi_p - c)h(z)^{-1}$$

for some $L' \in \mathcal{D}_{\mathscr{H}}$. □

Lemma 21. *Assume that $\mathcal{S}(L) \subset \mathbb{R}^+$. Then, for any $c \in \mathbb{C}^\times$ and any $h(z) \in \mathscr{H}^{\times}$ tangent to the identity, we have*

$$\mathcal{N}(L(\phi_p - c)h(z)^{-1}) = ([1, p] \times [\mathrm{val}_z(a_0(z)), +\infty[) \bigcup \varphi(\mathcal{N}(L)), \tag{22}$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is the map given by $\varphi(x, y) = (px, y)$. It follows that

$$\mathcal{S}(L(\phi_p - c)h(z)^{-1}) = p^{-1}\mathcal{S}(L) \cup \{0\}$$

and, hence, for any $\lambda \in \mathcal{S}(L(\phi_p - c)h(z)^{-1})$,

$$r(\lambda, L(\phi_p - c)h(z)^{-1}) = \begin{cases} r(p\lambda, L) \text{ if } \lambda \neq 0, \\ r(0, L) + 1 \text{ if } \lambda = 0. \end{cases} \tag{23}$$

Moreover, for any $\lambda \in \mathcal{S}(L(\phi_p - c)h(z)^{-1})$, we have

$$\chi(\lambda, L(\phi_p - c)h(z)^{-1}; X) = \begin{cases} \chi(p\lambda, L; X) \text{ if } \lambda \neq 0, \\ \chi(0, L; X)(X - c) \text{ if } \lambda = 0. \end{cases} \tag{24}$$

Proof. We will only treat the case when
— 0 is a slope of L
— and L has at least one nonzero slope,
the other cases being similar.

Without loss of generality, one can assume that $a_0(z) = 1$. Moreover, combining the equality

$$((L(\phi_p - c)h(z)^{-1})^{[e_c]})^{[h(z)]} = h(z)^{-1}cL^{[e_c]}(\phi_p - 1)$$

with Lemma 18 and Lemma 19, we see that it is sufficient to treat the case $h(z) = 1$ and $c = 1$.

We have

$$L(\phi_p - 1) = b_{n+1}(z)\phi_p^{n+1} + b_n(z)\phi_p^n + \cdots + b_1(z)\phi_p + b_0(z)$$

with

$$b_0(z) = -a_0(z), b_1(z) = a_0(z) - a_1(z), \ldots$$
$$\ldots, b_n(z) = a_{n-1}(z) - a_n(z), b_{n+1}(z) = a_n(z).$$

In order to prove (22), we have to show :

(i) $\mathrm{val}_z(b_0(z)) = \mathrm{val}_z(a_0(z))$ and, for all $i \in \{1, \ldots, k\}$, $\mathrm{val}_z(b_{\alpha_i+1}(z)) = \mathrm{val}_z(a_{\alpha_i}(z))$;

(ii) for $j \in \{0, \ldots, \alpha_1 + 1\}$, $\mathrm{val}_z(b_j(z)) \geq 0$;

(iii) for $i \in \{1, \ldots, k-1\}$, for $j \in \{\alpha_i + 1, \ldots, \alpha_{i+1}\}$, we have

$$\frac{\mathrm{val}_z(b_j(z)) - \mathrm{val}_z(b_{\alpha_i+1}(z))}{p^j - p^{\alpha_i+1}} \geq \frac{\mathrm{val}_z(b_{\alpha_{i+1}+1}(z)) - \mathrm{val}_z(b_{\alpha_i+1}(z))}{p^{\alpha_{i+1}+1} - p^{\alpha_i+1}}. \tag{25}$$

We have used the notations

$$v_0 = (p^{\alpha_0}, \mathrm{val}_z(a_{\alpha_0}(z))) = (p^0, \mathrm{val}_z(a_0(z))), \ldots$$
$$\ldots, v_i = (p^{\alpha_i}, \mathrm{val}_z(a_{\alpha_i}(z))), \ldots$$
$$\ldots, v_k = (p^{\alpha_k}, \mathrm{val}_z(a_{\alpha_k}(z))) = (p^n, \mathrm{val}_z(a_n(z)))$$

introduced at the very beginning of Section 4.3 for the vertices, ordered by increasing abscissa, of the polygon $\mathcal{N}(L)$.

We now proceed to the proof of the above properties (i), (ii) and (iii).

It is clear that $\mathrm{val}_z(b_0(z)) = 0$ and $\mathrm{val}_z(b_1(z)) \geq 0, \ldots, \mathrm{val}_z(b_{\alpha_1}(z)) \geq 0$ because $b_0(z) = a_0(z) = 1$ and because all the $a_i(z)$ have valuation ≥ 0 (since $\mathcal{S}(L) \subset \mathbb{R}^+$). Also, for $i \in \{1, \ldots, k-1\}$, we have $\mathrm{val}_z(b_{\alpha_i+1}(z)) = \mathrm{val}_z(a_{\alpha_i}(z))$ because $b_{\alpha_i+1}(z) = a_{\alpha_i}(z) - a_{\alpha_i+1}(z)$ and $\mathrm{val}_z(a_{\alpha_i+1}(z)) > \mathrm{val}_z(a_{\alpha_i}(z))$ because $e_i = (p^{\alpha_i}, \mathrm{val}_z(a_{\alpha_i}(z)))$ is a vertex of $\mathcal{N}(L)$. The equality $\mathrm{val}_z(b_{\alpha_k+1}(z)) = \mathrm{val}_z(a_{\alpha_k}(z))$ also holds true because $b_{\alpha_k+1}(z) = b_{n+1}(z) = a_n(z) = a_{\alpha_k}(z)$. This ensures that the above first two properties (i) and (ii) hold true.

Moreover, for $i \in \{1, \ldots, k-1\}$ and $j \in \{\alpha_i + 1, \ldots, \alpha_{i+1}\}$, the inequality (25) holds true because

$$\frac{\mathrm{val}_z(b_j(z)) - \mathrm{val}_z(b_{\alpha_i+1}(z))}{p^j - p^{\alpha_i+1}} = \frac{\mathrm{val}_z(b_j(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^j - p^{\alpha_i+1}}$$
$$\geq \min\left\{\frac{\mathrm{val}_z(a_j(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^j - p^{\alpha_i+1}}, \frac{\mathrm{val}_z(a_{j-1}(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^j - p^{\alpha_i+1}}\right\}$$

(we have used the fact that

$$\mathrm{val}_z(b_j(z)) \geq \min\{\mathrm{val}_z(a_{j-1}(z)), \mathrm{val}_z(a_j(z))\}).$$

But, on the one hand, we have

$$\frac{\mathrm{val}_z(a_{j-1}(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^j - p^{\alpha_i+1}} = \frac{1}{p}\frac{\mathrm{val}_z(a_{j-1}(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^{j-1} - p^{\alpha_i}}$$
$$\geq \frac{1}{p}\frac{\mathrm{val}_z(a_{\alpha_{i+1}}(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^{\alpha_{i+1}} - p^{\alpha_i}} = \frac{\mathrm{val}_z(b_{\alpha_{i+1}+1}(z)) - \mathrm{val}_z(b_{\alpha_i+1})(z)}{p^{\alpha_{i+1}+1} - p^{\alpha_i+1}},$$

the latter inequality being true because $e_i = (p^{\alpha_i}, \mathrm{val}_z(a_{\alpha_i}(z)))$ and $e_{i+1} = (p^{\alpha_{i+1}}, \mathrm{val}_z(a_{\alpha_{i+1}}(z)))$ are vertices of $\mathcal{N}(L)$. On the other hand, we have

$$\frac{\mathrm{val}_z(a_j(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^j - p^{\alpha_i+1}} \geq \frac{\mathrm{val}_z(a_j(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^j - p^{\alpha_i}}$$
$$\geq \frac{\mathrm{val}_z(a_{\alpha_{i+1}}(z)) - \mathrm{val}_z(a_{\alpha_i}(z))}{p^{\alpha_{i+1}} - p^{\alpha_i}} = p\frac{\mathrm{val}_z(b_{\alpha_{i+1}+1}(z)) - \mathrm{val}_z(b_{\alpha_i+1}(z))}{p^{\alpha_{i+1}+1} - p^{\alpha_i+1}}$$
$$\geq \frac{\mathrm{val}_z(b_{\alpha_{i+1}+1}(z)) - \mathrm{val}_z(b_{\alpha_i+1}(z))}{p^{\alpha_{i+1}+1} - p^{\alpha_i+1}},$$

the second inequality above being true because $e_i = (p^{\alpha_i}, \mathrm{val}_z(a_{\alpha_i}(z)))$ and $e_{i+1} = (p^{\alpha_{i+1}}, \mathrm{val}_z(a_{\alpha_{i+1}}(z)))$ are vertices of $\mathcal{N}(L)$. This concludes the proof of (25) and, hence, of (iii) and of (22).

