ON THE NATURE OF THE GENERATING SERIES OF WALKS IN THE QUARTER PLANE

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Abstract. In the present paper, we introduce a new approach, relying on differential Galois theory of difference equations, to study the nature of the generating series of walks in the quarter plane. Using this approach, we are not only able to recover many of the recent results about these series, but also to go beyond them. For instance, we give for the first time hypertranscendency results, *i.e.*, we prove that certain of these generating series do not satisfy any nontrivial nonlinear algebraic differential equation with rational coefficients.

Contents

1. Introduction

In the recent years, the nature of the generating series of walks in the quarter plane $\mathbb{Z}^2_{\geq 0}$ has attracted the attention of many authors, see [BMM10, BvHK10, FIM99, KR12, MM14, Ras12] and the references therein.

To be concrete, let us consider a walk with small steps in the quarter plane $\mathbb{Z}^2_{\geq 0}$. The set of authorized steps, denoted by $\mathcal{D}$, is a subset of $\{0, \pm 1\}^2 \backslash \{(0,0)\}$. For

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$i,j,k \in \mathbb{Z}_{\geq 0}$, we let $q_{\mathcal{D},i,j,k}$ be the number of walks in $\mathbb{Z}^2_{\geq 0}$ with steps in $\mathcal{D}$ starting at $(0,0)$ and ending at (i,j) in k steps and we consider the corresponding trivariate generating series

$$Q_{\mathcal{D}}(x,y,t) := \sum_{i,j,k\geq 0} q_{\mathcal{D},i,j,k} x^i y^j t^k.$$

The main questions considered in the literature are:

- is $Q_{\mathcal{D}}(x,y,t)$ algebraic over $\mathbb{Q}(x,y,t)$?
- is $Q_{\mathcal{D}}(x,y,t)$ holonomic, *i.e.*, is the $\mathbb{Q}(x,y,t)$-vector space generated by all the derivatives of $Q_{\mathcal{D}}(x,y,t)$ finite dimensional?
- is $Q_{\mathcal{D}}(x,y,t)$ x-hyperalgebraic (resp. y-hyperalgebraic), *i.e.*, is $Q_{\mathcal{D}}(x,y,t)$ seen as a function of x solution of some nonzero nonlinear polynomial differential equations with coefficients in $\mathbb{Q}(x,y,t)$? In case of a negative answer, we say that $Q_{\mathcal{D}}(x,y,t)$ is x-hypertranscendental (resp. y-hypertranscendental).

We shall now make a brief overview of some recent works on these questions. Random walks in the quarter plane were extensively considered in [FIM99]. These authors attached a group to any such walk and introduced powerful analytic tools to study the generating series of such a walk. In [BMM10], Bousquet-Mélou and Mishna give a detailed study of the various walks and make the conjecture that a walk has a holonomic generating series if and only if the associated group is finite. They proved that, if the group of the walk is finite, then the generating series is holonomic, except, maybe, in one case, which was solved positively by Bostan, van Hoeij and Kauers in [BvHK10] (see also [FR10]). In [MR09] Mishna and Rechnitzer showed that two of the walks with infinite groups have nonholonomic generating series. Kurkova and Raschel proved in [KR12] (see also [BRS14, Ras12]) that for all of the 51 nonsingular walks with infinite group the corresponding generating series is not holonomic. This work is very delicate and technical, and relies on the explicit uniformization of certain elliptic curves. Recently, Bernadi, Bousquet-Mélou and Raschel [BBMR16] have shown that the generating series for 9 of the nonsingular walks satisfy nonlinear differential equations despite the fact that they are not holonomic.

In the present paper, we introduce a new, more algebraic approach to study the nature of the generating series of walks. Using this approach, we are not only able to recover the above mentioned remarkable results, but also to go beyond them. For instance, the following theorem, proved in Section 5, is one of the main result of this paper.

Theorem 1. *Except for the 9 walks considered in* [BBMR16]*, the generating series of all nonsingular walks with infinite group are x- and y-hypertranscendental. In particular, they are nonholonomic.*

From a technical point of view, one of our main new idea is that results from the differential Galois theory of difference equations (explained in Section 3) allow to reduce the question of showing that the generating series is hypertranscendental to showing that a certain linear differential equation defined on an elliptic curve has no solutions in the function field of that curve. It turns out that the last question can be answered by some elementary considerations about the polar divisor of some elliptic functions. Note that the fact that difference equations come into play in the present context is classical and due to the fact that certain specializations of the generating series of a fixed walk can be analytically continued to a multivalued meromorphic function on an elliptic curve and that, once they are lifted to the universal covering of this curve, these specializations satisfy explicit difference equations (this was shown in [KR12] and [Ras12] in our perticuliar context, but similar results were already proved in [FIM99]). The novelty of our approach consists in the *algebraic* way we exploit these functional equations, in the light of the differential Galois theory of difference equations.

Our techniques also allow us to study the 9 exceptional cases and to recover some of the results of [BBMR16], namely :

Theorem 2. *In the 9 exceptional cases treated in* [BBMR16]*, the generating series* $Q_{\mathcal{D}}(x,y,t)$ *is* x*- and* y*-hyperalgebraic.*

It is very likely that our method can be used to study the generating series of *weighted* walks in the quarter-plane as well as *singular* walks. This is explained in more details in Section 8. We hope to come back on this in future publications.

The paper is organized as follows. In Section 2 we review several useful facts and ideas that form the basis of this paper as well as previous investigations concerning the generating series of walks in the quarter plane: the functional equation, the elliptic curve associated to this equation together with certain involutions and automorphisms and the method by which one reduces the question of hypertranscendence to a similar question for a multivalued meromorphic function on the associated curve. In Section 3, we present the criteria based on the differential Galois theory of difference equations which we will use to determine if a function is hypertranscendental. In Sections 4 and 5 we present the calculations that show that for all but the nine exceptional cases, the generating series of nonsingular walks are hypertranscendental. In Section 6 we show that for the nine exceptional cases, the generating series have specializations that are hyperalgebraic and in Section 7 we show these series are not holonomic. The appendix, Section A, contains useful necessary and sufficient conditions for certain linear differential equations on elliptic curves (similar to the telescopers appearing in the verification of combinatorial identities) to have solutions in the function field of the curve. These are the criteria which, together with the results of Section 3, are used in Sections 4 and 5 to determine hypertranscendency.

2. Fundamental properties of the walks with small steps

We start by recalling some basic facts about random walks in the quarter-plane, see [BMM10, FIM99, KY15, MM14] for more details.

2.1. **The generating series.** We consider a walk with small steps in the quarter plane $\mathbb{Z}^2_{\geq 0}$. The set of authorized steps $\mathcal{D}$ is a subset of $\{0,\pm 1\}^2\backslash\{(0,0)\}$. For $i,j,k \in \mathbb{Z}_{\geq 0}$, we let $q_{\mathcal{D},i,j,k}$ be the number of walks in $\mathbb{Z}^2_{\geq 0}$ with steps in $\mathcal{D}$ starting at $(0,0)$ and ending at (i,j) in k steps and we consider the corresponding trivariate generating series

$$Q_{\mathcal{D}}(x,y,t) := \sum_{i,j,k\geq 0} q_{\mathcal{D},i,j,k} x^i y^j t^k.$$

It can be deduced from [FIM99] that $Q_{\mathcal{D}}(x,y,t)$ converges for all $(x,y,t) \in \mathbb{C}^2 \times \mathbb{R}$ such that $|x| \leq 1$, $|y| \leq 1$ and $0 < t < 1/|\mathcal{D}|$.

2.2. **Kernel and functional equation.** The generating series $Q_{\mathcal{D}}(x,y,t)$ satisfies a functional equation that we shall now recall. The *Kernel* of the walk is defined by

$$K_{\mathcal{D}}(x,y,t) := xy(1 - tS_{\mathcal{D}}(x,y))$$

where

$$S_{\mathcal{D}}(x,y) = \sum_{(i,j)\in\{0,\pm 1\}^2} d_{i,j} x^i y^j$$

with $d_{i,j}$ is equal to 1 if $(i,j) \in \mathcal{D}$, and to 0 otherwise.

One can consider $S_{\mathcal{D}}(x,y)$ as a Laurent polynomial in x with coefficients that are Laurent polynomial in y and *vice versa*. Using the notations of [BMM10, KY15], we write

$$S_{\mathcal{D}}(x,y) = A_{\mathcal{D},-1}(x)\frac{1}{y} + A_{\mathcal{D},0}(x) + A_{\mathcal{D},1}(x)y = B_{\mathcal{D},-1}(y)\frac{1}{x} + B_{\mathcal{D},0}(y) + B_{\mathcal{D},1}(y)x$$

where $A_{\mathcal{D},i}(x) \in x^{-1}\mathbb{Q}[x]$ and $B_{\mathcal{D},i}(y) \in y^{-1}\mathbb{Q}[y]$. The generating series $Q_{\mathcal{D}}(x,y,t)$ satisfies the following functional equation:

$$K_{\mathcal{D}}(x,y,t)xyQ_{\mathcal{D}}(x,y,t) = xy - F^1_{\mathcal{D}}(x,t) - F^2_{\mathcal{D}}(y,t) + td_{-1,-1}Q_{\mathcal{D}}(0,0,t) \tag{2.1}$$

where

$$F^1_{\mathcal{D}}(x,t) := txA_{\mathcal{D},-1}(x)Q_{\mathcal{D}}(x,0,t), \quad F^2_{\mathcal{D}}(y,t) := tyB_{\mathcal{D},-1}(y)Q_{\mathcal{D}}(0,y,t).$$

2.3. **Classification of the walks with small steps.** There are *a priori* $2^8 = 256$ possible walks with small steps in the quarter plane $\mathbb{Z}^2_{\geq 0}$, but, as explained in [BMM10, §2], only 138 of them are truly worthy of consideration. Moreover, taking into account natural symmetries, we are finally left with 79 inherently different walks to study; see [BMM10, Figures 1 to 4].

Following [BMM10, Section 3] or [KY15, Section 3], we attach to any walk in the quarter plane its group, which is by definition the group $\langle i_1, i_2\rangle$ generated by the involutive birational transformations of $\mathbb{C}^2$ given by

$$i_1(x,y) = \left(x, \frac{A_{\mathcal{D},-1}(x)}{A_{\mathcal{D},1}(x)y}\right) \text{ and } i_2(x,y) = \left(\frac{B_{\mathcal{D},-1}(y)}{B_{\mathcal{D},1}(y)x}, y\right).$$

The kernel $K_{\mathcal{D}}(x,y,t)$ is left invariant by the natural action of this group. Amongst the 79 walks mentioned above, 23 have a finite group and 56 have an infinite group; see [BMM10, Theorem 3].

In the finite group case, the generating series $Q_{\mathcal{D}}(x,y,t)$ is holonomic. This has been proved in [BMM10] for 22 walks, and in [BvHK10] for the remaining walk, the so-called Gessel walk (its generating series is actually algebraic; see also [FR10]).

Amongst the walks having an infinite group, we distinguish the *singular* and *nonsingular* walks: amongst the 56 walks under consideration, there are 51 nonsingular walks and 5 singular walks. These nonsingular walks, on which this paper focuses, are listed in Figure 1. We reproduce the table [KR12, Figure 17] and maintains their notations for the convenience of the reader and for the ease of reference. In [KR12], the authors show that the generating series for all 51 nonsingular walks are nonholonomic. In [BBMR16], the authors show that 9 of these are hyperalgebraic. In Figure 2, we list these exceptional walks with numbers corresponding to both [BBMR16, Tab. 2] and Figure 1. As the authors of [BMM10] point out, interchanging x and y in the steps leads to equivalent counting problems. The notation "IIB.2 (after $x \leftrightarrow y$)" refers to the second walk (from the left) among the walks labelled IIB with the x and y axes interchanged.

2.4. **The algebraic curve $\overline{E_t}$ defined by the kernel $K(x,y,t)$.** We fix $\mathcal{D}$ a set of weights and $0 < t < 1/|\mathcal{D}|$. We consider the homogeneous polynomial given by

$$\overline{K}(x_0,x_1,y_0,y_1,t) = x_0x_1y_0y_1 - t\sum_{i,j=0}^{2} d_{i-1,j-1}x_0^i x_1^{2-i} y_0^j y_1^{2-j}. \tag{2.2}$$

We let $\overline{E_t}$ be the algebraic curve in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ defined by $\overline{K}$, *i.e.*,

$$\overline{E_t} = \{(x,y) = ([x_0 : x_1],[y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid \overline{K}(x_0,x_1,y_0,y_1,t) = 0\}.$$

The intersection of $\overline{E_t}$ with $(\mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}) \times (\mathbb{P}^1(\mathbb{C}) \setminus \{\infty\})$, where $\infty = [1 : 0]$, will be denoted by E_t and identified with a subset of $\mathbb{C} \times \mathbb{C}$, *i.e.*,

$$E_t = \{(x,y) \in \mathbb{C} \times \mathbb{C} \mid K(x,y,t) = 0\}.$$

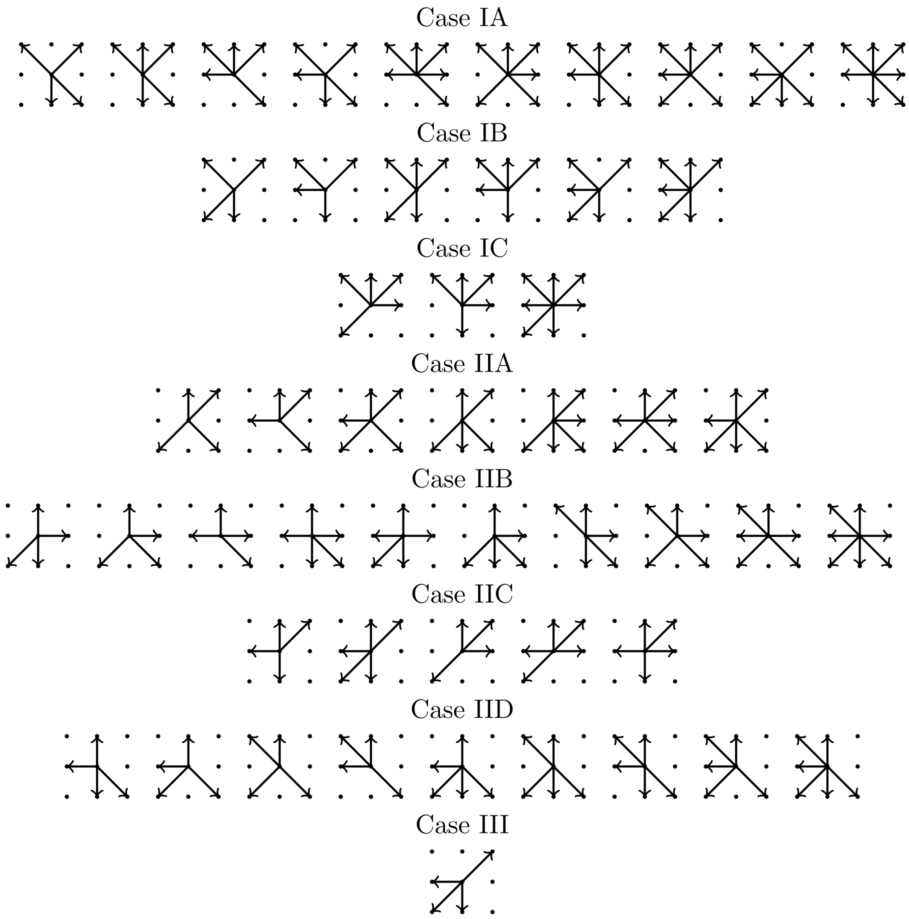


FIGURE 1. The nonsingular walks with small steps and infinite group

[BBMR16, Tab 2]	Figure 1.
1	IIB.1 (after $x \leftrightarrow y$)
2	IIB.2 (after $x \leftrightarrow y$)
3	IIC.1
4	IIB.3
5	IIC.4
6	IIC.2
7	IIB.6 (after $x \leftrightarrow y$)
8	IIC.5
9	IIB.7

FIGURE 2. The exceptional nonsingular walks with small steps and infinite group

Such curves have been studied in detail by Duistermaat in [Dui10, Chapter 2].

