Galois groups of the basic hypergeometric equations [1]

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Abstract. *In this paper we compute the Galois groups of basic hypergeometric equations.*

In this paper q is a complex number such that $0 < |q| < 1$.

1 Basic hypergeometric series and equations

The theory of hypergeometric functions and equations dates back at least as far as Gauss. It has long been and is still an integral part of the mathematical literature. In particular, the Galois theory of (generalized) hypergeometric equations attracted the attention of many authors. For this issue, we refer the reader to [2, 3, 13] and to the references therein. We also single out the papers [8, 14], devoted to the calculation of some Galois groups by means of a density theorem (Ramis theorem).

In this paper we focus our attention on the Galois theory of the basic hypergeometric equations, the later being natural q-analogues of the hypergeometric equations.

The *basic hypergeometric series* $\phi(z) = {}_2\phi_1(a,b;c;z)$ with parameters $(a,b,c) \in (\mathbb{C}^*)^3$ defined by :

$$\begin{aligned} {}_2\phi_1(a,b;c;z) &= \sum_{n=0}^{+\infty} \frac{(a,b;q)_n}{(c,q;q)_n} z^n \\ &= \sum_{n=0}^{+\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-b)(1-bq)\cdots(1-bq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)(1-c)(1-cq)\cdots(1-cq^{n-1})} z^n \end{aligned}$$

was first introduced by Heine and was later generalized by Ramanujan. As regards functional equations, the basic hypergeometric series provides us with a solution of the following second order q-difference equation, called the *basic hypergeometric equation* with parameters (a,b,c) :

$$\phi(q^2z) - \frac{(a+b)z-(1+c/q)}{abz-c/q}\phi(qz) + \frac{z-1}{abz-c/q}\phi(z) = 0. \tag{1}$$

This functional equation is equivalent to a functional system. Indeed, with the notations :

$$\lambda(a,b;c;z) = \frac{(a+b)z-(1+c/q)}{abz-c/q}, \qquad \mu(a,b;c;z) = \frac{z-1}{abz-c/q},$$

a function ϕ is solution of (1) if and only if the vector $\Phi(z) = \binom{\phi(z)}{\phi(qz)}$ satisfies the functional system :

$$\Phi(qz) = A(a,b;c;z)\Phi(z) \tag{2}$$

[1] A definitive version of this paper will appear in *Pacific Journal of Mathematics*.

with :

$$A(a,b;c;z) = \begin{pmatrix} 0 & 1 \\ -\mu(a,b;c;z) & \lambda(a,b;c;z) \end{pmatrix}.$$

The present paper focuses on the calculation of the Galois group of the q-difference equation (1) or, equivalently, that of the q-difference system (2). A number of authors have developed q-difference Galois theories over the past years, among whom Franke [10], Etingof [9], Van der Put and Singer [16], Van der Put and Reversat [17], Chatzidakis and Hrushovski [5], Sauloy [15], André [1], etc. The exact relations between the existing Galois theories for q-difference equations are partially understood. For this question, we refer the reader to [4], and also to our Remark section 2.2.

In this paper we follow the approach of Sauloy (initiated by Etingof in the regular case). Our method for computing the Galois groups of the basic hypergeometric equations is based on a q-analogue of Schlesinger's density theorem stated and established in [15]. Note that some of these groups were previously computed by Hendriks in [12] using a radically different method (actually, the author dealt with the Galois groups defined by Van der Put and Singer, but these do coincide with those defined by Sauloy : see our Remark section 2.2). On a related topic, we also point out the appendix of [7] which contains the q-analogue of Schwarz's list.

The paper is organized as follows. In a first part, we give a brief overview of some results from [15]. In a second part, we compute the Galois groups of the basic hypergeometric equations in all non-resonant (but possibly logarithmic) cases.

2 Galois theory for regular singular q-difference equations

Using analytic tools together with Tannakian duality, Sauloy developed in [15] a Galois theory for regular singular q-difference systems. In this section, we shall first recall the principal notions used in [15], mainly the Birkhoff matrix and the twisted Birkhoff matrix. Then we shall explain briefly that this lead to a Galois theory for regular singular q-difference systems. Last, we shall state a density theorem for these Galois groups, which will be of main importance in our calculations.

2.1 Basic notions

Let us consider $A \in \mathrm{Gl}_n(\mathbb{C}(\{z\}))$. Following Sauloy in [15], the q-difference system :

$$Y(qz) = A(z)Y(z) \tag{3}$$

is said to be *Fuchsian* at 0 if A is holomorphic at 0 and if $A(0) \in \mathrm{Gl}_n(\mathbb{C})$. Such a system is non-resonant at 0 if, in addition, $Sp(A(0)) \cap q^{\mathbb{Z}^*} Sp(A(0)) = \emptyset$. Last we say that the above q-difference system is *regular singular* at 0 if there exists $R^{(0)} \in \mathrm{Gl}_n(\mathbb{C}(\{z\}))$ such that the q-difference system defined by $(R^{(0)}(qz))^{-1}A(z)R^{(0)}(z)$ is Fuchsian at 0. We have similar notions at ∞ using the change of variable $z \leftarrow 1/z$.

In the case of a global system, that is $A \in \mathrm{Gl}_n(\mathbb{C}(z))$, we will use the following terminology. If $A \in \mathrm{Gl}_n(\mathbb{C}(z))$, then the system (3) is called *Fuchsian* (resp. *Fuchsian and non-resonnant*, *regular singular*) if it is Fuchsian (resp. Fuchsian and non-resonnant, regular singular) both at 0 and at ∞.

For instance, the basic hypergeometric system (2) is Fuchsian.

Local fundamental system of solutions at 0. Suppose that (3) is Fuchsian and non-resonant at 0 and consider $J^{(0)}$ a Jordan normal form of $A(0)$. According to [15] there exists $F^{(0)} \in \mathrm{Gl}_n(\mathbb{C}\{z\})$ such that :

$$F^{(0)}(qz)J^{(0)} = A(z)F^{(0)}(z). \tag{4}$$

Therefore, if $e^{(0)}_{J^{(0)}}$ denotes a fundamental system of solutions of the q-difference system with constant coefficients $X(qz) = J^{(0)}X(z)$, the matrix-valued function $Y^{(0)} = F^{(0)}e^{(0)}_{J^{(0)}}$ is a fundamental system of solutions of (3). We are going to describe a possible choice for $e^{(0)}_{J^{(0)}}$. We denote by θ_q the Jacobi theta function defined by $\theta_q(z) = (q;q)_\infty (z;q)_\infty (q/z;q)_\infty$. This is a meromorphic function over $\mathbb{C}^*$ whose zeros are simple and located on the discrete logarithmic spiral $q^{\mathbb{Z}}$. Moreover, the functional equation $\theta_q(qz) = -z^{-1}\theta_q(z)$ holds. Now we introduce, for all $\lambda \in \mathbb{C}^*$ such that $|q| \leq |\lambda| < 1$, the q-character $e^{(0)}_\lambda = \frac{\theta_q}{\theta_{q,\lambda}}$ with $\theta_{q,\lambda}(z) = \theta_q(\lambda z)$ and we extend this definition to an arbitrary non-zero complex number $\lambda \in \mathbb{C}^*$ requiring the equality $e^{(0)}_{q\lambda} = ze^{(0)}_\lambda$. If $D = P\mathrm{diag}(\lambda_1, ..., \lambda_n)P^{-1}$ is a semisimple matrix then we set $e^{(0)}_D := P\mathrm{diag}(e^{(0)}_{\lambda_1}, ..., e^{(0)}_{\lambda_n})P^{-1}$. It is easily seen that this does not depend on the chosen diagonalization. Furthermore, consider $\ell_q(z) = -z\frac{\theta'_q(z)}{\theta_q(z)}$ and, if U is a unipotent matrix, $e^{(0)}_U = \Sigma^n_{k=0}\ell^{(k)}_q(U - I_n)^k$ with $\ell^{(k)}_q = \binom{\ell_q}{k}$. If $J^{(0)} = D^{(0)}U^{(0)}$ is the multiplicative Dunford decomposition of $J^{(0)}$, with $D^{(0)}$ semi-simple and $U^{(0)}$ unipotent, we set $e^{(0)}_{J^{(0)}} = e^{(0)}_{D^{(0)}}e^{(0)}_{U^{(0)}}$.