It remains to justify the equalities (23) and (24). Let

$$\lambda_1 = p^{-1}\mu_1 = 0 < \lambda_2 = p^{-1}\mu_2 < \cdots < \lambda_k = p^{-1}\mu_k$$

be the slopes of $L(\phi_p - 1)$. Consider a nonzero slope λ_j of $L(\phi_p - 1)$. So, $p\lambda_j = \mu_j$ is a slope of L and, setting

$$L^{[\theta_{-(p-1)p\lambda_j}(z)]} = \sum_{i=0}^{n} c_i(z)\phi_p^i,$$

we have $\mathrm{val}_z(c_{\alpha_j}(z)) = \mathrm{val}_z(c_{\alpha_{j+1}}(z))$, $\mathrm{val}_z(c_i(z)) \geq \mathrm{val}_z(c_{\alpha_j}(z))$ for $i \in \{\alpha_j, \ldots, \alpha_{j+1}\}$ and $\mathrm{val}_z(c_i) > \mathrm{val}_z(c_{\alpha_j}(z))$ for $i \in \{0, \ldots, n\} \setminus \{\alpha_j, \ldots, \alpha_{j+1}\}$. The characteristic polynomial attached to the slope $p\lambda_j = \mu_j$ of L is given by

$$\chi(p\lambda_j, L; X) = \left(z^{-\mathrm{val}_z(L^{[\theta_{-(p-1)p\lambda_j}(z)]})} \sum_{i=\alpha_{j-1}}^{\alpha_j} c_i(z)X^i\right)_{|z=0}.$$

On the other hand, we have

$$(L(\phi_p - 1))^{[\theta_{-(p-1)\lambda_j}(z)]} = (L\phi_p)^{[\theta_{-(p-1)\lambda_j}(z)]} - L^{[\theta_{-(p-1)\lambda_j}(z)]}.$$

But,

$$(L\phi_p)^{[\theta_{-(p-1)\lambda_j}(z)]} = \theta_{(p-1)(1-p)\lambda_j}(z)L^{[\theta_{-(p-1)p\lambda_j}(z)]}\phi_p$$

and

$$L^{[\theta_{-(p-1)\lambda_j}(z)]} = (L^{[\theta_{-(p-1)p\lambda_j}(z)]})^{[\theta_{-(p-1)(1-p)\lambda_j}(z)]}.$$

It follows from this that

$$\mathrm{val}_z((L\phi_p)^{[\theta_{-(p-1)\lambda_j}(z)]}) < \mathrm{val}_z(L^{[\theta_{-(p-1)\lambda_j}(z)]}).$$

So,

$$\mathrm{val}_z((L(\phi_p - 1))^{[\theta_{-(p-1)\lambda_j}(z)]})$$
$$= \mathrm{val}_z((L\phi_p)^{[\theta_{-(p-1)\lambda_j}(z)]}) = \mathrm{val}_z(\theta_{(p-1)(1-p)\lambda_j}(z)L^{[\theta_{-(p-1)p\lambda_j}(z)]})$$

and, hence,

$$\chi(\lambda_j, L(\phi_p - 1); X) = \\ \left(z^{-\operatorname{val}_z(L^{[\theta-(p-1)p\lambda_j(z)]})} \sum_{i=\alpha_{j-1}+1}^{\alpha_j+1} c_{i-1}(z) X^i \right)_{|z=0} = \chi(p\lambda_j, L; X).$$

Last, the proof of the equality $\chi(0, L(\phi_p - 1); X) = \chi(0, L; X)(X - 1)$ is easy and left to the reader. □

7.3. **Proof of Proposition 15.** We set

$$M = L^{[\theta-(p-1)\mu_1(z)]}.$$

Lemma 17 ensures that

— $\mathcal{S}(M) = \{\mu_1 - \mu_1 = 0, \mu_2 - \mu_1, \ldots, \mu_k - \mu_1\}$;
— for $i \in \{1, \ldots, k\}$, $r(\mu_i - \mu_1, M) = r_i$;
— for $i \in \{1, \ldots, k\}$, the exponents of M (counted with multiplicities) attached to the slope $\mu_i - \mu_1$ coincide with those of L attached to the slope μ_i, *i.e.*, $c_{i,1}, \ldots, c_{i,r_i}$.

We claim that M has a decomposition of the form

$$M = M_{r_1}(\phi_p - c_{1,r_1})h_{1,r_1}(z)^{-1} \cdots (\phi_p - c_{1,1})h_{1,1}(z)^{-1} \tag{26}$$

for some $h_{1,1}(z), \ldots, h_{1,r_1}(z) \in \mathscr{H}^\times$ tangent to the identity and some $M_{r_1} \in \mathcal{D}_{\mathscr{H}}$ such that

— $\mathcal{S}(M_{r_1}) = \{p^{r_1}(\mu_2 - \mu_1), \ldots, p^{r_1}(\mu_k - \mu_1)\}$;
— for $i \in \{2, \ldots, k\}$, $r(p^{r_1}(\mu_i - \mu_1), M_{r_1}) = r(\mu_i, L)$;
— for $i \in \{2, \ldots, k\}$, the exponents of M_{r_1} (counted with multiplicities) attached to the slope $\mu_i - \mu_1$ are $c_{i,1}, \ldots, c_{i,r_i}$.

Indeed, Lemma 20 ensures that there exists $h_{1,1} \in \mathscr{H}^\times$ tangent to the identity and $M_1 \in \mathcal{D}_{\mathscr{H}}$ such that

$$M = M_1(\phi_p - c_{1,1})h_{1,1}(z)^{-1}.$$

If $r_1 = 1$ then Lemma 21 ensures that

— $\mathcal{S}(M_1) = \{p(\mu_2 - \mu_1), \ldots, p(\mu_k - \mu_1)\}$;
— for $i \in \{2, \ldots, k\}$, $r(p(\mu_i - \mu_1), M_1) = r_i$;
— for $i \in \{2, \ldots, k\}$, the exponents of M_1 (counted with multiplicities) attached to the slope $p(\mu_i - \mu_1)$ are $c_{i,1}, \ldots, c_{i,r_i}$.

This proves our claim when $r_1 = 1$.

If $r_1 \geq 2$ then Lemma 21 ensures that

— $\mathcal{S}(M_1) = \{0, p(\mu_2 - \mu_1), \ldots, p(\mu_k - \mu_1)\}$;
— $r(0, M_1) = r_1 - 1$;
— for $i \in \{2, \ldots, k\}$, $r(p(\mu_i - \mu_1), M_1) = r_i$;
— the exponents of M_1 (counted with multiplicities) attached to the slope 0 are $c_{1,2}, \ldots, c_{1,r_1}$ whereas those associated to the slope $p(\mu_i - \mu_1)$, for $i \in \{2, \ldots, k\}$, are $c_{i,1}, \ldots, c_{i,r_i}$.

Arguing as above, we get a decomposition of the form

$$M = M_2(\phi_p - c_{1,2})h_{1,2}(z)^{-1}(\phi_p - c_{1,1})h_{1,1}(z)^{-1}$$

for some $h_{1,2}(z) \in \mathscr{H}^\times$ tangent to the identity and some $M_2 \in \mathcal{D}_{\mathscr{H}}$.

If $r_1 = 2$, then Lemma 21 ensures that

— $\mathcal{S}(M_2) = \{p^2(\mu_2 - \mu_1), \dots, p^2(\mu_k - \mu_1)\}$;
— for $i \in \{2, \dots, k\}$, $r(p^2(\mu_i - \mu_1), M_2) = r_i$;
— for $i \in \{2, \dots, k\}$, the exponents of M_2 (counted with multiplicities) attached to the slope $p^2(\mu_i - \mu_1)$ are $c_{i,3}, \dots, c_{i,r_i}$.

This proves our claim when $r_1 = 2$

In the general case (arbitrary r_1), our claim follows by an obvious iteration of the above arguments.

Using (26) and the identities $L = M^{[\theta_{(p-1)\mu_1}(z)]}$ and $(\phi_p - c)^{[\theta_{(p-1)\mu_1}(z)]} = z^{(p-1)\mu_1}\phi_p - c$, we obtain a decomposition of the form

$$L = N_1 L_1$$

where

$$L_1 = (z^{(p-1)\mu_1}\phi_p - c_{1,r_1})h_{1,r_1}(z)^{-1} \cdots (z^{(p-1)\mu_1}\phi_p - c_{1,1})h_{1,1}(z)^{-1}$$

for some $h_{1,1}(z), \dots, h_{1,r_1}(z) \in \mathscr{H}^\times$ tangent to the identity and for some $N_1 \in \mathcal{D}_{\mathscr{H}}$ such that

— $\mathcal{S}(N_1) = \{p^{r_1}(\mu_2 - \mu_1) + \mu_1, \dots, p^{r_1}(\mu_k - \mu_1) + \mu_1\}$;
— for $i \in \{2, \dots, k\}$, $r(p^{r_1}(\mu_i - \mu_1) + \mu_1, N_1) = r_i$;
— for $i \in \{2, \dots, k\}$, the exponents of N_1 (counted with multiplicities) attached to the slope $p^{r_1}(\mu_i - \mu_1) + \mu_1$ are $c_{i,1}, \dots, c_{i,r_i}$.

These properties of $N_1 = M_{r_1}^{[\theta_{(p-1)\mu_1}(z)]}$ follow from the properties of M_{r_1} listed above and from Lemma 17.