2.5. **Smoothness and genus of $\overline{E_t}$.** We fix $0 < t < 1/|\mathcal{D}|$. For any $[x_0 : x_1]$ and $[y_0 : y_1]$ in $\mathbb{P}^1(\mathbb{C})$, we denote by $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ the discriminants of the degree 2 homogeneous polynomials given by $y \mapsto \overline{K}(x_0, x_1, y, t)$ and $x \mapsto \overline{K}(x, y_0, y_1, t)$

respectively, *i.e.*,

$$\Delta^x_{[x_0:x_1]} = t^2 \left((d_{-1,0}x_1^2 - \frac{1}{t}x_0x_1 + d_{1,0}x_0^2)^2 \right.$$
$$\left. -4(d_{-1,1}x_1^2 + d_{0,1}x_0x_1 + d_{1,1}x_0^2)(d_{-1,-1}x_1^2 + d_{0,-1}x_0x_1 + d_{1,-1}x_0^2) \right)$$

and

$$\Delta^y_{[y_0:y_1]} = t^2 \left((d_{0,-1}y_1^2 - \frac{1}{t}y_0y_1 + d_{0,1}y_0^2)^2 \right.$$
$$\left. -4(d_{1,-1}y_1^2 + d_{1,0}y_0y_1 + d_{1,1}y_0^2)(d_{-1,-1}y_1^2 + d_{-1,0}y_0y_1 + d_{-1,1}y_0^2) \right).$$

Proposition 2.1 ([Dui10, §2.4.1, especially Proposition 2.4.3])**.** *The curve $\overline{E_t}$ is smooth if and only if the discriminant $\Delta^x_{[x_0:x_1]}$ has simple roots in $\mathbb{C}^2 \setminus \{(0,0)\}$ if and only if the discriminant $\Delta^y_{[y_0:y_1]}$ has simple roots in $\mathbb{C}^2 \setminus \{(0,0)\}$. Moreover, if $\overline{E_t}$ is smooth, then it has genus 1, i.e., it is an elliptic curve.*

Remark 2.2. *In* [FIM99, Section 2.3.2]*, other explicit equivalent conditions on the $d_{i,j}$ are given ensuring that the discriminants have simple roots.*

Corollary 2.3. *For any walk listed in Figure 1, the algebraic curve $\overline{E_t}$ is an elliptic curve.*

Proof. This is a direct consequence of Proposition 2.1 in virtue of [KR12, Section 2.1] and [FIM99, Part 2.3]. ☐

Remark 2.4. *We work in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ instead of the projective plane over $\mathbb{C}$ in order to get a smooth curve. Moreover the smoothness criteria of Proposition 2.1 can be encoded as follows. Following* [Dui10, Proposition 2.4.3]*, the curve E_t is smooth if and only if the Eisenstein invariants F_x (resp. F_y) of $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$) are non zero (see* [Dui10, Section 2.3.5]*). The Eisenstein invariants F_x and F_y are given by explicit polynomial formulas in t and the $d_{i,j}$. Furthermore, if E_t is smooth then it is an elliptic curve with modulus J that can be explicitly computed thanks to the Eisenstein invariants (see* [Dui10, 2.3.23]*).*

2.6. **The involutions ι_1 and ι_2 and the QRT mapping τ of $\overline{E_t}$.** In this section, we fix $0 < t < 1/|\mathcal{D}|$ and we assume that $\overline{E_t}$ is an elliptic curve.

We let ι_1 and ι_2 be the involutions of $\overline{E_t}$ induced by i_1 and i_2, *i.e.*,

$$\iota_1(x,y) = \left(\frac{x_0}{x_1}, \frac{A_{-1}(\frac{x_0}{x_1})}{A_1(\frac{x_0}{x_1})\frac{y_0}{y_1}} \right) \text{ and } \iota_2(x,y) = \left(\frac{B_{-1}(\frac{y_0}{y_1})}{B_1(\frac{y_0}{y_1})\frac{x_0}{x_1}}, \frac{y_0}{y_1} \right).$$

These formulas define ι_1 and ι_2 as rational maps from $\overline{E_t}$ to itself, but, since $\overline{E_t}$ is a smooth projective curve, they are actually endomorphisms of $\overline{E_t}$. They are nothing but the vertical and horizontal switches of $\overline{E_t}$, *i.e.*, for any $P = (x,y) \in \overline{E_t}$, we have

$$\{P, \iota_1(P)\} = \overline{E_t} \cap (\{x\} \times \mathbb{P}^1(\mathbb{C})) \text{ and } \{P, \iota_2(P)\} = \overline{E_t} \cap (\mathbb{P}^1(\mathbb{C}) \times \{y\}).$$

The following result is a straightforward consequence of the definitions.

Lemma 2.5. *A point $P = ([x_0 : x_1], [y_0 : y_1]) \in \overline{E_t}$ is fixed by ι_1 (resp. ι_2) if and only if $\Delta^x_{[x_0:x_1]} = 0$ (resp. $\Delta^y_{[y_0:y_1]} = 0$).*

The automorphism τ of $\overline{E_t}$ given by

$$\tau = \iota_2 \circ \iota_1,$$

which will play a central role in this paper, is called the QRT mapping of $\overline{E_t}$. According to [Dui10, Proposition 2.5.2], τ is the addition by a point of the elliptic curve $\overline{E_t}$.

We denote by G_t the subgroup of $\mathrm{Aut}(\overline{E_t})$, that is the group formed of the automorphisms of $\overline{E_t}$, generated by ι_1 and ι_2.

If the group of the walk is finite then G_t is finite as well. However, if the group of the walk is infinite, there are *a priori* no reason why G_t should be infinite for all $0 < t < 1/|\mathcal{D}|$ (and this is false in general). However, we have the following result.

Proposition 2.6. *For any walk with infinite group, the set of $t \in \mathbb{C}$ such that G_t is infinite, i.e., such that τ has infinite order, has denumerable complement in $\mathbb{C}$.*

Proof. This follows from [KR12, Remark 6, Proposition 14]. However, we give an alternate more elementary proof below.

We will show that the set of $t \in \mathbb{C}$ such that τ has finite order on $\overline{E_t}$ is denumerable. To do this it suffices to show that, for each positive integer n, the set of $t \in \mathbb{C}$ such that τ^n is the identity on $\overline{E_t}$ is finite. We know that the walks under consideration have an infinite group, *i.e.*, that the birational transformation of $\mathbb{C}^2$ given by

$$f = \iota_2 \circ \iota_1$$

has infinite order. Fix a value of $n > 0$. Using the formulas for ι_1 and ι_2 in Section 2.3, one sees that

$$f^n(x, y) = \left(\frac{p_1(x, y)}{p_2(x, y)}, \frac{q_1(x, y)}{q_2(x, y)}\right)$$

where $p_1, p_2, q_1, q_2 \in \mathbb{Q}[x, y]$ are independent of t. This birational map is well defined on the complement of the curve $Z \subset \mathbb{C}^2$ defined by $p_2(x, y)q_2(x, y) = 0$. For any $t \in \mathbb{C}$, Bézout's theorem ensures that either E_t and Z have an irreducible component in common or $E_t \cap Z$ is finite. Let $\mathcal{S}$ be the set of $t \in \mathbb{C}$ such that E_t and Z have an irreducible component in common. We claim that $\mathcal{S}$ is finite. Assume not. For $t \in \mathcal{S}$, $K(x, y, t) = xy(1-tS(x, y))$ and $p_2(x, y)q_2(x, y)$ have a nonconstant factor in common. Since $\mathcal{S}$ is infinite, there are two values of t, t_1 and t_2, such that $K(x, y, t_1) = xy(1 - t_1S(x, y))$ and $K(x, y, t_2) = xy(1 - t_2S(x, y))$ will both be divisible by the same nonconstant factor $d(x, y) \in \mathbb{C}[x, y]$ of $p_2(x, y)q_2(x, y)$. This implies that $d(x, y)$ is a factor of xy and $xyS(x, y)$, and, hence, of $K(x, y, t)$. But, for the walks we consider $K(x, y, t)$ is irreducible, so, up to a multiplicative non zero constant, $d(x, y) = K(x, y, t)$, and this is a contradiction.

Let X_n be the set of $(x, y, t) \in \mathbb{C}^3$ such that

- $t \notin \mathcal{S}$,
- $(x, y) \in E_t$,
- $p_2(x, y)q_2(x, y) \neq 0$,
- $f^n(x, y) = \tau^n(x, y) = (x, y)$.

One sees that X_n is a constructible set[1]. The projection of a constructible set onto a subset of its coordinates is again constructible [CLO05, Ch.5.6, Cor.2, Ex.1] so the set

$$Y_n = \{t \in \mathbb{C} \mid \exists (x, y) \in \mathbb{C}^2 \text{ s.t. } (x, y, t) \in X_n\}$$

is also constructible. If $t \in Y_n$, then there is a point $(x, y) \in E_t$ such that f^n is defined at this point and $f^n(x, y) = \tau^n(x, y) = (x, y)$. Since τ^n is given by the addition of a point on $\overline{E_t}$, it must leave all of $\overline{E_t}$ fixed. Conversely, if $t \notin \mathcal{S}$ and τ^n is the identity on $\overline{E_t}$ then for some $(x, y) \in \mathbb{C}^2$, $(x, y, t) \in X_n$ and so $t \in Y_n$. We will have completed the proof once we show that Y_n is finite. If Y_n is not finite

[1] A subset of $\mathbb{C}^m$ is constructible if it lies in the boolean algebra generated by the Zariski closed sets.

then, since constructible subsets of $\mathbb{C}$ are either finite or cofinite, it must contain an open set U. The set of points $\{E_t \mid t \in Y_n\}$ will then contain an open subset V of $\mathbb{C} \times \mathbb{C}$ such that f^n leaves V pointwise fixed. Therefore, f^n would be the identity on $\mathbb{C} \times \mathbb{C}$, a contradiction. Therefore Y_n is finite. □

2.7. **Functional equations satisfied by $F^1(x,t)$ and $F^2(y,t)$.** In this section, following [KR12], we describe how one identifies the formal series $F^1(x,t)$ and $F^2(y,t)$ with meromorphic functions on a suitable domain and give the functional equations satisfied by these functions for the walks listed in Figure 1.

We recall that the generating series $Q_{\mathcal{D}}(x,y,t)$ satisfies the functional equation (2.1). This equation is formal yet, but for $|x|,|y| < 1$ and $0 < t < \frac{1}{|\mathcal{D}|}$, the series $Q_{\mathcal{D}}$, $F^1_{\mathcal{D}}(x,t)$ and $F^2_{\mathcal{D}}(y,t)$ are convergent. Therefore we have

$$0 = xy - F^1_{\mathcal{D}}(x,t) - F^2_{\mathcal{D}}(y,t) + td_{-1,-1}Q_{\mathcal{D}}(0,0,t) \tag{2.3}$$

for all $x, y \in V := E_t \cap \{|x|, |y| < 1\}$. This V is a non empty open subset of E_t as explained in [KR12, Section 4.1]. In particular, $F^1(x,t)$ and $F^2(y,t)$ yield analytic functions on some small pieces of the elliptic curve E_t. Thanks to uniformization, we can identify E_t with $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ via a map

$$\begin{array}{ccc} \mathbb{C} & \rightarrow & \overline{E}_t \\ \omega & \mapsto & (\mathfrak{q}_1(\omega), \mathfrak{q}_2(\omega)), \end{array}$$

where $\mathfrak{q}_1, \mathfrak{q}_2$ are rational functions of $\mathfrak{p}$ and its derivative $d\mathfrak{p}/d\omega$, $\mathfrak{p}$ the Weirestrass function associated with the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (cf. [KR12, Section 3.2]). Therefore one can lift the functions $F^1(x,t)$ and $F^2(y,t)$ to functions $r_x(\omega) = F^1(\mathfrak{q}_1(\omega), t)$ and $r_y(\omega) = F^2(\mathfrak{q}_2(\omega), t)$, each defined on a suitable open subset of $\mathbb{C}$. Note that the involutions ι_1, ι_2 and the map τ can also be lifted to the universal cover $\mathbb{C}$ of $\overline{E}_t$. We shall abuse notation and again denote these functions as $\iota_1(\omega), \iota_2(\omega)$ and $\tau(\omega)$. Furthermore, we have that $\tau(\omega) = \omega + \omega_3$ on $\mathbb{C}$ where no nonzero integer multiple of ω_3 belongs to the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ if τ has infinite order.

Remark 2.7. *In* [KR12, Section 3.2]*, explicit expressions for ω_1, ω_2, and ω_3 are given. Furthermore, ω_1 is a purely imaginary number, whereas ω_2 and ω_3 are real numbers.*

One can be deduce from [KR12, Theorems 3 and 4], that the functions $r_x(\omega)$ and $r_y(\omega)$ can be continued meromorphically as univalent functions on the universal cover $\mathbb{C}$. Furthermore, for any $\omega \in \mathbb{C}$, we have

$$\tau(r_x(\omega)) - r_x(\omega) = b_1, \text{ where } b_1 = \iota_1(\mathfrak{q}_2(\omega))(\tau(\mathfrak{q}_1(\omega)) - \mathfrak{q}_1(\omega)) \tag{2.4}$$

$$\tau(r_y(\omega)) - r_y(\omega) = b_2, \text{ where } b_2 = \mathfrak{q}_1(\omega)(\iota_1(\mathfrak{q}_2(\omega)) - \mathfrak{q}_2(\omega)) \tag{2.5}$$

$$r_x(\omega + \omega_1) = r_x(\omega) \tag{2.6}$$

$$r_y(\omega + \omega_1) = r_y(\omega). \tag{2.7}$$

Remark 2.8. *1. In the statement of* [KR12, Theorem 4] *Kurkova and Rashel give equations that are different from equations* (2.5) *and* (2.4)*. The above equations are presented in the proof of* [KR12, Theorem 4] *and are shown to be equivalent to those in the statement of their theorem.*

2. The functions r_x and r_y are not ω_2 periodic and therefore only define multi-valued functions on $\bar{E}_t$.

3. Equation (2.4) *gives meaning to the formula $\tau(F^1(x,t)) - F^1(x,t) = \iota_1(y)(\tau(x) - x)$ and* (2.5) *can be interpreted as giving meaning to the formula $\tau(F^2(y,t)) - F^2(y,t) = x(\iota_1(y) - y)$. Although the right hand sides of these equations are well defined on $\overline{E_t}$, further analysis is needed to give meaning to "$\tau(F^1(x,t))$" and "$\tau(F^2(y,t))$". This is one reason for lifting the various functions to the universal cover of this curve.*

From now on and until the end of the paper, we shall make the following assumptions on $\mathcal{D}$ and on t.

Assumption 2.9. *We assume that the set* $\mathcal{D}$ *and* $0 < t < 1/|\mathcal{D}|$ *are such that*

- *the curve* E_t *is smooth,*
- *the group* G_t *is infinite,*
- *the functions* $x \mapsto F^1(x,t)$ *and* $y \mapsto F^2(y,t)$*, each analytic on some open subset of* $\overline{E_t}$*, can be lifted and continued to functions* r_x *and* r_y *meromorphic on the universal cover of* $\overline{E_t}$ *such that these functions satisfy equations* (2.4)*,* (2.5)*,* (2.6)*, and* (2.7)*;*
- t *is not algebraic over* $\mathbb{Q}$*.*

For all 51 walks in Figure 1, using Corollary 2.3, Proposition 2.1 and Proposition 2.6, we see that the set of t such that the assumptions 2.9 are satisfied has denumerable complement in $]0, 1/|\mathcal{D}|[$.

3. Hypertranscendancy Criteria

In this section, we derive hypertranscendency criteria for $F^1(x,t)$ and $F^2(y,t)$. The related functions r_x and r_y satisfy "difference" equations of the form $\tau(Y) - Y = b$. Galois theoretic methods to study the differential properties of such functions have been developed in [HS08] and [DHR15] (see also [Har16]). In this section we describe a consequence of this latter theory and how it will be used to show that 42 of the 51 nonsingular unweighted walks are hypertranscendental. We will begin by making precise the differential situation.

3.1. **A derivation on $\overline{E_t}$ commuting with τ.** We assume that $\overline{E_t}$ is an elliptic curve. We denote its function field by $\mathbb{C}(\overline{E_t})$. The space of meromorphic differential forms on $\overline{E_t}$ forms a one dimensional vector space over $\mathbb{C}(\overline{E_t})$ and the space of regular differential forms is a one dimensional vector space over $\mathbb{C}$. We let Ω be a nonzero regular differential form on $\overline{E_t}$.

