REVIEW OF THE BOOK "DIFFERENTIAL GALOIS THEORY THROUGH RIEMANN-HILBERT CORRESPONDENCE: AN ELEMENTARY APPROACH" BY JACQUES SAULOY

JULIEN ROQUES

The linear differential equations are at the crossroads between several areas of mathematics. The book under review starts with an exploration of the analytic theory of (regular singular) linear differential equations and concludes with an invitation to their differential Galois theory. Let me stress that Sauloy's intention is not to expose the differential Galois theory in its full generality, for linear differential equations with coefficients in an arbitrary differential field K, but rather to focus on the case $K = \mathbb{C}(z)$, taking advantage of the analytic tools at our disposal in this context, and making apparent some beautiful interplay between analysis and algebra. The intention of Sauloy is clearly to explain the meaning of the adage "differential Galois theory is what the algebra can say about the dynamics" and to convince the reader that the modern differential Galois theory takes its roots in the analytic theory of differential equations (and, more precisely, in Riemann's *monodromy representations*). And it is a success.

I shall now give an idea of the content of this book.

Part I is an introduction to complex analysis, and a prelude to the analytic study of linear differential equations undertaken in Parts II and III.

Parts II and III are mainly concerned with the analytic theory of the regular singular differential equations on $\mathbb{P}^1(\mathbb{C})$, say of the form

$$a_n(z)y^{(n)}(z) + \cdots + a_1(z)y'(z) + a_0(z)y(z) = 0, \tag{1}$$

with $a_0(z), \ldots, a_n(z) \in \mathbb{C}(z)$, $a_n(z) \neq 0$, and more specifically with their monodromy representation. Before describing in more details what can be found in Parts II and III, I would like to say a few words about the concept of monodromy, which is the hero of the book under review. The staring point is the so-called Cauchy theorem. Denoting by $\mathcal{S}$ the set of singularities in $\mathbb{P}^1(\mathbb{C})$ of (1) and by U its complement in $\mathbb{P}^1(\mathbb{C})$, Cauchy theorem reads: for any $z_0 \in U$, the complex vector space V_{z_0} of germs of analytic functions at z_0 solutions of (1) has dimension n. These solutions possess a marvelous property : any $f \in V_{z_0}$ can be continued analytically along any loop γ in U based at z_0 and the germ of analytic function f^γ resulting from this process is again a solution of (1), *i.e.*, $f^\gamma \in V_{z_0}$. In this way, we get a $\mathbb{C}$-linear action of the fundamental group $\pi_1(U, z_0)$ on V_{z_0} given, for all $[\gamma] \in \pi_1(U, z_0)$ and $f \in V_{z_0}$, by $[\gamma]f := f^\gamma$. In other words, we get a finite dimensional linear representation

$$\rho : \pi_1(U, z_0) \to \mathrm{GL}(V_{z_0}).$$

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This is the monodromy representation attached to (1). The following two questions are natural:

— is the sole knowledge of the monodromy representation sufficient to reconstruct the differential equation we started with ?
— is any finite dimensional linear representation of $\pi_1(U, z_0)$ the monodromy representation of some differential equation ?

In whole generality the answer to the first question is negative but becomes positive if we restrict our attention to a special class of differential equations, namely to those having only regular singularities. The answer to the second question is positive, even if we restrict our attention to differential equations having only regular singularities. These facts are usually formalized as a certain equivalence of categories, known as the Riemann-Hilbert correspondance. I am now in a position to describe the content of Parts II and III. Part II starts with Cauchy theorem, continues with the proof of the analytic continuation property of the solutions mentioned above, and introduces the monodromy representation. Part III starts with the notion of regular singularity and with a detailed study of this type of singularity. A local version of the Riemann-Hilbert correspondance is then stated and proved. The statement and a sketch of proof of the (global) Riemann-Hilbert correspondance (over $\mathbb{P}^1(\mathbb{C})$) are then given. It is worth noticing that Part III also contains the determination of the monodromy representation of the so-called hypergeometric differential equations (following Riemann's original method).

With Part IV, the reader enters in the word of differential Galois theory. The first few pages of this chapter contain the definition of the local differential Galois groups and a proof of the fact that they are complex linear algebraic groups. The transition from Parts II and III to Part IV is done in a very natural and smooth way via the so-called Schlesinger density theorem: the monodromy is galoisian and gives rise to Zariski-dense subgroups of the above mentionned local differential Galois groups in the regular singular case. The rest of Part IV is essentially concerned with the regular singular local universal Galois groups; roughly speaking, this universal Galois group is the algebraic hull of the local topological fundamental group and it can be used to formulate an "algebraic Riemann-Hilbert correspondance".

The book ends with a selection of further developments and readings.

This is an accessible book, well-suited for students. It is mainly self-contained. Some "advanced concepts" such as sheaves, categories, linear algebraic groups, etc., are used. These concepts are introduced progressively all along the book, when needed, without any attempt of systematic presentation or maximal generality. This is a nice and efficient choice. Several exercises are given along the book.

I warmly recommand this book for those looking for an accessible text about the analytic theory of regular singular differential equations and their differential Galois theory.

0.1. **About the author.** Julien Roques is maître de conférences at the Institut Fourier of the Université Grenoble Alpes. He is notably working on algebra, number theory and functional equations, on their interactions and applications.

Université Grenoble Alpes, Institut Fourier, CNRS UMR 5582, CS 40700, 38058 Grenoble Cedex 09
E-mail address: Julien.Roques@univ-grenoble-alpes.fr