Applying what precedes to N_1 instead of L, we find a decomposition of the form

$$N_1 = N_2 L_2$$

where

$$L_2 = (z^{(p-1)(p^{r_1}(\mu_2-\mu_1)+\mu_1)}\phi_p - c_{2,r_2})h_{2,r_2}(z)^{-1} \cdots \\ \cdots (z^{(p-1)(p^{r_1}(\mu_2-\mu_1)+\mu_1)}\phi_p - c_{2,1})h_{2,1}(z)^{-1}$$

for some $h_{2,1}(z), \dots, h_{2,r_2}(z) \in \mathscr{H}^\times$ tangent to the identity and for some $N_2 \in \mathcal{D}_{\mathscr{H}}$ such that

— $\mathcal{S}(N_2) = \{p^{r_1+r_2}(\mu_3 - \mu_2) + p^{r_1}(\mu_2 - \mu_1) + \mu_1, \dots, p^{r_1+r_2}(\mu_k - \mu_2) + p^{r_1}(\mu_2 - \mu_1) + \mu_1\}$;
— for $i \in \{3, \dots, k\}$, $r(p^{r_1+r_2}(\mu_i - \mu_2) + p^{r_1}(\mu_2 - \mu_1) + \mu_1, N_2) = r_i$;
— for $i \in \{3, \dots, k\}$, the exponents of N_2 (and their multiplicities) attached to the slope $p^{r_1+r_2}(\mu_i - \mu_2) + p^{r_1}(\mu_2 - \mu_1) + \mu_1$ are $c_{i,1}, \dots, c_{i,r_i}$.

Iterating the previous construction, we find $L_1, \dots, L_k \in \mathcal{D}_{\mathscr{H}}$ of the form announced in the statement of Proposition 15 such that

$$L = a(z) L_k \cdots L_1$$

for some $a(z) \in \mathscr{H}^\times$. Equating the coefficients of degree 0 in the latter equation, we find $a(z)\prod_{i=1}^{k}\prod_{j=1}^{r_i}(-c_{i,j}h_{i,j}(z)^{-1}) = a_0(z)$, whence $\mathrm{val}_z(a(z)) = \mathrm{val}_z(a_0(z))$ and

$$(\mathrm{cld}_z\, a(z)) \prod_{i=1}^{k}\prod_{j=1}^{r_i}(-c_{i,j}) = \mathrm{cld}_z\, a_0(z).$$

8. Forbenius Method : first justifications

The aim of this section is to prove Proposition 22 below, which is relative to a Mahler operator

$$L = a_n(z)\phi_p^n + a_{n-1}(z)\phi_p^{n-1} + \cdots + a_0(z) \in \mathcal{D}_{\mathscr{H}}. \tag{27}$$

In the rest of this section, we let $\mu_1 < \cdots < \mu_k$ be the slopes of L with respective multiplicities $r_1, \dots, r_k$. Moreover, the multiplicity of an exponent c of L attached to the slope μ_j will be denoted by $m_{c,j} \in \mathbb{Z}_{\geq 1}$. If c is a nonzero complex number that is not an exponent attached to the slope μ_j, we set $m_{c,j} = 0$.

Proposition 22. *For any slope μ_j, for any exponent c of L associated to the slope μ_j, there exists a unique $g_{c,j}(\lambda, z) \in \mathscr{H}_{\mathbb{C}(\lambda)}$ such that*

$$L(g_{c,j}(\lambda, z)e_\lambda) = z^{\mathrm{val}_z(a_0(z)) - \frac{\nu_j}{p-1}}(\lambda - c)^{s_{c,j}+m_{c,j}} e_\lambda \tag{28}$$

where

$$s_{c,j} = m_{c,1} + \cdots + m_{c,j-1}.$$

and

$$\nu_j = (p-1)(p^{r_1+\cdots+r_{j-1}}(\mu_j - \mu_{j-1}) + \cdots + p^{r_1}(\mu_2 - \mu_1) + \mu_1).$$

The coefficients of $g_{c,j}(\lambda, z)$ have no pole at $\lambda = c$. Moreover, we have

$$\mathrm{val}_z(g_{c,j}(\lambda, z)) = -\mu_j \tag{29}$$

and

$$\mathrm{cld}_z\, g_{c,j}(\lambda, z) = \lambda^{-r_1 - \cdots - r_{j-1}} \frac{\prod_{i=1}^{j}\prod_{j=1}^{r_i}(-c_{i,j})}{\mathrm{cld}_z\, a_0} \frac{(\lambda - c)^{s_{c,j}+m_{c,j}}}{\prod_{l=1}^{r_j}(\lambda - c_{j,l})}. \tag{30}$$

The proof of Proposition 22, given in Section 8.2, uses some preliminary results gathered in the following section.

8.1. **Preliminary results.** We say that a family $(f_i(\lambda,z))_{i\in I}$ of elements of $\mathscr{H}_{\mathbb{C}(\lambda)}$ is summable if the following properties are satisfied :

— the set $\cup_{i\in I}\operatorname{supp}(f_i(\lambda,z))$ is well-ordered;

— for any $\gamma\in\mathbb{Q}$, the set

$$\{i\in I \mid \gamma\in\operatorname{supp}(f_i(\lambda,z))\}$$

is finite.

In this case, we define

$$\sum_{i\in I} f_i(\lambda,z)=\sum_{i\in I}(\sum_{\gamma\in\mathbb{Q}} f_{i,\gamma}(\lambda))z^{\gamma}\in\mathscr{H}_{\mathbb{C}(\lambda)}$$

where $f_i(\lambda,z)=\sum_{\gamma\in\mathbb{Q}} f_{i,\gamma}(\lambda)z^{\gamma}$.

In what follows, we let $\mathscr{H}_{\mathbb{C}(\lambda)}^{<0}$ (resp. $\mathscr{H}_{\mathbb{C}(\lambda)}^{>0}$) be the set made of the $f(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}$ such that $\operatorname{supp}(f(\lambda,z))\subset\mathbb{Q}_{<0}$ (resp. $\operatorname{supp}(f(\lambda,z))\subset\mathbb{Q}_{>0}$).

Lemma 23. *We have :*

— *if $g(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}^{<0}$ then $(\phi_p^k(g(\lambda,z)))_{k\leq -1}$ is summable;*

— *if $g(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}^{>0}$ then $(\phi_p^k(g(\lambda,z)))_{k\geq 0}$ is summable.*

Proof. Let us first assume that $g(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}^{<0}$. We set

$$E=\cup_{k\leq -1}E_k$$

where

$$E_k=\operatorname{supp}(\phi_p^k(g(\lambda,z)))=p^k\operatorname{supp}(g(\lambda,z)).$$

Let us first prove that E is well-ordered. Let F be a nonempty subset of E. Since $E\subset\mathbb{Q}_{<0}$ and $\inf E_k\xrightarrow[k\to-\infty]{}0$, we have

$$\inf F=\inf F\cap\cup_{k=k_0}^{-1}E_k$$

for some $k_0\leq -1$. Since $\operatorname{supp}(g(\lambda,z))$ is well-ordered, each $E_k=p^k\operatorname{supp}(g(\lambda,z))$ is well-ordered and, hence, $\cup_{k=k_0}^{-1}E_k$ is well-ordered. It follows that $F\cap\cup_{k=k_0}^{-1}E_k$ and, hence, F have a least element.

In order to prove that $(\phi_p^k(g(\lambda,z)))_{k\leq -1}$ is summable, it remains to prove that, for any $\gamma\in\mathbb{Q}$, the set $\{k\leq -1 \mid \gamma\in E_k\}$ is finite. This is clear since $E_k\subset\mathbb{Q}_{<0}$ and $\inf E_k\xrightarrow[k\to-\infty]{}0$.

The proof in the case $g(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}^{>0}$ is similar. □

Lemma 24. *For any $g(\lambda,z)=\sum_{\gamma\in\mathbb{Q}} g_{\gamma}(\lambda)z^{\gamma}\in\mathscr{H}_{\mathbb{C}(\lambda)}$, there exists a unique $f(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}$ such that*

$$(\lambda\phi_p-1)(f(\lambda,z))=g(\lambda,z). \tag{31}$$

If

$$g(\lambda,z)=g_-(\lambda,z)+g_0(\lambda)+g_+(\lambda,z)$$

with $g_-(\lambda,z)\in \mathscr{H}^{<0}_{\mathbb{C}(\lambda)}$, $g_0(\lambda)\in\mathbb{C}$, $g_+(\lambda,z)\in \mathscr{H}^{>0}_{\mathbb{C}(\lambda)}$, *then we have*

$$f(\lambda,z)=\sum_{k\leq -1}\lambda^k\phi_p^k(g_-(\lambda,z))+\frac{g_0(\lambda)}{\lambda-1}-\sum_{k\geq 0}\lambda^k\phi_p^k(g_+(\lambda,z)). \tag{32}$$

Proof. Lemma 23 ensures that the families $(\lambda^k\phi_p^k(g_-(\lambda,z)))_{k\leq -1}$ and $(\lambda^k\phi_p^k(g_+(\lambda,z)))_{k\geq 0}$ are summable. The right-hand side of (32) is thus meaningful and defines an element of $\mathscr{H}_{\mathbb{C}(\lambda)}$. A straightforward calculation shows that this $f(\lambda,z)$ satisfies (31).