Lemma 3.1. *The derivation* δ *of* $\mathbb{C}(\overline{E_t})$ *such that* $d(f) = \delta(f)\Omega$ *commutes with* τ*, that is*

$$\tau \circ \delta = \delta \circ \tau.$$

Proof. According to [Dui10, Lemma 2.5.1, Proposition 2.5.2] or to [Sil09, Proposition 5.1], we have $\tau^*(\Omega) = \Omega$. It follows that, for any f in $\mathbb{C}(\overline{E_t})$, we have

$$\delta(\tau(f))\Omega = d(\tau(f)) = \tau^*(df) = \tau^*(\delta(f)\Omega) = \tau(\delta(f))\tau^*(\Omega) = \tau(\delta(f))\Omega.$$

Whence the equality $\delta(\tau(f)) = \tau(\delta(f))$. □

Lemma 3.2. *Let* $P \in \overline{E_t}$ *and let* v_P *be the associated valuation on* $\mathbb{C}(\overline{E_t})$*. Then, for any* $f \in \mathbb{C}(\overline{E_t})$*, we have*

- *if* $v_P(f) \geq 0$ *then* $v_P(\delta(f)) \geq 0$*;*
- *if* $v_P(f) < 0$ *then* $v_P(\delta(f)) = v_P(f) - 1$*.*

Proof. We recall that ω has valuation 0 at any point of $\overline{E_t}$. Let u be a local parameter of $\overline{E_t}$ at P. Since $d(u) = \delta(u)\Omega$ and since both du and Ω have valuation 0 at P, we get that $v_P(\delta(u)) = 0$.

For $f = \sum_{i=v_u(f)}^{+\infty} a_i u^i \in \mathbb{C}(\overline{E_t})$, we have $\delta(f) = \sum_{i=v_u(f)}^{+\infty} a_i i \delta(u) u^{i-1}$ and we see that

- if $v_P(f) \geq 0$ then $v_P(\delta(f)) \geq 0$;
- if $v_P(f) < 0$ then $v_P(\delta(f)) = v_P(f) - 1$.

□

Remark 3.3. *In Section 2.7, we discussed the universal covering space map $\mathbb{C} \to \overline{E_t}$ and use it to lift functions on $\overline{E_t}$ to $\mathbb{C}$. In particular the elements of $\mathbb{C}(\overline{E_t})$ lift to doubly periodic meromorphic functions on $\mathbb{C}$ and so we can consider $\mathbb{C}(\overline{E_t}) \subset \mathcal{M}(\mathbb{C})$ where $\mathcal{M}(\mathbb{C})$ is the field of meromorphic functions on $\mathbb{C}$. One then sees that τ corresponds to the map $\omega \mapsto \omega + \omega_3$, Ω corresponds (up to constant multiple) to the regular differential form $d\omega$ and the τ-invariant derivation δ corresponds to $\frac{d}{d\omega}$.*

3.2. **Hypertranscendancy criteria.** We will reduce questions concerning the hypertranscendence of $F^1(x,t)$ and $F^2(y,t)$ to questions about the differential behavior of elements of $\mathbb{C}(\overline{E_t})$. Criteria derived by using the differential Galois theory of difference equations allow us to do this. We will not describe in detail this theory but rather just explain the criteria. We will start with an abstract formulation but quickly specialize to the present situation.

Definition 3.4. *A $\delta\tau$-field is a triple (K, δ, τ) where K is a field, δ is a derivation on K, τ is an automorphism of K and δ and τ commute on K. The* τ-constants *K^τ of K is the set $\{c \in K \mid \tau(c) = c\}$.*

The triples $(\mathbb{C}(\overline{E_t}), \delta, \tau)$ and $(\mathcal{M}(\mathbb{C}), \frac{d}{d\omega}, \tau : \omega \to \omega + \omega_3)$ are examples. We say that a $\delta\tau$-field is a subfield of another $\delta\tau$-field if the derivation and automorphism of the smaller field are just the restrictions of the derivation and automorphism of the larger field. If τ has infinite order as an automorphism of $\mathbb{C}(\overline{E_t})$, then $\mathbb{C}(\overline{E_t})^\tau = \mathbb{C}$. This follows from the fact that $\tau(f(X)) = f(X \oplus P)$ where P is a point of infinite order. If $\tau(f) = f$, then $f(Q) - f(Q \oplus nP) = 0$ for all $n \in \mathbb{Z}$ and Q a regular point of f. This implies that $f(X) = f(Q)$ on a dense subset of $\overline{E_t}$ and so $f(X)$ must be constant on $\overline{E_t}$.

The following formalizes the notion of holonomic, hyperalgebraic and hypertranscendental.

Definition 3.5. *Let $(E, \delta) \subset (F, \delta)$ be δ-fields We say that $f \in F$ is* hyperalgebraic *over E if it satisfies a non trivial algebraic differential equation with coefficients in E, i.e., if for some m there exists a nonzero polynomial $P(y_0, \dots, y_m) \in F[y_0, \dots, y_m]$ such that*

$$P(f, \delta(f), \dots, \delta^m(f)) = 0.$$

We say that f is honolomic *over E if in addition, the equation is linear. We say that f is* hypertranscendental *over E if it is not hyperalgebraic.*

Other terms have been used for the above concepts: hypotranscendental or differentially algebraic or δ-algebraic for hyperalgebraic and differentially transcendental or transcendentally transcendental for hypertranscendental.

Proposition 2.6 of [DHR15] give criteria for hypertranscendence in this general setting. We will only state this result in our situation. As in Remark 3.3, we may consider $(\mathbb{C}(\overline{E_t}), \delta, \tau)$ as a subfield of $(\mathcal{M}(\mathbb{C}), \frac{d}{d\omega}, \tau : \omega \to \omega + \omega_3)$. Given $f \in \mathcal{M}(\mathbb{C})$ we denote by $\mathbb{C}(\overline{E_t}) < f >_{\delta\tau}$ the smallest subfield of $\mathcal{M}(\mathbb{C})$ containing $\mathbb{C}(\overline{E_t})$ and $\{\tau^i(\delta^j(f)) \mid i, j \in \mathbb{Z}\}$. Note that this is a $\delta\tau$-field.

Proposition 3.6. *Let $b \in \mathbb{C}(\overline{E_t})$ and $f \in \mathcal{M}(\mathbb{C})$ and assume that*

$$\tau(f) - f = b.$$

If f is hyperalgebraic over $\mathbb{C}(\overline{E_t})$, then, there exist an integer $n \geq 1$, $c_0, \dots, c_{n-1} \in \mathbb{C}$ and $g \in \mathbb{C}(\overline{E_t})$ such that

$$\delta^n(b) + c_{n-1}\delta^{n-1}(b) + \cdots + c_1\delta(b) + c_0 b = \tau(g) - g. \tag{3.1}$$

Conversely, if b satisfies such an equality and if $(\mathbb{C}(\overline{E_t}) < f >_{\delta\tau})^\tau = \mathbb{C}$, then f is holonomic over $\mathbb{C}(\overline{E_t})$.

The condition $(\mathbb{C}(\overline{E_t}) < f >_{\delta\tau})^\tau = \mathbb{C}$ is not superfluous. We will reconfirm (cf. [BBMR16]) in Section 6 that for the nine exceptional cases, specializations of the generating series are hyperalgebraic but we know that they are nonetheless not holonomic. This situation arises because when one tries to apply the last part of Proposition 3.6 one is forced to add new τ-constants. This will become apparent in Section 6.

In the appendix, we give necessary and sufficient conditions on the poles of b to guarantee the existence of an equation of the form of (3.1). An example of a consequence of these conditions is the following result.

Corollary 3.7 (Corollary A.3)**.** *Assume that b has a pole $P \in \overline{E_t}$ of order $m \geq 1$ such that none of the $\tau^k(P)$ with $k \in \mathbb{Z} \setminus \{0\}$ is a pole of order $\geq m$ of b. Then, f is hypertranscendental.*

In Sections 5 and 6, we will verify conditions like this to show that various generating series are hypertranscendental.

3.3. **Applications to $F^1(x,t)$ and $F^2(y,t)$.** In this section, we assume that $\mathcal{D}$ and $0 < t < 1/|\mathcal{D}|$ satisfy the assumptions 2.9. We shall now apply the results of Section 3.2 to $F^1(x,t)$ and $F^2(y,t)$.

We begin by considering $F^1(x,t)$ as a formal power series with coefficients in $\mathbb{C}$. We wish to show that $F^1(x,t)$ does not satisfy a polynomial differential equation

$$P(x, F^1(x,t), \frac{dF^1(x,t)}{dx}, \ldots, \frac{d^nF^1(x,t)}{dx^n}) = 0$$

where $P \in \mathbb{C}[x, Y_0, \ldots, Y_n]$. Let us assume that such an equation existed. The derivation $\frac{d}{dx}$ extends uniquely to a derivation on $\mathbb{C}(\overline{E_t})$ which we again denote by $\frac{d}{dx}$. The derivations on $\mathbb{C}(\overline{E_t})$ form a one dimensional vector space over $\mathbb{C}(\overline{E_t})$. Therefore δ on $\overline{E_t}$ can be written as $\delta = h\frac{d}{dx}$ for some $h \in \mathbb{C}(\overline{E_t})$. In particular this implies that when we consider $F^1(x,t)$ as an analytic function on an open set of $\overline{E_t}$ it will satisfy a polynomial differential equation $\tilde{P}(F^1(x,t), \delta(F^1(x,t)), \ldots, \delta^n(F^1(x,t))) = 0$ where $\tilde{P}$ has coefficients in $\mathbb{C}(\overline{E_t})$. When we lift this equation to the universal cover, we see that $r_x \in \mathcal{M}(\mathbb{C})$ is hyperalgebraic over $\mathbb{C}(\overline{E_t})$. By assumption 2.9, r_x satisfies $\tau(r_x) - r_x = b_1$ where $b_1 \in \mathbb{C}(\overline{E_t})$. We can therefore apply Proposition 3.6 and conclude that there exist an integer $n \geq 1$, $c_0, \ldots, c_{n-1} \in \mathbb{C}$ and $g \in \mathbb{C}(\overline{E_t})$ such that

$$\delta^n(b_1) + c_{n-1}\delta^{n-1}(b_1) + \cdots + c_1\delta(b_1) + c_0 b_1 = \tau(g) - g. \tag{3.2}$$

Therefore to show $F^1(x,t)$ is hypertranscendental, it is enough to show that such an equation does not exist. Notice that this last condition only involves elements in $\mathbb{C}(\overline{E_t})$. Similar reasoning (where we replace $\frac{d}{dx}$ with $\frac{d}{dy}$) shows that $F^2(y,t)$ is hypertranscendental over $\mathbb{C}(\overline{E_t})$ if there is no relation such as (3.2) with b_1 replaced by b_2. Therefore we have

Proposition 3.8. *Let $i \in \{1,2\}$. The function F^i is hypertranscendental over $\mathbb{C}(\overline{E_t})$ if there does not exist an integer $n \geq 1$, $c_0, \ldots, c_{n-1} \in \mathbb{C}$ and $g \in \mathbb{C}(\overline{E_t})$ such that*

$$\delta^n(b_i) + c_{n-1}\delta^{n-1}(b_i) + \cdots + c_1\delta(b_i) + c_0 b_i = \tau(g) - g. \tag{3.3}$$

Once again, using results from the appendix we have

Corollary 3.9. *Let $i \in \{1,2\}$. Assume that b_i has a pole $P \in \overline{E_t}$ of order $m \geq 1$ such that none of the $\tau^k(P)$ with $k \in \mathbb{Z} \setminus \{0\}$ is a pole of order $\geq m$ of b_i. Then, F^i is hypertranscendental over $\mathbb{C}(\overline{E_t})$.*

The following additional result will also be useful.

Proposition 3.10. *The function $F^1(x,t)$ is hypertranscendental over $\mathbb{C}(\overline{E_t})$ if and only if $F^2(y,t)$ is hypertranscendental over $\mathbb{C}(\overline{E_t})$.*

Proof. We note that $b_1 + b_2 = \tau(xy) - xy$. So, for all $k \in \mathbb{Z}_{\geq 0}$,

$$\delta^k(b_2) \quad = \quad -\delta^k(b_1) + \tau(\delta^k(xy)) - \delta^k(xy). \tag{3.4}$$

It follows that an equation of the form (3.3) holds true for $i = 1$ if and only if such an equation holds true for $i = 2$. □

4. Preliminary results on the elliptic curve $\overline{E_t}$

In this and the next section we will show that the 42 nonsingular unweighted nonexceptional walks, have generating series that are hypertranscendental. As we have indicated, an examination of the poles of the $b_i, i = 1, 2$, and the orbits of these poles under τ will yield these results.

To get a sense of our techniques, we will outline in the following example how we show that $F^2(y,t)$ is hypertranscendental for the unweighted walk IA.1.

Example 4.1. *For the walk IA.1, we have $d_{-1,1} = d_{1,1} = d_{1,-1} = d_{0,-1} = 1$ and all other $d_{i,j} = 0$. The curve $\overline{E_t}$ is defined by*

$$\overline{K}(x_0, x_1, y_0, y_1, t) = x_0x_1y_0y_1 - t(x_1^2y_0^2 + x_0^2y_0^2 + x_0^2y_1^2 + x_0x_1y_1^2).$$

Recalling that $x = x_0/x_1$ and $y = y_0/y_1$ one sees that the poles of $b_2 = x(\iota_1(y) - y)$ are among the poles of x, y, and $\iota_1(y)$, that is, among the points

$$S_2 = \{P_1, P_2, Q_1, Q_2, \iota_1(Q_1), \iota_1(Q_2)\} = \{P_1, P_2, Q_1, Q_2, \tau^{-1}(Q_1), \tau^{-1}(Q_2)\}$$

where

$$P_1 = ([1:0], [\sqrt{-1}:1]), \qquad P_2 = ([1:0], [-\sqrt{-1}, 1]),$$
$$Q_1 = ([1:\sqrt{-1}], [1:0]), \qquad Q_2 = ([1:\sqrt{-1}], [1:0]).$$

One sees that b_2 has a pole at P_1. We claim that b_2 has no pole of the form $\tau^k(P_1)$ with $k \in \mathbb{Z}\backslash\{0\}$. Once this is shown the result will follow from Corollary 3.9. Therefore we must show that $\tau^k(P_1) \neq P_2$ for any $k \in \mathbb{Z}\backslash\{0\}$ and $\tau^k \neq Q_1, Q_2$ for any $k \in \mathbb{Z}$.

To verify this, first note that P_1, P_2, Q_1, Q_2 all have coordinates in $L = \mathbb{Q}(\sqrt{-1})$. Let σ be the automorphism of L defined by $\sigma(\sqrt{-1}) = -\sqrt{-1}$. A calculation shows that for any point $Q \in \overline{E_t}(L)$ one has that $\iota_1(Q), \iota_2(Q), \tau(Q) \in \overline{E_t}(L)$ and that $\iota_k \circ \sigma = \sigma \circ \iota_k, k = 1, 2$, and $\tau\sigma = \sigma\tau$ (cf. Proposition 4.8). Therefore all points in S_2 have coordinates in L. Furthermore $P_2 = \sigma(P_1)$ and $Q_2 = \sigma(Q_1)$.

If $P_2 = \tau^k(P_1)$, then $P_1 = \sigma(P_2) = \sigma(\tau^k(P_1)) = \tau^k(\sigma(P_1)) = \tau^k(P_2) = \tau^{2k}(P_1)$. Since τ corresponds to addition by a point of infinite order, we get a contradiction if $k \neq 0$.

If $Q_1 = \tau^k(P_1)$ for some $k \in \mathbb{Z}$, then $\tau^k(P_2) = \sigma(\tau^k(P_1)) = \sigma(Q_1) = Q_2$. Since $P_1 = \iota_1(P_2)$ and $\tau = \iota_2\iota_1$, we have that

$$\tau^{-k+1}(P_1) = \tau^{-k+1}\iota_1(P_2) = \iota_2\tau^k(P_2) = Q_1 = \tau^k(P_1).$$

Therefore $\tau^{-2k+1}(P_1) = P_1$, again a contradiction, since $k \in \mathbb{Z}$. The proof that $Q_2 \neq \tau^k(P_1)$ for any $k \in \mathbb{Z}$ is similar.

