Generalized basic hypergeometric equations

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Abstract. *This paper deals with regular singular generalized q-hypergeometric equations with either "large" or "small" Galois groups. In particular, we consider the fundamental problem of finding appropriate Galoisian substitutes for the usual notion of local monodromy.*

Contents

In this paper, q denotes a non zero complex number such that $|q| < 1$.

1 Introduction

The generalized hypergeometric operator $\mathrm{L}(\underline{\alpha};\underline{\beta};\lambda)$ with parameters $\underline{\alpha} = (\alpha_1, ..., \alpha_r) \in \mathbb{C}^r$, $\underline{\beta} = (\beta_1, ..., \beta_s) \in \mathbb{C}^s$ and $\lambda \in \mathbb{C}^*$ is given by :

$$\mathrm{L}(\underline{\alpha};\underline{\beta};\lambda) = \prod_{j=1}^{s}(\delta + \beta_j - 1) - z\lambda\prod_{i=1}^{r}(\delta + \alpha_i), \qquad \delta = z\frac{d}{dz}. \tag{1}$$

An historical account of the hypergeometric theory can be found in [22]. The hypergeometric operators are classically quantized as follows. We denote by $\mathcal{D}_q$ the non commutative algebra $\mathbb{C}(z)\langle\boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1}\rangle$ of non commutative polynomials with coefficients in $\mathbb{C}(z)$ satisfying to the relation $\boldsymbol{\sigma}_q z = qz\boldsymbol{\sigma}_q$. The generalized q-hypergeometric operator $\mathcal{L}_q(\underline{a};\underline{b};\lambda)$ with parameters $\underline{a} = (a_1, ..., a_r) \in (\mathbb{C}^*)^r$, $\underline{b} = (b_1, ..., b_s) \in (\mathbb{C}^*)^s$ and $\lambda \in \mathbb{C}^*$ is given by :

$$\mathcal{L}_q(\underline{a};\underline{b};\lambda) = \prod_{j=1}^{s}(\frac{b_j}{q}\boldsymbol{\sigma}_q - 1) - z\lambda\prod_{i=1}^{r}(a_i\boldsymbol{\sigma}_q - 1) \in \mathcal{D}_q. \tag{2}$$

It is regular singular both at 0 and at ∞ if and only if $r = s$ (see section 2).

In the rest of the paper, we will assume that $r = s =: n$.

Important results concerning the Galoisian aspects of the generalized hypergeometric operators were obtained by F. Beukers and G. Heckman in [6], F. Beukers, W. D. Brownawell and G. Heckman in [5] and N. M. Katz and O. Gabber in [21]. For an approach relying on Ramis' density theorem, we refer to the work of A. Duval and C. Mitschi in [15] and to C. Mitschi's paper [27].

The fact that the local monodromy associated to (1) at the singular point $z = 1/\lambda \in \mathbb{C}^*$ is a pseudo-reflection is crucial ([6, 21]). As pointed out by Y. André in [1] (with $\lambda = 1$), "l'outil essentiel dans la détermination du groupe de Galois différentiel hypergéométrique, à savoir la pseudo-réflexion donnée par la monodromie locale au point 1, ne se transporte pas au cas q-hypergéométrique" [1]. This leads us to the following central problem for the theory of q-difference operators over $\mathbb{P}^1_{\mathbb{C}}$:

Problem : *Find relevant Galoisian substitutes for the missing monodromy.*

This paper presents a solution of this problem. We use an infinitesimal version of a meromorphic family of Galoisian morphisms build up from twisted Birkhoff matrices which were introduced by J. Sauloy in [36]. We show that, in the q-hypergeometric case, the singularities of

1. "the essential tool for the calculation of the hypergeometric differential Galois group, namely the pseudo-reflection given by the monodromy around the point 1, is no longer available in the q-hypergeometric case".

this family give rise, *via* Taylor expansion, to infinitesimal Galoisian morphisms providing the missing geometric information. Then, we are in a position to determine the Galois groups of all Lie irreducible regular singular generalized q-hypergeometric operators, without restriction on q. Our calculations of q-hypergeometric Galois groups are generalizations of anterior results obtained, by a different method, by Y. André. Indeed, Y. André proved in [1] a specialization theorem for differential/difference Galois groups allowing the calculation of q-difference Galois groups for generic q.

In this paper, we are also interested in operators having finite Galois groups. We recall that H. A. Schwarz determined the list of Gauss' hypergeometric operators (this corresponds to the case $r = s = n = 2$) with finite differential Galois groups; see [38, 19, 4]. This list was extended to irreducible generalized hypergeometric operators by F. Beukers and G. Heckman in [6] and by N. M. Katz in [21] by using p-curvatures (Grothendieck's conjecture). In the present paper, we draw up the list of all generalized q-hypergeometric operators having a finite Galois group. This extends the case $n = 2$ tackled by L. Di Vizio in [12]. Our original approach has been greatly improved thanks to observations of one of the referees.

As suggested by one of the referees, it is interesting to include in this introduction some illustrations of the role of the q-hypergeometric theory in mathematics. L. Euler, C. F. Gauss and C. G. J. Jacobi already used q-hypergeometric objects but the general q-hypergeometric theory started with E. Heine; see [39] and [18] for historical informations. Since these pioneer works, the q-hypergeometric theory in one or several variables occurred in many branches of mathematics. In V. Tarasov and A. Varchenko's paper [40], the q-hypergeometric theory appears geometrically as a bridge between quantum affine algebras and elliptic quantum groups. In particular, these authors present a q-analogue of Kohno-Drinfeld theorem on the monodromy group of the differential KZ equation [14, 23, 24]. More precisely, a connection between representation theories of the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ and of the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ is provided by transition functions between asymptotic solutions of the qKZ equation associated with the quantum group $U_q(\mathfrak{sl}_2)$. This work crucially relies on the geometric interpretation of the qKZ equation as a discrete Gauss-Manin connection in the spirit of K. Oamoto's work [2, 3] and on the associated q-hypergeometric integrals (pairing cohomology/homology). The q-hypergeometric theory is also frequently used in analytic number theory. For instance, C. Krattenthaler, T. Rivoal and W. Zudilin used q-hypergeometric series in order to study the arithmetic nature of values of q-analogues of Riemann zeta function (related to modular forms); see [26] and the references therein. We finish by mentioning the occurrence of the q-hypergeometric theory in the framework of the discrete Morales-Ramis theory developed by G. Casale and the author in [8, 7] in order to study the non integrability of concrete discrete dynamical systems (e.g. q-Painlevé equations). This list, which is far from being exhaustive, illustrates the prominent role of the q-hypergeometric theory in mathematics.

We shall now give the organization of this paper. In section 2 we recall general definitions and properties regarding regular singular q-difference modules. In section 3 we introduce the notions of Birkhoff matrix and of twisted Birkhoff matrix. We also introduce the logarithmic derivative of the twisted Birkhof matrix. We give their links with q-difference Galois theory.

In section 4 we show that, in the q-hypergeometric case, the singularities of the logarithmic derivative of the twisted Birkhoff matrix provide interesting infinitesimal Galoisian morphisms seen as avatars of the usual local monodromy. In section 5 we study fundamental properties of the irreducible generalized q-hypergeometric modules, we investigate their Lie irreducibility and we compute the corresponding Galois groups. In section 6 we draw up the list of all generalized q-hypergeometric modules with finite Galois groups. We also give an application to generalized hypergeometric operators using Y. André' specialization theorem ([1]).

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2 Regular singular q-difference modules

The main references for what follows are M. van der Put and M. Singer's book [29] and J. Sauloy's paper [36]. In the whole paper, σ_q denotes the operator acting on any function (or germ of function) f of the complex variable z by $(\sigma_q f)(z) = f(qz)$.

We denote by $\mathbb{C}\{z\}$ the local ring of germs of analytic functions at $0 \in \mathbb{C}$ and by $\mathbb{C}(\{z\})$ its field of fractions, namely the field of germs of meromorphic functions at $0 \in \mathbb{C}$. Let $\mathbb{C}(\{z\})\langle \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1}\rangle$ be the non commutative algebra of non commutative polynomials with coefficients in $\mathbb{C}(\{z\})$ satisfying to the relation $\boldsymbol{\sigma}_q f = (\sigma_q f)\boldsymbol{\sigma}_q$ for any $f \in \mathbb{C}(\{z\})$. We denote by $\mathcal{F}^{(0)}$ the neutral Tannakian category over $\mathbb{C}$ of q-difference modules over $\mathbb{C}(\{z\})$: it is the full subcategory of the category of left $\mathbb{C}(\{z\})\langle \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1}\rangle$-modules whose objects are the left $\mathbb{C}(\{z\})\langle \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1}\rangle$-modules which are finite dimensional as $\mathbb{C}(\{z\})$-vector spaces. The objects of $\mathcal{F}^{(0)}$ can be considered as finite dimensional $\mathbb{C}(\{z\})$-vector spaces endowed with a σ_q-linear automorphism. In accordance with the tradition, two isomorphic objects of $\mathcal{F}^{(0)}$ will be called meromorphically equivalent at 0. For details, see section 1.4 of [29] or section 1.1 of [36].

An object of $\mathcal{F}^{(0)}$ is regular singular if it has a lattice over $\mathbb{C}\{z\}$ which is invariant under the action of $\boldsymbol{\sigma}_q$ and $\boldsymbol{\sigma}_q^{-1}$. We denote by $\mathcal{E}^{(0)}$ the full subcategory of $\mathcal{F}^{(0)}$ made of its regular singular objects; it is a neutral Tannakian subcategory of $\mathcal{F}^{(0)}$. For any object M of $\mathcal{E}^{(0)}$, there exist a $\mathbb{C}$-vector space $V^{(0)}$ and a $\mathbb{C}$-linear automorphism $\Phi^{(0)}$ of $V^{(0)}$ with eigenvalues in $\{c \in \mathbb{C}^* \mid |q| \leq |c| < 1\}$ such that M is meromorphically equivalent to the q-difference module $\mathbb{C}(\{z\}) \otimes_{\mathbb{C}} V^{(0)}$, the action of $\boldsymbol{\sigma}_q$ being given by $\sigma_q \otimes \Phi^{(0)}$. For details, see sections 12.1 and 12.2.1 of [29] or section 1.2 of [36].

Let M be an object of $\mathcal{E}^{(0)}$ and let $A \in \mathrm{GL}_n(\mathbb{C}(\{z\}))$ be the inverse of the matrix representing the action of $\boldsymbol{\sigma}_q$ on M with respect to some $\mathbb{C}(\{z\})$-basis. We now recall the construction of local solutions at 0 for the q-difference system $\sigma_q Y = AY$. The above reminders, translated in terms of matrices, ensure that there exist $A^{(0)} \in \mathrm{GL}_n(\mathbb{C})$ with eigenvalues in $\{c \in \mathbb{C}^* \mid |q| \leq |c| < 1\}$ and $F^{(0)} \in \mathrm{GL}_n(\mathbb{C}(\{z\}))$ such that :

$$(\sigma_q F^{(0)})A^{(0)} = AF^{(0)}. \tag{3}$$

Therefore, if $e^{(0)}_{A^{(0)}}$ is a fundamental system of solutions of $\sigma_q Y = A^{(0)}Y$ then $Y^{(0)} = F^{(0)}e^{(0)}_{A^{(0)}}$

is a fundamental system of solutions of $\sigma_q Y = AY$. In [36], $e^{(0)}_{A^{(0)}}$ is built as follows. Let $A^{(0)} = D^{(0)}U^{(0)}$ be the multiplicative Dunford decomposition of $A^{(0)}$ (i.e. $D^{(0)} \in \mathrm{GL}_n(\mathbb{C})$ is semi-simple, $U^{(0)} \in \mathrm{GL}_n(\mathbb{C})$ is unipotent and $D^{(0)}U^{(0)} = U^{(0)}D^{(0)}$). Let θ_q be the Jacobi theta function defined by $\theta_q(z) = (q;q)_\infty (z;q)_\infty (q/z;q)_\infty$ where, for any $a \in \mathbb{C}^*$, $(a;q)_\infty = \prod_{k=1}^{\infty}(1 - aq^{k-1})$. We have $\sigma_q \theta_q = -z^{-1}\theta_q$. For all $\lambda \in \mathbb{C}^*$ such that $|q| \leq |\lambda| < 1$, we introduce the "q-character" $e^{(0)}_{\lambda} = \frac{\theta_q}{\theta_{q,\lambda}}$ where $\theta_{q,\lambda}(z) = \theta_q(\lambda z)$. We extend the definition of $e^{(0)}_{\lambda}$ to any $\lambda \in \mathbb{C}^*$ by requiring that $e^{(0)}_{q\lambda} = z e^{(0)}_{\lambda}$. If $D^{(0)} = Q\mathrm{diag}(\lambda_1, ..., \lambda_n)Q^{-1}$ then we set $e^{(0)}_{D^{(0)}} = Q\mathrm{diag}(e^{(0)}_{\lambda_1}, ..., e^{(0)}_{\lambda_n})Q^{-1}$. It is easily seen that $e^{(0)}_{D^{(0)}}$ does not depend on the chosen diagonalization. We also introduce the "q-logarithm" $\ell_q = -z\frac{\theta_q'}{\theta_q}$ and we set $e^{(0)}_{U^{(0)}} = \Sigma_{k=0}^{n}\binom{\ell_q}{k}(U^{(0)} - I_n)^k$. Then $e^{(0)}_{A^{(0)}} = e^{(0)}_{D^{(0)}}e^{(0)}_{U^{(0)}}$ depends only on $A^{(0)}$ and we have $\sigma_q e^{(0)}_{A^{(0)}} = A^{(0)}e^{(0)}_{A^{(0)}}$ as expected.

We have similar constructions ($\mathcal{F}^{(\infty)}$, $\mathcal{E}^{(\infty)}$, etc) and results at ∞.

We denote by $\mathcal{F}$ the neutral Tannakian category over $\mathbb{C}$ of q-difference modules over $\mathbb{C}(z)$: its definition is similar to that of $\mathcal{F}^{(0)}$ by replacing the difference field $(\mathbb{C}(\{z\}), \sigma_q)$ by $(\mathbb{C}(z), \sigma_q)$. In accordance with the tradition, two isomorphic objects of $\mathcal{F}$ will be called rationally equivalent. For the sake of conciseness, we set $\mathcal{D}_q = \mathbb{C}(z)\langle \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1}\rangle$.