In order to prove that (32) is the unique element of $\mathscr{H}_{\mathbb{C}(\lambda)}$ satisfying (31), it is necessary and sufficient to prove that there is no nonzero $h(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}$ such that

$$(\lambda\phi_p-1)(h(\lambda,z))=0. \tag{33}$$

If $h(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}$ satisfies (33) then we have $\lambda\phi_p(h(\lambda,z))=h(\lambda,z)$. It follows that $p\,\mathrm{supp}(h(\lambda,z))=\mathrm{supp}(\phi_p(h(\lambda,z)))=\mathrm{supp}(h(\lambda,z))$ and, hence, for all $k\in\mathbb{Z}$, $p^k\,\mathrm{supp}(h(\lambda,z))=\mathrm{supp}(h(\lambda,z))$. Since $\mathrm{supp}(h(\lambda,z))$ is well-ordered, the latter equalities imply $\mathrm{supp}(h(\lambda,z))\subset\{0\}$; indeed, if there were $\gamma\in\mathrm{supp}(h(\lambda,z))\cap\mathbb{Q}_{>0}$ (resp. $\gamma\in\mathrm{supp}(h(\lambda,z))\cap\mathbb{Q}_{<0}$) then $p^{\mathbb{Z}_{\leq 0}}\gamma$ (resp. $p^{\mathbb{Z}_{\geq 0}}\gamma$) would be a subset of $\mathrm{supp}(f(\lambda,z))$ with no least element, contradicting the fact that $\mathrm{supp}(f(\lambda,z))$ is well-ordered. So $h(\lambda,z)=h_0(\lambda)\in\mathbb{C}(\lambda)$. Inserting $h(\lambda,z)=h_0(\lambda)$ in (33), we find $h(\lambda,z)=h_0(\lambda)=0$, as expected. □

Lemma 25. *Consider* $\mu\in\mathbb{Q}$ *and* $c\in\mathbb{C}^\times$. *For any* $g(\lambda,z)=\sum_{\gamma\in\mathbb{Q}}g_\gamma(\lambda)z^\gamma\in\mathscr{H}_{\mathbb{C}(\lambda)}$, *there exists a unique* $f(\lambda,z)=\sum_{\gamma\in\mathbb{Q}}f_\gamma(\lambda)z^\gamma\in\mathscr{H}_{\mathbb{C}(\lambda)}$ *such that*

$$(z^{-\mu}\lambda\phi_p-c)(f(\lambda,z))=g(\lambda,z). \tag{34}$$

Moreover,
- *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda,z))\geq 0$ *then* $\mathrm{val}_z(f(\lambda,z))=\mathrm{val}_z(g(\lambda,z))$;
- *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda,z))<0$ *then* $\mathrm{val}_z(\theta_{-\mu}(z)f(\lambda,z))=\frac{\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda,z))}{p}$;

and
- *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda,z))<0$, *then :*
  - $\mathrm{cld}_z(f(\lambda,z))=\lambda^{-1}\,\mathrm{cld}_z(g(\lambda,z))$;
  - *if the* $g_\gamma(\lambda)$ *have at most poles of order* ρ *at* c, *then the* $f_\gamma(\lambda)$ *have at most poles of order* $\rho+1$ *at* c;
- *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda,z))=0$, *then :*
  - $\mathrm{cld}_z(f(\lambda,z))=(\lambda-c)^{-1}\,\mathrm{cld}_z(g(\lambda,z))$;
  - *if the* $g_\gamma(\lambda)$ *have no pole at* c *and if* $\mathrm{cld}_z(g(\lambda,z))$ *vanishes at* c, *then the* $f_\gamma(\lambda)$ *have no pole at* c;
- *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda,z))>0$, *then :*
  - $\mathrm{cld}_z(f(\lambda,z))=(-c)^{-1}\,\mathrm{cld}_z(g(\lambda,z))$;

— *if the* $g_\gamma(\lambda)$ *have no pole at* c*, then the* $f_\gamma(\lambda)$ *have no pole at* c*.*

Last, if the $g_\gamma(\lambda)$ *have at most poles of order* ρ *at* $c' \in \mathbb{C}^\times \setminus \{c\}$*, then the* $f_\gamma(\lambda)$ *have at most poles of order* ρ *at* c'*.*

Proof. Replacing λ by $c\lambda$, we can rewrite equation (34) as follows

$$(\lambda\phi_p - 1)(\theta_{-\mu}(z)f(c\lambda, z)) = \theta_{-\mu}(z)c^{-1}g(c\lambda, z).$$

Combining this with Lemma 24, we get that there exists a unique $f(\lambda, z) \in \mathscr{H}_{\mathbb{C}(\lambda)}$ satisfying (34) and that it is given by

$$(35)\quad \theta_{-\mu}f(\lambda, z) = \sum_{k\leq -1}(c^{-1}\lambda)^k\phi_p^k(\widetilde{g}_-(c^{-1}\lambda, z)) + \frac{\widetilde{g}_0(c^{-1}\lambda)}{c^{-1}\lambda - 1} - \sum_{k\geq 0}(c^{-1}\lambda)^k\phi_p^k(\widetilde{g}_+(c^{-1}\lambda, z))$$

where $\widetilde{g}_-(\lambda, z) \in \mathscr{H}_{\mathbb{C}(\lambda)}^{<0}$, $\widetilde{g}_0(\lambda) \in \mathbb{C}$, $\widetilde{g}_+(\lambda, z), \in \mathscr{H}_{\mathbb{C}(\lambda)}^{>0}$ are such that

$$\theta_{-\mu}(z)c^{-1}g(c\lambda, z) = \widetilde{g}_-(\lambda, z) + \widetilde{g}_0(\lambda) + \widetilde{g}_+(\lambda, z),$$

i.e.,

$$\widetilde{g}_-(\lambda, z) = \theta_{-\mu}(z)\sum_{\gamma\in\mathbb{Q}_{<\frac{\mu}{p-1}}} c^{-1}g_\gamma(c\lambda)z^\gamma \in \mathscr{H}_{\mathbb{C}(\lambda)}^{<0},$$

$$\widetilde{g}_0(\lambda) = c^{-1}g_{\frac{\mu}{p-1}}(c\lambda) \in \mathbb{C}(\lambda)$$

and

$$\widetilde{g}_+(\lambda, z) = \theta_{-\mu}(z)\sum_{\gamma\in\mathbb{Q}_{>\frac{\mu}{p-1}}} c^{-1}g_\gamma(c\lambda)z^\gamma \in \mathscr{H}_{\mathbb{C}(\lambda)}^{>0}.$$

The properties of $f(\lambda, z)$ listed in the lemma follow by direct inspection of (35). □

Lemma 26. *Consider* $\mu \in \mathbb{Q}$*,* $c_1, \ldots, c_r \in \mathbb{C}^\times$ *and* $h_1(z), \ldots, h_r(z) \in \mathscr{H}^\times$ *tangent to the identity. For any* $g(\lambda, z) \in \mathscr{H}_{\mathbb{C}(\lambda)}$*, there exists a unique* $f(\lambda, z) \in \mathscr{H}_{\mathbb{C}(\lambda)}$ *such that*

$$(36)\quad (z^{-\mu}\lambda\phi_p - c_r)h_r(z)^{-1}\cdots(z^{-\mu}\lambda\phi_p - c_1)h_1(z)^{-1}(f(\lambda, z)) = g(\lambda, z).$$

Moreover,

— *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda, z)) \geq 0$ *then* $\mathrm{val}_z(f(\lambda, z)) = \mathrm{val}_z(g(\lambda, z))$*;*

— *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda, z)) < 0$ *then* $\mathrm{val}_z(\theta_{-\mu}(z)f(\lambda, z)) = \frac{\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda,z))}{p^r}$*;*

and

— *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda, z)) < 0$*, then :*
  — $\mathrm{cld}_z(f(\lambda, z)) = \lambda^{-r}\,\mathrm{cld}_z(g(\lambda, z))$*;*
  — *if the* $g_\gamma(\lambda)$ *have at most poles of order* ρ *at* $c \in \mathbb{C}^\times$*, then the* $f_\gamma(\lambda)$ *have at most poles of order* $\rho + m$ *at* c *where* $m = \sharp\{i \in \{1, \ldots, r\} \mid c_i = c\}$*;*

— *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda, z)) = 0$*, then :*

— $\mathrm{cld}_z(f(\lambda,z)) = (\lambda - c_r)^{-1}\cdots(\lambda-c_1)^{-1}\,\mathrm{cld}_z(g(\lambda,z))$;
— *if the* $g_\gamma(\lambda)$ *and* $\mathrm{cld}_z(f(\lambda,z))$ *have no pole at* $c \in \mathbb{C}^\times$*, then the* $f_\gamma(\lambda)$ *have no pole at* c*;*

— *if* $\mathrm{val}_z(\theta_{-\mu}(z)g(\lambda,z)) > 0$*, then :*
— $\mathrm{cld}_z(f(\lambda,z)) = (-c_r)^{-1}\cdots(-c_1)^{-1}\,\mathrm{cld}_z(g(\lambda,z))$;
— *if the* $g_\gamma(\lambda)$ *have no pole at* $c \in \mathbb{C}^\times$*, then the* $f_\gamma(\lambda)$ *have no pole at* c*.*