A similar proof combining arithmetic (*e.g.*, Galois theory) with geometry (the behavior of points under ι_1, ι_2, τ) shows that b_1 or b_2 for the walks corresponding to the steps IA, IB, IC, and IIA has a pole that is unique in its τ-orbit. In the remaining cases, the arguments become more complicated. The poles lie in $\mathbb{Q}$ and so we do not have a field automorphism at our disposal. In addition we may need to look more carefully at which poles lie in which orbits[2] and consider their orders as well as the expansions at these poles. Nonetheless, the interplay of arithmetic and geometry will be the key to the arguments.

In this section we examine points that are possible poles of b_1 and b_2 and develop the properties needed in the next sections. In Section 5 we will apply these together with results from the appendix and Proposition 3.8 to conclude hypertranscendence in the above 42 cases. As before, we assume that t and $\mathcal{D}$ satisfy the assumptions 2.9.

4.1. **On the base points.**

Definition 4.2. *The points $P = ([x_0 : x_1], [y_0 : y_1])$ of $\overline{E_t}$ such that $x_0x_1y_0y_1 = 0$ will be called the base points of $\overline{E_t}$.*

Remark 4.3. *This terminology comes from the fact that these points are the base points of a natural pencil of elliptic curves.*

Let us recall that the notation $[a : b] \in \mathbb{P}^1(\mathbb{C})$ represents a ray of points of the form $\{(\alpha a, \alpha b) \mid 0 \neq \alpha \in \mathbb{C}\}$. Let $\mathbb{L}$ be a subfield of $\mathbb{C}$. We say that $[a : b] \in \mathbb{P}^1(\mathbb{L})$ if there is some element of this ray with coordinates in $\mathbb{L}$. A similar notation concerns points in $\overline{E_t}(\mathbb{L})$, the elements of which will be called the $\mathbb{L}$-points of $\overline{E_t}$.

Lemma 4.4. *We assume that for all $i \in \{-1, 1\}$ there exists $j \in \{0, \pm 1\}$ such that $d_{i,j} \neq 0$ and vice versa (which is the case for the 51 nonsingular walks with small steps and infinite group listed in Figure 1). Any base point of $\overline{E_t}$ belongs to $\overline{E_t}(\overline{\mathbb{Q}})$.*

Proof. Let $P = ([x_0 : x_1], [y_0 : y_1])$ be a base point of $\overline{E_t}$. Let us assume for instance that $x_0 = 0$ (the other cases being similar). Then, we obviously have $[x_0 : x_1] = [0 : 1] \in \mathbb{P}^1(\overline{\mathbb{Q}})$. Moreover, we have

$$\overline{K}(x_0, x_1, y_0, y_1, t) = x_0x_1y_0y_1 - t\sum_{i,j=0}^{2} d_{i-1,j-1}x_0^i x_1^{2-i} y_0^j y_1^{2-j}$$
$$= -tx_1^2 \sum_{j=0}^{2} d_{-1,j-1}y_0^j y_1^{2-j}$$

which is equal to 0 if and only if $\sum_{j=0}^{2} d_{-1,j-1}y_0^j y_1^{2-j} = 0$, whence $[y_0 : y_1] \in \mathbb{P}^1(\overline{\mathbb{Q}})$. $\square$

Lemma 4.5. *We assume that the walk under consideration is one of the walks listed in Figure 1. Then, the following arrays describe precisely what are the possible base points fixed by ι_1 or ι_2 and the conditions on the $d_{i,j}$:*

Points	*Fixed by ι_1*	
$x_0 = 0$	$([0:1],[1:0])$ *iff* $d_{-1,0} = d_{-1,1} = 0$	$([0:1],[0:1])$ *iff* $d_{-1,0} = d_{-1,-1} = 0$
$y_0 = 0$	$([1:0],[0:1])$ *iff* $d_{1,0} = d_{1,-1} = 0$	$([0:1],[0:1])$ *iff* $d_{-1,-1} = d_{-1,0} = 0$
$x_1 = 0$	$([1:0],[1:0])$ *iff* $d_{1,0} = d_{1,1} = 0$	$([1:0],[0:1])$ *iff* $d_{1,0} = d_{1,-1} = 0$
$y_1 = 0$	$([1:0],[1:0])$ *iff* $d_{1,0} = d_{1,1} = 0$	$([0:1],[1:0])$ *iff* $d_{-1,0} = d_{-1,1} = 0$

Points	*Fixed by ι_2*	
$x_0 = 0$	$([0:1],[1:0])$ *iff* $d_{0,1} = d_{-1,1} = 0$	$([0:1],[0:1])$ *iff* $d_{-1,-1} = d_{0,-1} = 0$
$y_0 = 0$	$([1:0],[0:1])$ *iff* $d_{0,-1} = d_{1,-1} = 0$	$([0:1],[0:1])$ *iff* $d_{0,-1} = d_{-1,-1} = 0$
$x_1 = 0$	$([1:0],[1:0])$ *iff* $d_{0,1} = d_{1,1} = 0$	$([1:0],[0:1])$ *iff* $d_{0,-1} = d_{1,-1} = 0$
$y_1 = 0$	$([1:0],[1:0])$ *iff* $d_{0,1} = d_{1,1} = 0$	$([0:1],[1:0])$ *iff* $d_{0,1} = d_{-1,1} = 0$

[2]We note that given points Q_1 and Q_2 on an elliptic curve $\overline{E_t}$, a general procedure is given in [Mas88] to determine if there is an integer n such that $Q_1 = Q_2 \oplus nP$ where $\tau(Q) = Q \oplus P$. Our more elementary, direct approach is independent of [Mas88].

Proof. Taking into consideration obvious symmetries, we see that it is sufficient to prove the Lemma for a base point of the form $P = ([0:1],[\beta_0:\beta_1])$.

Assume that P is fixed by ι_1. By Lemma 2.5, this is equivalent to

$$\Delta^x_{[0:1]}/t^2 = d_{-1,0}^2 - 4d_{-1,1}d_{-1,-1} = 0.$$

Since the $d_{i,j}$ belong to $\{0,1\}$, the latter condition is equivalent to the equality $d_{-1,0} = 0 = d_{-1,-1}d_{-1,1}$.

If $d_{-1,0} = d_{-1,-1} = 0$, then we have $d_{-1,1} \neq 0$ and the fact that P belongs to $\overline{E_t}$ simply means that $d_{-1,1}\beta_1^2 = 0$. Therefore, we have $P = ([0:1],[1:0])$.

If $d_{-1,0} = d_{-1,1} = 0$, then we have $d_{-1,-1} \neq 0$ and the fact that P belongs to $\overline{E_t}$ simply means that $d_{-1,-1}\beta_0^2 = 0$. Therefore, we have $P = ([0:1],[0:1])$.

This is precisely the first line of the first array.

Assume now that P is fixed by ι_2. Since P and $\iota_2(P)$ belong to the curve E_t, the x-coordinates of both P and $\iota_2(P)$ must satisfy the homogeneous equation of degree 2 in x_0 and x_1 given by $x_0^2A + Bx_0x_1 + Cx_1^2 = 0$ with

$$\begin{array}{rcl} A &=& d_{1,-1}\beta_1^2 + d_{1,0}\beta_0\beta_1 + d_{1,1}\beta_0^2, \\ B &=& \beta_0\beta_1 - t(d_{0,-1}\beta_1^2 + d_{0,1}\beta_0^2), \\ C &=& d_{-1,-1}\beta_1^2 + d_{-1,0}\beta_0\beta_1 + d_{-1,1}\beta_0^2. \end{array}$$

Since the x-coordinate of P is equal to $[0:1]$, we see that $C = 0$. Moreover, the fact that $\iota_2(P) = P$ ensures that $[0:1]$ is the only solution in $\mathbb{P}^1(\mathbb{C})$ of the homogeneous equation $x_0^2A + Bx_0x_1 + Cx_1^2 = x_0^2A + Bx_0x_1 = 0$. This ensures that $B = 0$. Thus, we have obtained the equalities

$$B = \beta_0\beta_1 - t(d_{0,-1}\beta_1^2 + d_{0,1}\beta_0^2) = 0 \tag{4.1}$$

$$C = d_{-1,-1}\beta_1^2 + d_{-1,0}\beta_0\beta_1 + d_{-1,1}\beta_0^2 = 0. \tag{4.2}$$

But, according to Lemma 4.4, $[\beta_0:\beta_1]$ belongs to $\mathbb{P}^1(\overline{\mathbb{Q}})$ and, by hypothesis, t is transcendental. Therefore, equation (4.1) is equivalent to

$$\beta_0\beta_1 = 0 = d_{0,-1}\beta_1^2 + d_{0,1}\beta_0^2,$$

i.e., $\beta_0 = 0$ and $d_{0,-1} = 0$ or $\beta_1 = 0$ and $d_{0,1} = 0$.

If $\beta_0 = 0$ and $d_{0,-1} = 0$, then $P = ([0:1],[0:1])$. This point belongs to $\overline{E_t}$ if and only if $d_{-1,-1} = 0$.

If $\beta_1 = 0$ and $d_{0,1} = 0$, then $P = ([0:1],[1:0])$. This point belongs to $\overline{E_t}$ if and only if $d_{-1,1} = 0$.

This gives the first line of the second array. □

Lemma 4.6. *We assume that the walk under consideration is one of the walks listed in Figure 1 and that $t \in \mathbb{C} \setminus \overline{\mathbb{Q}}$. Let $P = ([\alpha_0:1],[\beta_0:1]) \in \overline{E_t}(\mathbb{Q}(t))$. The following properties hold true :*

- *if $\iota_1(P) = P$ then $P = ([0:1],[0:1])$ and $d_{-1,0} = d_{-1,-1} = 0$;*
- *if $\iota_2(P) = P$ then $P = ([0:1],[0:1])$ and $d_{0,-1} = d_{-1,-1} = 0$.*

Proof. Taking into consideration obvious symmetries, we see that it is sufficient to prove the first statement. Since t is transcendental, we can and will identify $\mathbb{Q}(t)$ with a field of rational functions. Assume that $P = ([\alpha_0:1],[\beta_0:1]) \in \overline{E_t}(\mathbb{Q}(t))$ is fixed by ι_1. By Lemma 2.5, we must have $\Delta^x_{[\alpha_0:1]} = 0$, *i.e.*,

$$(d_{-1,0} - \frac{\alpha_0}{t} + d_{1,0}\alpha_0^2)^2 = 4(d_{-1,1} + d_{0,1}\alpha_0 + d_{1,1}\alpha_0^2)(d_{-1,-1} + d_{0,-1}\alpha_0 + d_{1,-1}\alpha_0^2). \tag{4.3}$$

Step 1: Case $\alpha_0 \in \mathbb{Q}$.

If $\alpha_0 \in \mathbb{Q}$, then, comparing the t-adic valuations of both sides of (4.3), we get $\alpha_0 = 0$, *i.e.*, $P = ([0 : 1], [\beta_0 : 1])$. In this case, equation (4.3) is simply $d_{-1,0}^2 = 4d_{-1,1}d_{-1,1}$. Since $d_{i,j} \in \{0, 1\}$, we get $d_{-1,0} = 0 = d_{-1,1}d_{-1,-1}$. This leads us to consider the two cases $d_{-1,0} = d_{-1,-1} = 0$ and $d_{-1,0} = d_{-1,1} = 0$.

If $d_{-1,0} = d_{-1,-1} = 0$, then we have $d_{-1,1} \neq 0$ and the fact that P belongs to $\overline{E_t}$ simply means that $d_{-1,1}\beta_0^2 = 0$. Therefore, we have $P = ([0 : 1], [0 : 1])$, as desired.

If $d_{-1,0} = d_{-1,1} = 0$, the fact that P belongs to $\overline{E_t}$ simply means that $d_{-1,-1} = 0$. But for every walks under consideration, $d_{-1,-1} = d_{-1,0} = d_{-1,1} = 0$ never occur.

We shall now prove that $\alpha_0 \in \mathbb{Q}(t) \setminus \mathbb{Q}$ is impossible. A key fact that is used several times in the proof is that in the 51 cases under consideration, each set of steps contain elements lying on both sides of each of the lines: $i = 0, i + j = 0, j = 0, i - j = 0$. For example, the following condition $d_{-1,0} = d_{-1,1} = d_{0,1} = 0$ is never realized since this condition would imply all steps lie on or below the line $i - j = 0$. We argue by contradiction. Equation (4.3) ensures that α_0 must have either a pole or a zero at $t = 0$. Indeed, otherwise, the left hand side of (4.3) would have a pole at $t = 0$ but not the right hand side.

Let us write $\alpha_0 = t^l P(t)$ with $l \in \mathbb{Z}^*$ and $P(t) \in \mathbb{Q}(t)$ without zero and pole at $t = 0$.

Step 2: Case $P(t)$ constant.

If $P(t) = c \in \mathbb{Q}^*$ then (4.3) becomes

$$(4.4)\quad (d_{-1,0} - ct^{l-1} + d_{1,0}c^2t^{2l})^2 = 4(d_{-1,1} + d_{0,1}ct^l + d_{1,1}c^2t^{2l})(d_{-1,-1,} + d_{0,-1}ct^l + d_{1,-1}c^2t^{2l}).$$

If $l < 0$, then, equating the coefficients of t^0 in (4.4), we find the equality $d_{-1,0}^2 = 4d_{-1,1}d_{-1,-1}$. This gives $d_{-1,0} = 0$ and $d_{-1,1}d_{-1,-1} = 0$.

If $d_{-1,0} = 0$ and $d_{-1,1} = 0$, then $d_{-1,-1} \neq 0$. Moreover, equation (4.4) simplifies as follows.

$$(4.5)\quad (-ct^{l-1} + d_{1,0}c^2t^{2l})^2 = 4(d_{0,1}ct^l + d_{1,1}c^2t^{2l})(d_{-1,-1,} + d_{0,-1}ct^l + d_{1,-1}c^2t^{2l}).$$

On the right hand side of (4.5), we find the monomial $4cd_{0,1}d_{-1,-1}t^l$, but the non trivial monomials appearing on the left hand side have degree $2l - 2$, $3l - 1$ or $4l$ and none of them is equal to l. Therefore, we must have $d_{0,1}d_{-1,-1} = 0$ and, hence, $d_{0,1} = 0$. The condition $d_{-1,0} = d_{-1,1} = d_{0,1} = 0$ is never realized.

If $d_{-1,0} = 0$ and $d_{-1,-1} = 0$, then $d_{-1,1} \neq 0$. Moreover, equation (4.4) simplifies as follows.

$$(4.6)\quad (-ct^{l-1} + d_{1,0}c^2t^{2l})^2 = 4(d_{-1,1} + d_{0,1}ct^l + d_{1,1}c^2t^{2l})(d_{0,-1}ct^l + d_{1,-1}c^2t^{2l}).$$

On the right hand side of (4.6), we find the monomial $4cd_{-1,1}d_{0,-1}t^l$, but the non trivial monomials appearing on the left hand side have degree $2l - 2$, $3l - 1$ or $4l$ and none of them is equal to l. Therefore, we must have $d_{-1,1}d_{0,-1} = 0$ and, hence, $d_{0,-1} = 0$. The condition $d_{-1,0} = d_{-1,-1} = d_{0,-1} = 0$ is never realized.

If $l > 0$, then, equating the coefficients of t^{4l} in (4.4), we find the equality $d_{1,0}^2 = 4d_{1,1}d_{1,-1}$. This gives $d_{1,0} = 0$ and $d_{1,1}d_{1,-1} = 0$. We claim that this is impossible.

We first assume that $d_{1,0} = 0$ and $d_{1,-1} = 0$. In every walks under consideration, we must have $d_{1,1} \neq 0$. Equation (4.4) simplifies as follows:

$$(4.7)\quad (d_{-1,0} - ct^{l-1})^2 = d_{-1,0}^2 - 2cd_{-1,0}t^{l-1} + c^2t^{2l-2} = 4(d_{-1,1} + d_{0,1}ct^l + d_{1,1}c^2t^{2l})(d_{-1,-1,} + d_{0,-1}ct^l).$$

On the right side of (4.7), we find the monomial $4c^3 d_{1,1} d_{0,-1} t^{3l}$. But, since $l > 0$, we have $3l \neq 0, l-1, 2l-2$. It follows that $d_{1,1} d_{0,-1} = 0$ and, hence, $d_{0,-1} = 0$. The condition $d_{1,0} = d_{1,-1} = d_{0,-1} = 0$ is never realized.