We have natural functors $\mathcal{F} \rightsquigarrow \mathcal{F}^{(0)}$ and $\mathcal{F} \rightsquigarrow \mathcal{F}^{(\infty)}$. We say that an object of $\mathcal{F}$ is regular singular both at 0 and at ∞ if it is regular singular when viewed both in $\mathcal{F}^{(0)}$ and in $\mathcal{F}^{(\infty)}$. We denote by $\mathcal{E}$ the full subcategory of $\mathcal{F}$ made of its regular singular objects. It is a neutral Tannakian subcategory of $\mathcal{F}$.

For instance, it is easily seen that the generalized q-hypergeometric module $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ with parameters $\underline{a} = (a_1, ..., a_n) \in (\mathbb{C}^*)^n$, $\underline{b} = (b_1, ..., b_n) \in (\mathbb{C}^*)^n$ and $\lambda \in \mathbb{C}^*$ given by

$$\mathcal{H}_q(\underline{a}; \underline{b}; \lambda) = \mathcal{D}_q/\mathcal{D}_q\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) \tag{4}$$

is regular singular both at 0 and at ∞.

As explained above, choosing a basis, we can attach to any object M of $\mathcal{E}$ two fundamental systems of solutions $Y^{(0)} = F^{(0)}e^{(0)}_{A^{(0)}}$ with $F^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ and $A^{(0)} \in \mathrm{GL}_n(\mathbb{C})$ and $Y^{(\infty)} = F^{(\infty)}e^{(\infty)}_{A^{(\infty)}}$ with $F^{(\infty)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^* \cup \{\infty\}))$ and $A^{(\infty)} \in \mathrm{GL}_n(\mathbb{C})$. We have denoted by $\mathcal{M}(X)$ the field of meromorphic functions on the Riemann surface X.

3 Galois theory for q-difference modules

A number of authors have contributed to the development of Galois theories for q-difference equations over the past years, among whom C. H. Franke [17], P. Etingof [16], M. van der Put and M. Singer [29], M. van der Put and M. Reversat [28], Z. Chatzidakis and E. Hrushovski [10], J.-P. Ramis, J. Sauloy [36], Y. André [1], etc. The relations between the existing Galois theories for q-difference equations are partially understood (see in particular [9] and [13]).

3.1 The Tannakian formalism

Let $\otimes$ be the natural tensor product on $\mathcal{E}$ and let ω be a complex valued fiber functor on $\mathcal{E}$ (see [29, 36] for details). Following the general formalism of the theory of Tannakian categories (see [11]), the *absolute Galois group* of $\mathcal{E}$ is the complex proalgebraic group $\mathrm{Gal}(\mathcal{E},\omega) = \mathrm{Aut}^{\otimes}(\omega)$ and the *Galois group of an object* M of $\mathcal{E}$ is the complex linear algebraic group $\mathrm{Gal}(M,\omega) = \mathrm{Aut}^{\otimes}(\omega_{|\langle M\rangle})$ where $\langle M\rangle$ denotes the Tannakian subcategory of $\mathcal{E}$ generated by M. For the sake of conciseness, we shall omit in what follows the "base point" ω.

The fiber functor ω induces equivalences of Tannakian categories between $\mathcal{E}$ and the category of finite dimensional rational linear $\mathbb{C}$-representations of $\mathrm{Gal}(\mathcal{E})$ and between $\langle M\rangle$ and the category of finite dimensional rational linear $\mathbb{C}$-representations of $\mathrm{Gal}(M)$.

3.2 Galois groups and Birkhoff matrices

Let M be an object of $\mathcal{E}$ and let $A \in \mathrm{GL}_n(\mathbb{C}(z))$ be the inverse of the matrix representing the action of $\boldsymbol{\sigma}_q$ on M with respect to some $\mathbb{C}(z)$-basis. We denote by $\mathrm{G} \subset \mathrm{GL}_n(\mathbb{C})$ the corresponding Galois group. We maintain the notations introduced in section 2.

Birkhoff matrix. The Birkhoff matrix, also called connection matrix, is defined by $P = (Y^{(\infty)})^{-1}Y^{(0)}$. Its entries are elliptic functions i.e. meromorphic functions on the elliptic curve $\mathbb{E}_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

A link between Birkhoff matrices and difference Galois theory was discovered by P. Etingof in [16] for *regular* q-difference modules over $\mathbb{C}(z)$. Recall that a q-difference module over $\mathbb{C}(z)$ is regular if it is trivial when viewed both in $\mathcal{F}^{(0)}$ and in $\mathcal{F}^{(\infty)}$. The extension of P. Etingof's work to the *regular singular* case is not obvious. The reason is that in the regular singular case the construction of the Birkhoff matrix is in general not compatible with the tensor product. The fundamental obstruction to tensor compatibility comes from the fact that, in general, $e^{(0)}_\lambda e^{(0)}_\mu \neq e^{(0)}_{\lambda\mu}$. This problem was first overcome "algebraically" by M. van der Put and M. Singer in [29] by introducing symbolic solutions. Later, J. Sauloy solved "analytically" this problem in [36] by introducing twisted Birkhoff matrices.

Twisted Birkhoff matrix. Let $\tau \in \mathbb{C}$ be such that $q = e^{-2\pi i\tau}$. For all $y \in \mathbb{C}$, we set $q^y = e^{-2\pi i\tau y}$. Let $\widetilde{\mathbb{C}^*}$ be the Riemann surface of log (universal covering of $\mathbb{C}^*$); $\widetilde{z}$ will denote a variable on $\widetilde{\mathbb{C}^*}$.

For any $\lambda \in \mathbb{C}^*$, we introduce the analytic map on $\widetilde{\mathbb{C}^*}$ defined by $\widetilde{z} \mapsto g_{\widetilde{z}}(\lambda) = \widetilde{z}^{\omega}$ where ω is the unique element of $\mathbb{R}$ such that λ/q^{ω} belongs to the unit circle $\mathbb{U} \subset \mathbb{C}^*$.

For any $\lambda \in \mathbb{C}^*$, we introduce the meromorphic map on $\widetilde{\mathbb{C}^*}$ defined by $\widetilde{z} \mapsto \psi^{(0)}_{\widetilde{z}}(\lambda) = \frac{e^{(0)}_\lambda(\widetilde{z}_*)}{g_{\widetilde{z}}(\lambda)}$, where $\widetilde{z}_*$ denotes the projection of $\widetilde{z} \in \widetilde{\mathbb{C}^*}$ on $\mathbb{C}^*$. We define the *twisting factor* at 0 as the meromorphic map on $\widetilde{\mathbb{C}^*}$ defined by $\widetilde{z} \mapsto \psi^{(0)}_{\widetilde{z}}\left(D^{(0)}\right) = Q\mathrm{diag}(\psi^{(0)}_{\widetilde{z}}(\lambda_1), ..., \psi^{(0)}_{\widetilde{z}}(\lambda_n))Q^{-1}$ where $Q \in \mathrm{GL}_n(\mathbb{C})$ and $(\lambda_1, ..., \lambda_n) \in (\mathbb{C}^*)^n$ are such that $D^{(0)} = Q\mathrm{diag}(\lambda_1, ..., \lambda_n)Q^{-1}$.

We have a similar twisting factor $\psi^{(\infty)}_{\widetilde{z}}(D^{(\infty)})$ at ∞.

The twisted connection matrix is the meromorphic function $\breve{P}$ on $\widetilde{\mathbb{C}^*}$ defined by $\widetilde{z} \mapsto \breve{P}(\widetilde{z}) =$

$\psi_{\widetilde{z}}^{(\infty)}\left(D^{(\infty)}\right)P(\widetilde{z}_*)\psi_{\widetilde{z}}^{(0)}\left(D^{(0)}\right)^{-1}$.

Notation. *For the sake of conciseness, we will write z for both $\widetilde{z}$ and $\widetilde{z}_*$. Note that, in section 4, which is at the heart of the paper, we will not need $\breve{P}$ globally : we will choose an arbitrary branch of $\breve{P}$ and we will consider $\breve{P}$ locally near some point of $\mathbb{C}^*$.*

A density theorem. Fix a point $y_0 \in \Omega = \widetilde{\mathbb{C}^*} \setminus \{\text{zeros of } \det(\breve{P}) \text{ or poles of } \breve{P}\}$. J. Sauloy exhibited in [36] the following elements of the Galois group G of M :

1.a. $D_1^{(0)} = \gamma_1(D^{(0)})$ and $D_2^{(0)} = \gamma_2(D^{(0)})$ where, using the q-polar decomposition of $\mathbb{C}^*$ (that is identifying $\mathbb{C}^*$ with $\mathbb{U} \times q^{\mathbb{R}}$ via the map $\mathbb{U} \times q^{\mathbb{R}} \to \mathbb{C}^*$, $(u, q^\omega) \mapsto uq^\omega$), $\gamma_1 : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{U}$ is the projection on the first factor and $\gamma_2 : \mathbb{C}^* = \mathbb{U} \times q^{\mathbb{R}} \to \mathbb{C}^*$ is defined by $\gamma_2(uq^\omega) = e^{2\pi i\omega}$;

1.b. $U^{(0)}$;

2.a. $\breve{P}(y_0)^{-1}D_1^{(\infty)}\breve{P}(y_0) = \breve{P}(y_0)^{-1}\gamma_1(D^{(\infty)})\breve{P}(y_0)$ and $\breve{P}(y_0)^{-1}D_2^{(\infty)}\breve{P}(y_0) = \breve{P}(y_0)^{-1}\gamma_2(D^{(\infty)})\breve{P}(y_0)$;

2.b. $\breve{P}(y_0)^{-1}U^{(\infty)}\breve{P}(y_0)$;

3. $\breve{P}(y_0)^{-1}\breve{P}(z)$, $z \in \Omega$.

The following result, due to J. Sauloy in [36], is a generalization of a density theorem due to P. Etingof in [16]. We refer to Theorem 12.14 in section 12.3.3 of [29] for a more "algebraic" density theorem due to M. van der Put and M. Singer.

Theorem 1. *The Galois group G is generated, as an algebraic group, by the matrices 1. to 3.*

We denote by C and call *connection component* of G the complex linear algebraic subgroup of G generated by the matrices involved in 3. : $\mathrm{C} = \overline{\langle \breve{P}(y)^{-1}\breve{P}(z) \mid y, z \in \Omega \rangle} \subset \mathrm{G}$. The following result is obvious but useful.

Proposition 1. *The connection component* C *is an algebraic subgroup of the neutral component G^0 of G.*

Note that $\breve{P}$ is no longer elliptic; nevertheless, it has the following properties (proofs are immediate) :

$$(\gamma\breve{P})(z) = (D_2^{(\infty)})^{-1}\breve{P}(z)D_2^{(0)} \quad (5)$$

$$\breve{P}(qz) = (D_1^{(\infty)})^{-1}\breve{P}(z)D_1^{(0)} \quad (6)$$

where $\gamma\breve{P}$ is obtained from $\breve{P}$ by analytic continuation when $z \in \widetilde{\mathbb{C}^*}$ turns counterclockwise one time around the origin.

Corollary 1. *If M is non logarithmic both at* 0 *and at* ∞ *then G is generated, as an algebraic group, by $D_1^{(0)}, D_2^{(0)}$ and* C.

Proof. Let H be the algebraic subgroup of G generated by $D_1^{(0)}, D_2^{(0)}$ and C. Let us consider $y, z \in \Omega$. Formula (5) entails that $\breve{P}(y)^{-1}(\gamma\breve{P})(z) = \breve{P}(y)^{-1}(D_2^{(\infty)})^{-1}\breve{P}(y)\breve{P}(y)^{-1}\breve{P}(z)D_2^{(0)}$ but $\breve{P}(y)^{-1}(\gamma\breve{P})(z)$, $\breve{P}(y)^{-1}\breve{P}(z)$ and $D_2^{(0)}$ belong to H so $\breve{P}(y)^{-1}(D_2^{(\infty)})^{-1}\breve{P}(y)$ belongs to H. Using formula (6) and a similar argument, we see that $\breve{P}(y)^{-1}(D_1^{(\infty)})^{-1}\breve{P}(y)$ belongs to H. Theorem 1 ensures that $H = \mathrm{G}$. □

Corollary 2. *The Galois group G is generated, as an algebraic group, by $D_1^{(0)}, D_2^{(0)}$ and G^0.*

Proof. Similar to the proof of Corollary 1. □

We end this section with the following fundamental remark.

Proposition 2. *For any $z \in \Omega$, $\breve{P}^{-1}(z)\breve{P}'(z)$ belongs to the Lie algebra $\mathfrak{g}$ of G.*

Proof. Indeed, let us consider $z \in \Omega$. The map $\breve{P}^{-1}(z)\breve{P}(\cdot)$ is analytic near z, takes its values in G^0 and its evaluation at z is equal to I_n. Therefore, its derivative at z belongs to $\mathfrak{g}$ that is $\breve{P}^{-1}(z)\breve{P}'(z)$ belongs to $\mathfrak{g}$. □

We will see in the next section that, under additional hypotheses, $\breve{P}^{-1}\breve{P}'$ is sufficient for describing the whole Lie algebra $\mathfrak{g}$.

3.3 The case of q-rational exponents

We maintain the notations of the previous subsection. We assume that the exponents of M both at 0 and at ∞ belong to $q^{\mathbb{Q}}$. Let $d \in \mathbb{N}^*$ be such that they belong to $q^{d^{-1}\mathbb{Z}}$. For clarity of exposition, we also assume that M is non logarithmic both at 0 and at ∞. We denote by $\mathbb{E}_{q;d}$ the quotient of the Riemann surface of $z^{1/d}$ by the natural action of $q^{\mathbb{Z}}$. The twisted Birkhoff matrix $\breve{P}$ can be interpreted as a morphism of quasiprojective varieties from $\Omega_d = \mathbb{E}_{q;d} \setminus \{\text{zeros of } \det(\breve{P}) \text{ or poles of } \breve{P}\}$ to $GL_n(\mathbb{C})$.

Proposition 3. *The connection component* C *is generated as an abstract group by $\{\breve{P}(y)^{-1}\breve{P}(z) \mid y, z \in \Omega_d\}$.*

Proof. Similar the proof of Proposition 3.2 of [16]. □

The following result is due to P. Etingof (private communication).

Proposition 4 (P. Etingof)**.** *The Lie algebra $\mathfrak{g}$ of G is generated by $\{\breve{P}(z)^{-1}\breve{P}'(z) \mid z \in \Omega_d\}$.*

Proof. Let us first prove that $\mathrm{G} = \cup_{i \in [\![0,d-1]\!]} \mathrm{C}(D_2^{(0)})^i$ and that $\mathrm{G}^0 = \mathrm{C}$.