Last, if the $g_\gamma(\lambda)$ *have at most poles of order* ρ *at* $c' \in \mathbb{C}^\times \setminus \{c_1,\ldots,c_r\}$*, then the* $f_\gamma(\lambda)$ *have at most poles of order* ρ *at* c'*.*

Proof. The equation (36) is equivalent to the system of equations

$$\begin{cases} (z^{-\mu}\lambda\phi_p - c_r)h_r(z)^{-1}(f_r(\lambda,z)) = g(\lambda,z) \\ (z^{-\mu}\lambda\phi_p - c_{r-1})h_{r-1}(z)^{-1}(f_{r-1}(\lambda,z)) = f_r(\lambda,z) \\ \cdots \\ (z^{-\mu}\lambda\phi_p - c_1)h_1(z)^{-1}(f_1(\lambda,z)) = f_2(\lambda,z) \\ f(\lambda,z) = f_1(\lambda,z). \end{cases}$$

The result follows by r successive applications of Lemma 25. □

8.2. **Proof of Proposition 22.** Using the fact that $g(\lambda,z) \in \mathscr{H}_{\mathbb{C}(\lambda)}$ satisfies

(37) $$L(g(\lambda,z)e_\lambda) = z^{\mathrm{val}_z(a_0(z)) - \frac{\nu_j}{p-1}}(\lambda - c)^{s_{c,j}+m_{c,j}}e_\lambda$$

if and only if $f(\lambda,z) = (\lambda-c)^{-s_{c,j}}g(\lambda,z) \in \mathscr{H}_{\mathbb{C}(\lambda)}$ satisfies

(38) $$\begin{aligned} L(f(\lambda,z)e_\lambda) &= z^{\mathrm{val}_z(a_0(z)) - \frac{\nu_j}{p-1}}(\lambda-c)^{m_{c,j}}e_\lambda \\ &= z^{\mathrm{val}_z(a_0(z))}\theta_{-\nu_j}(z)(\lambda-c)^{m_{c,j}}e_\lambda, \end{aligned}$$

we see that, in order to prove Proposition 22, it is sufficient to prove that :

(i) there exists a unique $f(\lambda,z)$ in $\mathscr{H}_{\mathbb{C}(\lambda)}$ satisfying (37);

and that this $f(\lambda,z)$ has the following properties :

(ii) $\mathrm{val}_z(f(\lambda,z)) = -\mu_j$;

(iii) $\mathrm{cld}_z\, f(\lambda,z) = \lambda^{-r_1-\cdots-r_{j-1}}\frac{\prod_{i=1}^{j}\prod_{j=1}^{r_i}(-c_{i,j})}{\mathrm{cld}_z\, a_0(z)}\frac{(\lambda-c)^{m_{c,j}}}{\prod_{l=1}^{r_j}(\lambda-c_{j,l})}$;

(iv) the coefficients of $f(\lambda,z)$ have poles of order at most $s_{c,j}$ at c.

In order to prove these claims, let us first note that (38) can be rewritten as

(39) $$M(\theta_{\nu_j}(z)f(\lambda,z)) = z^{\mathrm{val}_z(a_0(z))}(\lambda-c)^{m_{c,j}}$$

where

$$M = (L^{[\theta_{-\nu_j}(z)]})^{[e_\lambda]}.$$

But, from the factorization

$$L = a(z)L_k\cdots L_1$$

given by Proposition 15, we get the factorization

$$M = a(z) M_k \cdots M_1$$

where $a(z) \in \mathscr{H}^\times$ is such that $\mathrm{val}_z(a(z)) = \mathrm{val}_z(a_0(z))$ and where

$$\begin{aligned} M_i &= (L_i^{[\theta_{-\nu_j}(z)]})^{[e_\lambda]} \\ &= (z^{\nu_i - \nu_j}\lambda\phi_p - c_{i,r_i}) h_{i,r_i}(z)^{-1} \cdots (z^{\nu_i-\nu_j}\lambda\phi_p - c_{i,1}) h_{i,1}(z)^{-1}. \end{aligned}$$

So, the equation (39) can be rewritten as follows :

$$\begin{cases} M_k(f_k(\lambda,z)) = \frac{z^{\mathrm{val}_z(a_0(z))}}{a(z)}(\lambda - c)^{m_{c,j}} \\ M_{k-1}(f_{k-1}(\lambda,z)) = f_k(\lambda,z) \\ \cdots \\ M_1(f_1(\lambda,z)) = f_2(\lambda,z) \\ \theta_{\nu_j}(z) f(\lambda,z) = f_1(\lambda,z). \end{cases} \tag{40}$$

Now, Lemma 26 ensures that there exists a unique k-uple

$$(f_1(\lambda,z),\dots,f_k(\lambda,z)) \in \mathscr{H}^k_{\mathbb{C}(\lambda)}$$

satisfying the first k equations of (40), whence the existence and the uniqueness of $f(\lambda,z) \in \mathscr{H}$ satisfying (37); it is given by $f(\lambda,z) = \theta_{-\nu_j}(z) f_1(\lambda,z)$. This proves claim (i).

Note that $\mathrm{val}_z(\frac{z^{\mathrm{val}_z(a_0(z))}}{a(z)}) = 0$ and set

$$\alpha = \mathrm{cld_z}\, \frac{z^{\mathrm{val}_z(a_0(z))}}{a(z)} = \mathrm{cld_z}\, a(z)^{-1}.$$

For $i \in \{j+1,\dots,k\}$, we have $\nu_i - \nu_j > 0$, so Lemma 26 ensures that $f_k(\lambda,z),\dots,f_{j+1}(\lambda,z)$ have z-adic valuation 0 with constant terms

$$\begin{aligned} \mathrm{cld_z}\, f_k(\lambda,z) &= \left(\prod_{l=1}^{r_k}(-c_{k,l})^{-1}\right)\alpha(\lambda-c)^{m_{c,j}}, \\ \mathrm{cld_z}\, f_{k-1}(\lambda,z) &= \left(\prod_{i=k-1}^{k}\prod_{l=1}^{r_i}(-c_{i,l})^{-1}\right)\alpha(\lambda-c)^{m_{c,j}}, \\ \cdots &\quad \cdots \quad \cdots \\ \mathrm{cld_z}\, f_{j+1}(\lambda,z) &= \left(\prod_{i=j+1}^{k}\prod_{l=1}^{r_i}(-c_{i,l})^{-1}\right)\alpha(\lambda-c)^{m_{c,j}} \end{aligned}$$

and also that the coefficients of $f_k(\lambda,z),\dots,f_{j+1}(\lambda,z)$ have no pole at $\lambda = c$.

For $i = j$, we have $\nu_i - \nu_j = 0$, so Lemma 26 ensures that $f_j(\lambda,z)$ has z-adic valuation 0 with constant term

$$\mathrm{cld_z}\, f_j(\lambda,z) = \left(\prod_{i=j+1}^{k}\prod_{l=1}^{r_i}(-c_{i,l})^{-1}\right)\alpha\frac{(\lambda-c)^{m_{c,j}}}{\prod_{l=1}^{r_j}(\lambda - c_{j,l})}$$

and also that the coefficients of $f_j(\lambda,z)$ have no pole at $\lambda = c$.

We have $\mathrm{val}_z(\theta_{\nu_{j-1}-\nu_j}(z)f_j(\lambda,z)) = \mathrm{val}_z(\theta_{\nu_{j-1}-\nu_j}(z)) < 0$, so Lemma 26 ensures that

$$\mathrm{val}_z(\theta_{\nu_{j-1}-\nu_j}(z)h_{j-1}(\lambda,z)) = \frac{\mathrm{val}_z(\theta_{\nu_{j-1}-\nu_j}(z)h_j(\lambda,z))}{p^{r_{j-1}}} < 0.$$

Therefore, we have

$$\begin{aligned}&\mathrm{val}_z(\theta_{\nu_{j-2}-\nu_j}(z)f_{j-1}(\lambda,z))\\ &\qquad= \mathrm{val}_z(\theta_{\nu_{j-2}-\nu_{j-1}}(z)) + \mathrm{val}_z(\theta_{\nu_{j-1}-\nu_j}(z)f_{j-1}(\lambda,z)) < 0\end{aligned}$$

and Lemma 26 ensures that

$$\mathrm{val}_z(\theta_{\nu_{j-2}-\nu_j}(z)f_{j-2}(\lambda,z)) = \frac{\mathrm{val}_z(\theta_{\nu_{j-2}-\nu_j}(z)h_{j-1}(\lambda,z))}{p^{r_{j-2}}} < 0.$$

An obvious iteration of this argument leads to the fact that, for $i \in \{1,\dots,j-1\}$, we have

$$\mathrm{val}_z(\theta_{\nu_i-\nu_j}(z)h_i(\lambda,z)) = \frac{\mathrm{val}_z(\theta_{\nu_i-\nu_j}(z)f_{i+1}(\lambda,z))}{p^{r_i}} < 0.$$