We now assume that $d_{1,0} = 0$ and $d_{1,1} = 0$. In every walks under consideration, we must have $d_{1,-1} \neq 0$. Equation (4.4) simplifies as follows:

$$(4.8)\quad (d_{-1,0} - ct^{l-1})^2 = d_{-1,0}^2 - 2cd_{-1,0}t^{l-1} + c^2 t^{2l-2} = 4(d_{-1,1} + d_{0,1}ct^l)(d_{-1,-1}, + d_{0,-1}ct^l + d_{1,-1}c^2 t^{2l}).$$

On the right side of (4.8), we find the monomial $4c^3 d_{0,1} d_{1,-1} t^{3l}$. But, since $l > 0$, we have $3l \neq 0, l-1, 2l-2$. It follows that $d_{0,1} d_{1,-1} = 0$ and, hence, $d_{0,1} = 0$. The condition $d_{1,0} = d_{1,1} = d_{0,1} = 0$ is never realized.

So, it remains to study the case when $P(t)$ is not constant. In this case, it has at least one zero or pole at some $t_0 \in \mathbb{C}^*$.

Step 3: Poles of $P(t)$.

Assume that $P(t)$ has a pole $t_0 \in \mathbb{C}^*$ of order $\kappa \geq 1$. Multiplying both sides of (4.3) by $(t-t_0)^{4\kappa}$ and evaluating at t_0, we find that $d_{1,0}^2 = 4d_{1,1}d_{1,-1}$, which implies that $d_{1,0} = 0 = d_{1,1}d_{1,-1}$.

We first assume that $d_{1,0} = 0$ and $d_{1,1} = 0$. Then, (4.3) simplifies as follows:

$$(4.9)\qquad (d_{-1,0} - \frac{\alpha_0}{t})^2 = 4(d_{-1,1} + d_{0,1}\alpha_0)(d_{-1,-1} + d_{0,-1}\alpha_0 + d_{1,-1}\alpha_0^2).$$

Therefore, the term in α_0^3 of the right hand side of (4.9) must be equal to 0, *i.e.*, $d_{0,1}d_{1,-1} = 0$. So, we have either $d_{1,0} = d_{1,1} = d_{0,1} = 0$ or $d_{1,0} = d_{1,1} = d_{1,-1} = 0$. These conditions are never realized.

We now assume that $d_{1,0} = 0$ and $d_{1,-1} = 0$. Then, (4.3) simplifies as follows:

$$(4.10)\qquad (d_{-1,0} - \frac{\alpha_0}{t})^2 = 4(d_{-1,1} + d_{0,1}\alpha_0 + d_{1,1}\alpha_0^2)(d_{-1,-1} + d_{0,-1}\alpha_0).$$

Multiplying both sides of (4.10) by $(t-t_0)^{3\kappa}$ and evaluating at t_0, we find that the term in α_0^3 of the right hand side of (4.10) must be equal to 0, *i.e.*, $d_{1,1}d_{0,-1} = 0$. So, we have either $d_{1,0} = d_{1,-1} = d_{1,1} = 0$ or $d_{1,0} = d_{1,-1} = d_{0,-1} = 0$. These conditions are never realized.

Step 4: Zeros of $P(t)$.

Assume that $P(t)$ has a zero $t_0 \in \mathbb{C}^*$ of order $\nu \geq 1$. Evaluating (4.3) at t_0, we get $d_{-1,0}^2 = 4d_{-1,1}d_{-1,-1}$, which implies $d_{-1,0} = 0 = d_{-1,1}d_{-1,-1}$.

We first assume that $d_{-1,0} = 0$ and $d_{-1,1} = 0$. Then (4.3) simplifies as follows:

$$(4.11)\qquad (-\frac{\alpha_0}{t} + d_{1,0}\alpha_0^2)^2 = 4(d_{0,1}\alpha_0 + d_{1,1}\alpha_0^2)(d_{-1,-1} + d_{0,-1}\alpha_0 + d_{1,-1}\alpha_0^2).$$

Since the walk is non degenerate, we have $d_{-1,-1} \neq 0$. Let $\alpha \neq 0$ be the value of $(t-t_0)^{-\nu}\alpha_0$ at t_0. Dividing both sides of (4.11) by $(t-t_0)^{\nu}$ and evaluating at t_0 we find $0 = 4\alpha d_{0,1}d_{-1,-1}$, which implies $d_{0,1} = 0$. So, we have obtained $d_{-1,0} = d_{-1,1} = d_{0,1} = 0$. This condition are never realized.

We now assume that $d_{-1,0} = 0$ and $d_{-1,-1} = 0$. Then (4.3) simplifies as follows:

$$(4.12)\qquad (-\frac{\alpha_0}{t} + d_{1,0}\alpha_0^2)^2 = 4(d_{-1,1} + d_{0,1}\alpha_0 + d_{1,1}\alpha_0^2)(d_{0,-1}\alpha_0 + d_{1,-1}\alpha_0^2).$$

Since the walk is non degenerate, we have $d_{-1,1} \neq 0$. Let $\alpha \neq 0$ be the value of $(t-t_0)^{-\nu}\alpha_0$ at t_0. Dividing both sides of (4.12) by $(t-t_0)^{\nu}$ and evaluating at t_0 we find $0 = 4\alpha d_{-1,1}d_{0,-1}$, which implies $d_{0,-1} = 0$. So, we have obtained $d_{-1,0} = d_{-1,-1} = d_{0,-1} = 0$. This condition are never realized. □

In the following lemma, we focus our attention on the base points $([x_0 : x_1], [y_0 : y_1])$ of $\overline{E_t}$ corresponding to the equation $x_1 y_1 = 0$, namely:

$$P_1 = ([1:0],[\beta_0:\beta_1]), \quad P_2 = \iota_1(P_1) = ([1:0],[\beta_0':\beta_1']),$$
$$Q_1 = ([\alpha_0:\alpha_1],[1:0]), \quad Q_2 = \iota_2(Q_1) = ([\alpha_0':\alpha_1'],[1:0]).$$

We will use the following notations :

$$L_x = \mathbb{Q}\left(\sqrt{\Delta^x_{[1:0]}/t^2}\right) \text{ and } L_y = \mathbb{Q}\left(\sqrt{\Delta^y_{[1:0]}/t^2}\right).$$

Lemma 4.7. *The points P_1 and P_2 (resp. Q_1 and Q_2) are L_x-points (resp. L_y-points) of $\overline{E_t}$. They are $\mathbb{Q}$-points of $\overline{E_t}$ if and only if $\Delta^x_{[1:0]}/t^2$ (resp. $\Delta^y_{[1:0]}/t^2$) is a square in $\mathbb{Q}$. If the walk is one of the walks listed in in Figure 1, then the following properties hold true :*

- *$\Delta^x_{[1:0]}/t^2$ is a square in $\mathbb{Q}$ if and only if $d_{1,-1}d_{1,1} = 0$; moreover :*
  - *$d_{1,1} = 0$ if and only if there exist $i,j \in \{1,2\}$ such that $P_i = Q_j = ([1:0],[1:0])$;*
  - *$d_{1,-1} = 0$ if and only if there exists $i \in \{1,2\}$ such that $P_i = ([1:0],[0:1])$;*
- *$\Delta^y_{[1:0]}/t^2$ is a square in $\mathbb{Q}$ if and only if $d_{-1,1}d_{1,1} = 0$; moreover :*
  - *$d_{1,1} = 0$ if and only if there exist $i,j \in \{1,2\}$ such that $P_i = Q_j = ([1:0],[1:0])$;*
  - *$d_{-1,1} = 0$ if and only if there exists $j \in \{1,2\}$ such that $Q_j = ([0:1],[1:0])$.*

Proof. Taking into consideration the obvious symmetry in x and y, we see that it is sufficient to prove the result for P_1 and P_2 and the first of the last two statements of the Lemma.

The y-coordinates $[\beta_0 : \beta_1]$ and $[\beta_0' : \beta_1']$ of P_1 and P_2 are the roots in $\mathbb{P}^1(\mathbb{C})$ of the homogeneous polynomial in y_0 and y_1 given by

$$d_{1,-1}y_1^2 + d_{1,0}y_0y_1 + d_{1,1}y_0^2 = 0. \tag{4.13}$$

Therefore, $[\beta_0 : \beta_1]$ and $[\beta_0' : \beta_1']$ belong to $\mathbb{P}^1(L_x)$. Moreover, we see that they belong to $\mathbb{P}^1(\mathbb{Q})$ if and only if $\Delta^x_{[1:0]}/t^2 = d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ is a square in $\mathbb{Q}$. If the walk in unweighted, then the $d_{i,j}$ are in $\{0,1\}$ and $d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ is a square in $\mathbb{Q}$ if and only if $d_{1,-1}d_{1,1} = 0$.

The fact that $d_{1,1} = 0$ is equivalent to the fact that $[1 : 0]$ is a root of equation (4.13) and this is equivalent to the fact that there exist $i,j \in \{1,2\}$ such that $P_i = Q_j = ([1:0],[1:0])$.

Similarly, the fact that $d_{1,-1} = 0$ is equivalent to the fact that $[0 : 1]$ is a root of equation (4.13) and this is equivalent to the fact that $P_1 = ([1:0],[0:1])$ or $P_2 = ([1:0],[0:1])$. □

4.2. **Galois action on $\overline{E_t}$.** Let $\mathbb{Q}(t) \subset L \subset \mathbb{C}$ be a field extension. For any L-point $P = ([x_0 : x_1],[y_0 : y_1])$ of $\overline{E_t}$, with $x_0, x_1, y_0, y_1 \in L$, and any $\sigma \in \mathrm{Aut}(L/\mathbb{Q}(t))$, we set

$$\sigma(P) = ([\sigma(x_0) : \sigma(x_1)], [\sigma(y_0) : \sigma(y_1)]).$$

Since $\overline{E_t}$ is defined over $\mathbb{Q}(t)$, $\sigma(P)$ is an L-point of $\overline{E_t}$.

Proposition 4.8. *Let $\mathbb{Q}(t) \subset L \subset \mathbb{C}$ be a field extension and let $\sigma \in \mathrm{Aut}(L/\mathbb{Q}(t))$. Let P be a L-point of $\overline{E_t}$. Then, the following properties hold true :*

- *$\iota_1(P), \iota_2(P)$ and, hence, $\tau^n(P)$ for any $n \in \mathbb{Z}$, are L-points of $\overline{E_t}$;*
- *for any $k \in \{1,2\}$, $\iota_k \circ \sigma = \sigma \circ \iota_k$ on $\overline{E_t}(L)$ and, hence, $\tau \circ \sigma = \sigma \circ \tau$ on $\overline{E_t}(L)$.*

Proof. We only prove the assertions concerning ι_1. The proofs for ι_2 are similar and the assertions concerning τ follow from those about ι_1 and ι_2 since $\tau = \iota_2 \circ \iota_1$.

We set $P = ([a_0 : a_1], [b_0 : b_1]) \in \overline{E_t}(L)$ (with $a_0, a_1, b_0, b_1 \in L$) and $\iota_1(P) = ([a_0 : a_1], [b'_0 : b'_1])$. The point $[b'_0 : b'_1]$ is characterized by the fact that $[b_0 : b_1]$ and $[b'_0 : b'_1]$ are the roots in $\mathbb{P}^1(\mathbb{C})$ of the homogeneous polynomial in y_0 and y_1 given by

$$A(a_0, a_1)y_0^2 + B(a_0, a_1)y_0y_1 + C(a_0, a_1)y_1^2 \tag{4.14}$$

where $A(a_0, a_1) = d_{-1,1}a_1^2 + d_{0,1}a_0a_1 + d_{1,1}a_0^2$, $B(a_0, a_1) = d_{-1,0}a_1^2 - \frac{1}{t}a_0a_1 + d_{1,0}a_0^2$ and $C(a_0, a_1) = d_{-1,-1}a_1^2 + d_{0,-1}a_0a_1 + d_{1,-1}a_0^2$ (these are not all 0). Since (4.14) has coefficients in L and $b_0, b_1 \in L$, we can assume that $b'_0, b'_1 \in L$ as well. Hence, $[b'_0 : b'_1] \in \mathbb{P}^1(L)$ and $\iota_1(P) \in \overline{E_t}(L)$, as desired.

Moreover, $[\sigma(b_0) : \sigma(b_1)]$ and $[\sigma(b'_0) : \sigma(b'_1)]$ are the roots in $\mathbb{P}^1(\mathbb{C})$ of

$$A(\sigma(a_0), \sigma(a_1))y_0^2 + B(\sigma(a_0), \sigma(a_1))y_0y_1 + C(\sigma(a_0), \sigma(a_1))y_1^2.$$

Therefore, $\iota_1(\sigma(P)) = ([\sigma(a_0) : \sigma(a_1)], [\sigma(b'_0) : \sigma(b'_1)]) = \sigma(\iota_1(P))$. □

4.3. **On the τ-orbits.**

Definition 4.9. *We define an equivalence relation $\sim$ on $\overline{E_t}$ by*

$$P \sim Q \Leftrightarrow \exists n \in \mathbb{Z}, \tau^n(P) = Q.$$

If $P \sim Q$ is not true, we shall write $P \nsim Q$. An equivalence class for $\sim$ will be called a τ-orbit.

Lemma 4.10. *Let us recall the notations $L_x = \mathbb{Q}(t)\left(\sqrt{\Delta^x_{[1:0]}}\right)$ and $L_y = \mathbb{Q}(t)\left(\sqrt{\Delta^y_{[1:0]}}\right)$. Moreover, we denote by $L = \mathbb{Q}(t)\left(\sqrt{\Delta^x_{[1:0]}}, \sqrt{\Delta^y_{[1:0]}}\right)$ the compositum of L_x and L_y. The following properties hold true :*

- *if $\mathbb{Q}(t) \subsetneq L_x$ or $\mathbb{Q}(t) \subsetneq L_y$ then, for all $i, j \in \{1, 2\}$, we have $P_i \nsim Q_j$;*
- *if $\mathbb{Q}(t) \subsetneq L_x$ (resp. $\mathbb{Q}(t) \subsetneq L_y$) then $P_i \nsim P_j$ (resp. $Q_i \nsim Q_j$) for $i \neq j$.*

Proof. We recall that due to the assumption 2.9, τ has infinite order (so that, it corresponds to a translation by a non torsion point). Let us prove the first assertion. Suppose to the contrary that, for instance, $P_1 \sim Q_1$ and that $\mathbb{Q}(t) \subsetneq L_x$, the other cases being similar. So, there exists $n \in \mathbb{Z}$ such that $\tau^n(P_1) = Q_1$. The fact that $P_1 = ([1 : 0], [\beta_0 : \beta_1])$ belongs to $\overline{E_t}$ means that

$$d_{1,-1}\beta_1^2 + d_{1,0}\beta_0\beta_1 + d_{1,1}\beta_0^2 = 0. \tag{4.15}$$

Since $\mathbb{Q}(t) \subsetneq L_x$, we have that $\Delta^x_{[1:0]}/t^2 = d_{1,0}^2 - 4d_{1,1}d_{1,-1}$ is not a square in $\mathbb{Q}(t)$. It follows that $d_{1,1}d_{1,-1} \neq 0$ and that $P_1 \in \overline{E_t}(L_x) \setminus \overline{E_t}(\mathbb{Q}(t))$. On the other hand, the fact that $Q_1 = ([\alpha_0 : \alpha_1], [1 : 0])$ belongs to $\overline{E_t}$ means that

$$d_{-1,1}\alpha_1^2 + d_{0,1}\alpha_0\alpha_1 + d_{1,1}\alpha_0^2 = 0.$$

So, Q_1 belongs to $\overline{E_t}(L_y)$. Since $\tau^{-n}(Q_1) = P_1$, Proposition 4.8 ensures that $P_1 \in \overline{E_t}(L_y)$ as well. Therefore, $P_1 \in (\overline{E_t}(L_x) \setminus \overline{E_t}(\mathbb{Q}(t))) \cap \overline{E_t}(L_y)$. In particular, $L_x \cap L_y$ is not reduced to $\mathbb{Q}(t)$. Since L_x and L_y are fields extensions of degree at most 2 of $\mathbb{Q}(t)$, we get $L_x = L_y$. Let $\sigma \in \mathrm{Gal}(L_x/\mathbb{Q}(t)) = \mathrm{Gal}(L_y/\mathbb{Q}(t))$ be an element of order 2. We obviously have $\sigma(P_1) = P_2$ and $\sigma(Q_1) = Q_2$. Using Proposition 4.8, it follows that $\tau^n(P_2) = \tau^n(\sigma(P_1)) = \sigma(\tau^n(P_1)) = \sigma(Q_1) = Q_2$. Therefore, $\iota_2\tau^n(P_2) = \iota_2(Q_2) = Q_1$. But, we have $\iota_2\tau^n = \tau^{-n+1}\iota_1$ (because $\tau = \iota_2\iota_1$ and the ι_k are involutions). Then, we find $\tau^{-n+1}(P_1) = \tau^{-n+1}\iota_1(P_2) = \iota_2\tau^n(P_2) = Q_1 = \tau^n(P_1)$. This gives $\tau^{-2n+1}(P_1) = P_1$. Since τ is a translation by a non torsion point of $\overline{E_t}$, this implies that $-2n + 1 = 0$. This yields a contradiction because $n \in \mathbb{Z}$.