Local fundamental system of solutions at ∞. Using the variable change $z \leftarrow 1/z$, we have a similar construction at ∞. The corresponding fundamental system of solutions is denoted by $Y^{(\infty)} = F^{(\infty)}e^{(\infty)}_{J^{(\infty)}}$.

Throughout this section we assume that the system (3) is global and that it is Fuchsian and non-resonant.

Birkhoff matrix. The linear relations between the two fundamental systems of solutions introduced above are given by the Birkhoff matrix (also called connection matrix) $P = (Y^{(\infty)})^{-1}Y^{(0)}$. Its entries are elliptic functions *i.e.* meromorphic functions over the elliptic curve $\mathbb{E}_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

Twisted Birkhoff matrix. In order to describe a Zariki-dense set of generators of the Galois group associated to the system (3), we introduce a "twisted" connection matrix. According to [15], we choose for all $z \in \mathbb{C}^*$ a group endomorphism g_z of $\mathbb{C}^*$ sending q to z. Before giving an explicit example, we have to introduce more notations. Let us, for any fixed $\tau \in \mathbb{C}$ such that $q = e^{-2\pi i\tau}$, write $q^y = e^{-2\pi i\tau y}$ for all $y \in \mathbb{C}$. We also define the (non continuous) function $\log_q$ on the whole punctured complex plane $\mathbb{C}^*$ by $\log_q(q^y) = y$ if $y \in \mathbb{C}^* \setminus \mathbb{R}^+$ and we require that its discontinuity is located just before the cut (that is $\mathbb{R}^+$) when turning counterclockwise around 0. We can now give an explicit example of endomorphism g_z namely the function $g_z : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \rightarrow \mathbb{C}^*$ sending uq^ω to $g_z(uq^\omega) = z^\omega = e^{-2\pi i\tau \log_q(z)\omega}$ for $(u, \omega) \in \mathbb{U} \times \mathbb{R}$, where $\mathbb{U} \subset \mathbb{C}$ is the unit circle.

Then we set, for all z in $\mathbb{C}^*$, $\psi_z^{(0)}(\lambda) = \frac{e_{q,\lambda}(z)}{g_z(\lambda)}$ and we define $\psi_z^{(0)}\left(D^{(0)}\right)$, the *twisted factor* at 0, by $\psi_z^{(0)}\left(D^{(0)}\right) = P\mathrm{diag}(\psi_z^{(0)}(\lambda_1), ..., \psi_z^{(0)}(\lambda_n))P^{-1}$ with $D^{(0)} = P\mathrm{diag}(\lambda_1, ..., \lambda_n)P^{-1}$. We have a similar construction at ∞ by using the variable change $z \leftarrow 1/z$. The corresponding twisting factor is denoted by $\psi_z^{(\infty)}(J^{(\infty)})$.

Finally, the twisted connection matrix $\breve{P}(z)$ is :

$$\breve{P}(z) \quad = \quad \psi_z^{(\infty)}\left(D^{(\infty)}\right)P(z)\psi_z^{(0)}\left(D^{(0)}\right)^{-1}.$$

2.2 Definition of the Galois groups

The definition of the Galois groups of regular singular q-difference systems given by Sauloy in [15] is somewhat technical and long. Here we do no more than describe the underlying idea.

(Global) Galois group. Let us denote by $\mathcal{E}$ the category of regular singular q-difference systems with coefficients in $\mathbb{C}(z)$ (so, the base field is $\mathbb{C}(z)$; the difference field is $(\mathbb{C}(z), f(z) \mapsto f(qz))$). This category is naturally equipped with a tensor product $\otimes$ such that $(\mathcal{E}, \otimes)$ satisfies all the axioms defining a Tannakian category over $\mathbb{C}$ except the existence of a *fiber functor* which is not obvious. This problem can be overcome using an analogue of the Riemann-Hilbert correspondance.

The Riemann-Hilbert correspondance for regular singular q-difference systems entails that $\mathcal{E}$ is equivalent to the category $\mathcal{C}$ of connection triples whose objects are triples $(A^{(0)}, P, A^{(\infty)}) \in \mathrm{Gl}_n(\mathbb{C})\times\mathrm{Gl}_n(\mathcal{M}(\mathbb{E}_q))\times\mathrm{Gl}_n(\mathbb{C})$ (we refer to [15] for the complete definition of $\mathcal{C}$). Furthermore $\mathcal{C}$ can be endowed with a tensor product $\underline{\otimes}$ making the above equivalence of categories compatible with the tensor products. Let us emphasize that $\underline{\otimes}$ is not the usual tensor product for matrices. Indeed some twisting factors appear because of the bad multiplicative properties of the q-characters $e_{q,c}$: in general $e_{q,c}e_{q,d} \neq e_{q,cd}$.

The category $\mathcal{C}$ allows us to define a Galois group : $\mathcal{C}$ is a Tannakian category over $\mathbb{C}$. The functor ω_0 from $\mathcal{C}$ to $Vect_{\mathbb{C}}$ sending an object $(A^{(0)}, P, A^{(\infty)})$ to the underlying vector space $\mathbb{C}^n$ on which $A^{(0)}$ acts is a fiber functor. Let us remark that there is a similar fiber functor ω_∞ at ∞. Following the general formalism of the theory of Tannakian categories (see [6]), the *absolute Galois group* of $\mathcal{C}$ (or, using the above equivalence of categories, of $\mathcal{E}$) is defined as the pro-algebraic group $Aut^{\underline{\otimes}}(\omega_0)$ and the *global Galois group of an object* χ of $\mathcal{C}$ (or, using the above equivalence of categories, of an object of $\mathcal{E}$) is the complex linear algebraic group $Aut^{\underline{\otimes}}(\omega_{0|\langle\chi\rangle})$ where $\langle\chi\rangle$ denotes the Tannakian subcategory of $\mathcal{C}$ generated by χ. For the sake of simplicity, we will often call $Aut^{\underline{\otimes}}(\omega_{0|\langle\chi\rangle})$ the *Galois group* of χ (or, using the above equivalence of categories, of the corresponding object of $\mathcal{E}$).

Local Galois groups. Let us point out that notions of local Galois groups at 0 and at ∞ are also available (here the difference fields are respectively $(\mathbb{C}(\{z\}), f(z) \mapsto f(qz))$ and $(\mathbb{C}(\{z^{-1}\}), f(z) \mapsto f(qz))$). As expected, they are subgroups of the (global) Galois group. Nevertheless, since these groups are of second importance in what follows, we omit the details and we refer the interesting reader to [15].

Remark. In [16], Van der Put and Singer showed that the Galois groups defined using a Picard-Vessiot theory can be recovered by means of Tannakian duality : it is the group of tensor automorphisms of some suitable complex valued fiber functor over $\mathcal{E}$. Since two complex valued fiber functors on a same Tannakian category are necessarily isomorphic, we conclude that Sauloy's and Van der Put and Singer's theories coincide.

In the rest of this section we exhibit some natural elements of the Galois group of a given Fuchsian q-difference system and we state the density theorem of Sauloy.

2.3 The density theorem

Fix a "base point" $y_0 \in \Omega = \mathbb{C}^* \setminus \{\text{zeros of } \det(P(z)) \text{ or poles of } P(z)\}$. Sauloy exhibits in [15] the following elements of the (global) Galois group associated to the q-difference system (3) :

1.a) $\gamma_1(D^{(0)})$ and $\gamma_2(D^{(0)})$ where :

$$\gamma_1 : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{U}$$

is the projection over the first factor and :

$$\gamma_2 : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{C}^*$$

is defined by $\gamma_2(uq^{\omega}) = e^{2\pi i \omega}$.