Therefore, we have

$$\begin{aligned}\mathrm{val}_z(\theta_{\nu_1-\nu_j}(z)f_1(\lambda,z)) &= \frac{\mathrm{val}_z(\theta_{\nu_1-\nu_j}(z)h_2(\lambda,z))}{p^{r_1}}\\ &= \frac{\mathrm{val}_z(\theta_{\nu_1-\nu_2}(z))}{p^{r_1}} + \frac{\mathrm{val}_z(\theta_{\nu_2-\nu_j}(z)h_2(\lambda,z))}{p^{r_1}}\\ &= \frac{\mathrm{val}_z(\theta_{\nu_1-\nu_2}(z))}{p^{r_1}} + \frac{\mathrm{val}_z(\theta_{\nu_2-\nu_j}(z)h_3(\lambda,z))}{p^{r_1+r_2}}\\ &= \cdots\\ &= \frac{\mathrm{val}_z(\theta_{\nu_1-\nu_2}(z))}{p^{r_1}} + \frac{\mathrm{val}_z(\theta_{\nu_2-\nu_3}(z))}{p^{r_1+r_2}} + \cdots + \frac{\mathrm{val}_z(\theta_{\nu_{j-1}-\nu_j}(z))}{p^{r_1+r_2+\cdots+r_{j-1}}}\\ &= \frac{1}{p-1}\left(\frac{\nu_1-\nu_2}{p^{r_1}} + \frac{\nu_2-\nu_3}{p^{r_1+r_2}} + \cdots + \frac{\nu_{j-1}-\nu_j}{p^{r_1+r_2+\cdots+r_{j-1}}}\right)\\ &= \frac{p^{r_1}(\mu_1-\mu_2)}{p^{r_1}} + \frac{p^{r_1+r_2}(\mu_2-\mu_3)}{p^{r_1+r_2}} + \cdots + \frac{p^{r_1+r_2+\cdots+r_{j-1}}(\mu_{j-1}-\mu_j)}{p^{r_1+r_2+\cdots+r_{j-1}}}\\ &= \mu_1-\mu_j\end{aligned}$$

so

$$\mathrm{val}_z(\theta_{-\nu_j}(z)f_1(\lambda,z)) = -\mu_j.$$

This proves (ii).

It follows also from Lemma 26 that

$$\mathrm{cld}_z(f_1(\lambda,z)) = \lambda^{-r_1-\cdots-r_{j-1}}\left(\prod_{i=j+1}^{k}\prod_{l=1}^{r_i}(-c_{i,l})^{-1}\right)\alpha\frac{(\lambda-c)^{m_{c,j}}}{\prod_{l=1}^{r_j}(\lambda-c_{j,l})}$$

$$= \lambda^{-r_1-\cdots-r_{j-1}}\left(\prod_{i=j+1}^{k}\prod_{l=1}^{r_i}(-c_{i,l})^{-1}\right)\frac{\prod_{i=1}^{k}\prod_{j=1}^{r_i}(-c_{i,j})}{\mathrm{cld}_z\, a_0(z)}\frac{(\lambda-c)^{m_{c,j}}}{\prod_{l=1}^{r_j}(\lambda-c_{j,l})}$$

$$= \lambda^{-r_1-\cdots-r_{j-1}}\frac{\prod_{i=1}^{j}\prod_{j=1}^{r_i}(-c_{i,j})}{\mathrm{cld}_z\, a_0(z)}\frac{(\lambda-c)^{m_{c,j}}}{\prod_{l=1}^{r_j}(\lambda-c_{j,l})}$$

(we have used the formula for $\mathrm{cld}_z\, a(z)$ given by Proposition 15 for the second equality) and that the coefficients of $f_{j-1}(\lambda,z),\ldots,f_1(\lambda,z)$ have poles order at most $m_{c,j-1}, m_{c,j-1}+m_{c,j-2},\ldots,m_{c,j-1}+m_{c,j-2}+\cdots+m_{c,1}$ at c respectively. This prove (iii) and (iv).

9. Frobenius method : last justifications

We use the notations introduced at the very beginning of Section 8. We have seen in Proposition 22 that, for any exponent c associated to the slope μ_j of L, there exists a unique $g_{c,j}(\lambda,z)\in\mathscr{H}_{\mathbb{C}(\lambda)}$ such that

$$L(g_{c,j}(\lambda,z)e_\lambda) = z^{\mathrm{val}_z(a_0(z))-\frac{\nu_j}{p-1}}(\lambda-c)^{s_{c,j}+m_{c,j}}e_\lambda$$

where

$$s_{c,j} = m_{c,1}+\cdots+m_{c,j-1}.$$

and

$$\nu_j = (p-1)(p^{r_1+\cdots+r_{j-1}}(\mu_j-\mu_{j-1})+\cdots+p^{r_1}(\mu_2-\mu_1)+\mu_1)$$

and that the coefficients of $g_{c,j}(\lambda,z)$ have no pole at $\lambda=c$ and, hence, $g_{c,j}(\lambda,z)e_\lambda$ belongs to $\mathscr{R}_{\lambda,c}$.

We first prove:

Proposition 27. *For any $m\in\{0,\ldots,m_{c,j}-1\}$,*

$$y_{c,j,m} = \mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}+m}(g_{c,j}(\lambda,z)e_\lambda))$$

is a solution of $L(y)=0$.

Proof. Using the fact that $\mathrm{ev}_{\lambda=c}$ and ∂_λ are $\mathscr{H}$-linear and commute with ϕ_p, we see that, for any $m\in\{0,\ldots,m_{c,j}-1\}$,

$$\begin{aligned}L(y_{c,j,m})\\ &= L(\mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}+m}(g_{c,j}(\lambda,z)e_\lambda))) = \mathrm{ev}_{\lambda=c}\,\partial_\lambda^{s_{c,j}+m}(L(g_{c,j}(\lambda,z)e_\lambda))\\ &= z^{\mathrm{val}_z(a_0(z))-\frac{\nu_j}{p-1}}\,\mathrm{ev}_{\lambda=c}\,\partial_\lambda^{s_{c,j}+m}((\lambda-c)^{s_{c,j}+m_{c,j}}e_\lambda)).\end{aligned}$$

Since $m\in\{0,\ldots,m_{c,j}-1\}$, we have $s_{c,j}+m<s_{c,j}+m_{c,j}$ and, hence,

$$\mathrm{ev}_{\lambda=c}\,\partial_\lambda^{s_{c,j}+m}((\lambda-c)^{s_{c,j}+m_{c,j}}e_\lambda)) = 0.$$

This proves that $y_{c,j,m}$ is a solution of L as claimed. □

It remains to prove the following result.

Theorem 28. *We have attached to any slope μ_j, to any exponent c attached to the slope μ_j and to any $m \in \{0, \dots, m_{c,j} - 1\}$, a solution*

$$y_{c,j,m} = \mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}+m}(g_{c,j}(\lambda, z)e_\lambda))$$

of $L(y) = 0$. These n solutions are $\mathbb{C}$-linearly independent.

The proof is given in the following Section.

9.1. **Proof of Theorem 28.** Using the Leibniz rule, we see that

$$\begin{aligned} y_{c,j,m} &\in \mathrm{Span}_{\mathscr{H}}(\mathrm{ev}_{\lambda=c}(\partial_\lambda^0(e_\lambda)), \mathrm{ev}_{\lambda=c}(\partial_\lambda^1(e_\lambda)), \dots \\ &\qquad\qquad \dots, \mathrm{ev}_{\lambda=c}(\partial_\lambda^{s_{c,j}+m_{c,j}-1}(e_\lambda))) \\ &= \mathrm{Span}_{\mathscr{H}}(e_c, \ell_{c,1}, \dots, \ell_{c,s_{c,j}+m_{c,j}-1}) \subset \mathrm{Span}_{\mathscr{H}}(\ell_{c,j})_{j\geq 0}. \end{aligned}$$

But, Lemma 30 proven below guarantees that the family $(\ell_{c,j})_{c\in\mathbb{C}^\times, j\geq 0}$ is $\mathscr{H}$-linearly independent. So, in order to prove Theorem 28, it is sufficient to prove that, for any exponent c of L, the family $(y_{c,j,m})_{j\in\{1,\dots,k\}, m\in\{0,\dots,m_{c,j}-1\}}$ is $\mathbb{C}$-linearly independent. Let us prove this. Fix such a c and consider a family $(a_{c,j,m})_{j\in\{1,\dots,k\}, m\in\{0,\dots,m_{c,j}-1\}}$ of complex numbers such that

$$\sum_{j\in\{1,\dots,k\}, m\in\{0,\dots,m_{c,j}-1\}} a_{c,j,m} y_{c,j,m} = 0. \tag{41}$$

We have to prove that the $a_{c,j,m}$ are all 0. In this respect, we will use the following result.