We shall now prove the second assertion. Assume that $\mathbb{Q}(t) \subsetneq L_x$. In particular $\Delta^x_{[1:0]} \neq 0$ and, hence, $P_1 \neq \iota_1(P_1) = P_2$. Suppose to the contrary that $P_1 \sim P_2$, that is, that there exists $n \in \mathbb{Z}^*$ such that $\tau^n(P_1) = P_2$. Let $\sigma \in \mathrm{Gal}(L_x/\mathbb{Q}(t))$ be an element of order 2. We obviously have $\sigma(P_1) = P_2$. Using Proposition 4.8, we get $\tau^n(P_2) = \tau^n(\sigma(P_1)) = \sigma(\tau^n(P_1)) = \sigma(P_2) = P_1$. Therefore, $\tau^{2n}(P_1) = P_1$. Since τ is a translation by a non torsion point of $\overline{E_t}$, this implies that $n = 0$ and hence, $P_1 = P_2$. This yields a contradiction. □

4.4. The poles of b_1 and b_2.

Lemma 4.11. *The set of poles of $b_1 = \iota_1(y)\,(\tau(x) - x)$ in $\overline{E_t}$ is contained in*

$$\mathcal{S}_1 = \{\iota_1(Q_1), \iota_1(Q_2), P_1, P_2, \tau^{-1}(P_1), \tau^{-1}(P_2)\}.$$

Similarly, the set of poles of $b_2 = x(\iota_1(y) - y)$ in $\overline{E_t}$ is contained in

$$\mathcal{S}_2 = \{P_1, P_2, Q_1, Q_2, \iota_1(Q_1), \iota_1(Q_2)\} = \{P_1, P_2, Q_1, Q_2, \tau^{-1}(Q_1), \tau^{-1}(Q_2)\}.$$

Moreover, we have

$$(b_2)^2 = \frac{x_0^2 \Delta^x_{[x_0:x_1]}}{x_1^2 (\sum_{i=0}^2 x_0^i x_1^{2-i} t d_{i-1,1})^2}. \tag{4.16}$$

Proof. The proof of the assertions about the localization of the poles of b_1 and b_2 are straightforward. Let us prove (4.16). By definition, $\iota_1(\frac{y_0}{y_1})$ and $\frac{y_0}{y_1}$ are the two roots of the polynomial $y \mapsto \overline{K}(x_0, x_1, y, t)$. The square of their difference equals to the discriminant divided by the square of the leading term. Then, we have

$$\left(\iota_1(\frac{y_0}{y_1}) - \frac{y_0}{y_1}\right)^2 = \frac{\Delta^x_{[x_0:x_1]}}{(\sum_i x_0^i x_1^{2-i} t d_{i-1,1})^2}.$$

Therefore, we find

$$b_2(\frac{x_0}{x_1}, \frac{y_0}{y_1})^2 = \frac{x_0^2 \Delta^x_{[x_0:x_1]}}{x_1^2 (\sum_i x_0^i x_1^{2-i} t d_{i-1,1})^2}.$$

□

5. Hypertranscendance of the nonexceptional generating series

In the whole section, we assume that t and $\mathcal{D}$ satisfy the assumptions 2.9.

5.1. Generic cases.

The following proposition gives a diophantine criteria on $\mathcal{D}$ ensuring that the corresponding generating series are hypertranscendental.

Proposition 5.1. *We assume that the set $\mathcal{D}$ and t satisfy the assumptions 2.9. If $\Delta^x_{[1:0]}/t^2 = d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ or $\Delta^y_{[1:0]}/t^2 = d_{0,1}^2 - 4d_{-1,1}d_{1,1}$ is not a square in $\mathbb{Q}$ then $F^1(x,t)$ and $F^2(y,t)$ are δ-hypertranscendental over $\mathbb{C}(\overline{E_t})$.*

Proof. Assume for instance that $\Delta^x_{[1:0]}/t^2$ is not a square in $\mathbb{Q}$, the other case being similar. Combining Corollary 3.9 and Proposition 3.10, we see that it is sufficient to prove that P_1 is a pole of b_2 and that it is the only pole of b_2 of the form $\tau^n(P_1)$ with $n \in \mathbb{Z}$. The fact that P_1 is a pole of b_2 is clear (indeed, on the one hand, P_1 is a pole of x and, on the other hand, the y-coordinates of P_1 and $\iota_1(P_1)$ are distinct because $\Delta^x_{[1:0]} \neq 0$, and, hence, P_1 is not a zero of $\iota_1(y) - y$). Moreover, Lemma 4.10 implies that $P_1 \nsim P_2$ and $P_1 \nsim Q_i$ for $i = 1, 2$. The latter also implies that $P_1 \nsim \tau^{-1}(Q_i)$ for $i = 1, 2$. But, Lemma 4.11 ensures that the set of poles of b_2 is included in $\{P_1, P_2, Q_1, Q_2, \tau^{-1}(Q_1), \tau^{-1}(Q_2)\}$. So, P_1 is the only pole of b_2 of the form $\tau^n(P_1)$ with $n \in \mathbb{Z}$, as desired. □

From now on and until the end of the paper, we shall focus our attention on the walks listed in Figure 1.

Proposition 5.2. *For any of the walks listed in IA, IB, IC or IIA and for any* $t \in]0, 1/|\mathcal{D}|[\backslash\overline{\mathbb{Q}}$ *such that* G_t *is infinite, the generating series* $F^1(x,t)$ *and* $F^2(y,t)$ *are* δ*-hypertranscendental over* $\mathbb{C}(\overline{E_t})$.

Proof. It is easily seen that either the discriminant $\Delta^x_{[1:0]}/t^2 = d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ or $\Delta^y_{[1:0]}/t^2 = d_{0,1}^2 - 4d_{-1,1}d_{1,1}$ is not a square in $\mathbb{Q}$. □

5.2. **Non generic Cases.** In this subsection, we shall focus our attention on the walks listed in Figure 1 such that the discriminants $\Delta^x_{[1:0]}/t^2 = d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ and $\Delta^y_{[1:0]}/t^2 = d_{0,1}^2 - 4d_{-1,1}d_{1,1}$ are both squares in $\mathbb{Q}$. The latter condition is equivalent to the fact that $d_{1,1}d_{1,-1} = 0 = d_{1,1}d_{-1,1}$. This situation corresponds precisely to the cases IIB, IIC, IID and III. In each of these cases we will show that the nonexceptional walks have hypertranscendental generating series.

5.2.1. *The subcases IIB.* In that situation, we have

$$d_{1,1} = 0,\ d_{1,0} \neq 0 \text{ and } d_{0,1} \neq 0.$$

The following properties hold :

- according to Lemma 4.7, there exist $i, j \in \{1, 2\}$ such that $P_i = Q_j$; up to renumbering, we can and will assume that $P_1 = Q_1 = ([1:0],[1:0])$;
- $P_1 \neq P_2 = \iota_1(P_1)$ because $\Delta^x_{[1:0]} = d_{1,0}^2 - 4d_{1,1}d_{1,-1} = d_{1,0}^2 \neq 0$;
- $P_1 = Q_1 = ([1:0],[1:0]) \neq Q_2 = \iota_2(Q_1)$ in virtue of Lemma 4.5 (or simply because $\Delta^y_{[1:0]} \neq 0$);
- $Q_1 \neq \iota_1(Q_2)$ because $Q_1 \neq Q_2$ so Q_1 and Q_2 do not have the same x-coordinates;

In particular, we see that

- $P_1 = Q_1 = ([1:0],[1:0])$;
- $\iota_1(P_1) = P_2 = ([1:0],[\beta_1' : \beta_2'])$ with $[\beta_1' : \beta_2'] \neq [1:0]$;
- $\iota_1(P_2) = P_1 = ([1:0],[1:0])$;
- $Q_2 = ([\alpha_0' : \alpha_1'],[1:0])$ with $[\alpha_0' : \alpha_1'] \neq [1:0]$;
- $\iota_1(Q_2) = ([\alpha_0' : \alpha_1'], *)$ with $[\alpha_0' : \alpha_1'] \neq [1:0]$.

Theorem 5.3. *For any of the walks IIB.4, IIB.5, IIB.8, IIB.9, IIB.10 and* $t \in]0, 1/|\mathcal{D}|[\ \backslash\overline{\mathbb{Q}}$ *be such that* G_t *is infinite, i.e., such that* τ *has infinite order,* $F^1(x,t)$ *and* $F^2(y,t)$ *are hypertranscendental.*

Proof. Lemma 4.11 ensures that the set of poles of b_2 is included in $\{P_1, P_2, Q_1, Q_2, \iota_1(Q_1), \iota_1(Q_2)\} = \{P_1, P_2, Q_2, \iota_1(Q_2)\}$. Using §4.4, we see that :

- P_1 is a pole of order 2 of b_2 because
  - P_1 is a pole of order 1 of x_0/x_1;
  - P_1 is a pole of order 1 of y_0/y_1;
  - P_1 is not a pole of $\iota_1(y_0/y_1)$.
- P_2 is a pole of order 2 of b_2 because
  - P_2 is a pole of order 1 of x_0/x_1;
  - P_2 is not a pole of y_0/y_1;
  - P_2 is a pole of order 1 of $\iota_1(y_0/y_1)$.

There are at least two double poles. Since x_0/x_1, y_0/y_1 and $\iota_1(y_0/y_1)$ have at most two poles counted with multiplicities, we find that $b_2 = x_0/x_1(\iota_1(y_0/y_1) - y_0/y_1)$ has at most 6 poles, counted with multiplicities. So there are at most 3 double poles (in fact P_1 and P_2 are the only double poles in this situation but this fact will not be used).

If P_1 and P_2 are the only double, combining Corollary 3.9 and Proposition 3.10, we see that, in order to conclude, it is sufficient to show that $P_1 \not\sim P_2$. Assume that there exists a third double pole P_3 and that $P_1 \not\sim P_2$. Then there should exists $j \in \{1,2\}$ such that $P_3 \not\sim P_j$. Combining Corollary 3.9 and Proposition 3.10, we see that we have the conclusion in this case. So it is sufficient to prove that $P_1 \not\sim P_2$.

Suppose to the contrary that $P_1 \sim P_2$, *i.e.*, that there exists $n \in \mathbb{Z}$ such that $\tau^n(P_1) = P_2$.

On the one hand, since $P_1 \in \overline{E_t}(\mathbb{Q}) \subset \overline{E_t}(\mathbb{Q}(t))$, Proposition 4.8 implies that $\tau^j(P_1) \in \overline{E_t}(\mathbb{Q}(t))$ for any $j \in \mathbb{Z}$. Moreover, Proposition 4.8 implies that $P_2 = \iota_1(P_1) \in \overline{E_t}(\mathbb{Q}(t))$ is such that, for any $j \in \mathbb{Z}$, $\tau^j(\iota_2(P_2)) \in \overline{E_t}(\mathbb{Q}(t))$.

On the other hand, it is easily seen that the equality $\tau^n(P_1) = P_2$, together with the fact that $\tau = \iota_2 \circ \iota_1$ is the composition of two involutions, imply,

- if $n = 2k$, then $\tau^k(P_1) = \tau^{-k}(P_2)$ and thus
$$\iota_1\tau^k(P_1) = \iota_1\tau^{-k}(P_2) = \iota_1\tau^{-k}\iota_1(P_1) = \tau^k(P_1).$$
- if $n = 2k+1$, then $\tau^{k+1}(P_1) = \tau^{-k}(P_2)$ and thus
$$\tau^k\iota_2(P_2) = \tau^{k+1}(P_1) = \tau^{-k}(P_2) = \iota_2\tau^k\iota_2 P_2.$$

For $n = 2k$, we get that $\tau^k(P_1)$ is fixed by the involution ι_1. The fact that we are considering the case IIB avoiding the walks number 1, 2, 3, 6 and 7 combined with Lemma 4.5 ensures that none of the base points of $\overline{E_t}$ is fixed by ι_1. Therefore, $\tau^k(P_1)$ is not a base point. By Lemma 4.6, we conclude that $\tau^k(P_1) \notin \overline{E_t}(\mathbb{Q}(t))$. This yields a contradiction. For $n = 2k+1$, we get that $\tau^k(\iota_2(P_2))$ is fixed by the involution ι_2. The fact that we are considering the case IIB avoiding the walks number 1, 2, 3, 6 and 7 combined with Lemma 4.5 ensures that none of the base points of $\overline{E_t}$ is fixed by ι_2. Therefore, $\tau^k(\iota_2(P_2))$ is not a base point. By Lemma 4.6, we conclude that $\tau^k(\iota_2(P_2)) \notin \overline{E_t}(\mathbb{Q}(t))$. This yields a contradiction. □

5.2.2. *The subcases IID.* In that situation, we have
$$d_{1,1} = 0,\ d_{1,0} = 0 \text{ and } d_{0,1} \neq 0.$$
In particular, $\Delta^x_{[1:0]} = 0$ and, hence, $P_1 = P_2$. Moreover, Lemma 4.7 ensures that $P_i = Q_j = ([1:0],[1:0])$ for some $i,j \in \{1,2\}$. Up to renumbering, we can and will assume that $P_1 = Q_1$. Since $\Delta^y_{[1:0]} = d_{0,1}^2 \neq 0$, we have $P_1 = Q_1 \neq Q_2$. Since $P_1 = \iota_1(P_1)$, we also have $P_1 = Q_1 \neq \iota_1(Q_2)$. Last, using Lemma 4.5, we see that $Q_2 \neq \iota_1(Q_2)$.

Theorem 5.4. *For any of the walks listed in the case IID and $t \in]0,1/|\mathcal{D}|[\setminus\overline{\mathbb{Q}}$ such that G_t is infinite, $F^1(x,t)$ and $F^2(y,t)$ are hypertranscendental.*

Proof. We recall the formula (4.16)
$$(b_2)^2 = \frac{x_0^2\Delta^x_{[x_0:x_1]}}{x_1^2(\sum_i x_0^i x_1^{2-i} t d_{i-1,1})^2} = \frac{x_0^2}{x_1^4}\frac{\Delta^x_{[x_0:x_1]}}{t^2(x_1 d_{-1,1} + x_0 d_{0,1})^2}.$$
Since the curve $\overline{E_t}$ is nonsingular, the point P_1 is a simple zero of $\Delta^x_{[x_0:x_1]}$ seen as a rational function on $\mathbb{P}^1(\mathbb{C})$ (cf Proposition 2.1 and Corollary 2.3). Using additionally $d_{0,1} \neq 1$, it is easily seen that the polar divisor of b_2 is of the form
$$3[P_1] + i[Q_2] + j[\iota_1(Q_2)],$$
for some $i,j \in \{0,1,2\}$. Thus, P_1 is the only pole of b_2 of order ≥ 3. The result is now a consequence of Corollary 3.9 and Proposition 3.10. □

5.2.3. *The case III.* This walk is symmetric in the sense of the following definition.