1.b) $U^{(0)}$.

2.a) $\breve{P}(y_0)^{-1}\gamma_1(D^{(\infty)})\breve{P}(y_0)$ and $\breve{P}(y_0)^{-1}\gamma_2(D^{(\infty)})\breve{P}(y_0)$.

2.b) $\breve{P}(y_0)^{-1}U^{(\infty)}\breve{P}(y_0)$.

3) $\breve{P}(y_0)^{-1}\breve{P}(z)$, $z \in \Omega$.

The following result is due to Sauloy [15].

Theorem 1. *The algebraic group generated by the matrices 1.a. to 3. is the (global) Galois group G of the q-difference system (3). The algebraic group generated by the matrices 1.a) and 1.b) is the local Galois group at 0 of the q-difference system (3). The algebraic group generated by the matrices 2.a) and 2.b) is the local Galois group at ∞ of the q-difference system (3).*

The algebraic group generated by the matrices 3) is called the *connection component* of the Galois group G. The following result is easy but very useful. Its proof is left to the reader.

Lemma 1. *The connection component of the Galois group G of a regular singular q-difference system is a subgroup of the identity component G^I of G.*

3 Galois groups of the basic hypergeometric equations : non-resonant and non-logarithmic cases

We write $a = uq^{\alpha}$, $b = vq^{\beta}$ and $c = wq^{\gamma}$ with $u, v, w \in \mathbb{U}$ and $\alpha, \beta, \gamma \in \mathbb{R}$ (we choose a logarithm of q).

In this section we are aiming to compute the Galois group of the basic hypergeometric system (2) under the following assumptions :

$$a/b \notin q^{\mathbb{Z}} \text{ and } c \notin q^{\mathbb{Z}}.$$

First, we give explicit formulas for the generators of the Galois group of (2) involved in Theorem 1.

Local fundamental system of solutions at 0. We have :

$$A(a,b;c;0) = \begin{pmatrix} 1 & 1 \\ 1 & q/c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q/c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & q/c \end{pmatrix}^{-1}.$$

Hence the system (2) is non-resonant, and non-logarithmic at 0 since $A(a,b;c;0)$ is semi-simple. A fundamental system of solutions at 0 of (2) as described in section 2.1 is given by $Y^{(0)}(a,b;c;z) = F^{(0)}(a,b;c;z)e^{(0)}_{J^{(0)}(c)}(z)$ with $J^{(0)}(c) = \mathrm{diag}(1, q/c)$ and :

$$F^{(0)}(a,b;c;z) = \begin{pmatrix} {}_2\phi_1\left(a,b;c;z\right) & {}_2\phi_1\left(aq/c, bq/c; q^2/c; z\right) \\ {}_2\phi_1\left(a,b;c;qz\right) & (q/c){}_2\phi_1\left(aq/c, bq/c; q^2/c; qz\right) \end{pmatrix}.$$

Generators of the local Galois group at 0. We have two generators :

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\gamma} \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}.$$

Local fundamental system of solutions at ∞. We have :

$$A(a,b;c;\infty) = \begin{pmatrix} 1 & 1 \\ 1/a & 1/b \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1/a & 1/b \end{pmatrix}^{-1}.$$

Hence the system (2) is non-resonant and non-logarithmic at ∞ and a fundamental system of solutions at ∞ of (2) as described in section 2.1 is given by $Y^{(\infty)}(a,b;c;z) = F^{(\infty)}(a,b;c;z)e^{(\infty)}_{J^{(\infty)}(a,b)}(z)$ with $J^{(\infty)}(a,b) = \mathrm{diag}(1/a, 1/b)$ and :

$$F^{(\infty)}(a,b;c;z) = \begin{pmatrix} {}_2\phi_1\left(a, aq/c; aq/b; \frac{cq}{ab}z^{-1}\right) & {}_2\phi_1\left(b, bq/c; bq/a; \frac{cq}{ab}z^{-1}\right) \\ \frac{1}{a}{}_2\phi_1\left(a, aq/c; aq/b; \frac{c}{ab}z^{-1}\right) & \frac{1}{b}{}_2\phi_1\left(b, bq/c; bq/a; \frac{c}{ab}z^{-1}\right) \end{pmatrix}.$$

Generators of the local Galois group at ∞. We have two generators :

$$\breve{P}(y_0)^{-1}\begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\beta}\end{pmatrix}\breve{P}(y_0) \text{ and } \breve{P}(y_0)^{-1}\begin{pmatrix} u & 0 \\ 0 & v\end{pmatrix}\breve{P}(y_0).$$

Birkhoff matrix. The Barnes-Mellin-Watson formula (*cf.* [11]) entails that :

$$P(z)=(e^{(\infty)}_{J^{(\infty)}(a,b)}(z))^{-1}\begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)} \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}\end{pmatrix} e^{(0)}_{J^{(0)}(c)}(z).$$

Twisted Birkhoff matrix. We have :

$$\breve{P}(z)=\begin{pmatrix}(1/z)^{-\alpha} & 0 \\ 0 & (1/z)^{-\beta}\end{pmatrix}\begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)} \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & z^{1-\gamma}\end{pmatrix}.$$

We need to consider different cases.

(**Case 1**) $\underline{a,b,c,a/b,a/c,b/c\notin q^{\mathbb{Z}} \text{ and } a/b \text{ or } c\notin \pm q^{\mathbb{Z}/2}}$.

Under this assumption the four numbers $\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}$, $\frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}$, $\frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}$ and $\frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}$ are non-zero.

Proposition 1. *Suppose that (**Case 1**) holds. Then the natural action of G^I on $\mathbb{C}^2$ is irreducible.*

Proof. Suppose, at the contrary, that the action of G^I is reducible and let $L\subset\mathbb{C}^2$ be an invariant line.

Remark that L is distinct from $\mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$. Indeed, assume at the contrary that $L=\mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$ (the case $L=\mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$ is similar). The line $L=\mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$ being in particular invariant by the connection component, we see that the line generated by $\breve{P}(z)\begin{pmatrix}1\\0\end{pmatrix}$ does not depend on $z\in\Omega$. This yields a contradiction because the ratio of the components of $\breve{P}(z)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta}\end{pmatrix}$ depends on z (remember that $a/b\notin q^{\mathbb{Z}}$).

On the other hand, since, for all $n \in \mathbb{N}$, both matrices $\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\gamma n} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & w^n \end{pmatrix}$ belong to G and since G^I is a normal subgroup of G, both lines $L_n := \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\gamma n} \end{pmatrix} L$ and $L'_n := \begin{pmatrix} 1 & 0 \\ 0 & w^n \end{pmatrix} L$ are also invariant by G^I.

Note that because (**Case 1**) holds, at least one of the complex numbers $w, e^{2\pi i\gamma}, u/v, e^{2\pi i(\alpha-\beta)}$ is distinct from ± 1.

First, suppose that $w \neq \pm 1$. We have seen that $L \neq \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, hence L_0, L_1, L_2 are three distinct lines invariant by the action of G^I. This implies that G^I consists of scalar matrices : this is a contradiction (because, for instance, $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not invariant for the action of G^I). Hence, if $w \neq \pm 1$ we have proved that G^I acts irreducibly.

The case $e^{2\pi i\gamma} \neq \pm 1$ is similar.

Last, the proof is analogous if $u/v \neq \pm 1$ or $e^{2\pi i(\alpha-\beta)} \neq \pm 1$ (we then use the fact that, for all $z \in \Omega$, G^I is normalized by $\breve{P}(z)^{-1}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\breve{P}(z)$ and $\breve{P}(z)^{-1}\begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\beta} \end{pmatrix}\breve{P}(z)$ and that there exists $z \in \Omega$ such that $\breve{P}(z)L$ is distinct from $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$). □

We have the following theorem.