Since, for all $y, z \in \Omega_d$, $D_2^{(0)}(\gamma\breve{P})(y)^{-1}(\gamma\breve{P})(z) = \breve{P}(y)^{-1}\breve{P}(z)D_2^{(0)}$ (consequence of formula (5) in section 3.2), the abstract group K generated by the connection component C and by $D_2^{(0)}$ coincides with $\cup_{i \in \mathbb{Z}} \mathrm{C}(D_2^{(0)})^i = \cup_{i \in [\![0,d-1]\!]} \mathrm{C}(D_2^{(0)})^i$. So K is Zariski closed and hence coincides with G in virtue of Corollary 1. Consequently, C has finite index in $\mathrm{G} = K = \cup_{i \in [\![0,d-1]\!]} \mathrm{C}(D_2^{(0)})^i$ and hence $\mathrm{G}^0 \subset \mathrm{C}$. The converse inclusion also holds in virtue of Proposition 1.

We now prove the proposition. Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ (in virtue of Proposition 2) generated by $\{\breve{P}^{-1}(z)\breve{P}'(z) \mid z \in \Omega_d\}$ and let H be the embedded connected Lie subgroup of G^0 corresponding to $\mathfrak{h}$. Let z be an element of Ω_d. Note that $\breve{P}^{-1}(\cdot)\breve{P}(z)$ is the fundamental system of solutions with values in G^0 and with initial condition $Y(z) = I_n$ of the differential system $Y' = \left[-\breve{P}^{-1}\breve{P}'\right]Y$. But $-\breve{P}^{-1}\breve{P}'$ takes its values in $\mathfrak{h}$. So $\breve{P}^{-1}(\cdot)\breve{P}(z)$ takes its values in H. Hence H contains $\{\breve{P}(y)^{-1}\breve{P}(z) \mid y, z \in \Omega_d\}$ which is, in virtue of Proposition 3, a set of generators of $\mathrm{C} = \mathrm{G}^0$ (as an abstract group) so $H = \mathrm{G}^0$ and $\mathfrak{h} = \mathfrak{g}$. □

4 Substitutes for the missing monodromy

In this section, we solve the "monodromy problem" described in section 1. The polar parts of the logarithmic derivative of the twisted Birkhoff matrix will play a crucial role. Remark that, in the local analytic theory of q-difference equations at 0 or ∞, the notion of pole plays a special role; see for instance [32, 33, 30, 31, 35].

We consider $\underline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{Q}^n$, $\underline{\beta} = (\beta_1, ..., \beta_n) \in \mathbb{Q}^n$ and $\lambda \in \mathbb{Q}^*$. We set $\underline{a} = (a_1, ..., a_n) = (q^{\alpha_1}, ..., q^{\alpha_n}) \in (\mathbb{C}^*)^n$ and $\underline{b} = (b_1, ..., b_n) = (q^{\beta_1}, ..., q^{\beta_n}) \in (\mathbb{C}^*)^n$.

Let $f_0, ..., f_n$ be the coefficients of $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$:

$$\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) = f_0 \boldsymbol{\sigma}_q^n + f_1 \boldsymbol{\sigma}_q^{n-1} + \cdots + f_n.$$

Note that $f_0, f_1, ..., f_n$ are degree one polynomials with complex coefficients and that $f_0 = \prod_{j=1}^n \frac{b_j}{q} - z\lambda \prod_{i=1}^n a_i$ and $f_n = (-1)^n(1 - \lambda z)$. The inverse A of the matrix representing the action of $\boldsymbol{\sigma}_q$ on $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ with respect to the basis $\overline{1}, \overline{\boldsymbol{\sigma}_q}, ..., \overline{\boldsymbol{\sigma}_q^{n-1}}$ is given by :

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{f_n}{f_0} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{f_{n-1}}{f_0} \\ 0 & 1 & 0 & \cdots & 0 & -\frac{f_{n-2}}{f_0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{f_1}{f_0} \end{pmatrix}^{-1} \in GL_n(\mathbb{C}(z)). \tag{7}$$

For the sake of conciseness, we denote by $z_0 = (\prod_{j=1}^n \frac{b_j}{q})(\lambda \prod_{i=1}^n a_i)^{-1}$ and by $z_n = \lambda^{-1}$ the respective roots of f_0 and of f_n.

We maintain the notations ($F^{(0)}$, $F^{(\infty)}$, $\breve{P}$, etc) introduced in sections 2 and 3.

Convention. *In what follows, we choose an arbitrary branch of $\breve{P}$ and we consider $\breve{P}$ locally near $z_n \in \mathbb{C}^*$.*

Lemma 1. *If $q^{\mathbb{Z}} z_0 \cap q^{\mathbb{Z}} z_n = \emptyset$ then the residue of $\breve{P}^{-1}\breve{P}'$ at z_n is conjugate to $diag(c, 0, ..., 0)$ for some $c \in \mathbb{C}^*$.*

Proof. We recall (see section 2) that $F^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ is such that $(\sigma_q F^{(0)})A^{(0)} = AF^{(0)}$ and that $F^{(\infty)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^* \cup \{\infty\}))$ is such that $(\sigma_q F^{(\infty)})A^{(\infty)} = AF^{(\infty)}$. This entails that, for all $m \in \mathbb{N}^*$, we have, over $\mathbb{C}^*$:

$$(F^{(0)})^{-1}F^{(\infty)} = (A^{(0)})^{-m}(\sigma_q^m F^{(0)})^{-1}(\sigma_q^{m-1}A) \cdots A(\sigma_q^{-1}A) \cdots (\sigma_q^{-m}A)(\sigma_q^{-m}F^{(\infty)})(A^{(\infty)})^{-m}. \tag{8}$$

But, for $m \in \mathbb{N}^*$ large enough, $\sigma_q^m F^{(0)} \in \mathrm{GL}_n(\mathbb{C}\{z - z_n\})$ and $\sigma_q^{-m} F^{(\infty)} \in \mathrm{GL}_n(\mathbb{C}\{z - z_n\})$. Moreover, the hypotheses on z_0 and z_n show that, for any $k \in \mathbb{Z}^*$, $\sigma_q^k A \in \mathrm{GL}_n(\mathbb{C}\{z - z_n\})$ and that $A = \frac{R}{z - z_n}$ mod. $M_n(\mathbb{C}\{z - z_n\})$ for some $R \in M_n(\mathbb{C})$ with rank at most one. Therefore, there exists $R_1 \in M_n(\mathbb{C})$ with rank at most one such that :

$$(F^{(0)})^{-1}F^{(\infty)} = \frac{R_1}{z - z_n} \text{ mod. } M_n(\mathbb{C}\{z - z_n\}). \tag{9}$$

But, by construction (see section 3.2), there exist N and M in $\mathrm{GL}_n(\mathbb{C}\{z-z_n\})$ such that :

$$\breve{P}^{-1} = N(F^{(0)})^{-1}F^{(\infty)}M. \tag{10}$$

So, there exists $R_2 \in M_n(\mathbb{C})$ with rank at most one such that :

$$\breve{P}^{-1} = \frac{R_2}{z-z_n} \text{ mod. } M_n(\mathbb{C}\{z-z_n\}). \tag{11}$$

On the other hand, note that the hypothesis on z_0 and z_n implies that, for any $k \in \mathbb{Z}$, $\sigma_q^k A^{-1} \in M_n(\mathbb{C}\{z-z_n\})$. Arguing as above, we get that :

$$\breve{P} \in M_n(\mathbb{C}\{z-z_n\}). \tag{12}$$

Formulas (11) and (12) imply that there exists $R_3 \in M_n(\mathbb{C})$ with rank at most one such that :

$$\breve{P}^{-1}\breve{P}' = \frac{R_3}{z-z_n} \text{ mod. } M_n(\mathbb{C}\{z-z_n\}).$$

It remains to prove that R_3 has a non zero trace. Note that $\det(\breve{P}) \in \mathbb{C}\{z-z_n\}$ has a simple zero at z_n -the proof of this assertion is similar to that of formula (11)- so its derivative is non zero in a neighborhood of z_n. But $\det(\breve{P})' = -\det(\breve{P})\mathrm{tr}(\breve{P}^{-1}\breve{P}')$ so $\mathrm{tr}(\breve{P}^{-1}\breve{P}')$ has a non zero residue at z_n. Therefore $\mathrm{tr}(R_3)$ is non zero. □

Lemma 2. *We assume that $q^{\mathbb{Z}}z_0 \cap q^{\mathbb{Z}}z_n \neq \emptyset$ and that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is not rationally equivalent to some q-difference module of the form $\mathbb{C}(z) \otimes_{\mathbb{C}} V$, the action of $\boldsymbol{\sigma}_q$ being given by $\sigma_q \otimes \Phi$, where Φ is a $\mathbb{C}$-linear automorphism of a $\mathbb{C}$-vector space V. Then either the residue at z_n of $(z-z_n)^{-1}\breve{P}^{-1}\breve{P}'$ is traceless with rank one or the residue at z_n of $\breve{P}^{-1}\breve{P}'$ is conjugate to $diag(c,-c,0,...,0)$ for some $c \in \mathbb{C}^*$.*

Proof. Arguing as for the proof of Lemma 1, we see that there exist $R_4 \in M_n(\mathbb{C})$ with rank at most one and $S_4, T_4 \in M_n(\mathbb{C})$ such that :

$$\breve{P}^{-1} = \frac{R_4}{z-z_n} + S_4 + (z-z_n)T_4 \text{ mod. } (z-z_n)^2 M_n(\mathbb{C}\{z-z_n\}). \tag{13}$$

Similarly, there exist $R_5 \in M_n(\mathbb{C})$ with rank at most one and $S_5, T_5 \in M_n(\mathbb{C})$ such that :

$$\breve{P} = \frac{R_5}{z-z_n} + S_5 + (z-z_n)T_5 \text{ mod. } (z-z_n)^2 M_n(\mathbb{C}\{z-z_n\}). \tag{14}$$

Note that the equality $\breve{P}^{-1}\breve{P} = I_n$ implies that $R_4R_5 = 0$. Therefore :

$$\breve{P}^{-1}\breve{P}' = \frac{Q_6}{(z-z_n)^2} + \frac{R_6}{z-z_n} \text{ mod. } M_n(\mathbb{C}\{z-z_n\}) \tag{15}$$

where $Q_6 = -S_4R_5$ and $R_6 = R_4T_5 - T_4R_5$.

We claim that $R_4 \neq 0$ and $R_5 \neq 0$ i.e. that z_n is a pole of $\breve{P}^{-1}$ and $\breve{P}$. Indeed, assume at the contrary that z_n is a regular point for $\breve{P}^{-1}$ (the other case is similar). Then equation (10) shows that $Q := (F^{(0)})^{-1}F^{(\infty)}$ is regular at z_n and hence on $q^{\mathbb{Z}}z_n$ because $\sigma_q Q = A^{(0)}Q(A^{(\infty)})^{-1}$. But equation (8) shows that the poles of Q belong to $q^{\mathbb{Z}}z_n$. Therefore Q is analytic on $\mathbb{C}^*$. Converting the equation $\sigma_q Q = A^{0)}Q(A^{(\infty)})^{-1}$ on the Taylor coefficients of Q at 0, we easily see that the entries of Q are Laurent polynomials so $Q \in \mathrm{GL}_n(\mathbb{C}(z))$. Then the equality $F^{(0)} = F^{(\infty)}Q^{-1}$ shows that $F^{(0)}$ is meromorphic at ∞ and hence on $\mathbb{P}^1_{\mathbb{C}}$: $F^{(0)}$ belongs to $\mathrm{GL}_n(\mathbb{C}(z))$. This implies that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is rationally equivalent to $\mathbb{C}(z)\otimes_{\mathbb{C}}\mathbb{C}^n$, the action of $\boldsymbol{\sigma}_q$ being given by $\sigma_q \otimes (A^{(0)})^{-1}$: this contradicts our hypotheses and proves our claim.

We claim that $Q_6 \neq 0$ or $R_6 \neq 0$ i.e. that z_n is a pole of $\breve{P}^{-1}\breve{P}'$. Indeed, since z_n is a pole of $\breve{P}^{-1}$, we get from the equality $(\breve{P}^{-1})' = -(\breve{P}^{-1}\breve{P}')\breve{P}^{-1}$ that z_n is a pole of $\breve{P}^{-1}\breve{P}'$ (because the order of z_n as a pole of $(\breve{P}^{-1})'$ is greater than the order of z_n as a pole of $\breve{P}^{-1}$).

Moreover, since the ranks of R_4 and R_5 are not more than one, we get :

$$\mathrm{rk}Q_6 \leq 1 \text{ and } \mathrm{rk}R_6 \leq 2.$$

So, if $Q_6 \neq 0$ then Q_6 has rank one.

Assume that $Q_6 = 0$. Then, we have seen that $R_6 \neq 0$. We claim that R_6 is not nilpotent. Indeed, since $Q_6 = 0$, the equality $(\breve{P}^{-1})' = -(\breve{P}^{-1}\breve{P}')\breve{P}^{-1}$ implies that :

$$\begin{aligned} &\frac{-R_4}{(z-z_n)^2} \text{ mod. } M_n(\mathbb{C}\{z-z_n\}) \\ = &-\left(\frac{R_6}{z-z_n} \text{ mod. } M_n(\mathbb{C}\{z-z_n\})\right)\left(\frac{R_4}{z-z_n} \text{ mod. } M_n(\mathbb{C}\{z-z_n\})\right). \end{aligned}$$

So $R_4 = R_6R_4$ that is $(R_6 - I_n)R_4 = 0$. This implies that R_6 is not nilpotent because, otherwise, $R_6 - I_n$ would belong to $GL_n(\mathbb{C})$ and hence R_4 would be 0 : contradiction.

In order to finish the proof it is clearly sufficient to prove that R_6 and Q_6 belong to $\mathfrak{sl}_n(\mathbb{C})$ and hence it is sufficient to prove that $\mathrm{tr}(\breve{P}^{-1}\breve{P}') = 0$. This is indeed the case because $\det(\breve{P})$ is a non zero constant so $0 = \det(\breve{P})' = \det(\breve{P})\mathrm{tr}(\breve{P}^{-1}\breve{P}')$. □

Corollary 3. *Assume that the hypotheses of Lemma 1 or of Lemma 2 are satisfied. Then the Lie algebra $\mathfrak{g}$ of the Galois group G of $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ contains either a rank one matrix or, in a suitable basis, $diag(1,-1,0,\cdots,0)$.*

Proof. Proposition 2 shows that $\breve{P}^{-1}\breve{P}'$ takes its values in $\mathfrak{g}$. Therefore, $\mathfrak{g}$ contains the coefficients of the Taylor expansion of $\breve{P}^{-1}\breve{P}'$ at z_n. The result is now a direct consequence of Lemma 1 and Lemma 2. □

Let us recall useful results. In what follows E denotes a finite dimensional $\mathbb{C}$-vector space and $End_{\mathbb{C}}(E)$ stands for the Lie algebra of $\mathbb{C}$-linear endomorphisms of E.