Lemma 29. *We have a decomposition of the form*

$$\begin{aligned} y_{c,j,m} &= \sum_{u=0}^{s_{c,j}+m} h_{c,j,m,u}(z)\ell_{c,u} \\ &= \sum_{u=0}^{m-1} h_{c,j,m,u}(z)\ell_{c,u} + h_{c,j,m,m}\ell_{c,m} + \sum_{u=m+1}^{s_{c,j}+m} h_{c,j,m,u}(z)\ell_{c,u} \end{aligned} \tag{42}$$

for some $h_{c,j,m,u}(z) \in \mathscr{H}$ such that

$$\begin{cases} \mathrm{val}_z(h_{c,j,m,u}(z)) \geq -\mu_j \text{ for } u \in \{0, \dots, m-1\}, \\ \mathrm{val}_z(h_{c,j,m,m}(z)) = -\mu_j, \\ \mathrm{val}_z(h_{c,j,m,u}(z)) > -\mu_j \text{ for } u \in \{m+1 \dots, s_{c,j} + m\}. \end{cases} \tag{43}$$

Proof. Using Leibniz rule, we obtain

$$\partial_\lambda^{s_{c,j}+m}(g_{c,j}(\lambda, z)e_\lambda) = \sum_{u=0}^{s_{c,j}+m} u! \binom{s_{c,j}+m}{u} \partial_\lambda^{s_{c,j}+m-u}(g_{c,j}(\lambda, z))\ell_{\lambda,u}$$

and, hence,

$$y_{c,j,m} = \mathrm{ev}_{\lambda=c}\,\partial_\lambda^{s_{c,j}+m}(g_{c,j}(\lambda,z)e_\lambda) = \sum_{u=0}^{s_{c,j}+m} h_{c,j,m,u}\ell_{c,u}.$$

with

$$h_{c,j,m,u}(z) = u!\binom{s_{c,j}+m}{u}\,\mathrm{ev}_{\lambda=c}\,\partial_\lambda^{s_{c,j}+m-u}(g_{c,j}(\lambda,z)) \in \mathscr{H}.$$

Accordingly to (29), we have

$$\mathrm{val}_z(g_{c,j}(\lambda,z)) = -\mu_j$$

and, hence,

$$\mathrm{val}_z(h_{c,j,m,u}(z)) \geq \mathrm{val}_z(g_{c,j}(\lambda,z)) = -\mu_j.$$

Moreover, the latter inequality is an equality if and only if

$$\mathrm{ev}_{\lambda=c}\,\partial_\lambda^{s_{c,j}+m-u}(\mathrm{cld_z}\, g_{c,j}(\lambda,z)) \neq 0 \tag{44}$$

and it is a strict inequality if and only if

$$\mathrm{ev}_{\lambda=c}\,\partial_\lambda^{s_{c,j}+m-u}(\mathrm{cld_z}\, g_{c,j}(\lambda,z)) = 0. \tag{45}$$

But, using (30), we see that $\mathrm{cld_z}\, g_{c,j}(\lambda,z)$ is a rational function in λ with $(\lambda-c)$-adic valuation $s_{c,j}$. So, (44) holds true if $s_{c,j}+m-u = s_{c,j}$ and that (45) holds true if $s_{c,j}+m-u < s_{c,j}$. Whence the result. □

Inserting (42) in (41), we get

$$\sum_{j\in\{1,\ldots,k\},m\in\{0,\ldots,m_{c,j}-1\}} \sum_{u=0}^{s_{c,j}+m} a_{c,j,m}h_{c,j,m,u}(z)\ell_{c,u}. \tag{46}$$

Using the fact that the family $(\ell_{c,j})_{j\geq 0}$ is $\mathscr{H}$-linearly independent (see Lemma 30 below), we get, for all $u \in \{0,\ldots,s_{c,k+1}-1\}$,

$$\sum_{\substack{j\in\{1,\ldots,k\},m\in\{0,\ldots,m_{c,j}-1\}\\ \text{such that } u\in\{0,\ldots,s_{c,j}+m\}}} a_{c,j,m}h_{c,j,m,u}(z) = 0. \tag{47}$$

But, using (43), we see that

— for $u = m_{c,k}-1$, all the terms in (47), with the possible exception of the term $a_{c,k,m_{c,k}-1}h_{c,k,m_{c,k}-1,m_{c,k}-1}(z)$ corresponding to $j = k$ and $m = m_{c,k}-1$, have z-adic valuation $> -\mu_k$;

— $\mathrm{val}_z(h_{c,k,m_{c,k}-1,m_{c,k}-1}(z)) = -\mu_k$.

It follows that $a_{c,k,m_k-1} = 0$.

Similarly, using (43), we see that

— for $u = m_{c,k}-2$, all the nonzero terms in (47), with the possible exception of the term $a_{c,k,m_{c,k}-2}h_{c,k,m_{c,k}-2,m_{c,k}-2}(z)$ corresponding to $j = k$ and $m = m_{c,k}-2$ have z-adic valuation $> -\mu_k$;

— $\mathrm{val}_z(h_{c,k,m_k-2,m_k-2}(z)) = -\mu_k$.

It follows that $a_{c,k,m_k-2} = 0$.

Iterating this procedure, we find $a_{c,k,m} = 0$ for $m \in \{0, \dots, m_{c,k}-1\}$.

An obvious iteration of what precedes yields to $a_{c,j,m} = 0$ for all $j \in \{1, \dots, k\}$ and all $m \in \{0, \dots, m_{c,j} - 1\}$, as expected.

In order to complete the proof, it remains to state and prove the following two lemmas used above.

Lemma 30. *The family $(\ell_{c,j})_{c \in \mathbb{C}^\times, j \geq 0}$ is $\mathscr{H}$-linearly independent.*

Proof. Assume on the contrary that the family $(\ell_{c,j})_{c \in \mathbb{C}^\times, j \geq 0}$ is $\mathscr{H}$-linearly dependent. Consider a $\mathscr{H}$-linearly dependent family $(\ell_{c,j})_{c \in C, j \geq 0}$ with $C \subset \mathbb{C}^\times$ finite (nonempty) of minimal cardinality. There exist $c \in C$ and $j \geq 0$ such that $\ell_{c,j}$ is a $\mathscr{H}$-linear combinaison of the $\ell_{d,k}$ with $d \in C$ such that $d \neq c$ or ($d = c$ and $k < j$).

Let us first assume that $j = 0$. So, we have

$$e_c = \sum_{d \in C \setminus \{c\}, k \geq 0} \alpha_{d,k}(z) \ell_{d,k} \tag{48}$$

for some $\alpha_{d,k}(z) \in \mathscr{H}$. Applying ϕ_p to this equality, we obtain:

$$c e_c = \sum_{d \in C \setminus \{c\}, k \geq 0} \phi_p(\alpha_{d,k}(z))(d\ell_{d,k} + \ell_{d,k-1}). \tag{49}$$

Considering the linear combinaison (49) $- c$(48), we find

(50)

$$\begin{aligned} 0 = & \sum_{d \in C \setminus \{c\}, k \geq 0} (d\phi_p(\alpha_{d,k}(z)) - c\alpha_{d,k}(z))\ell_{d,k} + \sum_{d \in C \setminus \{c\}, k \geq 0} \phi_p(\alpha_{d,k}(z))\ell_{d,k-1} \\ & = \sum_{d \in C \setminus \{c\}, k \geq 0} (d\phi_p(\alpha_{d,k}(z)) - c\alpha_{d,k}(z) + \phi_p(\alpha_{d,k+1}(z)))\ell_{d,k}. \end{aligned}$$

We claim that this $\mathscr{H}$-linear relation is nontrivial. Indeed, assume at the contrary that, for all $d \in C \setminus \{c\}$ and all $k \geq 0$, we have

$$d\phi_p(\alpha_{d,k}(z)) - c\alpha_{d,k}(z) + \phi_p(\alpha_{d,k+1}(z)) = 0.$$

For k large enough, we have $\alpha_{d,k+1}(z) = 0$. But, if $\alpha_{d,k+1}(z) = 0$ then it follows from Lemma 31 below that $\alpha_{d,k}(z) = 0$. Iterating this, we find that all the $\alpha_{d,k}(z)$ with $d \in C \setminus \{c\}$ and $k \geq 0$ are zero, whence a contradiction. So, the linear combinaison (50) is non trivial; this contradicts the minimality of C.