Theorem 2. *Suppose that (**Case 1**) holds. Then we have the following dichotomy :*

- *if $abq/c \notin q^{\mathbb{Z}}$ then $G = Gl_2(\mathbb{C})$;*
- *if $abq/c \in q^{\mathbb{Z}}$ then $G = \overline{\langle Sl_2(\mathbb{C}), \sqrt{w}I, e^{\pi i\gamma}I \rangle}$.*

Proof. Since G^I acts irreducibly on $\mathbb{C}^2$, the general theory of algebraic groups entails that G^I is generated by its center $Z(G^I)$ together with its derived subgroup $G^{I,der}$ and that $Z(G^I)$ acts as scalars. Hence, $G^{I,der} \subset \mathrm{Sl}_2(\mathbb{C})$ also acts irreducibly on $\mathbb{C}^2$. Therefore $G^{I,der} = \mathrm{Sl}_2(\mathbb{C})$ (a connected algebraic group of dimension less than or equal to 2 is solvable hence $\dim(G^{I,der}) = 3$ and $G^{I,der} = \mathrm{Sl}_2(\mathbb{C})$). In order to complete the proof, it is sufficient to determine $\det(G)$. We have :

$$\det(\breve{P}(z)) = \frac{(1/z)^{-(\alpha+\beta)} z^{1-\gamma}}{(q^2/c, a/b, c, b/a; q)_\infty} \left(\underbrace{\theta_q(b)\theta_q(c/a)\frac{\theta_q(az)}{\theta_q(z)}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)} - \theta_q(c/b)\theta_q(a)\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)}\frac{\theta_q(bz)}{\theta_q(z)}}_{\psi(z)} \right).$$

A straightforward calculation shows that the function :

$$\theta_q(b)\theta_q(c/a)\theta_q(az)\theta_q(\frac{bq}{c}z) - \theta_q(c/b)\theta_q(a)\theta_q(\frac{aq}{c}z)\theta_q(bz)$$

vanishes for $z \in q^{\mathbb{Z}}$ and for $z \in \frac{c}{abq}q^{\mathbb{Z}}$. On the other hand ψ is a solution of the first order q-difference equation $y(qz) = \frac{c}{abq}y(z)$. Hence, if we suppose that $abq/c \notin q^{\mathbb{Z}}$, we deduce that the ratio $\chi(z) = \frac{\psi(z)}{\frac{\theta_q(\frac{abq}{c}z)}{\theta_q(z)}}$ defines an holomorphic elliptic function over $\mathbb{C}^*$. Therefore χ is constant and, evaluating χ at $z = 1/b$, we get :

$$\chi = -b\theta_q(a/b)\theta_q(c).$$

Finally, we obtain the identity :

$$\det(\breve{P}(z)) \quad = \quad \frac{1-q/c}{1/a-1/b}(1/z)^{-(\alpha+\beta)}z^{1-\gamma}\frac{\theta_q(\frac{abq}{c}z)}{\theta_q(z)}. \tag{5}$$

By analytic continuation (with respect to the parameters) we see that this formula also holds if $abq/c \in q^{\mathbb{Z}}$.

Consequently, if $abq/c \notin q^{\mathbb{Z}}$, for any fixed $y_0 \in \Omega$, $\det(\breve{P}(y_0)^{-1}\breve{P}(z))$ is a non constant holomorphic function (with respect to z). This implies that $G = G^I = \mathrm{Gl}_2(\mathbb{C})$. On the other hand, if $abq/c \in q^{\mathbb{Z}}$, then we have that $\det(\breve{P}(y_0)^{-1}\breve{P}(z)) = 1$, so that the connection component of the Galois group is a subgroup of $\mathrm{Sl}_2(\mathbb{C})$ and the Galois group G is the smallest algebraic group which contains $\mathrm{Sl}_2(\mathbb{C})$ and $\{\sqrt{w}I, e^{\pi i\gamma}I\}$. □

We are going to study the case $a, b, c, a/b, a/c, b/c \notin q^{\mathbb{Z}}$ and $a/b, c \in \pm q^{\mathbb{Z}+1/2}$ in two steps.

(**Case 2**) $\underline{a, b, c, a/b, a/c, b/c \notin q^{\mathbb{Z}} \text{ and } q^{\mathbb{Z}}a \cup q^{\mathbb{Z}}b \cup q^{\mathbb{Z}}aq/c \cup q^{\mathbb{Z}}bq/c = q^{\mathbb{Z}}a \cup -q^{\mathbb{Z}}a \cup q^{\mathbb{Z}+1/2}a \cup -q^{\mathbb{Z}+1/2}a}$

We first establish a preliminary result.

Lemma 2. *Suppose that (**Case 2**) holds. Then any functional equation of the form* $Az^{n/2}\theta_q(q^N az)+Bz^{m/2}\theta_q(-q^M az)+Cz^{l/2}\theta_q(q^L q^{1/2}az)+Dz^{k/2}\theta_q(-q^K q^{1/2}az) = 0$ *with* $A, B, C, D \in \mathbb{C}$, $n, m, l, k, N, M, L, K \in \mathbb{Z}$ *is trivial, that is* $A = B = C = D = 0$.

Proof. Using the non-trivial monodromy of $z^{1/2}$, we reduce the problem to the case of n, m, l, k being odd numbers. In this case, using the functional equation $\theta_q(qz) = -z^{-1}\theta_q(z)$, we can assume without loss of generality that $n = l = m = k = 0$. The expansion of θ_q as an infinite Laurent series $\theta_q(z) = \sum_{j\in\mathbb{Z}} q^{\frac{j(j-1)}{2}}(-z)^j$ ensures that, for all $j \in \mathbb{Z}$, the following equality holds :

$$A(q^N)^j + B(-q^M)^j + C(q^{L+1/2})^j + D(-q^{K+1/2})^j = 0.$$

Considering the associated generating series, this implies that :

$$\frac{A}{1-q^N z} + \frac{B}{1+q^M z} + \frac{C}{1-q^{L+1/2}z} + \frac{D}{1+q^{K+1/2}z} = 0.$$

Hence, considering the poles of this rational fraction, we obtain $A = B = C = D = 0$. □

Proposition 2. *Suppose that (**Case 2**) holds. Then the natural action of* G^I *on* $\mathbb{C}^2$ *is irreducible.*

Proof. Suppose, at the contrary, that the action of G^I is reducible and consider an invariant line $L \subset \mathbb{C}^2$. In particular, L is invariant under the action of the connection component. Consequently, the line $\breve{P}(z)L$ does not depend on $z \in \Omega$. This is impossible using Lemma 2 (the cases $L = \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are excluded by direct calculation; for the remaining cases consider the ratio of the coordinates of a generator of L and apply Lemma 2). We get a contraction, hence prove that G^I acts irreducibly. □

Theorem 3. *If (**Case 2**) holds then we have the following dichotomy :*

- *if $abq/c \notin q^{\mathbb{Z}}$ then $G = Gl_2(\mathbb{C})$;*
- *if $abq/c \in q^{\mathbb{Z}}$ then $G = \overline{\langle Sl_2(\mathbb{C}), \sqrt{w}I, e^{\pi i\gamma}I\rangle}$.*

Proof. The proof follows the same lines as that of theorem 2. □

The remaining subcases are $b \in -aq^{\mathbb{Z}}$ and $c \in -q^{\mathbb{Z}}$; $b \in -aq^{\mathbb{Z}+1/2}$ and $c \in -q^{\mathbb{Z}+1/2}$; $b \in aq^{\mathbb{Z}+1/2}$ and $c \in q^{\mathbb{Z}+1/2}$.