Theorem 2 (see [6] or chapter 1 of [21])**.** *Let $\mathfrak{g}$ be a semisimple Lie subalgebra of $End_{\mathbb{C}}(E)$ which acts irreducibly on E. Assume that $\mathfrak{g}$ is normalized by some pseudo-reflection in $GL(E)$. Then $\mathfrak{g}$ is either $\mathfrak{sl}(E)$ or $\mathfrak{so}(E)$ or $\mathfrak{sp}(E)$.*

Theorem 3 (see [25], [41] or chapter 1 of [21])**.** *Let $\mathfrak{g}$ be a semisimple Lie subalgebra of $End_{\mathbb{C}}(E)$ which acts irreducibly on E. Assume that, with respect to some basis, $\mathfrak{g}$ contains $diag(1,-1,0,\cdots,0)$. Then $\mathfrak{g}$ is either $\mathfrak{sl}(E)$ or $\mathfrak{so}(E)$ or $\mathfrak{sp}(E)$.*

Recall that an object M of $\mathcal{E}$ is *Lie irreducible* if the restriction to $\mathrm{Gal}(M)^0$ of the representation of $\mathrm{Gal}(M)$ corresponding to M by Tannakian duality (see section 3.1) is irreducible.

Theorem 4. *Assume that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is Lie irreducible. Let G be the Galois group of $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$. Then, the derived subgroup $[G^0, G^0]$ of the neutral component G^0 of G is either $SL_n(\mathbb{C})$ or $SO_n(\mathbb{C})$ or $Sp_n(\mathbb{C})$.*

Proof. Let us denote by $\mathfrak{g} \subset End_{\mathbb{C}}(\mathbb{C}^n)$ the Lie algebra of G. By hypothesis $\mathfrak{g}$ acts irreducibly on $\mathbb{C}^n$ so $\mathfrak{g} = Z(\mathfrak{g})+[\mathfrak{g},\mathfrak{g}]$ where $Z(\mathfrak{g})$ denotes the center of $\mathfrak{g}$ and where $[\mathfrak{g},\mathfrak{g}]$ denotes the derived Lie subalgebra of $\mathfrak{g}$. Moreover $Z(\mathfrak{g})$ acts as scalars on $\mathbb{C}^n$ and $[\mathfrak{g},\mathfrak{g}]$ is a semisimple Lie subalgebra of $End_{\mathbb{C}}(\mathbb{C}^n)$ acting irreducibly on $\mathbb{C}^n$. The Lie irreducibility condition clearly implies that either the hypotheses of Lemma 1 or of Lemma 2 are satisfied. So Corollary 3 ensures that $\mathfrak{g}$ contains either a rank one element or an element whose representative matrix is, in a suitable basis, $\mathrm{diag}(1,-1,0,...,0)$. In the first case, we get that $\mathfrak{g}$, and hence $[\mathfrak{g},\mathfrak{g}]$ is normalized by a pseudo-reflection. In the second case, $[\mathfrak{g},\mathfrak{g}]$ contains an element whose representative matrix is, in a suitable basis, $\mathrm{diag}(1,-1,0,...,0)$. The result follows from Theorem 2 and Theorem 3. □

5 Lie irreducible generalized q-hypergeometric modules

In this section, we consider $\underline{\alpha} = (\alpha_1,...,\alpha_n) \in \mathbb{Q}^n$, $\underline{\beta} = (\beta_1,...,\beta_n) \in \mathbb{Q}^n$ and $\lambda \in \mathbb{Q}^*$. We set $\underline{a} = (a_1,...,a_n) = (q^{\alpha_1},...,q^{\alpha_n}) \in (\mathbb{C}^*)^n$ and $\underline{b} = (b_1,...,b_n) = (q^{\beta_1},...,q^{\beta_n}) \in (\mathbb{C}^*)^n$.

5.1 Basic results

We recall that, for any $L \in \mathcal{D}_q\backslash\{0\}$, the dual $(\mathcal{D}_q/\mathcal{D}_qL)^*$ of the q-difference module $\mathcal{D}_q/\mathcal{D}_qL$ is rationally equivalent to $\mathcal{D}_q/\mathcal{D}_qL^*$ where $.^* : \mathcal{D}_q \to \mathcal{D}_q^{\mathrm{op}}$ denotes the involutive morphism of $\mathcal{D}_q$-modules defined by $(z^i\boldsymbol{\sigma}_q^j)^* = \boldsymbol{\sigma}_q^{-j}z^i$. A slightly modified version of this result is Proposition 2.1.10 of [37]; the link with our assertion is easy and left to the reader.

Proposition 5. *The dual $\mathcal{H}_q(\underline{a};\underline{b};\lambda)^*$ of $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is rationally equivalent to $\mathcal{H}_q(q/\underline{a};q^2/\underline{b};\lambda\frac{\prod_{i=1}^n a_i}{\prod_{j=1}^n b_j})$.*

Proof. The result is a consequence of the following computation :

$$\begin{aligned}
&\left(\prod_{j=1}^n(\frac{b_j}{q}\boldsymbol{\sigma}_q-1)-\lambda z\prod_{i=1}^n(a_i\boldsymbol{\sigma}_q-1)\right)^* \\
&= \prod_{j=1}^n(\boldsymbol{\sigma}_q^{-1}\frac{b_j}{q}-1)-\prod_{i=1}^n(\boldsymbol{\sigma}_q^{-1}a_i-1)\lambda z \\
&= (-1)^n\boldsymbol{\sigma}_q^{-n}\frac{\prod_{j=1}^n b_j}{q^n}\left(\prod_{j=1}^n(\frac{q}{b_j}\boldsymbol{\sigma}_q-1)-\lambda\frac{\prod_{i=1}^n a_i}{\prod_{j=1}^n b_j}z\prod_{i=1}^n(\frac{q}{a_i}\boldsymbol{\sigma}_q-1)\right).
\end{aligned}$$

□

Proposition 6. *Assume that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible. If $\underline{a}' = (a_1',...,a_n') \in (\mathbb{C}^*)^n$, $\underline{b}' = (b_1',...,b_n') \in (\mathbb{C}^*)^n$ and $\lambda' \in \mathbb{C}^*$ are such that :*

- *there exist two permutations μ, ν of $[\![1,n]\!]$ such that, for all $i \in [\![1,n]\!]$, $a'_i = a_{\mu(i)}$ mod. $q^{\mathbb{Z}}$ and $b'_i = b_{\nu(i)}$ mod. $q^{\mathbb{Z}}$;*
- *$\lambda' = \lambda$ mod. $q^{\mathbb{Z}}$;*

then $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ and $\mathcal{H}_q(\underline{a}';\underline{b}';\lambda')$ are rationally equivalent.

Proof. Let us consider $j \in [\![1,n]\!]$ and set $\underline{b}^j = (b_1, ..., b_{j-1}, qb_j, b_{j+1}, ..., b_n)$. Since $\mathcal{L}_q(\underline{a},\underline{b},\lambda)(b_j\boldsymbol{\sigma}_q - 1) = (\frac{b_j}{q}\boldsymbol{\sigma}_q - 1)\mathcal{L}_q(\underline{a},\underline{b}^j,\lambda)$, the map $\varphi : \mathcal{D}_q \to \mathcal{D}_q, L \mapsto L(b_j\boldsymbol{\sigma}_q - 1)$ induces a morphism of q-difference modules $\overline{\varphi} : \mathcal{H}_q(\underline{a};\underline{b};\lambda) \to \mathcal{H}_q(\underline{a};\underline{b}^j;\lambda)$. This morphism is non zero and hence injective because $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible. Since $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ and $\mathcal{H}_q(\underline{a};\underline{b}^j;\lambda)$ have the same dimension on $\mathbb{C}(z)$, $\overline{\varphi}$ is an isomorphism : $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ and $\mathcal{H}_q(\underline{a};\underline{b}^j;\lambda)$ are rationally equivalent. Similarly, for any $j \in [\![1,n]\!]$, $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ and $\mathcal{H}_q(\underline{a};\underline{b}^{-j};\lambda)$ are rationally equivalent where $\underline{b}^{-j} = (b_1, ..., b_{j-1}, q^{-1}b_j, b_{j+1}, ..., b_n)$. It is now clear that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ and $\mathcal{H}_q(\underline{a}';\underline{b}';\lambda)$ are rationally equivalent. Moreover, it is easily seen that, for all $k \in \mathbb{Z}$, the map $\psi_k : \mathcal{D}_q \to \mathcal{D}_q$, $L \mapsto L\boldsymbol{\sigma}_q^{-k}$ induces an isomorphism of q-difference modules $\overline{\psi_k} : \mathcal{H}_q(\underline{a}';\underline{b}';\lambda) \to \mathcal{H}_q(\underline{a}';\underline{b}';q^k\lambda)$ and hence $\mathcal{H}_q(\underline{a}';\underline{b}';\lambda)$ and $\mathcal{H}_q(\underline{a}';\underline{b}';q^k\lambda)$ are rationally equivalent. So $\mathcal{H}_q(\underline{a}';\underline{b}';\lambda)$ and $\mathcal{H}_q(\underline{a}';\underline{b}';\lambda')$ are rationally equivalent. □

Proposition 7. *If $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible then, for all $(i,j) \in [\![1,n]\!]^2$, $a_i \neq b_j$ mod. $q^{\mathbb{Z}}$.*

Proof. Assume at the contrary that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible and that there exists $(i,j) \in [\![1,n]\!]^2$ such that $a_i = b_j$ mod. $q^{\mathbb{Z}}$. Then Proposition 6 implies that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ and $\mathcal{H}_q(\underline{a}';\underline{b};\lambda)$ are rationally equivalent where $a'_i = b_j$ and, for all $k \in [\![1,n]\!] \setminus \{i\}$, $a'_k = a_k$. But $(\frac{b_j}{q}\boldsymbol{\sigma}_q - 1)$ is a left factor of $\mathcal{L}_q(\underline{a}',\underline{b},\lambda)$ so $\mathcal{H}_q(\underline{a}';\underline{b};\lambda)$ is reducible : contradiction. □

The converse also holds.

Proposition 8. *If, for all $(i,j) \in [\![1,n]\!]^2$, $a_i \neq b_j$ mod. $q^{\mathbb{Z}}$ then $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible.*

Proof. Let us consider $N = \mathbb{C}[z,z^{-1}]\langle\boldsymbol{\sigma}_q,\boldsymbol{\sigma}_q^{-1}\rangle/\mathbb{C}[z,z^{-1}]\langle\boldsymbol{\sigma}_q,\boldsymbol{\sigma}_q^{-1}\rangle\mathcal{L}_q(\underline{a};\underline{b};\lambda)$ and let $\pi : N \to \mathcal{H}_q(\underline{a};\underline{b};\lambda)$ be the natural morphism of $\mathbb{C}[z,z^{-1}]\langle\boldsymbol{\sigma}_q,\boldsymbol{\sigma}_q^{-1}\rangle$-modules. Let M be a subobject of $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$. It is proved in [34] that N is an irreducible left $\mathbb{C}[z,z^{-1}]\langle\boldsymbol{\sigma}_q,\boldsymbol{\sigma}_q^{-1}\rangle$-module. So the sub-$\mathbb{C}[z,z^{-1}]\langle\boldsymbol{\sigma}_q,\boldsymbol{\sigma}_q^{-1}\rangle$-module $\pi^{-1}(M)$ of N is either $\{0\}$ or N. Therefore $M = \mathbb{C}(z)\pi(\pi^{-1}(M))$ is either $\{0\}$ or $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$. □

The following result will be used twice in this paper.

Lemma 3. *Let M and M' be rationally equivalent objects of $\mathcal{E}$. Let A be the inverse of the matrix representing the action of $\boldsymbol{\sigma}_q$ on M with respect to some basis and let $A^{(0)} \in GL_n(\mathbb{C})$ and $F^{(0)} \in GL_n(\mathbb{C}(\{z\}))$ be such that $(\sigma_q F^{(0)})A^{(0)} = AF^{(0)}$. We consider similar objects A', $A'^{(0)}$ and $F'^{(0)}$ for M'. Then there exists $(R,C) \in GL_n(\mathbb{C}(z)) \times GL_n(\mathbb{C}(z))$ such that $F'^{(0)} = R(F^{(0)})C$.*

Proof. This lemma follows from Théorème 1.2 in [35]. □

The following result is the converse of Proposition 6.

Proposition 9. *Assume that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible. If $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is rationally equivalent to $\mathcal{H}_q(\underline{a}';\underline{b}';\lambda')$ for some $\underline{a}' = (a'_1, ..., a'_n) \in (\mathbb{C}^*)^n$, $\underline{b}' = (b'_1, ..., b'_n) \in (\mathbb{C}^*)^n$ and $\lambda' \in \mathbb{C}^*$ then :*

– *there exist two permutations* μ, ν *of* $[\![1, n]\!]$ *such that, for all* $i \in [\![1, n]\!]$, $a'_i = a_{\mu(i)}$ *mod.* $q^{\mathbb{Z}}$ *and* $b'_i = b_{\nu(i)}$ *mod.* $q^{\mathbb{Z}}$*;*
– $\lambda' = \lambda$ *mod.* $q^{\mathbb{Z}}$*.*

Proof. The first assertion of the proposition is immediate by considering the exponents of $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ and $\mathcal{H}_q(\underline{a}'; \underline{b}'; \lambda')$.