We now assume that $j \geq 1$. We have

$$\ell_{c,j} = \sum_{k=0}^{j-1} \alpha_{c,k}(z)\ell_{c,k} + \sum_{d \neq c, k \geq 0} \alpha_{d,k}(z)\ell_{d,k} \tag{51}$$

for some $\alpha_{d,k}(z) \in \mathscr{H}$. Applying ϕ_p to this equality, we find:

$$(52)\quad c\ell_{c,j} + \ell_{c,j-1} = \sum_{k=0}^{j-1} \phi_p(\alpha_{c,k}(z))(c\ell_{c,k} + \ell_{c,k-1}) + \sum_{d\neq c, k\geq 0} \phi_p(\alpha_{d,k}(z))(d\ell_{d,k} + \ell_{d,k-1}).$$

Considering the linear combination (52)$-c$(51), we get

$$(53)\quad \ell_{c,j-1} = \sum_{k=0}^{j-1} c(\phi_p(\alpha_{c,k}(z)) - \alpha_{c,k}(z))\ell_{c,k} + \sum_{k=0}^{j-1} \phi_p(\alpha_{c,k}(z))\ell_{c,k-1} + \sum_{d\neq c, k\geq 0} (d\phi_p(\alpha_{d,k}(z)) - c\alpha_{d,k}(z))\ell_{d,k} + \sum_{d\neq c, k\geq 0} \phi_p(\alpha_{d,k}(z))\ell_{d,k-1}.$$

The above equality can be rewritten as

$$(54)\quad (1 - c(\phi_p(\alpha_{c,j-1}(z)) - \alpha_{c,j-1}(z)))\ell_{c,j-1} = \sum_{k=0}^{j-2} c(\phi_p(\alpha_{c,k}(z)) - \alpha_{c,k}(z))\ell_{c,k} + \sum_{k=0}^{j-1} \phi_p(\alpha_{c,k}(z))\ell_{c,k-1} + \sum_{d\neq c, k\geq 0} (d\phi_p(\alpha_{d,k}(z)) - c\alpha_{d,k}(z))\ell_{d,k} + \sum_{d\neq c, k\geq 0} \phi_p(\alpha_{d,k}(z))\ell_{d,k-1}.$$

But $1-c(\phi_p(\alpha_{c,j-1}(z))-\alpha_{c,j-1}(z)) \neq 0$ (follows from Lemma 31 below), so we obtain that $\ell_{c,j-1}$ is a $\mathscr{H}$-linear combinaison of the $\ell_{d,k}$ with $d \in C$ such that $d \neq c$ or ($d = c$ and $k < j-1$). Iterating this, we get that $e_{c,j}$ is a $\mathscr{H}$-linear combinaison of the $\ell_{d,k}$ with $d \in C \setminus \{c\}$ and, hence, we are reduced to the first case considered at the beginning of this proof. □

Lemma 31. *Let R be a ring. If $g(z) \in \mathscr{H}_R$ has a nonzero constant term, then the equation*

$$\phi_p(f(z)) - f(z) = g(z)$$

has no solution $f(z) \in \mathscr{H}_R$.

Assume that R is an integral domain. If c, d are distinct nonzero elements of R, then the equation

$$c\phi_p(f(z)) - df(z) = 0$$

has no nonzero solution $f(z) \in \mathscr{H}_R$.

Proof. The first assertion follows from the fact that, for any $f(z) = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}_R$, the constant term of $\phi_p(f(z)) - f(z) = \sum_{\gamma\in\mathbb{Q}}(f_{\gamma/p} - f_\gamma)z^\gamma$ is equal to $f_{0/p} - f_0 = 0$.

Let us prove the second assertion. Consider c, d as in the statement of the lemma and let $f(z) = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}_R$ be such that $c\phi_p(f) - df =$

0, *i.e.*, such that, for all $\gamma \in \mathbb{Q}$, $cf_{\gamma/p} - df_\gamma = 0$. If $f(z) \neq 0$, then there exists $\gamma \in \mathbb{Q}^\times$ such that $f_\gamma \neq 0$ and the latter equation implies that $f_{p^k\gamma} \neq 0$ for all $k \in \mathbb{Z}$, *i.e.*, that $p^{\mathbb{Z}}\gamma \subset \operatorname{supp}(f(z))$. This contradicts the fact that $\operatorname{supp}(f(z))$ is well-ordered. □

References

[AB17] B. Adamczewski and J. P. Bell. A problem about Mahler functions. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 17(4):1301–1355, 2017.

[AF17] B. Adamczewski and C. Faverjon. Méthode de Mahler: relations linéaires, transcendance et applications aux nombres automatiques. *Proc. Lond. Math. Soc. (3)*, 115(1):55–90, 2017.

[AF18] B. Adamczewski and C. Faverjon. Méthode de Mahler, transcendance et relations linéaires: aspects effectifs. *J. Théor. Nombres Bordeaux*, 30(2):557–573, 2018.

[AS03] J.-P. Allouche and J. Shallit. *Automatic sequences*. Cambridge University Press, Cambridge, 2003.

[BCCD18] J. Bell, F. Chyzak, M. Coons, and P. Dumas. Becker's conjecture on Mahler functions. *arXiv:1802.08653*, 2018.

[BCZ16] R. P. Brent, M. Coons, and W. Zudilin. Algebraic independence of Mahler functions via radial asymptotics. *Int. Math. Res. Not. IMRN*, (2):571–603, 2016.

[Bec94] P.-G. Becker. k-Regular power series and Mahler-type functional equations. *J. Number Theory*, 49(3):269–286, 1994.

[CDDM18] F. Chyzak, T. Dreyfus, P. Dumas, and M. Mezzarobba. Computing solutions of linear Mahler equations. *Math. Comp.*, 87(314):2977–3021, 2018.

[CL55] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1955.

[CZ02] P. Corvaja and U. Zannier. Some new applications of the subspace theorem. *Compositio Math.*, 131(3):319–340, 2002.

[DF96] P. Dumas and P. Flajolet. Asymptotique des récurrences mahlériennes : le cas cyclotomique. *J. Théor. Nombres Bordeaux*, 8(1):1–30, 1996.

[DHR18] T. Dreyfus, C. Hardouin, and J. Roques. Hypertranscendence of solutions of Mahler equations. *J. Eur. Math. Soc. (JEMS)*, 20(9):2209–2238, 2018.

[Dum93] P. Dumas. Récurrences mahlériennes, suites automatiques : études asymptotiques. Thèse de doctorat. 1993.

[Fer18] G. Fernandes. Méthode de Mahler en caractéristique non nulle: un analogue du théorème de Ku. Nishioka. *Ann. Inst. Fourier (Grenoble)*, 68(6):2553–2580, 2018.

[Fro73] G. Frobenius. Ueber die Integration der linearen Differentialgleichungen durch Reihen. *J. Reine Angew. Math.*, 76:214–235, 1873.

[Hah07] H. Hahn. Über die nichtarchimedischen Größensysteme. *Wien. Ber.*, 116:601–655, 1907.

[HS99] P.-F. Hsieh and Y. Sibuya. *Basic theory of ordinary differential equations*. Universitext. Springer-Verlag, New York, 1999.

[IKSY91] K. Iwasaki, H. Kimura, S. Shimomura, and Masaaki Yoshida. *From Gauss to Painlevé*. Aspects of Mathematics, E16. Friedr. Vieweg & Sohn, Braunschweig, 1991.

[Kub77] K. K. Kubota. On the algebraic independence of holomorphic solutions of certain functional equations and their values. *Math. Ann.*, 227(1):9–50, 1977.

[LvdP78] J. H. Loxton and A. J. van der Poorten. Algebraic independence properties of the Fredholm series. *J. Austral. Math. Soc. Ser. A*, 26(1):31–45, 1978.

[Mah29] K. Mahler. Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen. *Math. Ann.*, 101(1):342–366, 1929.

[Mah30] K. Mahler. Uber das Verschwinden von Potenzreihen mehrerer Veränderlichen in speziellen Punktfolgen. *Math. Ann.*, 103(1):573–587, 1930.

[Mas82] D. W. Masser. A vanishing theorem for power series. *Invent. Math.*, 67(2):275–296, 1982.

[Ngu11] P. Nguyen. Hypertranscendance de fonctions de Mahler du premier ordre. *C. R. Math. Acad. Sci. Paris*, 349(17-18):943–946, 2011.

[Ngu12] P. Nguyen. Équations de Mahler et hypertranscendance. *Thèse de l'Institut de Mathématiques de Jussieu*, 2012.

[Nis96] K. Nishioka. *Mahler functions and transcendence*, volume 1631 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[oM] Encyclopedia of Mathematics. Frobenius method. `http://encyclopediaofmath.org/index.php?title=Frobenius_method&oldid=12220`. [Online].

[Pel09] F. Pellarin. An introduction to Mahler's method for transcendence and algebraic independence. In G. Boeckle, D. Goss, U. Hartl, and M. Papanikolas, editors, *to appear in the EMS proceedings of the conference "Hodge structures, transcendence and other motivic aspects"*, 2009.

[Per11] O. Perron. Über Lineare Differenzengleichungen. *Acta Math.*, 34(1):109–137, 1911.

[Phi15] P. Philippon. Groupes de Galois et nombres automatiques. *J. Lond. Math. Soc. (2)*, 92(3):596–614, 2015.

[Pou20] M. Poulet. A density theorem for the difference galois groups of regular singular mahler equations. arXiv:2012.14659, 2020.

[Ran92] B. Randé. Équations fonctionnelles de Mahler et applications aux suites p-régulières. *Thèse de l'Université Bordeaux I available at https://tel.archives-ouvertes.fr/tel-01183330*, 1992.

[Roq20] J. Roques. On the local structure of mahler systems. *Int. Math. Res. Not. IMRN*, 2020.

[SS16] R. Schäfke and M. F. Singer. Consistent systems of linear differential and difference equations. *arXiv:1605.02616, to appear in the Journal of the European Mathematical Society*, 2016.

[vdPS03] Marius van der Put and Michael F. Singer. *Galois theory of linear differential equations*, volume 328 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003.

[Zan98] U. Zannier. On a functional equation relating a Laurent series $f(x)$ to $f(x^m)$. *Aequationes Math.*, 55(1-2):15–43, 1998.

Université de Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, F-69622 Villeurbanne, France

Email address: Julien.Roques@univ-lyon1.fr