(**Case 3**) $\underline{a, b, c, a/b, a/c, b/c \notin q^{\mathbb{Z}} \text{ and } b \in -aq^{\mathbb{Z}} \text{ and } c \in -q^{\mathbb{Z}}}$.

We use the following notations : $b = -aq^\delta$ and $c = -q^\gamma$ with $\delta = \beta - \alpha, \gamma \in \mathbb{Z}$.

The twisted connection matrix takes the following form :

$$\begin{aligned}
\breve{P}(z) &= (1/z)^{-\alpha}\begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)}z^{1-\gamma} \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)}z^\delta & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}z^{1+\delta-\gamma} \end{pmatrix} \\
&= (1/z)^{-\alpha}\begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}q^{\frac{\gamma(1-\gamma)}{2}}a^{\gamma-1}\frac{\theta_q(-az)}{\theta_q(z)} \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}q^{\frac{-\delta(\delta-1)}{2}}a^{-\delta}\frac{\theta_q(-az)}{\theta_q(z)} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}q^{-\frac{(\delta-\gamma+1)(\delta-\gamma)}{2}}(-a)^{\gamma-\delta-1}\frac{\theta_q(az)}{\theta_q(z)} \end{pmatrix}
\end{aligned}$$

Theorem 4. *Suppose that (**Case 3**) holds. We have $G = R\begin{pmatrix} \mathbb{C}^* & 0 \\ 0 & \mathbb{C}^* \end{pmatrix}R^{-1} \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}R\begin{pmatrix} \mathbb{C}^* & 0 \\ 0 & \mathbb{C}^* \end{pmatrix}R^{-1}$ for some $R \in Gl_2(\mathbb{C})$ of the form $\begin{pmatrix} 1 & 1 \\ C & -C \end{pmatrix}$, $C \in \mathbb{C}^*$.*

Proof. Remark that there exist two nonzero constants A, B such that, for all $z \in \Omega$:

$$\begin{aligned}
\breve{P}(-1/a)^{-1}\breve{P}(z) &= (-a)^\alpha\frac{\theta_q(-1/a)}{\theta_q(-1)}(1/z)^{-\alpha}\begin{pmatrix} \frac{\theta_q(az)}{\theta_q(z)} & A\frac{\theta_q(-az)}{\theta_q(z)} \\ B\frac{\theta_q(-az)}{\theta_q(z)} & \frac{\theta_q(az)}{\theta_q(z)} \end{pmatrix} \\
&= (-a)^\alpha\frac{\theta_q(-1/a)}{\theta_q(-1)}(1/z)^{-\alpha}R\begin{pmatrix} \frac{\theta_q(az)}{\theta_q(z)} + \sqrt{BA}\frac{\theta_q(-az)}{\theta_q(z)} & 0 \\ 0 & \frac{\theta_q(az)}{\theta_q(z)} - \sqrt{BA}\frac{\theta_q(-az)}{\theta_q(z)} \end{pmatrix}R^{-1}
\end{aligned}$$

with $R = \begin{pmatrix} 1 & 1 \\ \sqrt{B/A} & -\sqrt{B/A} \end{pmatrix}$.

Furthermore, we claim that the functions $X(z) := (1/z)^{-\alpha}(\frac{\theta_q(az)}{\theta_q(z)} + \sqrt{BA}\frac{\theta_q(-az)}{\theta_q(z)})$ and $Y(z) := (1/z)^{-\alpha}(\frac{\theta_q(az)}{\theta_q(z)} - \sqrt{BA}\frac{\theta_q(-az)}{\theta_q(z)})$ do not satisfy any non-trivial relation of the form $X^rY^s = 1$ with $(r,s) \in \mathbb{Z}^2 \setminus \{(0,0\}$. Indeed, suppose on the contrary that such a relation holds. Then $\frac{((1/z)^{-\alpha}(\theta_q(az)+\sqrt{BA}\theta_q(-az)))^r}{((1/z)^{-\alpha}(\theta_q(az)-\sqrt{BA}\theta_q(-az)))^s} = \theta_q(z)^{s-r}$. Let us first exclude the case $r \neq s$. If $s > r$ then we conclude that $\theta_q(az) + \sqrt{BA}\theta_q(-az)$ must vanish on $q^{\mathbb{Z}}$. In particular, $\theta_q(a) + \sqrt{BA}\theta_q(-a) = 0$ and $\theta_q(aq) + \sqrt{BA}\theta_q(-aq) = -(az)^{-1}(\theta_q(az) - \sqrt{BA}\theta_q(-az)) = 0$, so $\theta_q(a) = 0$ that is $a \in q^{\mathbb{Z}}$. This yields a contradiction. The case $r > s$ is similar by symmetry. Hence we have $r = s$ so that $\left(\frac{\theta_q(az)+\sqrt{BA}\theta_q(-az)}{\theta_q(az)-\sqrt{BA}\theta_q(-az)}\right)^r = 1$. Therefore the function $\frac{\theta_q(az)+\sqrt{BA}\theta_q(-az)}{\theta_q(az)-\sqrt{BA}\theta_q(-az)}$ is constant. This is clearly impossible and our claim is proved.

This ensures that the connection component of G^I, generated by the matrices $\breve{P}(-1/a)^{-1}\breve{P}(z)$, $z \in \Omega$, is equal to $R\begin{pmatrix} \mathbb{C}^* & 0 \\ 0 & \mathbb{C}^* \end{pmatrix}R^{-1}$. Consequently, G is generated as an algebraic group by $R\begin{pmatrix} \mathbb{C}^* & 0 \\ 0 & \mathbb{C}^* \end{pmatrix}R^{-1}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\breve{P}(-1/a)^{-1}\begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}\breve{P}(-1/a) = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$ and $\breve{P}(-1/a)^{-1}\begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}\breve{P}(-1$ $\begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}$. The theorem follows. □

- Both cases ($b \in -aq^{\mathbb{Z}+1/2}$ and $c \in -q^{\mathbb{Z}+1/2}$) and ($b \in aq^{\mathbb{Z}+1/2}$ and $c \in q^{\mathbb{Z}+1/2}$) are similar.

(**Case 4**) $\underline{a \in q^{\mathbb{N}^*}}$.

In this case, the twisted connection matrix is lower triangular :

$$\breve{P}(z) = \begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}(-1)^\alpha q^{-\frac{\alpha(\alpha-1)}{2}} & 0 \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma} \end{pmatrix}$$

Theorem 5. *Suppose that (**Case 4**) holds. We have the following trichotomy :*

- *if $b/c \notin q^{\mathbb{Z}}$ then $G = \begin{pmatrix} 1 & 0 \\ \mathbb{C} & \mathbb{C}^* \end{pmatrix}$;*
- *if $c/b \in q^{\mathbb{N}^*}$ then $G = \begin{pmatrix} 1 & 0 \\ \mathbb{C} & \langle w, e^{2\pi i\gamma}\rangle \end{pmatrix}$;*
- *if $bq/c \in q^{\mathbb{N}^*}$ then $G = \begin{pmatrix} 1 & 0 \\ 0 & \langle w, e^{2\pi i\gamma}\rangle \end{pmatrix}$.*

Proof. Remark that in each case there exist two constants A, B with $B \neq 0$ such that, for all $z \in \Omega$, $\breve{P}(1/b)^{-1}\breve{P}(z) = \begin{pmatrix} 1 & 0 \\ A\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta} & B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma} \end{pmatrix}$. Hence, the connection component is a subgroup of $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & \mathbb{C}^* \end{pmatrix}$.