It remains to prove that $\lambda = \lambda'$ mod. $q^{\mathbb{Z}}$. Let A be the inverse of the matrix representing the action of $\boldsymbol{\sigma}_q$ on $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ with respect to the basis $\overline{1}, \overline{\boldsymbol{\sigma}_q}, ..., \overline{\boldsymbol{\sigma}_q^{n-1}}$; see formula (7). We also introduce the similar matrix A' for $\mathcal{H}_q(\underline{a}'; \underline{b}'; \lambda')$. Section 2 ensures that there exist $A^{(0)} \in \mathrm{GL}_n(\mathbb{C})$ and $F^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ such that :

$$(\sigma_q F^{(0)}) A^{(0)} = A F^{(0)}. \tag{16}$$

We first note that this functional equation ensures that the set of poles of $F^{(0)}$ on $\mathbb{C}^*$ is included in $q^{\mathbb{Z}_{\leq N}} z_0$ for some $N \in \mathbb{Z}$ where $z_0 = (\prod_{j=1}^n \frac{b_j}{q})(\lambda \prod_{i=1}^n a_i)^{-1}$. We claim that $F^{(0)}$ has infinitely many poles. Indeed, otherwise $F^{(0)}$ would be an element of $\mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ with at most polynomial growth at ∞ (this is a direct consequence of (16)) and hence would belong to $\mathrm{GL}_n(\mathbb{C}(z))$. In terms of modules, this entails that $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ would be rationally equivalent to the reducible q-difference module $\mathbb{C}(z) \otimes_{\mathbb{C}} \mathbb{C}^n$, the action of $\boldsymbol{\sigma}_q$ being given by $\sigma_q \otimes (A^{(0)})^{-1}$: contradiction.

Similarly, there exist $A'^{(0)} \in \mathrm{GL}_n(\mathbb{C})$ and $F'^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ such that :

$$(\sigma_q F'^{(0)}) A'^{(0)} = A' F'^{(0)}. \tag{17}$$

Arguing as above, we see that the set of poles in $\mathbb{C}^*$ of $F'^{(0)}$ is infinite and included in $q^{\mathbb{Z}_{\leq N'}} z'_0$ for some $N' \in \mathbb{Z}$ where $z'_0 = (\prod_{j=1}^n \frac{b'_j}{q})(\lambda' \prod_{i=1}^n a'_i)^{-1}$.

Since $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ and $\mathcal{H}_q(\underline{a}'; \underline{b}'; \lambda')$ are rationally equivalent, Lemma 3 ensures that there exists $(R, C) \in \mathrm{GL}_n(\mathbb{C}(z)) \times \mathrm{GL}_n(\mathbb{C}(z))$ such that $F'^{(0)} = R F^{(0)} C$. Considering the poles in $\mathbb{C}^*$ of the left hand term and of the right hand term of this equation, we get that $q^{\mathbb{Z}_{\leq N}} z_0 \cap q^{\mathbb{Z}_{\leq N'}} z'_0$ is infinite and hence $\lambda = \lambda'$ mod. $q^{\mathbb{Z}}$. □

Recall that a q-difference module is *selfdual* if it is isomorphic to its dual.

Corollary 4. *Suppose that* $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ *is irreducible. Then* $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ *is selfdual if and only if the following conditions hold :*

(i) $\frac{\prod_{i=1}^n a_i}{\prod_{j=1}^n b_j} \in q^{\mathbb{Z}}$*;*

(ii) there exist two permutations μ, ν *of* $[\![1, n]\!]$ *such that, for all* $i \in [\![1, n]\!]$ *:*

- $a_i a_{\mu(i)} \in q^{\mathbb{Z}}$*;*
- $b_i b_{\nu(i)} \in q^{\mathbb{Z}}$*.*

Moreover, in case that $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ *is selfdual, the corresponding selfduality is symmetric (resp. alternating) if and only if* n *is an even (resp. odd) number.*

Proof. The first part of the corollary is a consequence of Proposition 5, Proposition 6 and Proposition 9. Suppose that $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ is selfdual. Since $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ is irreducible, it has exactly one selfduality $\langle \cdot, \cdot \rangle_q$ up to a non zero scalar which is either symmetric or alternating.

Considering q as a parameter, we see that the fact that the selfduality is symmetric or alternating does not depend on $q \in \mathbb{C}^*$ and that the result is a consequence of the Duality Recognition Theorem 3.4 of [21]. $\square$

5.2 Lie irreducibility

Recall that an object M of $\mathcal{E}$ is *Lie irreducible* if the restriction to $\mathrm{Gal}(M)^0$ of the representation of $\mathrm{Gal}(M)$ corresponding to M by Tannakian duality (see section 3.1) is irreducible. Of course, Lie irreducibility implies irreducibility.

Definition 1. *The generalized q-hypergeometric module $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is q-Kummer induced if the following conditions hold :*

- *$\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible;*
- *there exists a divisor $d \in \mathbb{N} \setminus \{0,1\}$ of n and there exist two permutations μ, ν of $[\![1,n]\!]$ such that, for all $i \in [\![1,n]\!]$, $a_i = q^{1/d}a_{\mu(i)}$ mod. $q^{\mathbb{Z}}$ and $b_i = q^{1/d}b_{\nu(i)}$ mod. $q^{\mathbb{Z}}$.*

The second condition in the previous definition can be paraphrased as follows : there exists a divisor $d \in \mathbb{N} \setminus \{0,1\}$ of n and there exist $\underline{A} = (A_1,...,A_{n/d}) \in (\mathbb{C}^*)^{n/d}$ and $\underline{B} = (B_1,...,B_{n/d}) \in (\mathbb{C}^*)^{n/d}$ such that, up to order and mod. $q^{\mathbb{Z}}$, the list $a_1,...,a_n$ coincides with the list $A_1, A_1q^{-1/d},...,A_1q^{-(d-1)/d},...,A_{n/d},A_{n/d}q^{-1/d},...,A_{n/d}q^{-(d-1)/d}$ and the list $b_1,...,b_n$ coincides with the list $B_1,B_1q^{1/d},...,B_1q^{(d-1)/d},...,B_{n/d},B_{n/d}q^{1/d},...,B_{n/d}q^{(d-1)/d}$.

As suggested by one of the referees, the q-Kummer induced generalized q-hypergeometric modules are induced by $q^{1/d}$-difference modules of smaller dimensions. More precisely, denoting by $[d] : \mathbb{C}^* \to \mathbb{C}^*$ the étale morphism $z \mapsto z^d$ and denoting by μ some dth complex root of λ, the q-difference module $[d]_*\mathcal{H}_{q^{1/d}}(\underline{A};\underline{B};\mu)$ is isomorphic to $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$. The proof of this statement is similar to the proof of Lemma 3.5.6 in [21] and is left to the reader. This result will not be used in what follows.

Theorem 5. *If $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible and Lie reducible then it is q-Kummer induced.*

Proof. Using our preceding results, the proof is similar to that of Theorem 5.3. of [6]. We denote by G the Galois group of $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ and we use the notations of section 3.2. Let V be a minimal non trivial invariant subspace of $\mathbb{C}^n$ for the action of G^0. For all $k \in \mathbb{Z}$, G^0 is normalized by $(D_2^{(0)})^k$ (because G^0 is normalized by any element of G) so $(D_2^{(0)})^kV$ is an invariant subspace of $\mathbb{C}^n$ for the action of G^0. But, since the abstract group H generated by G^0 and $D_2^{(0)}$ is Zariski-dense in G (see Corollary 2), H acts irreducibly on $\mathbb{C}^n$. Therefore $\mathbb{C}^n = \Sigma_{k\in\mathbb{Z}}(D_2^{(0)})^kV$. Since V is minimal, for all $(k,l) \in \mathbb{Z}^2$, we have either $(D_2^{(0)})^kV = (D_2^{(0)})^lV$ or $(D_2^{(0)})^kV \cap (D_2^{(0)})^lV = \{0\}$. So $\mathbb{C}^n = \bigoplus_{k=0}^{d-1}(D_2^{(0)})^kV$ for some $d \in [\![2,\infty]\!]$. This implies that $D_2^{(0)}$ and $e^{\frac{2\pi i}{d}}D_2^{(0)}$ are conjugate. Considering the eigenvalues of $D_2^{(0)}$, we see that there exists a permutation ν of $[\![1,n]\!]$ such that, for all $j \in [\![1,n]\!]$, $e^{2\pi i\beta_j} = e^{\frac{2\pi i}{d}}e^{2\pi i\beta_{\nu(j)}}$ i.e. $b_j = q^{\frac{1}{d}}b_{\nu(j)}$ mod. $q^{\mathbb{Z}}$. Note that $n = d\dim_{\mathbb{C}}V$ so d divides n and depends only on n and $\dim_{\mathbb{C}}V$. A similar argument proves an analogous statement for $\underline{a}$. $\square$

5.3 Galois groups

Theorem 6. *Assume that $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is irreducible and non q-Kummer induced and denote by G its Galois group. Then, the derived subgroup $[G^0, G^0]$ of the neutral component G^0 of G is either $SL_n(\mathbb{C})$ or $SO_n(\mathbb{C})$ or $Sp_n(\mathbb{C})$. Moreover, $[G^0, G^0] = SO_n(\mathbb{C})$ (resp. $Sp_n(\mathbb{C})$) if and only if the following three conditions hold :*

(i) $\frac{\prod_{i=1}^n a_i}{\prod_{j=1}^n b_j} \in q^{\mathbb{Z}}$;

(ii) there exists $c \in \mathbb{C}^*$, there exist two permutations μ, ν of $[\![1, n]\!]$ such that :

- *for all $i \in [\![1, n]\!]$, $ca_i a_{\mu(i)} \in q^{\mathbb{Z}}$;*
- *for all $j \in [\![1, n]\!]$, $cb_j b_{\nu(j)} \in q^{\mathbb{Z}}$;*

(iii) n is odd (resp. even).

Furthermore :

- $G = G^0 = \mathbb{C}^*[G^0, G^0]$ *if* $\frac{\prod_{i=1}^n a_i}{\prod_{j=1}^n b_j} \notin q^{\mathbb{Z}}$;
- $G^0 = [G^0, G^0]$ *if* $\frac{\prod_{i=1}^n a_i}{\prod_{j=1}^n b_j} \in q^{\mathbb{Z}}$.

Proof. It is instructive to compare the following proof with that of a similar statement in the differential case in [21]; note in particular the role of poles in the q-difference case.

Theorem 5 implies that $M = \mathcal{H}_q(\underline{a};\underline{b};\lambda)$ is Lie irreducible. Theorem 4 ensures that $[\mathrm{G}^0, \mathrm{G}^0]$ is either $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SO}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$.

We first assume that $[\mathrm{G}^0, \mathrm{G}^0]$ is $\mathrm{SO}_n(\mathbb{C})$. We have to prove that n is odd and that conditions (i) to (iii) hold. It is equivalent, in virtue of Corollary 4, to prove that there exists $c \in \mathbb{C}^*$ (not the same c than in condition (ii)) such that $\mathcal{H}_q(c\underline{a};c\underline{b};\lambda)$, which is rationally equivalent to $M \otimes \mathcal{D}_q/\mathcal{D}_q(c\boldsymbol{\sigma}_q - 1)$, is symmetrically selfdual.

Since G is a subgroup of the normalizer $\mathbb{C}^*\mathrm{SO}_n(\mathbb{C})$ of $[\mathrm{G}^0, \mathrm{G}^0]$ in $\mathrm{GL}_n(\mathbb{C})$, there exists N a rank one object of $\mathcal{E}$ such that M^* is rationally equivalent to $M \otimes N$ (use Tannakian duality together with the fact that if ρ is a linear representation with values in $\mathbb{C}^*\mathrm{SO}_n(\mathbb{C})$ then its dual ρ^* is isomorphic to $\rho \otimes (\chi^{-1} \circ \rho)$ where χ is the character of $\mathbb{C}^*\mathrm{SO}_n(\mathbb{C})$ defined, for all $(t, X) \in \mathbb{C}^* \times \mathrm{SO}_n(\mathbb{C})$, by $\chi(tX) = t^2$).

Let $A \in \mathrm{GL}_n(\mathbb{C}(z))$ be the inverse of the matrix representing the action of $\boldsymbol{\sigma}_q$ on $\mathcal{H}_q(\underline{a};\underline{b};\lambda)$ with respect to the basis $\overline{1}, \overline{\boldsymbol{\sigma}_q}, ..., \overline{\boldsymbol{\sigma}_q^{n-1}}$. Section 2 ensures that there exist $F^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ and $A^{(0)} \in \mathrm{GL}_n(\mathbb{C})$ such that $(\sigma_q F^{(0)})A^{(0)} = AF^{(0)}$ so $(\sigma_q(F^{(0)})^{-t})(A^{(0)})^{-t} = A^{-t}(F^{(0)})^{-t}$.

Moreover, let $B \in \mathrm{GL}_1(\mathbb{C}(z))$ be the inverse of the matrix representing the action of $\boldsymbol{\sigma}_q$ on N with respect to some basis. Section 2 ensures that there exist $G^{(0)} \in \mathrm{GL}_1(\mathcal{M}(\mathbb{C}))$ and $B^{(0)} \in \mathrm{GL}_1(\mathbb{C})$ such that $(\sigma_q G^{(0)})B^{(0)} = BG^{(0)}$ so $(\sigma_q(F^{(0)}G^{(0)}))(A^{(0)}B^{(0)}) = (AB)(F^{(0)}G^{(0)})$.

Note that AB and A^{-t} are the inverses of the matrices representing, with respect to suitable basis, the action of $\boldsymbol{\sigma}_q$ on $M \otimes N$ and on M^* respectively. But $M \otimes N$ and M^* are rationally equivalent. So, in virtue of Lemma 3, there exists $(R, C) \in \mathrm{GL}_n(\mathbb{C}(z)) \times \mathrm{GL}_n(\mathbb{C}(z))$ such that $F^{(0)}G^{(0)} = R(F^{(0)})^{-t}C$. Therefore, $(G^{(0)})^n \det(F^{(0)})^2 \in \mathbb{C}(z)^\times$.

We claim that $\det(F^{(0)})$ has finitely many poles on $\mathbb{C}$. Indeed, otherwise, since $(G^{(0)})^n \det(F^{(0)})^2 \in \mathbb{C}(z)^\times$, $G^{(0)}$ would have infinitely many zeros on $\mathbb{C}$ and hence $(G^{(0)})^n$ would have infinitely many zeros of order at least n on $\mathbb{C}$. But the functional equation

$$(\sigma_q \det(F^{(0)}))\det(A^{(0)}) = \det(A)\det(F^{(0)}) = \frac{\prod_{j=1}^{n}\frac{b_j}{q} - z\lambda\prod_{i=1}^{n}a_i}{(-1)^n(1-\lambda z)}\det(F^{(0)}). \quad (18)$$

shows that $\det(F^{(0)})$ has at most simple poles on $\mathbb{C}^*$. So $(G^{(0)})^n \det(F^{(0)})^2$ would have infinitely many zeros on $\mathbb{C}$ and hence would not belong to $\mathbb{C}(z)^{\times}$: contradiction. Now we see by using the functional equation (18) that $\det(F^{(0)})$ has at most polynomial growth at ∞ so $\det(F^{(0)})$ and hence $G^{(0)}$ belong to $\mathbb{C}(z)^{\times}$. Consequently, N is rationally equivalent to $\mathcal{D}_q/\mathcal{D}_q(B^{(0)}\boldsymbol{\sigma}_q - 1)$. Choosing a square root c of $B^{(0)}$, we see that $M \otimes \mathcal{D}_q/\mathcal{D}_q(c\boldsymbol{\sigma}_q - 1)$ is selfdual. Since the derived subgroups of the neutral components of the Galois groups of M and of $M \otimes \mathcal{D}_q/\mathcal{D}_q(c\boldsymbol{\sigma}_q - 1)$ coincide, we get that the selfduality pairing is symmetric.