Definition 5.5. *We say that a walk is symmetric if $d_{i,j} = d_{j,i}$ for all $i, j \in \{0, \pm 1\}$.*

Note that the walk is symmetric if and only if

$$\overline{K}(x_0, x_1, y_0, y_1, t) = \overline{K}(y_0, y_1, x_0, x_1, t).$$

Therefore, the involutive morphism s of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$s(x, y) = (y, x)$$

induces an involutive morphism of $\overline{E_t}$ in the symmetric case, still denoted by s. Note that

$$s \circ \iota_1 = \iota_2 \circ s. \tag{5.1}$$

Indeed, on the one hand, for any point $P = (x, y) \in \overline{E_t}$, we have that $\{P, \iota_1(P)\} = \overline{E_t} \cap (\{x\} \times \mathbb{P}^1(\mathbb{C}))$. and, hence, $\{s(P), s(\iota_1(P))\} = \overline{E_t} \cap (\mathbb{P}^1(\mathbb{C}) \times \{x\})$. On the other hand, we find $\{s(P), \iota_2(s(P))\} = \overline{E_t} \cap (\mathbb{P}^1(\mathbb{C}) \times \{x\})$. Whence the desired equality $s(\iota_1(P)) = \iota_2(s(P))$.

Similarly, we have

$$\iota_1 \circ s = s \circ \iota_2. \tag{5.2}$$

It follows that

$$s \circ \tau = \tau^{-1} \circ s. \tag{5.3}$$

Lemma 5.6. *Assume that the walk under consideration is symmetric. Consider $R_1, R_2 \in \overline{E_t}(\mathbb{Q}(t))$ such that $s(R_1) = R_2$. If $R_1 \sim R_2$ then there exists $R_3 \in \overline{E_t}(\mathbb{Q}(t))$ such that $s(R_3) = R_3$.*

Proof. We have to prove that, if there exists $\ell \in \mathbb{Z}$ such that $\tau^\ell(R_1) = R_2$ for some $R_1, R_2 \in \overline{E_t}(\mathbb{Q}(t))$ such that $s(R_1) = R_2$, then there exists $R_3 \in \overline{E_t}(\mathbb{Q}(t))$ such that $s(R_3) = R_3$. Up to interchanging R_1 and R_2, we can assume that $\ell \geq 0$. We argue by induction on $\ell \geq 0$. The result is obvious for $\ell = 0$. Let us assume that the result is true for some $\ell \geq 0$. The equality $\tau^\ell(R_1) = R_2$ ensures that $s\tau^\ell(R_1) = s(R_2) = R_1$. Using (5.3), we get $\tau^{-\ell}s(R_1) = R_1$. Using the equality $\tau^{-\ell} = \iota_1\tau^{\ell-1}\iota_2$, we get $\iota_1\tau^{\ell-1}\iota_2 s(R_1) = \tau^{-\ell}s(R_1) = R_1$. Apply ι_1 in the both sides of the equality gives $\tau^{\ell-1}\iota_2 s(R_1) = \iota_1(R_1)$. With (5.1), we obtain $\tau^{\ell-1}s\iota_1(R_1) = \iota_1(R_1)$. Note that $\iota_1(R_1)$ belongs to $\overline{E_t}(\mathbb{Q}(t))$ in virtue of Proposition 4.8. The induction hypothesis leads to the desired result. □

Lemma 5.7. *We assume that $t \in \mathbb{C} \setminus \overline{\mathbb{Q}}$. For the walk of the case III of Figure 1, there are no $R_1, R_2 \in \overline{E_t}(\mathbb{Q}(t))$ such that $s(R_1) = R_2$ and $R_1 \sim R_2$.*

Proof. According to Lemma 5.6, it is sufficient to prove that s does not have fixed points in $\overline{E_t}(\mathbb{Q}(t))$. Suppose to the contrary that there exists $P \in \overline{E_t}(\mathbb{Q}(t))$ such that $s(P) = P$. So, $P = (x, x)$ for some $x = [x_0 : x_1] \in \mathbb{P}^1(\mathbb{Q}(t))$ such that

$$x_0^2 x_1^2 - t(x_1^4 + 2x_1^3 x_0 + x_0^4) = 0.$$

If $x_0 = 0$, then we can assume that $x_1 = 1$ and we see that (5.4) is impossible. Assume that $x_0 \neq 0$. Then, we can and will assume that $x_0 = 1$ and we have

$$x_1^2 = t(1 + 2x_1^3 + x_1^4). \tag{5.4}$$

Since t is transcendental, we can and will identify $\mathbb{Q}(t)$ with a field of rational functions. If x_1 has a pole of order $\mu \geq 1$ at some $t = t_0 \in \overline{\mathbb{Q}}$, then $t(1 + 2x_1^3 + x_1^4)$ has a pole of order 4μ or $4\mu - 1$ at t_0 (depending on whether t_0 is equal to 0) whereas x_1^2 has a pole of order 2μ at t_0. Equation (5.4) yields $2\mu = 4\mu$ or $4\mu - 1$, whence a contradiction.

If x_1 vanishes at some $t = t_0 \in \overline{\mathbb{Q}}$, then the equality (5.4) specialized at $t = t_0$ gives $0 = t_0$.

Therefore, we have proved that $x_1 = ct^m$ for some $c \in \overline{\mathbb{Q}}^{\times}$ and $m \in \mathbb{Z}_{\geq 0}$. Equation (5.4) becomes

$$c^2 t^{2m} = t(1 + 2c^3 t^{3m} + c^4 t^{4m}).$$

Equating the degrees of both sides of this equation, we get $m = 0$. Now, the equality $c^2 = t(1 + 2c^3 + c^4)$ implies that $c = 0$, whence a contradiction.

□

Theorem 5.8. *For the walk of case III and $t \in]0, 1/|\mathcal{D}|[\setminus\overline{\mathbb{Q}}$ be such that G_t is infinite, $F^1(x,t)$ and $F^2(y,t)$ are hypertranscendental.*

Proof. Here, we have

$$P_1 = P_2 = ([1:0],[0:1]) \quad \text{and} \quad Q_1 = Q_2 = ([0:1],[1:0]),$$
$$\iota_1(Q_1) = ([0:1],[-1:1]) \quad \text{and} \quad \iota_2(P_1) = ([-1:1],[0:1]).$$

The formula (4.16) applied in this setting gives

$$(b_2)^2 = \frac{x_0^2}{x_1^2}\frac{\Delta^x_{[x_0:x_1]}}{(x_0^2 t)^2}. \tag{5.5}$$

Since the curve $\overline{E_t}$ is nonsingular, the point P_1 is a simple zero of $\Delta^x_{[x_0:x_1]}$ seen as a rational function on $\mathbb{P}^1(\mathbb{C})$ (see Proposition 2.1 and Corollary 2.3). Then, it is easily seen that the polar divisor of b_2 is $[P_1] + [Q_1] + [\iota_1(Q_1)] = [P_1] + [Q_1] + [\tau(Q_1)]$. Using Corollary 3.9 and Proposition 3.10, we see that, in order to conclude the proof, it is sufficient to prove that $P_1 \not\sim Q_1$. Since $s(P_1) = Q_1$, this follows from Lemma 5.7. □

5.2.4. *The walk IIC.3.* We shall exploit the fact that this walk is symmetric in the sense of Definition 5.5.

Lemma 5.9. *We assume that $t \in \mathbb{C}\setminus\overline{\mathbb{Q}}$. For the 3rd walk of the case IIC of Section 2.3, there are no $R_1, R_2 \in \overline{E_t}(\mathbb{Q}(t))$ such that $s(R_1) = R_2$ and $R_1 \sim R_2$.*

Proof. According to Lemma 5.6, it is sufficient to prove that s does not have fixed points in $\overline{E_t}(\mathbb{Q}(t))$. Suppose to the contrary that there exists $P \in \overline{E_t}(\mathbb{Q}(t))$ such that $s(P) = P$. So, $P = (x,x)$ for some $x = [x_0 : x_1] \in \mathbb{P}^1(\mathbb{Q}(t))$ such that

$$x_0^2 x_1^2 - t(x_1^4 + 2x_0^3 x_1 + x_0^4) = 0.$$

If $x_1 = 0$, then we can assume that $x_0 = 1$ and we see that (5.6) is impossible. Assume that $x_1 \neq 0$. Then, we can and will assume that $x_1 = 1$ and we have

$$x_0^2 = t(1 + 2x_0^3 + x_0^4). \tag{5.6}$$

As we can see in the proof of Lemma 5.7, there are no such $x_0 \in \mathbb{Q}(t)$, whence a contradiction.

□

Theorem 5.10. *For the walk of case IIC.3 with $t \in]0, 1/|\mathcal{D}|[\setminus\overline{\mathbb{Q}}$ such that G_t is infinite, we have that $F^1(x,t)$ and $F^2(y,t)$ are hypertranscendental.*

Proof. In this case, we have

- $P_1 = ([1:0],[0:1])$;
- $P_2 = ([1:0],[-1:1]) = \iota_1(P_1)$;
- $\iota_2(P_1) = P_1$;
- $Q_1 = ([0:1],[1:0])$;
- $Q_2 = ([-1:1],[1:0]) = \iota_2(Q_1)$;
- $\iota_1(Q_1) = ([0:1],[1:0]) = Q_1$;

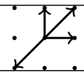
Walk		
Polar divisor of b_2	$([-1:1],[-t:t+1])$ $+([-1:1],[1:0])$	$([1:0],[-1:1])$ $+([1:0],[0:1])$
τ-Orbit of the poles of b_2	$([-1:1],[-t:t+1])$ $\downarrow\tau$ $([0:1],[1:0])$ $\not\sim$ $\downarrow\tau$ $([-1:1],[1:0])$	$([1:0],[-1:1])$ $\downarrow\tau$ $([1:0],[0:1])$

FIGURE 3. τ-Orbit of the poles in the case IIC.3

- $\iota_1(Q_2) = ([-1:1],[-t:t+1])$.

The polar divisor of b_2 is

$$[P_1] + [P_2] + [Q_2] + [\iota_1(Q_2)].$$

The poles P_1 and $\iota_1(Q_2)$ belong to the same τ-orbit:

$$\iota_1(Q_2) \xrightarrow{\iota_1} Q_2 \xrightarrow{\iota_2} Q_1 \xrightarrow{\iota_1} Q_1 \xrightarrow{\iota_2} Q_2.$$

Similarly, the poles P_2 and P_1 belong to the same τ-orbit :

$$P_2 \xrightarrow{\iota_1} P_1 \xrightarrow{\iota_2} P_1.$$

Since $s(Q_1) = P_1$, Lemma 5.9 implies that the τ-orbits of $\iota_1(Q_2)$ and P_2 are distinct. We refer to Figure 3 for a summary of these facts.

Since $\iota_1(b_2) = -b_2$, Lemma A.17 ensures that the residue of $b_2\omega$ at P_1 and P_2 are equal (and they are non zero). Therefore, the sum of the residues of $b_2\omega$ on the τ-orbit of P_2 is nonzero. Lemma A.8 together with Remark A.15 and Proposition A.1 lead to the desired result. □

In conclusion excluding the exceptional for the walks IIB.1, IIB.2, IIB.3, IIB.6. IIB.7, IIC.1, IIC.2, IIC.4, and IIC.5, we have shown that the series $F^1(x,t)$ and $F^2(y,t)$ for the walks in Figure 1 are hypertranscendental.

6. Hyperalgebraicity of the exceptional generating series

In each of the cases IIB.1, IIB.2, IIB.3, IIB.6, IIB.7, IIC.1, IIC.2, IIC.4, and IIC.5 we will show below that each of these satisfy the following

Condition 6.1. *For $i=1$ or $i=2$ there exist an integer $n \geq 1$, $c_0,\dots,c_{n-1} \in \mathbb{C}$ and $g \in \mathbb{C}(\overline{E_t})$*

$$\delta^n(b_i) + c_{n-1}\delta^{n-1}(b_i) + \cdots + c_1\delta(b_i) + c_0 b_i = \tau(g) - g. \tag{6.1}$$

As we have noticed in the proof Proposition 3.10, if Condition 6.1 is satisfied for $i=1$ or $i=2$, then it is satisfied for both values. Condition 6.1 precludes the possibility of using Proposition 3.8 to show that the corresponding generating series are hypertranscendental. As we have already mentioned, this does not immediately imply that the series F^1 and F^2 are hyperalgebraic. Nonetheless, we can use this data together with properties of the related r_x and r_y to show that they are indeed hyperalgebraic. We have the following result.

Proposition 6.2. *If a walk satisfies Assumption 2.9 and Condition 6.1 then the corresponding $F^1(x,t)$ and $F^2(y,t)$ are hyperalgebraic over $\mathbb{C}$.*

We will need the following lemma to prove this result.

Lemma 6.3. *Let U, V be open subsets of $\mathbb{C}$ and $f : V \to \mathbb{C}, g : U \to V$ be functions analytic in their domains. If $f(x)$ and $g(x)$ are hyperalgebraic over $\mathbb{C}$, then so is $f(g(x))$.*

Proof. One easily checks that a function h is hyperalgebraic over $\mathbb{C}$ if and only if the field $\mathbb{C}(h(x), h'(x), \ldots, h^{(n)}(x), \ldots)$ has finite transcendence degree over $\mathbb{C}$. Since f is hyperalgebraic, the field $\mathbb{C}(f(g(x)), f'(g(x)), \ldots, f^{(n)}(g(x)), \ldots)$ has finite transcendence degree over $\mathbb{C}$. Faà di Bruno's formula [Wik16] for the derivative of a composite function shows that for all m

$$(f(g(x))^{(m)} \in \mathbb{C}(f(g(x)), f'(g(x)), \ldots, g(x), g'(x), \ldots)$$

which is of finite transcendence degree over $\mathbb{C}$. □

We note that similar techniques show that if f and g are hyperalgebraic over $\mathbb{C}$ then so is any element of $\mathbb{C}(f, f', \ldots, g, g', \ldots)$.

Lemma 6.4. *Let $h : U \to V$ be a biholomorphism between open subsets of $\mathbb{C}$. If h is hyperalgebraic over $\mathbb{C}$, then h^{-1} is hyperalgebraic over $\mathbb{C}$.*

Proof. We know that $\mathbb{C}(h, h', \ldots, h^{(k)}, \ldots)$ has finite transcendence degree over $\mathbb{C}$, so $\mathbb{C}(h \circ h^{-1}, h' \circ h^{-1}, \ldots, h^{(k)} \circ h^{-1}, \ldots)$ has finite transcendence degree over $\mathbb{C}$ as well. But, the successive derivatives of h^{-1} are rational fractions in the $h^{(k)} \circ h^{-1}$ ($k \in \mathbb{Z}_{\geq 0}$), so $\mathbb{C}((h^{-1})', \ldots, (h^{-1})^{(k)}, \ldots)$ is a subfield of $\mathbb{C}(h' \circ h^{-1}, \ldots, h^{(k)} \circ h^{-1}, \ldots)$ and, hence, has finite transcendence degree over $\mathbb{C}$. □

Proof of Proposition 6.2. We will prove this for $F^1(x, t)$; the other case is similar. We set

$$L = \delta^n + c_{n-1}\delta^{n-1} + \cdots + c_1\delta + c_0.$$

Let $g \in \mathbb{C}(\overline{E_t})$ be as in Condition 6.1. We then have

$$\begin{aligned} \tau(L(r_x) - g) - (L(r_x) - g) &= L(\tau(r_x) - r_x) - (\tau(g) - g) \\ &= L(b_1) - (\tau(g) - g) \\ &= 0. \end{aligned}$$

Therefore $L(r_x) - g$ is τ-invariant. From Assumption 2.9, we have that r_x is invariant under $\omega \mapsto \omega + \omega_1$ and since $g \in \mathbb{C}(\overline{E_t})$ the same is true for g. Therefore $L(r_x) - g = R(\mathfrak{p}_{1,3}, \mathfrak{p}'_{1,3})$ where R is a rational function of two variables and $\mathfrak{p}_{1,3}$ is the Weierstrass $\mathfrak{p}$-function with periods ω_1 and ω_3. Since $\mathfrak{p}_{1,3}$ is hyperalgebraic over $\mathbb{C}$, we have that $L(r_x) - g$ is hyperalgebraic over $\mathbb{C}$. The element g is a rational function of a Weierstass $\mathfrak{p}$-function $\mathfrak{p}_{1,2}$ with periods ω_1 and ω_2 and its derivative, so it is also hyperalgebraic over $\mathbb{C}$. Therefore $L(r_x)$ is hyperalgebraic over $\mathbb{C}$ and thus the same holds for r_x. By definition, for some open set U, we have that $r_x(\omega) = F^1(\mathfrak{q}(\omega), t)$ where $\mathfrak{q}$ is a rational function of $\mathfrak{p}_{1,2}$ and $\mathfrak{p}'_{1,2}$. Let $U' \subset U$ and V' be nonempty open subsets of $\mathbb{C}$ such that $\mathfrak{p}_{1,2}$ induces a biholomorphism $U \to V$. Then, on V, we have $F^1(x, t) = r_x(\mathfrak{p}_{1,2}^{-1}(x))$ and we deduce from Lemma 6.4 and Lemma 6.3 that $F^1(x, t)$ is hyperalgebraic over $\mathbb{C}$. □

The remainder of this section is devoted to showing that each of the exceptional cases satisfy Condition 6.1 for b_2. These proofs rely heavily on the alternate characterizations of Condition 6.1 for b_2 given in the appendix. These characterizations are just in terms of the τ-orbits of the poles of b_2.