Assume $b/c \notin q^{\mathbb{Z}}$. Then $A \neq 0$ and we claim that the connection component is equal to $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & \mathbb{C}^* \end{pmatrix}$. Indeed, for all $n \in \mathbb{Z}$, the following matrix :

$$(\breve{P}(1/b)^{-1}\breve{P}(z))^n = \begin{pmatrix} 1 & 0 \\ A\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta}\frac{1-\left(B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}\right)^n}{1-B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}} & \left(B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}\right)^n \end{pmatrix}$$

belongs to the connection component. Consider a polynomial in two variables $K(X,Y) \in \mathbb{C}[X,Y]$ such that :

$$K(A\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta}\frac{1-\left(B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}\right)^n}{1-B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}}, \left(B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}\right)^n) = 0.$$

If K was non zero then we could assume that $K(X,0) \neq 0$. But, for all $z \in \Omega$ in a neighborhood of $\frac{c}{bq}$, we have $\left|B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}\right| < 1$, hence letting n tend to $+\infty$, we would get $K(A\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta}\frac{1}{1-B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}}, 0) = 0$ which would imply $K(X,0) = 0$. This proves that $K = 0$. In other words the only algebraic subvariety of $\mathbb{C} \times \mathbb{C}^*$ containing $(A\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta}\frac{1-\left(B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}\right)^n}{1-B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}}, \left(B\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}\right)^n)$ for all $n \in \mathbb{Z}$ is $\mathbb{C} \times \mathbb{C}^*$ itself. In particular, the algebraic group generated by the matrix $(\breve{P}(1/b)^{-1}\breve{P}(z))^n$ for all $n \in \mathbb{Z}$ is $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & \mathbb{C}^* \end{pmatrix}$, hence the connection component is equal to $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & \mathbb{C}^* \end{pmatrix}$.

It is now straightforward that $G = \begin{pmatrix} 1 & 0 \\ \mathbb{C} & \mathbb{C}^* \end{pmatrix}$.

Suppose that $c/b \in q^{\mathbb{N}^*}$. Then the function $\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}$ is constant. Hence the matrix $\breve{P}(1/b)^{-1}\breve{P}(z)$ simplifies as follows :

$$\breve{P}(1/b)^{-1}\breve{P}(z) = \begin{pmatrix} 1 & 0 \\ A\frac{\theta_q(bz)}{\theta_q(z)}(1/z)^{-\beta} & 1 \end{pmatrix}$$

with $A \neq 0$. The connection component is equal to $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & 1 \end{pmatrix}$ and the whole Galois group G is equal to $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & \langle w, e^{2\pi i\gamma}\rangle \end{pmatrix}$.

Last, suppose that $bq/c \in q^{\mathbb{N}^*}$. Then $A = 0$ and the function $\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma}$ is constant, hence $G = \begin{pmatrix} 1 & 0 \\ 0 & \langle w, e^{2\pi i\gamma}\rangle \end{pmatrix}$. □

(**Case 5**) $\underline{a \in q^{-\mathbb{N}}}$.

In this case, the twisted connection matrix is upper triangular :

$$\breve{P}(x) \quad = \quad \begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}(-1)^\alpha q^{-\frac{\alpha(\alpha-1)}{2}} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)}(1/z)^{-\alpha}z^{1-\gamma} \\ 0 & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}(1/z)^{-\beta}z^{1-\gamma} \end{pmatrix}$$

Theorem 6. *Suppose that (**Case 5**) holds. We have the following trichotomy :*

- *if* $b/c \notin q^{\mathbb{Z}}$ *then* $G = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & \mathbb{C}^* \end{pmatrix}$*;*
- *if* $bq/c \in q^{\mathbb{N}^*}$ *then* $G = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & \langle w, e^{2\pi i\gamma}\rangle \end{pmatrix}$*;*
- *if* $c/b \in q^{\mathbb{N}^*}$ *then* $G = \begin{pmatrix} 1 & 0 \\ 0 & \langle w, e^{2\pi i\gamma}\rangle \end{pmatrix}$.

Proof. We argue as for theorem 5. □

• The cases $\underline{b \in q^{\mathbb{Z}} \text{ or } a/c \in q^{\mathbb{Z}} \text{ or } b/c \in q^{\mathbb{Z}}}$ is similar to the case $a \in q^{\mathbb{Z}}$. We leave the details to the reader.

4 Galois groups of the basic hypergeometric equations : logarithmic cases

We write $a = uq^\alpha$, $b = vq^\beta$ and $c = wq^\gamma$ with $u, v, w \in \mathbb{U}$ and $\alpha, \beta, \gamma \in \mathbb{R}$ (we choose a logarithm of q).

4.1 $c = q$ and $a/b \notin q^{\mathbb{Z}}$

The aim of this section is to compute the Galois group of the basic hypergeometric system (2) under the assumption : $c = q$ and $a/b \notin q^{\mathbb{Z}}$.

Local fundamental system of solutions at 0. We have :

$$A(a, b; q; 0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Consequently, we are in the non-resonant logarithmic case at 0. We consider this situation as a degenerate case as c tends to q, $c \neq q$.

More precisely, we consider the limit as c tends to q, with $c \not{\mathbb{E}}_q q$, of the following matrix-valued function :

$$
\begin{aligned}
& F^{(0)}(a,b;c;z) \begin{pmatrix} 1 & 1 \\ 1 & q/c \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
= \; & \frac{-c}{c-q} \begin{pmatrix} (q/c-1) {}_2\phi_1(a,b;c;z) & {}_2\phi_1\left(aq/c, bq/c; q^2/c; z\right) - {}_2\phi_1(a,b;c;z) \\ (q/c-1) {}_2\phi_1(a,b;c;qz) & (q/c) {}_2\phi_1\left(aq/c, bq/c; q^2/c; qz\right) - {}_2\phi_1(a,b;c;qz) \end{pmatrix}
\end{aligned}
$$

Using the notations :

$$
\zeta(a,b;z) = \frac{d}{dc}_{|c=q} \left[{}_2\phi_1(a,b;c;z) \right] \text{ and } \xi(a,b;z) = \frac{d}{dc}_{|c=q} \left[{}_2\phi_1\left(aq/c, bq/c; q^2/c; z\right) \right]
$$

the above limit is equal to :

$$
F^{(0)}(a,b;q;z) \quad := \quad \begin{pmatrix} {}_2\phi_1(a,b;q;z) & -q(\xi(a,b;z) - \zeta(a,b;z)) \\ {}_2\phi_1(a,b;q;qz) & {}_2\phi_1(a,b;q;qz) - q(\xi(a,b;qz) - \zeta(a,b;qz)) \end{pmatrix}.
$$

From (4) we deduce that $F^{(0)}(a,b;q;z)$ satisfies $F^{(0)}(a,b;q;qz)J^{(0)}(q) = A(a,b;c;z)F^{(0)}(a,b;q;z)$ with $J^{(0)}(q) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence, this matrix being invertible as a matrix in the field of meromorphic functions, the matrix-valued function $Y^{(0)}(a,b;q;z) = F^{(0)}(a,b;q;z) e^{(0)}_{J^{(0)}(q)}(z)$ is a fundamental system of solutions of the basic hypergeometric equation with $c = q$. Let us recall that $e^{(0)}_{J^{(0)}(q)}(z) = \begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1 \end{pmatrix}$.