Conversely, suppose that n is odd and that conditions (i) to (iii) hold or, equivalently, in virtue of Corollary 4, that there exists $c \in \mathbb{C}^*$ such that $M \otimes \mathcal{D}_q/\mathcal{D}_q(c\boldsymbol{\sigma}_q - 1)$ is symmetrically selfdual. Then the Galois group G of M is a subgroup of the normalizer $\mathbb{C}^*\mathrm{SO}_n(\mathbb{C})$ of $\mathrm{SO}_n(\mathbb{C})$ in $\mathrm{GL}_n(\mathbb{C})$. Since $[\mathrm{G}^0, \mathrm{G}^0]$ is either $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$ or $\mathrm{SO}_n(\mathbb{C})$, we get that $[\mathrm{G}^0, \mathrm{G}^0]$ is $\mathrm{SO}_n(\mathbb{C})$ as expected.

The case that $[\mathrm{G}^0, \mathrm{G}^0]$ is $\mathrm{Sp}_n(\mathbb{C})$ is similar.

Note that, in any case, $[\mathrm{G}^0, \mathrm{G}^0] \subset \mathrm{G} \subset \mathbb{C}^*[\mathrm{G}^0, \mathrm{G}^0]$ so the last part of the theorem is a direct consequence of the fact that $\det M$ has finite order if and only if $\frac{\prod_{i=1}^{n} a_i}{\prod_{j=1}^{n} b_j} \in q^{\mathbb{Z}}$. □

Let us finish this section with a remark on the case $n = 3$.

Corollary 5. *With the hypotheses and notations of Theorem 6, if $n = 3$ then $[G^0, G^0] = SL_3(\mathbb{C})$.*

Proof. Assume at the contrary that $[\mathrm{G}^0, \mathrm{G}^0]$ is not $\mathrm{SL}_3(\mathbb{C})$. Theorem 6 ensures that $\frac{a_1a_2a_3}{b_1b_2b_3} \in q^{\mathbb{Z}}$ and that there exist $c \in \mathbb{C}^*$ and two permutations μ, ν of $\{1,2,3\}$ such that, for all $i \in \{1,2,3\}$, $ca_ia_{\mu(i)} \in q^{\mathbb{Z}}$ and $cb_ib_{\nu(i)} \in q^{\mathbb{Z}}$. The strategy of the proof is to prove that there exists $(i,j) \in \{1,2,3\}^2$ such that $a_i = b_j \mod. q^{\mathbb{Z}}$: this would provide a contradiction because in this case $\mathcal{H}_q(\underline{a}; \underline{b}; \lambda)$ is reducible (Proposition 7).

Since a_1, a_2 and a_3 belong to $q^{\mathbb{Q}}$, c also belongs to $q^{\mathbb{Q}}$. Let $\delta \in \mathbb{Q}$ be such that $c = q^{\delta}$. We set, for all $i \in \{1,2,3\}$, $\alpha'_i = \alpha_i + \frac{\delta}{2}$ and $\beta'_i = \beta_i + \frac{\delta}{2}$. We have :

(a) $\alpha'_1 + \alpha'_2 + \alpha'_3 - (\beta'_1 + \beta'_2 + \beta'_3) \in \mathbb{Z}$;

(b) $\forall i \in \{1,2,3\}$, $\alpha'_i + \alpha'_{\mu(i)} \in \mathbb{Z}$;

(c) $\forall i \in \{1,2,3\}$, $\beta'_i + \beta'_{\nu(i)} \in \mathbb{Z}$.

We claim that there exists $(i,j) \in \{1,2,3\}^2$ such that $\alpha'_i - \beta'_j$ belongs to $\mathbb{Z}$ (so $a_i = b_j \mod. q^{\mathbb{Z}}$). Up to renumbering, it is sufficient to consider the cases that μ and ν are one of the following permutations of $\{1,2,3\}$: Id, $(1,2)$, $(1,2,3)$. Since the roles of μ and ν are symmetric, our claim follows from the following discussion.

If $\mu = \mathrm{Id}$ and $\nu = \mathrm{Id}$ then we deduce from (b) and (c) that, for all $i \in \{1,2,3\}$, $\alpha'_i \in \mathbb{Z}/2$ and $\beta'_i \in \mathbb{Z}/2$. Condition (a) ensures that there exists $(i,j) \in \{1,2,3\}^2$ such that $\alpha'_i - \beta'_j \in \mathbb{Z}$.

If $\mu = (1,2)$ and $\nu = \mathrm{Id}$ then condition (b) implies that $\alpha'_1 + \alpha'_2 \in \mathbb{Z}$ and $\alpha'_3 \in \mathbb{Z}/2$, and condition (c) implies that, for all $i \in \{1,2,3\}$, $\beta'_i \in \mathbb{Z}/2$. Condition (a) ensures that there exists $j \in \{1,2,3\}$ such that $\alpha'_3 - \beta'_j \in \mathbb{Z}$.

If $\mu = (1,2,3)$ and $\nu = \mathrm{Id}$ then condition (b) implies that $\alpha'_1 + \alpha'_2 \in \mathbb{Z}$, $\alpha'_2 + \alpha'_3 \in \mathbb{Z}$ and $\alpha'_1 + \alpha'_3 \in \mathbb{Z}$, and condition (c) implies that, for all $i \in \{1,2,3\}$, $\beta'_i \in \mathbb{Z}/2$. Remark that, for all $(i,j) \in \{1,2,3\}^2$, $\alpha'_i - \alpha'_j \in \mathbb{Z}$ (because if $i \neq j$ and if $\{k\} = \{1,2,3\} \setminus \{i,j\}$ then $\alpha'_i - \alpha'_j = \alpha'_i + \alpha'_k - (\alpha'_j + \alpha'_k)$). So, for all $i \in \{1,2,3\}$, $\alpha'_i \in \mathbb{Z}/2$. We conclude as in the case $\mu = \mathrm{Id}$, $\nu = \mathrm{Id}$ treated above.

If $\mu = (1,2)$ and $\nu = (1,2)$ then condition (b) implies that $\alpha'_1 + \alpha'_2 \in \mathbb{Z}$, and condition (c) implies that $\beta'_1 + \beta'_2 \in \mathbb{Z}$. Condition (a) ensures that $\alpha'_3 - \beta'_3 \in \mathbb{Z}$.

The case $\mu = (1,2,3)$ and $\nu = (1,2)$ and the case $\mu = (1,2,3)$ and $\nu = (1,2,3)$ are similar to the case $\mu = (1,2)$ and $\nu = (1,2)$. □

The natural analogue of this Corollary for differential equations is false.

Note that calculations of q-difference Galois groups are interesting for hypertranscendancy; see C. Hardouin and M. Singer's paper [20].

6 Generalized q-hypergeometric modules with finite Galois groups

We fist state a general result regarding q-difference modules over $\mathbb{C}(z)$ with finite Galois groups.

Lemma 4. *Let M be an object of $\mathcal{E}$. Let $\breve{P}$ be a corresponding twisted Birkhoff matrix (see section 3.2). If M is non logarithmic both at 0 and at ∞ and if $\breve{P}$ is constant then there exists a semisimple $\mathbb{C}$-linear automorphism Φ of some $\mathbb{C}$-vector space V such that M is rationally equivalent to $\mathbb{C}(z) \otimes_{\mathbb{C}} V$, the action of $\boldsymbol{\sigma}_q$ being given by $\sigma_q \otimes \Phi$.*

Proof. We maintain the notations introduced in sections 2 and 3.2. Since M is non logarithmic both at 0 and at ∞, we have $\breve{P} = \breve{P}(z) = \left(F^{(\infty)}(z) g_z\left(D^{(\infty)}\right)\right)^{-1} F^{(0)}(z) g_z\left(D^{(0)}\right)$. Hence $F^{(0)}(z) = F^{(\infty)}(z) g_z\left(D^{(\infty)}\right) \breve{P} g_z\left(D^{(0)}\right)^{-1}$. This equality shows that $F^{(0)}$ is meromorphic at ∞ an hence on $\mathbb{P}^1_{\mathbb{C}}$. So $F^{(0)} \in \mathrm{GL}_n(\mathbb{C}(z))$. This entails that M is rationally equivalent to the the q-difference module $\mathbb{C}(z) \otimes_{\mathbb{C}} \mathbb{C}^n$, the action of $\boldsymbol{\sigma}_q$ being given by $\sigma_q \otimes (A^{(0)})^{-1} = \sigma_q \otimes (D^{(0)})^{-1}$. □

In what follows, for all $m \in \mathbb{N}^*$, $\mathbb{U}_m$ denotes the group of mth roots of the unity in $\mathbb{C}$ and $\mathbb{U}_\infty = \cup_{m \in \mathbb{N}^*} \mathbb{U}_m$ denotes the group of all roots of the unity in $\mathbb{C}$.

Proposition 10. *Let M be an object of $\mathcal{F}$. The following properties are equivalent :*

(i) M has a finite Galois group over $\mathbb{C}(z)$;

(ii) there exists a semisimple $\mathbb{C}$-linear automorphism Φ of some $\mathbb{C}$-vector space V with eigenvalues in $\mathbb{U}_\infty q^{\mathbb{Q}}$ such that M is rationally equivalent to $\mathbb{C}(z) \otimes_{\mathbb{C}} V$, the action of $\boldsymbol{\sigma}_q$ being given by $\sigma_q \otimes \Phi$.

Proof. The implication (ii) $\Rightarrow$ (i) is an obvious consequence of Theorem 1 stated in section 3.2. Alternatively, (ii) $\Rightarrow$ (i) is immediate by Tannakian duality because if (ii) is satisfied then M is the direct sum of n order 1 q-difference modules over $\mathbb{C}(z)$ whose mth $\otimes$-power are trivial for some $m \in \mathbb{N}^*$. Let us prove (i) $\Rightarrow$ (ii). So, we assume that M has a finite Galois group. We first remark that M is necessary regular singular both at 0 and at ∞ because, as it has been proved by M. van der Put and M. Reversat in [28], the Galois group of any irregular q-difference module contains an infinite "theta torus". We now use the notations and the terminologies introduced in section 3.2. Since the Galois group of M is finite, its neutral component and hence its connection component are trivial. So $\check{P}$ is constant. Moreover, M is non logarithmic both at 0 and at ∞ because any non trivial unipotent element of the Galois group of M has infinite order. The result follows from Lemma 4 and Theorem 1 (or of a simple $\otimes$-argument). □

The following corollary is immediate.

Corollary 6. *Let M be an object of $\mathcal{F}$. The following properties are equivalent :*

(i) M has a trivial Galois group over $\mathbb{C}(z)$;

(ii) M has a finite Galois group over $\mathbb{C}(z)$ and its exponents both at 0 and at ∞ belong to $q^{\mathbb{Z}}$.

In terms of operators, the above results have the following consequence :

Proposition 11. *Let $L \in \mathcal{D}_q \setminus \{0\}$ be a non zero regular singular q-difference operator of order n. The exponents of L at 0 (counted with multiplicity) are denoted by $c_{1,1}, ..., c_{1,n_1}, ..., c_{r,1}, ..., c_{r,n_r}$ in such a way that $c_{\mu,\nu} = c_{\mu',\nu'}$ mod. $q^{\mathbb{Z}}$ if and only if $\mu = \mu'$. Then L has a finite Galois group over $\mathbb{C}(z)$ if and only if the following conditions hold :*

1) $\forall \mu \in [\![1, r]\!]$, $\forall \nu \in [\![1, n_\mu]\!]$, $c_{\mu,\nu} \in \mathbb{U}_\infty q^{\mathbb{Q}}$;

2) the exponents of L at ∞ (counted with multiplicity) can be ordered as $d_{1,1}, ..., d_{1,n_1}, ..., d_{r,1}, ..., d_{r,n_r}$ in such a way that, $\forall \mu \in [\![1, r]\!]$, $\forall \nu \in [\![1, n_\mu]\!]$, $d_{\mu,\nu} = c_{\mu,\nu}$ mod. $q^{\mathbb{Z}}$;

3) $\forall \mu \in [\![1, r]\!]$, $\dim_{\mathbb{C}} Ker(L : \mathbb{C}(z)e^{(0)}_{c_{\mu,1}} \to \mathbb{C}(z)e^{(0)}_{c_{\mu,1}}) = n_\mu$.

Moreover, L has a trivial Galois group over $\mathbb{C}(z)$ if and only if L has a finite Galois group over $\mathbb{C}(z)$ and if its exponents both at 0 and at ∞ belong to $q^{\mathbb{Z}}$.

Of course, condition 2) is redundant but is included for later reference.

6.1 Finite Galois groups

The following lemma, due to one of the referees, simplifies greatly our original approach.

Lemma 5. *If $(\prod_{j=1}^n b_j)(\prod_{i=1}^n a_i)^{-1} \notin q^{-\mathbb{N}}$ then $Ker(\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) : \mathbb{C}(z) \to \mathbb{C}(z)) = Ker(\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) : \mathbb{C}[z, z^{-1}] \to \mathbb{C}(z))$.*

Proof. We maintain the notation $f_0, f_1, ..., f_n$ for the coefficients of $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$ introduced at the beginning of section 4. In order to prove the lemma it is equivalent to prove that if there exists $y \in \mathrm{Ker}(\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) : \mathbb{C}(z) \to \mathbb{C}(z))$ such that $y \notin \mathbb{C}[z, z^{-1}]$ then $(\prod_{j=1}^n b_j)(\prod_{i=1}^n a_i)^{-1} \in q^{-\mathbb{N}}$.

Assume that such a y exists and let $w \in \mathbb{C}^*$ be a pole of y. We introduce $M = \max\{k \in \mathbb{Z} \mid q^k w \text{ is a pole of } y\}$ and $w_M = q^M w$. Since $y \in \mathrm{Ker}(\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) : \mathbb{C}(z) \to \mathbb{C}(z))$, we have :

$$f_0(\sigma_q^n y) + f_1(\sigma_q^{n-1} y) + \cdots + f_{n-1}(\sigma_q y) = -f_n y. \tag{19}$$

The definition of w_M shows that w_M is not a pole of the left hand term of (19). Hence, w_M is not a pole of the right hand term $-f_n y$ of (19) but w_M is a pole of y so $f_n(w_M) = 0$ i.e. $w_M = \lambda^{-1}$. Using a similar argument, we see that if $m = \min\{k \in \mathbb{Z} \mid q^k w \text{ is a pole of } y\}$ and $w_m = q^m w$ then $f_0(q^{-n} w_m) = 0$ i.e. $w_m = \lambda^{-1}(\prod_{j=1}^n b_j)(\prod_{i=1}^n a_i)^{-1}$. Therefore, $w_m w_M^{-1} = (\prod_{j=1}^n b_j)(\prod_{i=1}^n a_i)^{-1}$. But the definitions of w_M and of w_m ensure that $w_m w_M^{-1}$ belongs to $q^{-\mathbb{N}}$ i.e. that $(\prod_{j=1}^n b_j)(\prod_{i=1}^n a_i)^{-1}$ belongs to $q^{-\mathbb{N}}$ as expected. □

We denote by $\mathbb{C}^{\mathbb{Z}}$ the $\mathbb{C}$-vector space of complex valued sequences indexed by $\mathbb{Z}$ and by $\mathbb{C}^{(\mathbb{Z})}$ its sub-$\mathbb{C}$-vector space made of the sequences with finite support. For any $(p_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, we set $Z((p_k)_{k\in\mathbb{Z}}) = \{k \in \mathbb{Z} \mid p_k = 0\}$.

Lemma 6. *Let us consider $(p_k)_{k\in\mathbb{Z}}$ and $(q_k)_{k\in\mathbb{Z}}$ in $\mathbb{C}^{\mathbb{Z}}$ such that both $Z((p_k)_{k\in\mathbb{Z}}) = \{k \in \mathbb{Z} \mid p_k = 0\}$ and $Z((q_k)_{k\in\mathbb{Z}}) = \{k \in \mathbb{Z} \mid q_k = 0\}$ have at most n elements. Let us denote by S the $\mathbb{C}$-vector space of solutions in $\mathbb{C}^{(\mathbb{Z})}$ of the following recurrent equation :*

$$\forall k \in \mathbb{Z}, \quad p_k y_k = q_k y_{k+1}. \tag{20}$$

Then $\dim_{\mathbb{C}} S = n$ if and only if the following conditions hold :
1) $Z((p_k)_{k\in\mathbb{Z}})$ has exactly n elements $j_1 + 1 < \cdots < j_n + 1$;
2) $Z((q_k)_{k\in\mathbb{Z}})$ has exactly n elements $i_1 < \cdots < i_n$;
3) $i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_n \leq j_n$.

Proof. The following notation will be convenient for the proof. We consider $k_1, ..., k_s \in \mathbb{Z}$ and $g_1, ..., g_s \in \mathbb{N}$ such that :

$$Z((q_k)_{k\in\mathbb{Z}}) = \{k_1, k_1+1, ..., k_1+g_1\} \cup \cdots \cup \{k_s, k_s+1, ..., k_s+g_s\}$$

and such that, for all $r \in [\![1, s-1]\!]$, $k_r + g_r + 2 \leq k_{r+1}$. Moreover, we set $k_{s+1} = +\infty$.

We first construct a $\mathbb{C}$-vector space E of dimension $\leq \sharp Z((q_k)_{k\in\mathbb{Z}}) \leq n$ containing S.

For all $r \in [\![1, s]\!]$, for all $l \in [\![1, g_r]\!]$, we introduce the sequence e_{k_r+l} with support in $\{k_r + l\}$ defined by $(e_{k_r+l})_{k_r+l} = 1$. For all $r \in [\![1, s]\!]$, we denote by $e_{k_r+g_r+1}$ the sequence with support in $[\![k_r + g_r + 1, k_{r+1}]\!]$ defined by $(e_{k_r+g_r+1})_{k_r+g_r+1} = 1$ and, for all $k \in [\![k_r + g_r + 2, k_{r+1}]\!]$, by $(e_{k_r+g_r+1})_k = \frac{p_{k-1}\cdots p_{k_r+g_r+1}}{q_{k-1}\cdots q_{k_r+g_r+1}}$. We define E as the $\mathbb{C}$-vector space of dimension $\leq \sharp Z((q_k)_{k\in\mathbb{Z}}) \leq n$ generated by $\{e_{k_r+l} \mid r \in [\![1, s]\!], l \in [\![1, g_r + 1]\!]\}$.

Let us consider $(y_k)_{k\in\mathbb{Z}} \in S$. We have, for all $k \in]\!] - \infty, k_1]\!]$, $y_k = \frac{p_{k-1}}{q_{k-1}} y_{k-1}$ but, for $j \ll 0$, $y_j = 0$ so, for all $k \in]\!] - \infty, k_1]\!]$, $y_k = 0$. Moreover, note that, for all $r \in [\![1, s]\!]$, for all $k \in [\![k_r + g_r + 2, k_{r+1}]\!]$, we have $y_k = \frac{p_{k-1}}{q_{k-1}} y_{k-1} = \cdots = \frac{p_{k-1}\cdots p_{k_r+g_r+1}}{q_{k-1}\cdots q_{k_r+g_r+1}} y_{k_r+g_r+1}$. This clearly implies that $(y_k)_{k\in\mathbb{Z}} = \sum_{r\in[\![1,s]\!], l\in[\![1,g_r+1]\!]} y_{k_r+l} e_{k_r+l} \in E$. So $S \subset E$ as expected.

Assume that $\dim_{\mathbb{C}} S = n$. This implies that $E = S$. A first consequence is that E has dimension n and hence that $\sharp Z((q_k)_{k\in\mathbb{Z}}) = n$. Another consequence is that any sequence e_{k_r+l} ($r \in [\![1, s]\!]$, $l \in [\![1, g_r + 1]\!]$) is a solution of (20). In particular, we have, for all $r \in [\![1, s]\!]$, for all $l \in [\![1, g_r]\!]$, $p_{k_r+l}(e_{k_r+l})_{k_r+l} = q_{k_r+l}(e_{k_r+l})_{k_r+l+1}$ i.e. $p_{k_r+l} = 0$. Moreover, for all $r \in [\![1, s-1]\!]$, we have $p_{k_{r+1}}(e_{k_r+g_r+1})_{k_{r+1}} = q_{k_{r+1}}(e_{k_r+g_r+1})_{k_{r+1}+1} = 0$ so either $p_{k_{r+1}} = 0$ or $(e_{k_r+g_r+1})_{k_{r+1}} = 0$ i.e. $p_l = 0$ for some $l \in [\![k_r + g_r + 1, k_{r+1} - 1]\!]$. Since $e_{k_s+g_s+1}$ belongs to S, it belongs to $\mathbb{C}^{(\mathbb{Z})}$ so $p_l = 0$ for some $l \in [\![k_s + g_s + 1, \infty]\!]$. These remarks show that $Z((p_k)_{k\in\mathbb{Z}})$ has cardinal n and that, for all $r \in [\![1, s]\!]$, there exists $l_r \in [\![k_r + g_r + 1, k_{r+1}]\!]$ such that :

$$Z((p_k)_{k\in\mathbb{Z}}) = \{k_1 + 1, ..., k_1 + g_1, l_1, k_2 + 1, ..., k_2 + g_2, l_2, ..., k_s + 1, ..., k_s + g_s, l_s\}.$$

Translated in terms of $i_1, ..., i_n$ and $j_1, ..., j_n$, the above discussion leads to conditions 1) to 3).

Conversely, it remains to verify that if conditions 1) to 3) are satisfied then $\dim_{\mathbb{C}} S = n$. This easy verification is left to the reader. □

Definition 2. *For $c, d \in \mathbb{C}^*$, the notation $c \preceq_q d$ (resp. $c \prec_q d$) means that $dc^{-1} \in q^{\mathbb{N}}$ (resp. $dc^{-1} \in q^{\mathbb{N}^*}$). For $\underline{c} = (c_1, ..., c_n) \in (\mathbb{C}^*)^n$ and $\underline{d} = (d_1, ..., d_n) \in (\mathbb{C}^*)^n$, the notation $\underline{c} \trianglelefteq_q \underline{d}$ means that :*

- *the list $c_1, ..., c_n$ can be rearranged as $c_{1,1}, ..., c_{1,n_1}, ..., c_{r,1}, ..., c_{r,n_r}$;*
- *the list $d_1, ..., d_n$ can be rearranged as $d_{1,1}, ..., d_{1,n_1}, ..., d_{r,1}, ..., d_{r,n_r}$;*

in such a way that :

- *$c_{\mu,\nu} = c_{\mu',\nu'}$ mod. $q^{\mathbb{Z}}$ if and only if $\mu = \mu'$;*
- *$\forall \mu \in [\![1, r]\!]$, $\forall \nu \in [\![1, n_\mu]\!]$, $d_{\mu,\nu} = c_{\mu,\nu}$ mod. $q^{\mathbb{Z}}$;*
- *$\forall \mu \in [\![1, r]\!], c_{\mu,1} \preceq_q d_{\mu,1} \prec_q c_{\mu,2} \preceq_q d_{\mu,2} \prec_q \cdots \prec_q c_{\mu,n_\mu} \preceq_q d_{\mu,n_\mu}$.*

Note that $n_1 + \cdots + n_r = n$. Moreover, if $\underline{c} \trianglelefteq_q \underline{d}$ then, for all $(i, j) \in [\![1, n]\!]^2$ with $i \neq j$, $c_i \neq c_j$ and $d_i \neq d_j$.

Theorem 7. *The generalized q-hypergeometric operator $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$ has a finite Galois group over $\mathbb{C}(z)$ if and only if $\underline{a} \in (\mathbb{U}_\infty q^{\mathbb{Q}})^n$, $\underline{b} \in (\mathbb{U}_\infty q^{\mathbb{Q}})^n$ and either $\underline{a} \trianglelefteq_q q^{-1}\underline{b}$ or $\underline{b} \trianglelefteq_q \underline{a}$.*

Proof. Let us first assume that $(\prod_{j=1}^n b_j)(\prod_{i=1}^n a_i)^{-1} \notin q^{-\mathbb{N}}$. We relabel the parameters $a_1, ..., a_n$ as $a_{1,1}, ..., a_{1,n_1}, ..., a_{r,1}, ..., a_{r,n_r}$ in such a way that $a_{\mu,\nu} = a_{\mu',\nu'}$ mod. $q^{\mathbb{Z}}$ if and only if $\mu = \mu'$.

Proposition 11 ensures that $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$ has a finite Galois group if and only the following conditions hold :

1) $\forall \mu \in [\![1, r]\!]$, $\forall \nu \in [\![1, n_r]\!]$, $a_{\mu,\nu} \in \mathbb{U}_\infty q^{\mathbb{Q}}$;
2) the parameters $b_1, ..., b_n$ can be relabeled as $b_{1,1}, ..., b_{1,n_1}, ..., b_{r,1}, ..., b_{r,n_r}$ in such a way that, $\forall \mu \in [\![1, r]\!]$, $\forall \nu \in [\![1, n_\mu]\!]$, $b_{\mu,\nu} = a_{\mu,\nu}$ mod. $q^{\mathbb{Z}}$;
3) $\forall \mu \in [\![1, r]\!]$, $\dim_{\mathbb{C}} \mathrm{Ker}(\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) : \mathbb{C}(z)e^{(0)}_{a^{-1}_{\mu,1}} \to \mathbb{C}(z)e^{(0)}_{a^{-1}_{\mu,1}}) = n_\mu$.

So we now assume that conditions 1) and 2) hold and we study condition 3).

Note that :

$$\begin{aligned} & \mathrm{Ker}(\mathcal{L}_q(\underline{a}; \underline{b}; \lambda) : \mathbb{C}(z)e^{(0)}_{a^{-1}_{\mu,1}} \to \mathbb{C}(z)e^{(0)}_{a^{-1}_{\mu,1}}) \\ = \ & \mathrm{Ker}(\mathcal{L}_q(a^{-1}_{\mu,1}\underline{a}; a^{-1}_{\mu,1}\underline{b}; \lambda) : \mathbb{C}(z) \to \mathbb{C}(z))e^{(0)}_{a^{-1}_{\mu,1}} \\ = \ & \mathrm{Ker}(\mathcal{L}_q(a^{-1}_{\mu,1}\underline{a}; a^{-1}_{\mu,1}\underline{b}; \lambda) : \mathbb{C}[z, z^{-1}] \to \mathbb{C}(z))e^{(0)}_{a^{-1}_{\mu,1}} \end{aligned}$$

(the first equality is immediate, the second one is Lemma 5). Moreover, an easy computation shows that, for any $\mu \in [\![1, r]\!]$, a Laurent polynomial $y = \sum_k y_k z^k \in \mathbb{C}[z, z^{-1}]$ satisfies $\mathcal{L}_q(a_{\mu,1}^{-1}\underline{a}; a_{\mu,1}^{-1}\underline{b}; \lambda)y = 0$ if and only if its sequence of coefficients $(y_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{(\mathbb{Z})}$ satisfies to the following recurrent equation :

$$\zeta_{\mu,k} \prod_{\nu=1}^{n_\mu}(a_{\mu,1}^{-1}b_{\mu,\nu}q^{k-1} - 1)y_k = \lambda\xi_{\mu,k} \prod_{\nu=1}^{n_\mu}(a_{\mu,1}^{-1}a_{\mu,\nu}q^{k-1} - 1)y_{k-1} \tag{21}$$

where

$$(\zeta_{\mu,k})_{k\in\mathbb{Z}} = \left(\prod_{\mu'\in[\![1,r]\!]\setminus\{\mu\}} \prod_{\nu=1}^{n_{\mu'}}(a_{\mu,1}^{-1}b_{\mu',\nu}q^{k-1} - 1) \right)_{k\in\mathbb{Z}}$$

and

$$(\xi_{\mu,k})_{k\in\mathbb{Z}} = \left(\prod_{\mu'\in[\![1,r]\!]\setminus\{\mu\}} \prod_{\nu=1}^{n_{\mu'}}(a_{\mu,1}^{-1}a_{\mu',\nu}q^{k-1} - 1) \right)_{k\in\mathbb{Z}}$$

are non vanishing sequences. So $\dim_{\mathbb{C}} \mathrm{Ker}(\mathcal{L}_q(a_{\mu,1}^{-1}\underline{a}; a_{\mu,1}^{-1}\underline{b}; \lambda) : \mathbb{C}[z, z^{-1}] \to \mathbb{C}(z)) = n_\mu$ if and only if the $\mathbb{C}$-vector space of solutions in $\mathbb{C}^{(\mathbb{Z})}$ of the recurrent equation (21) has dimension n_μ. Lemma 6 ensures that this happens if and only if $(a_{\mu,1}, ..., a_{\mu,n_\mu}) \trianglelefteq_q q^{-1}(b_{\mu,1}, ..., b_{\mu,n_\mu})$. Therefore, condition 3) holds if and only if $\underline{a} \trianglelefteq_q q^{-1}\underline{b}$.