Generators of the local Galois group at 0. We have the following generator :

$$
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
$$

Local fundamental system of solutions at ∞. The situation is as Section 3. Hence we are in the non-resonant and non-logarithmic case at ∞ and a fundamental system of solutions at ∞ of (2) as described in Section 2.1 is given by $Y^{(\infty)}(a,b;q;z) = F^{(\infty)}(a,b;q;z) e^{(\infty)}_{J^{(\infty)}(a,b)}(z)$ with $J^{(\infty)}(a,b) = \mathrm{diag}(1/a, 1/b)$ and :

$$
F^{(\infty)}(a,b;q;z) = \begin{pmatrix} {}_2\phi_1\left(a,a;aq/b; \frac{q^2}{ab} z^{-1}\right) & {}_2\phi_1\left(b,b;bq/a; \frac{q^2}{ab} z^{-1}\right) \\ \frac{1}{a} {}_2\phi_1\left(a,a;aq/b; \frac{q}{ab} z^{-1}\right) & \frac{1}{b} {}_2\phi_1\left(b,b;bq/a; \frac{q}{ab} z^{-1}\right) \end{pmatrix}.
$$

Generators of the local Galois group at ∞. We have two generators :

$$\breve{P}(y_0)^{-1}\begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\beta}\end{pmatrix}\breve{P}(y_0) \text{ and } \breve{P}(y_0)^{-1}\begin{pmatrix} u & 0 \\ 0 & v\end{pmatrix}\breve{P}(y_0).$$

Connection matrix. The connection matrix is the limit as c tends to q of :

$$(e^{(\infty)}_{J^{(\infty)}(a,b)}(z))^{-1}\begin{pmatrix} \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}\frac{\theta_q(az)}{\theta_q(z)} & \frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta_q(\frac{aq}{c}z)}{\theta_q(z)} \\ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{\theta_q(bz)}{\theta_q(z)} & \frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta_q(\frac{bq}{c}z)}{\theta_q(z)}\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & q/c\end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}e^{(0)}_{J^{(0)}(q)}(z)$$

which is equal to :

$$P(z) \quad := \quad (e^{(\infty)}_{J^{(\infty)}(a,b)}(z))^{-1}\begin{pmatrix} u(a,b;q)\frac{\theta_q(az)}{\theta_q(z)} & q(u_c(a,b;q)-v_c(a,b;q))\frac{\theta_q(az)}{\theta_q(z)}+azv(a,b;q)\frac{\theta_q'(az)}{\theta_q(z)} \\ w(a,b;q)\frac{\theta_q(bz)}{\theta_q(z)} & q(w_c(a,b;q)-y_c(a,b;q))\frac{\theta_q(bz)}{\theta_q(z)}+bzy(a,b;q)\frac{\theta_q'(bz)}{\theta_q(z)}\end{pmatrix}e^{(0)}_{J^{(0)}(q)}(z)$$

where :

$$u(a,b;c)=\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty};\quad v(a,b;c)=\frac{(bq/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty};$$
$$w(a,b;c)=\frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty};\quad y(a,b;c)=\frac{(aq/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}.$$

and where the subscript c means that we take the derivative with respect to the third variable c.

Twisted connection matrix.

$$\breve{P}(z) \quad = \quad \begin{pmatrix} (1/z)^{-\alpha} & 0 \\ 0 & (1/z)^{-\beta}\end{pmatrix}\begin{pmatrix} u(a,b;q)\frac{\theta_q(az)}{\theta_q(z)} & q(u_c(a,b;q)-v_c(a,b;q))\frac{\theta_q(az)}{\theta_q(z)}+azv(a,b;q)\frac{\theta_q'(az)}{\theta_q(z)} \\ w(a,b;q)\frac{\theta_q(bz)}{\theta_q(z)} & q(w_c(a,b;q)-y_c(a,b;q))\frac{\theta_q(bz)}{\theta_q(z)}+bzy(a,b;q)\frac{\theta_q'(bz)}{\theta_q(z)}\end{pmatrix}\begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1\end{pmatrix}$$

We need to consider different cases.

(**Case 6**) $a \notin q^{\mathbb{Z}}$ and $b \notin q^{\mathbb{Z}}$.

Subject to this condition, the complex numbers $u(a,b;q)$, $v(a,b;q)$, $w(a,b;q)$ and $y(a,b;q)$ are non-zero.

Proposition 3. *If* ***(Case 6)*** *holds then the natural action of G^I on $\mathbb{C}^2$ is irreducible.*

Proof. Assume, at the contrary, that the action of G^I is reducible and let L be an invariant line.

Let us fist remark that $L \neq \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (in particular, G^I does not consist of scalar matrices). Indeed, if not, $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ would be stabilized by the connection component and the line spanned by $\breve{P}(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ would be independent of $z \in \Omega$: this is clearly false.

The group G^I being normalized by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (since G^I is a normal subgroup of G), the lines $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n L$ are also invariant by the action of G^I. These lines being distinct (since $L \neq \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$) we conclude that G^I consists of scalar matrices and we get a contradiction. This proves that G^I acts irreducibly. □

As a consequence we have the following theorem.

Theorem 7. *Suppose that* ***(Case 6)*** *holds. Then we have the following dichotomy :*

- *if $ab \notin q^{\mathbb{Z}}$ then $G = Gl_2(\mathbb{C})$;*
- *if $ab \in q^{\mathbb{Z}}$ then $G = Sl_2(\mathbb{C})$.*

Proof. Using the irreducibility of the natural action of G^I and arguing as for the proof of theorem 2, we obtain the equality $G^{I,der} = \mathrm{Sl}_2(\mathbb{C})$. From the formula (5) we deduce that the determinant of the twisted connection matrices when $c = q$ is equal to the limit as c tends to q of $\frac{-1}{1/a-1/b}(1/z)^{-(\alpha+\beta)} z^{1-\gamma} \frac{\theta_q(\frac{abq}{c}z)}{\theta_q(z)}$ *i.e.* $\frac{-1}{1/a-1/b}(1/z)^{-(\alpha+\beta)} z^{1-\gamma} \frac{\theta_q(abz)}{\theta_q(z)}$.

If $ab \notin q^{\mathbb{Z}}$ then this determinant is a non-constant holomorphic function and consequently $G = \mathrm{Gl}_2(\mathbb{C})$.

If $ab \in q^{\mathbb{Z}}$ then this determinant does not depend on z. This implies that the connection component of the Galois group is a sub-group of $\mathrm{Sl}_2(\mathbb{C})$. Furthermore, $ab \in q^{\mathbb{Z}}$ entails that $uv = 1$ and $\alpha + \beta \in \mathbb{Z}$, that is, $e^{2\pi i(\alpha+\beta)} = 1$. Consequently, the local Galois groups are subgroups of $\mathrm{Sl}_2(\mathbb{C})$ and the global Galois group G is therefore a subgroup of $\mathrm{Sl}_2(\mathbb{C})$. □

(Case 7) $\underline{b \in q^{\mathbb{N}^*}}$.

Then the twisted connection matrix simplifies as follows :

$$\breve{P}(z) = \begin{pmatrix} u(a,b;q)\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & q(u_c(a,b;q) - v_c(a,b;q))\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} + azv(a,b;q)\frac{\theta_q'(az)}{\theta_q(z)}(1/z)^{-\alpha} \\ 0 & q(w_c(a,b;q) - y_c(a,b;q))(-1)^{\beta} q^{-\frac{\beta(\beta-1)}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1 \end{pmatrix}$$

Theorem 8. *Suppose that* (***Case 7***) *holds. Then we have* $G = \begin{pmatrix} \mathbb{C}^* & \mathbb{C} \\ 0 & 1 \end{pmatrix}$.

Proof. Fix a point $y_0 \in \Omega$ such that $\breve{P}(y_0)$ is of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with $A, C \neq 0$. There exists a constant $D \in \mathbb{C}^*$ such that :

$$\breve{P}(y_0)^{-1}\breve{P}(z) = \begin{pmatrix} D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & * \\ 0 & 1 \end{pmatrix}.$$

Since G^I is normalized by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (remember that G^I is a normal subgroup of G), it contains, for all $n \in \mathbb{Z}$, the matrix :

$$\begin{pmatrix} D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & * + n(D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} - 1) \\ 0 & 1 \end{pmatrix}.$$

Because $a \notin q^{\mathbb{Z}}$, the function $D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} - 1$ is not identically equal to zero over $\mathbb{C}^*$ and therefore G^I contains, for all $z \in \Omega$:

$$\begin{pmatrix} D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & \mathbb{C} \\ 0 & 1 \end{pmatrix}.$$

In particular, $\begin{pmatrix} D\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} & 0 \\ 0 & 1 \end{pmatrix}$ belongs to G^I, so that $\begin{pmatrix} \mathbb{C}^* & 0 \\ 0 & 1 \end{pmatrix}$ is a subgroup of G^I and $\begin{pmatrix} \mathbb{C}^* & \mathbb{C} \\ 0 & 1 \end{pmatrix} \subset G$. The converse inclusion is clear. □

(**Case 8**) $\underline{b \in q^{-\mathbb{N}}}$.

Using the identity :

$$bz\frac{\theta_q'(bz)}{\theta_q(z)} = (-\beta - \ell_q(z))(-1)^{\beta}q^{-\frac{\beta(\beta-1)}{2}}$$

we see that, in this case, the twisted connection matrix takes the form :

$$\breve{P}(z) \quad = \quad \begin{pmatrix} 0 & q(u_c(a,b;q) - v_c(a,b;q))\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} \\ w(a,b,q)(-1)^{\beta}q^{-\frac{\beta(\beta-1)}{2}} & q(w_c(a,b;q) - y_c(a,b;q) - \beta/q)(-1)^{\beta}q^{-\frac{\beta(\beta-1)}{2}} \end{pmatrix}$$

Theorem 9. *Suppose that* (***Case 8***) *holds. Then we have* $G = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & \mathbb{C}^* \end{pmatrix}$.

Proof. Fix a base point $y_0 \in \Omega$. There exist three constants $C, C', C'' \in \mathbb{C}$ with $C \neq 0$, such that the following identity holds for all $z \in \Omega$:

$$\breve{P}(y_0)^{-1}\breve{P}(z) = \begin{pmatrix} 1 & C'\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} + C'' \\ 0 & C\frac{\theta_q(az)}{\theta_q(z)}(1/z)^{-\alpha} \end{pmatrix}.$$

The proof is similar to the proof of Theorem 8. □

The remaining case $a \in q^{\mathbb{Z}}$ is similar to (**Case 8**) .
The case $a = b$ and $c \notin q^{\mathbb{Z}}$ is similar to the case treated in this section.

4.2 $a = b$ and $c = q$

The aim of this section is to compute the Galois group of the basic hypergeometric system (2) under the assumption : $a = b$ and $c = q$.

Local fundamental system of solutions at 0. The situation is the same as in the case $c = q$ and $a/b \notin q^{\mathbb{Z}}$.

Generator of the local Galois group at 0. We have the following generator :

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Local fundamental system of solutions at ∞. We have :

$$A(a, a; q; z) = \begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix} \begin{pmatrix} 1/a & 1 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix}^{-1}.$$

Consequently, we are in the non-resonant logarithmic case at ∞. We consider the case $a = b$ and $c = q$ as a degenerate case of the situation $c = q$ as a tends to b, $a/b \neq 1$.

We consider the following matrix valued function :

$$F^{(\infty)}(a, b; q; z) \begin{pmatrix} 1 & 1 \\ 1/a & 1/b \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix}.$$

A straightforward calculation, which we omit here because it is long although easy, shows that this matrix-valued function does admit a limit as a tends to b that we denote $F^{(\infty)}(a, a; q; z)$. A fundamental system of solutions at ∞ of (2) as described in section 2.1 is given by $Y^{(\infty)}(a, a; q; z) = F^{(\infty)}(a, a; q; z)e^{(\infty)}_{J^{(\infty)}(a,a)}(z)$ with $J^{(\infty)}(a, a) = \begin{pmatrix} 1/a & 1 \\ 0 & 1/a \end{pmatrix}$.

Generator of the local Galois group at ∞. We have the following generators :

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix} \text{ and } \breve{P}(y_0)^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \breve{P}(y_0).$$

Birkhoff matrix. The Birkhoff matrix is equal to $(e^{(\infty)}_{J^{(\infty)}(a,a)}(z))^{-1}Qe^{(0)}_{J^{(0)}(q)}(z)$ where Q is the limit as a tends to b of :

$$\begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 1/a & 1/b \end{pmatrix}\begin{pmatrix} u(a,b;q)\frac{\theta_q(az)}{\theta_q(z)} & q(u_c(a,b;q)-v_c(a,b;q))\frac{\theta_q(az)}{\theta_q(z)}+azv(a,b;q)\frac{\theta_q'(az)}{\theta_q(z)} \\ w(a,b;q)\frac{\theta_q(bz)}{\theta_q(z)} & q(w_c(a,b;q)-y_c(a,b;q))\frac{\theta_q(bz)}{\theta_q(z)}+bzy(a,b;q)\frac{\theta_q'(bz)}{\theta_q(z)} \end{pmatrix}.$$

It has the following form :

$$\begin{pmatrix} C\frac{\theta_q(az)}{\theta_q(z)}+az\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q'(az)}{\theta_q(z)} & * \\ -(1/a)\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q(az)}{\theta_q(z)} & C'\frac{\theta_q(az)}{\theta_q(z)}-z\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q'(az)}{\theta_q(z)} \end{pmatrix}$$

where $*$ denotes some meromorphic function.

Twisted Birkhoff matrix.

$$(1/z)^{-\alpha}\begin{pmatrix} 1 & -a\ell_q(z) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C\frac{\theta_q(az)}{\theta_q(z)}+az\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q'(az)}{\theta_q(z)} & * \\ -(1/a)\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q(az)}{\theta_q(z)} & C'\frac{\theta_q(az)}{\theta_q(z)}-z\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q'(az)}{\theta_q(z)} \end{pmatrix}\begin{pmatrix} 1 & \ell_q(z) \\ 0 & 1 \end{pmatrix}.$$

We need to consider different cases.

(**Case 9**) $\underline{a \notin q^{\mathbb{Z}}}$.

Proposition 4. *Suppose that* ***(Case 9)*** *holds. Then the natural action of G^I on $\mathbb{C}^2$ is irreducible.*

Proof. Remark that $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not an invariant line. Indeed, if not, this line would be invariant by the action of the connection component, hence the line spanned by $\breve{P}(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ would be independent of $z \in \Omega$. Considering the ratio of the coordinates of this line, this would imply the existence of some constant $A \in \mathbb{C}$ such that the following functional equation holds on $\mathbb{C}^*$:

$$C\frac{\theta_q(az)}{\theta_q(z)}+az\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q'(az)}{\theta_q(z)}+\frac{\theta_q(a)}{(q;q)_\infty^2}\frac{\theta_q(az)}{\theta_q(z)}\ell_q(z)=A\frac{\theta_q(az)}{\theta_q(z)}.$$

The fact that $\theta_q(az)$ vanishes exactly to the order one at $z=1/a$, yields a contradiction.

The end of the proof is similar to the proof of Proposition 3. □

Theorem 10. *If* **(Case 9)** *holds then we have the following dichotomy :*

- *if* $a^2 \notin q^{\mathbb{Z}}$ *then* $G = Gl_2(\mathbb{C})$*;*
- *if* $a^2 \in q^{\mathbb{Z}}$ *then* $G = Sl_2(\mathbb{C})$*.*

Proof. The proof follows the same line as that of theorem 2. $\square$

(**Case 10**) $\underline{a \in q^{\mathbb{Z}}}$.

Under this condition, the connection matrix simplifies as follows, for some constants $C, C' \in \mathbb{C}$:

$$\begin{pmatrix} C & * \\ 0 & C' \end{pmatrix}.$$

Theorem 11. *Suppose that* **(Case 10)** *holds. Then we have :* $G = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$.

Proof. The local Galois group at 0 is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, hence G contains $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$.

Since the twisted connection matrix is upper triangular with constant diagonal entries, the connection component is a subgroup of $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$. The generators of the local Galois group at 0 and at ∞ also lie in $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$. Therefore, G is a subgroup of $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$. $\square$

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