Assume that $(\prod_{j=1}^n b_j)(\prod_{i=1}^n a_i)^{-1} \in q^{-\mathbb{N}}$. The operator $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$ has a finite Galois group if and only if its dual $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)^*$ has a finite Galois group. But $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)^*$ is equal, up to a invertible left factor, to $\mathcal{L}_q(\underline{a}'; \underline{b}'; \lambda')$ where $\underline{a}' = q/\underline{a}$, $\underline{b}' = q^2/\underline{b}$ and $\lambda' = \lambda(\prod_{i=1}^n a_i)(\prod_{j=1}^n b_j)^{-1}$ (see the proof of Proposition 5) and we have $(\prod_{j=1}^n b'_j)(\prod_{i=1}^n a'_i)^{-1} = (\prod_{j=1}^n \frac{q^2}{b_j})(\prod_{i=1}^n \frac{q}{a_i})^{-1} = ((\prod_{j=1}^n b_j)(\prod_{i=1}^n a_i)^{-1})^{-1}q^n \notin q^{-\mathbb{N}}$. Hence, the above discussion ensures that $\mathcal{L}_q(\underline{a}'; \underline{b}'; \lambda')$ has a finite Galois group over $\mathbb{C}(z)$ if and only if $\underline{a}' \in (\mathbb{U}_\infty q^{\mathbb{Q}})^n$ i.e. $\underline{a} \in (\mathbb{U}_\infty q^{\mathbb{Q}})^n$, $\underline{b}' \in (\mathbb{U}_\infty q^{\mathbb{Q}})^n$ i.e. $\underline{b} \in (\mathbb{U}_\infty q^{\mathbb{Q}})^n$ and $\underline{a}' \trianglelefteq_q q^{-1}\underline{b}'$ i.e. $\underline{b} \trianglelefteq_q \underline{a}$. □

6.2 Trivial Galois groups

Theorem 8. *The generalized q-hypergeometric operator $\mathcal{L}_q(\underline{a}; \underline{b}; \lambda)$ has a trivial Galois group over $\mathbb{C}(z)$ if and only if $\underline{a} \in (q^{\mathbb{Z}})^n$, $\underline{b} \in (q^{\mathbb{Z}})^n$ and either $\underline{a} \trianglelefteq_q q^{-1}\underline{b}$ or $\underline{b} \trianglelefteq_q \underline{a}$.*

Proof. It is an immediate consequence of Theorem 7 and of the last assertion of Proposition 11. □

6.3 Generalized hypergeometric equations with finite differential Galois groups

The generalized hypergeometric operator $\mathrm{L}(\underline{\alpha}; \underline{\beta}; \lambda)$ was defined in section 1 by formula (1).

Let us consider $\underline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{Q}^n$, $\underline{\beta} = (\beta_1, ..., \beta_n) \in \mathbb{Q}^n$ and $\lambda \in \mathbb{Q}^*$ such that $\mathcal{L}_q(q^{\underline{\alpha}}; q^{\underline{\beta}}; \lambda)$ has a finite Galois group for all $q \in \mathbb{C}^*$ such that $|q| < 1$. Note that $\mathcal{L}_q(q^{\underline{\alpha}}; q^{\underline{\beta}}; \lambda)$ has a finite Galois group for at least one $q \in \mathbb{C}^*$ such that $|q| < 1$ if and only if it has a finite Galois group for all $q \in \mathbb{C}^*$ such that $|q| < 1$: this fact, which is not obvious a priori, is a direct consequence of Theorem 7. Moreover, since the exponents of $\mathcal{L}_q(q^{\underline{\alpha}}; q^{\underline{\beta}}; \lambda)$ belong to $q^{\mathbb{Q}}$, the Galois group of $\mathcal{L}_q(q^{\underline{\alpha}}; q^{\underline{\beta}}; \lambda)$ is finite cyclic : this is a direct consequence of Theorem 1. The specialization theorem for q-difference Galois groups due to Y. André in [1] entails that $\mathrm{L}(\underline{\alpha}; \underline{\beta}; \lambda)$ has a finite cyclic differential Galois group.

Similarly, if $\underline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{Q}^n$, $\underline{\beta} = (\beta_1, ..., \beta_n) \in \mathbb{Q}^n$ and $\lambda \in \mathbb{Q}$ are such that $\mathcal{L}_q(q^{\underline{\alpha}}; q^{\underline{\beta}}; \lambda)$ has a trivial Galois group for all $q \in \mathbb{C}^*$ such that $|q| < 1$ then $\mathrm{L}(\underline{\alpha}; \underline{\beta}; \lambda)$ has a trivial differential Galois group.

Hence, using Theorem 7 and Theorem 8, we get the following result.

Definition 3. *For $\underline{\gamma} = (\gamma_1, ..., \gamma_n) \in \mathbb{C}^n$ and $\underline{\delta} = (\delta_1, ..., \delta_n) \in \mathbb{C}^n$, the notation $\underline{\gamma} \trianglelefteq \underline{\delta}$ means that :*

- *the list $\gamma_1, ..., \gamma_n$ can be rearranged as $\gamma_{1,1}, ..., \gamma_{1,n_1}, ..., \gamma_{r,1}, ..., \gamma_{r,n_r}$;*
- *the list $\delta_1, ..., \delta_n$ can be rearranged as $\delta_{1,1}, ..., \delta_{1,n_1}, ..., \delta_{r,1}, ..., \delta_{r,n_r}$;*

in such a way that :

- *$\gamma_{\mu,\nu} = \gamma_{\mu',\nu'}$ mod. $\mathbb{Z}$ if and only if $\mu = \mu'$;*
- *$\forall \mu \in [\![1, r]\!]$, $\forall \nu \in [\![1, n_\mu]\!]$, $\delta_{\mu,\nu} = \gamma_{\mu,\nu}$ mod. $\mathbb{Z}$;*
- *$\forall \mu \in [\![1, r]\!], \gamma_{\mu,1} \leq \delta_{\mu,1} < \gamma_{\mu,2} \leq \delta_{\mu,2} < \cdots < \gamma_{\mu,n_\mu} \leq \delta_{\mu,n_\mu}$.*

Theorem 9. *If $\underline{\alpha} \in \mathbb{Q}^n$ and $\underline{\beta} \in \mathbb{Q}^n$ are such that either $\underline{\alpha} \trianglelefteq \underline{\beta} - (1, ..., 1)$ or $\underline{\beta} \trianglelefteq \underline{\alpha}$ then $L(\underline{\alpha}; \underline{\beta}; \lambda)$ has a finite cyclic differential Galois group over $\mathbb{C}(z)$. If, moreover, $\underline{\alpha} \in \mathbb{Z}^n$ and $\underline{\beta} \in \mathbb{Z}^n$ then $L(\underline{\alpha}; \underline{\beta}; \lambda)$ is trivial over $\mathbb{C}(z)$.*

References

[1] Y. André. Différentielles non commutatives et théorie de Galois différentielle ou aux différences. *Ann. Sci. École Norm. Sup. (4)*, 34(5):685–739, 2001.

[2] K. Aomoto. q-analogue of de Rham cohomology associated with Jackson integrals. I. *Proc. Japan Acad. Ser. A Math. Sci.*, 66(7):161–164, 1990.

[3] K. Aomoto. q-analogue of de Rham cohomology associated with Jackson integrals. II. *Proc. Japan Acad. Ser. A Math. Sci.*, 66(8):240–244, 1990.

[4] F. Beukers. Gauss' hypergeometric function. In *Arithmetic and geometry around hypergeometric functions*, volume 260 of *Progr. Math.*, pages 23–42. Birkhäuser, Basel, 2007.

[5] F. Beukers, W. D. Brownawell, and G. Heckman. Siegel normality. *Ann. of Math. (2)*, 127(2):279–308, 1988.

[6] F. Beukers and G. Heckman. Monodromy for the hypergeometric function ${}_nF_{n-1}$. *Invent. Math.*, 95(2):325–354, 1989.

[7] G. Casale and J. Roques. *Work in progress.*

[8] G. Casale and J. Roques. Dynamics of rational symplectic mappings and difference Galois theory. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 103, 23, 2008.

[9] Z. Chatzidakis, C. Hardouin, and M. F. Singer. On the definitions of difference Galois groups. In *Model theory with applications to algebra and analysis. Vol. 1*, volume 349 of *London Math. Soc. Lecture Note Ser.*, pages 73–109. Cambridge Univ. Press, Cambridge, 2008.

[10] Z. Chatzidakis and E. Hrushovski. Model theory of difference fields. *Trans. Amer. Math. Soc.*, 351(8):2997–3071, 1999.

[11] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.

[12] L. Di Vizio. Arithmetic theory of q-difference equations: the q-analogue of Grothendieck-Katz's conjecture on p-curvatures. *Invent. Math.*, 150(3):517–578, 2002.

[13] L. Di Vizio and C. Hardouin. Algebraic and differential galois groups of q-difference equations. *In progress*, 2010.

[14] V. G. Drinfel′d. Quasi-Hopf algebras. *Leningrad Journal*, 1(6):1419–1457, 1990.

[15] A. Duval and C. Mitschi. Matrices de Stokes et groupe de Galois des équations hypergéométriques confluentes généralisées. *Pacific J. Math.*, 138(1):25–56, 1989.

[16] P. I. Etingof. Galois groups and connection matrices of q-difference equations. *Electron. Res. Announc. Amer. Math. Soc.*, 1(1):1–9 (electronic), 1995.

[17] C. H. Franke. Picard-Vessiot theory of linear homogeneous difference equations. *Trans. Amer. Math. Soc.*, 108:491–515, 1963.

[18] G. Gasper and M. Rahman. *Basic hypergeometric series*, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Richard Askey.

[19] É. Goursat. *Leçons sur les séries hypergéométriques et sur quelques fonctions qui s'y rattachent*. Hermann, CambridgeParis, 1936.

[20] C. Hardouin and M. F. Singer. Differential Galois theory of linear difference equations. *Math. Ann.*, 342(2):333–377, 2008.

[21] N. M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1990.

[22] F. Klein. *Vorlesungen über die hypergeometrische Funktion*. Springer, Berlin-Heidelberg-New York, 1933.

[23] T. Kohno. Monodromy representations of braid groups and Yang-Baxter equations. *Ann. Inst. Fourier (Grenoble)*, 37(4):139–160, 1987.

[24] T. Kohno. Linear representations of braid groups and classical Yang-Baxter equations. In *Braids (Santa Cruz, CA, 1986)*, volume 78 of *Contemp. Math.*, pages 339–363. Amer. Math. Soc., Providence, RI, 1988.

[25] B. Kostant. A characterization of the classical groups. *Duke Math. J.*, 25:107–123, 1958.

[26] C. Krattenthaler, T. Rivoal, and W. Zudilin. Séries hypergéométriques basiques, q-analogues des valeurs de la fonction zêta et séries d'Eisenstein. *J. Inst. Math. Jussieu*, 5(1):53–79, 2006.

[27] C. Mitschi. Differential Galois groups of confluent generalized hypergeometric equations: an approach using Stokes multipliers. *Pacific J. Math.*, 176(2):365–405, 1996.

[28] M. Van der Put and M. Reversat. Galois theory of q-difference equations. *Preprint : http://arxiv.org/abs/math.QA/0507098*, 2005.

[29] M. Van der Put and M. F. Singer. *Galois theory of difference equations*, volume 1666 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[30] J.-P. Ramis and J. Sauloy. The q-analogue of the wild fundamental group. I. In *Algebraic, analytic and geometric aspects of complex differential equations and their deformations. Painlevé hierarchies*, RIMS Kôkyûroku Bessatsu, B2, pages 167–193. Res. Inst. Math. Sci. (RIMS), Kyoto, 2007.

[31] J.-P. Ramis and J. Sauloy. The q-analogue of the wild fundamental group. II. In *Singularities and Differential Equations (Congrés satellite de ICM 2006)*. To appear.

[32] J.-P. Ramis, J. Sauloy, and C. Zhang. La variété des classes analytiques d'équations aux q-différences dans une classe formelle. *C. R. Math. Acad. Sci. Paris*, 338(4):277–280, 2004.

[33] J.-P. Ramis, J. Sauloy, and C. Zhang. Développement asymptotique et sommabilité des solutions des équations linéaires aux q-différences. *C. R. Math. Acad. Sci. Paris*, 342(7):515–518, 2006.

[34] C. Sabbah. Systèmes holonomes d'équations aux q-différences. In *D-modules and microlocal geometry (Lisbon, 1990)*, pages 125–147. de Gruyter, Berlin, 1993.

[35] J. Sauloy. Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie. *Ann. Inst. Fourier (Grenoble)*, 50(4):1021–1071, 2000.

[36] J. Sauloy. Galois theory of Fuchsian q-difference equations. *Ann. Sci. École Norm. Sup. (4)*, 36(6):925–968 (2004), 2003.

[37] J. Sauloy. La filtration canonique par les pentes d'un module aux q-différences et le gradué associé. *Ann. Inst. Fourier (Grenoble)*, 54(1):181–210, 2004.

[38] H. A. Schwarz. über diejenigen fälle in welchen die gaussische hypergeometrische reiheeiner algebraische funktion ihres vierten elementes darstellt. *Crelle J*, 75:292–335, 1873.

[39] L. J. Slater. *Generalized hypergeometric functions*. Cambridge University Press, Cambridge, 1966.

[40] V. Tarasov and A. Varchenko. Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups. *Astérisque*, (246):vi+135, 1997.

[41] Yu. G. Zarkhin. Weights of simple Lie algebras in the cohomology of algebraic varieties. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(2):264–304, 1984.

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