In all cases we assume that t has been chosen such that Assumption 2.9 holds.

6.1. **The walks IIC.1, IIC.2 and IIC.4.**

Theorem 6.5. *The walks IIC.1, IIC.2 and IIC.4 satisfy Condition 6.1.*

Proof. We shall first give a detailed proof for IIC.1. In this case, we have

$$\overline{K}(x_0,x_1,y_0,y_1,t) = -ty_0y_1x_1^2 - tx_0x_1y_1^2 + x_0x_1y_0y_1 - tx_0x_1y_0^2 - tx_0^2y_0^2$$

and

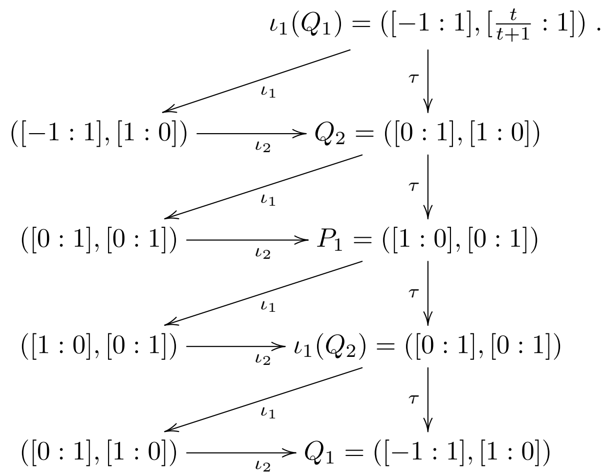
$$\begin{array}{rl} P_1 = ([1:0],[0:1]), & P_2 = \iota_1(P_1) = P_1 = ([1:0],[0:1]), \\ Q_1 = ([-1:1],[1:0]), & Q_2 = \iota_2(Q_1) = ([0:1],[1:0]) \neq Q_1. \end{array}$$

We see that

- P_1 is the only pole of x and it has order 2;
- the poles of y are Q_1 and Q_2 and they have order 1;
- the poles of $\iota_1(y)$ are $\iota_1(Q_1) = ([-1:1],[\frac{t}{t+1}:1]), \iota_1(Q_2) = ([0:1],[0:1])$, and they have order 1;
- since $Q_1, Q_2, \iota_1(Q_1), \iota_1(Q_2)$ are two by two distinct, these four points are the poles of $\iota_1(y) - y$ and they all have order 1;
- we have, see (4.16), $(b_2)^2 = \frac{\Delta^x_{[x_0:x_1]}}{x_1^2(x_0+x_1)^2}$, but P_1 is a zero of order 2 of $\frac{\Delta^x_{[x_0:x_1]}}{x_0^4}$ and a zero of order 4 of $\left(\frac{x_1}{x_0}\right)^2$, so P_1 is a pole of order 2 of b_2^2, and hence of order 1 of b_2;
- Q_2 and $\iota_1(Q_2)$ are zeros of x;
- Q_1, and $\iota_1(Q_1)$ are not zeros of x.

Finally, the polar divisor of b_2 is $[P_1] + [Q_1] + [\iota_1(Q_1)]$ and $P_1, Q_1, \iota_1(Q_1)$ are two by two distinct.

Moreover, the first 4 elements of the orbit of $\iota_1(Q_1)$ by the iterated action of τ are given by :



Therefore, all the poles of b_2 belong to the same τ-orbit. In summary, b_2 has only simple poles, and they all belong to the same τ-orbit. The result is now a direct consequence of Corollary A.3.

The other cases are similar. The polar divisor of b_2 and the first few terms of the τ-orbit of one of the poles of b_2 in the remaining cases are listed in Figure 4. □

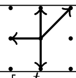
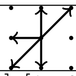
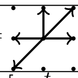
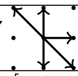
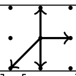
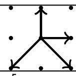
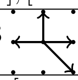
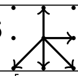
Walk	IIC.1 	IIC.2 	IIC.4 	IIB.7 
Polar divisor of b_2	$([-1:1],[\frac{t}{t+1}:1])$ $+([1:0],[0:1])$ $+([-1:1],[1:0])$	$([-1:1],[1:0])$ $+([1:0],[0:1])$ $+([-1:1],[0:1])$	$([-1:1],[\frac{-t}{2t+1}:1])$ $+([1:0],[-1:1])$ $+([1:0],[0:1])$ $+([-1:1],[1:0])$	$2([1:0],[-1:1])$ $+([-1:1],[1:0])$ $+([-1:1],[0:1])$ $+2([1:0],[1:0])$
τ-Orbit of one of the poles of b_2	$([-1:1],[\frac{t}{t+1}:1])$ $\downarrow \tau$ $([0:1],[1:0])$ $\downarrow \tau$ $([1:0],[0:1])$ $\downarrow \tau$ $([0:1],[0:1])$ $\downarrow \tau$ $([-1:1],[1:0])$	$([-1:1],[1:0])$ $\downarrow \tau$ $([1:0],[0:1])$ $\downarrow \tau$ $([-1:1],[0:1])$	$([-1:1],[\frac{-t}{2t+1}:1])$ $\downarrow \tau$ $([0:1],[1:0])$ $\downarrow \tau$ $([1:0],[-1:1])$ $\downarrow \tau$ $([1:0],[0:1])$ $\downarrow \tau$ $([0:1],[-1:1])$ $\downarrow \tau$ $([-1:1],[1:0])$	$([1:0],[-1:1])$ $\downarrow \tau$ $([-1:1],[1:0])$ $\downarrow \tau$ $([0:1],[0:1])$ $\downarrow \tau$ $([-1:1],[0:1])$ $\downarrow \tau$ $([1:0],[1:0])$
Walk	IIB.1 	IIB.2 	IIB.3 	IIB.6 
Polar divisor of b_2	$2([1:0],[0:1])$ $+2([1:0],[1:0])$	$2([1:0],[-1:1])$ $+2([1:0],[1:0])$	$2([1:0],[-1:1])$ $+2([1:0],[1:0])$	$2([1:0],[-1:1])$ $+2([1:0],[1:0])$
τ-Orbit of one of the poles of b_2	$([1:0],[0:1])$ $\downarrow \tau$ $([0:1],[1:0])$ $\downarrow \tau$ $([1:0],[1:0])$	$([1:0],[-1:1])$ $\downarrow \tau$ $([0:1],[1:0])$ $\downarrow \tau$ $([1:0],[1:0])$	$([1:0],[-1:1])$ $\downarrow \tau$ $([0:1],[1:0])$ $\downarrow \tau$ $([1:0],[1:0])$	$([1:0],[-1:1])$ $\downarrow \tau$ $([0:1],[1:0])$ $\downarrow \tau$ $([1:0],[1:0])$

FIGURE 4. Polar divisors of b_2 in cases when all poles belong to the same $\tau-$orbit

6.2. **The walks IIB.1, IIB.2, IIB.3, IIB.6 and IIB.7.**

Theorem 6.6. *The walks IIB.1, IIB.2, IIB.3, IIB.6 and IIB.7 satisfy Condition 6.1.*

Proof. In the 5 cases,

- $([1:0],[1:0])$ is a double pole of b_2,
- there exists $\alpha \in \{0,-1\}$ such that $([1:0],[\alpha:1])$ is a double pole of b_2,
- b_2 has at most two simple poles.

Furthermore, every pole belong to the same τ-orbit, see Figure 4. We consider a set of analytic local parameters as given by Lemma A.18 and we use the same notations. Since $\iota_1(b_2) = -b_2$, Lemma A.18 ensures that $\text{ores}_{([1:0],[1:0]),2}(b_2) = 0$. Moreover, since every pole of b_2 belong to the same τ-orbit, Lemma A.19 ensures that $\text{ores}_{Q,1}(b_2) = 0$ for all Q. Lemma A.8 together with Proposition A.1 lead to the desired result. □

6.3. **The walk IIC.5.** In this case, unlike the previous cases, the poles of b_2 form two distinct τ-orbits. To proceed, we will need the following

Lemma 6.7. *Let $R_1, R_2 \in \overline{E_t}(\mathbb{Q}(t))$ be such that $\iota_1(R_1) = R_2$ and $R_1 \sim R_2$. Then, there exist $j \in \{1,2\}$ and $R \in \overline{E_t}(\mathbb{Q}(t))$ such that $\iota_j(R) = R$.*

Proof. The reasonning is similar to a part of the proof of Theorem 5.3.

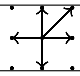
Walk			
Polar divisor of b_2	$([-1:1],[t:2t+1])$ $+([1:0],[0:1])$		$([1:0],[-1:1])$ $+([-1:1],[1:0])$
τ-Orbit of the poles of b_2	$([-1:1],[t:2t+1])$ $\downarrow\tau$ $([0:1],[1:0])$ $\downarrow\tau$ $([1:0],[0:1])$	$\not\sim$	$([1:0],[-1:1])$ $\downarrow\tau$ $([0:1],[0:1])$ $\downarrow\tau$ $([-1:1],[1:0])$

FIGURE 5. τ-Orbit of the poles in the case IIC.5

It is easily seen that the equality $\tau^n(R_1) = R_2$, together with the fact that $\tau = \iota_2 \circ \iota_1$ is the composition of two involutions, imply, if $n = 2k$, that $\iota_1(\tau^k(R_1)) = \tau^k(R_1)$, and, if $n = 2k+1$, that $\iota_2(\tau^k(\iota_2(R_2))) = \tau^k(\iota_2(R_2))$. Proposition 4.8 ensures that both $\tau^k(R_1)$ and $\tau^k(\iota_2(R_2))$ belong to $\overline{E_t}(\mathbb{Q}(t))$. Whence the desired result. □

Theorem 6.8. *The walk IIC.5 satisfies Condition 6.1.*

Proof. We have

- $P_1 = ([1:0],[0:1])$;
- $P_2 = ([1:0],[-1:1])$;
- $Q_1 = ([0:1],[1:0])$;
- $Q_2 = ([-1:1],[1:0])$;
- $\iota_1(Q_1) = ([0:1],[0:1])$;
- $\iota_1(Q_2) = ([-1:1],[t:2t+1])$.

The polar divisor of b_2 is

$$[P_1] + [P_2] + [Q_2] + [\iota_1(Q_2)].$$

The poles P_1 and $\iota_1(Q_2)$ belong to the same τ-orbit:

$$\iota_1(Q_2) = ([-1:1],[t:2t+1]) \xrightarrow{\tau} ([0:1],[1:0]) \xrightarrow{\tau} P_1 = ([1:0],[0:1])$$

Similarly, the poles P_2 and Q_2 belong to the same τ-orbit:

$$P_2 = ([1:0],[-1:1]) \xrightarrow{\tau} ([0:1],[0:1]) \xrightarrow{\tau} Q_2 = ([-1:1],[1:0]).$$

Moreover, the τ-orbits of $\iota_1(Q_2)$ and P_2 are distinct. Indeed, otherwise, since we have

$$\iota_1([0:1],[1:0]) = ([0:1],[0:1]),$$

Lemma 6.7 would imply the existence of $R \in \overline{E_t}(\mathbb{Q}(t))$ such that $\iota_j(R) = R$ for some $j \in \{1,2\}$. But, Lemma 4.6 ensures that R is a base point and Lemma 4.5 ensures that none of the base points are fixed by ι_j, whence a contradiction.

Lemma A.17 ensures that the residues of $b_2\omega$ at P_1 and $P_2 = \iota_1(P_1)$ are equal and will be denoted by a. Similarly, the residues of $b_2\omega$ at Q_2 and $\iota_1(Q_2)$ are the same and will be denoted by b. Since the sum of the residues over $\overline{E_t}$ of $b_2\omega$ is equal to 0, we have $2a + 2b = 0$, so $a + b = 0$. Therefore, the sum of the residues of $b_2\omega$ on any τ-orbit is equal to 0. Lemma A.8 together with Remark A.15 and Proposition A.1 lead to the desired result. □

7. Nonholonomicity in the exceptional cases

Theorem 7.1. *In each of the 9 exceptional cases the series $F^1(x,t)$ and $F^2(y,t)$ are not holonomic.*

Proof. We only present the proof for $F^2(y,t)$, the proof for $F^1(x,t)$ being similar. Assume that $F^2(y,t)$ is holonomic. Then, $F^2(y,t)$ could be analytically continued to a multivalued meromorphic function on $\overline{E_t}\backslash\{$ a finite set of points $\}$. This implies that r_y would be a meromorphic function on the universal cover of $\overline{E_t}$ whose singular points form a finite set modulo the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. In each of the exceptional 9 cases, we have that $F^2(y,t)$ satisfies Condition 6.1. Therefore as in the proof of Proposition 6.2, there is a $g \in \mathbb{C}(\overline{E_t})$ such that $\tau(L(r_y) - g) = L(r_y) - g$. Note that the poles of $L(r_y) - g$ also form a finite set modulo the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Since $L(r_y) - g$ is τ-periodic, this set is left invariant by $\omega \mapsto \omega + \omega_3$. Using the fact that the reduction of ω_3 modulo $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ has infinite order, we get that the set of poles of $L(r_y) - g$ is empty. Using Condition 6.1 again, we see that $L(r_y) - g$ is ω_1-periodic as well as being ω_3 periodic. Since it has no poles, we must have that $L(r_y) - g = c \in \mathbb{C}$.

We want to prove that this last fact leads to a contradiction. To do this we will use some notation from [KR12, Sec. 4.2]. Let Δ_x be the set of ω in $\omega_1\mathbb{R} +]0, \omega_2[$ corresponding to points on the elliptic curve with $|x| < 1$ and let Δ_y be the set of ω in $\omega_1\mathbb{R} + \omega_3/2 +]0, \omega_2[$ corresponding to points on the elliptic curve with $|y| < 1$. Let us state and prove two lemmas.

Lemma 7.2. *The function r_y has no poles in $\Delta_x \cup \Delta_y$.*

Proof of Lemma 7.2. From [KR12, Theorem 3], one sees that r_y has no poles on Δ_y and that r_x has no poles on Δ_x.

Using formula (4.5) in [KR12] and the fact that r_x has no poles on Δ_x, we find that the poles of r_y on Δ_x are poles of xy. It is therefore sufficient to prove that xy does not have poles for $|x| < 1$. Note that a pole of xy with $|x| < 1$ is a pole of y.

We claim that $([0:1],[1:0])$ is the only possible pole of xy with $|x| < 1$. We recall that the poles of y are Q_1 and Q_2. Let $\alpha_1, \alpha_2 \in \mathbb{P}^1(\mathbb{C})$ be the x-coordinates of Q_1 and Q_2 respectively. To prove the claim it is sufficient to prove that for $j \in \{1,2\}$, $|\alpha_j| < 1$ implies $\alpha_j = 0$. The x-coordinates $\{\alpha_1, \alpha_2\}$ are the two roots in $\mathbb{P}^1(\mathbb{C})$, counted with multiplicities, of the polynomial $d_{-1,1} + d_{0,1}X + d_{1,1}X^2$. For the 9 walks under consideration, we have $d_{0,1} = 1$. The following array summarizes the possible values of the pair (α_1, α_2) in the four situations:

	$d_{-1,1} = 0$	$d_{-1,1} = 1$
$d_{1,1} = 0$	$[0:1],[1:0]$	$[-1:1],[1:0]$
$d_{1,1} = 1$	$[0:1],[-1:1]$	$[\frac{-1-i\sqrt{3}}{2}:1],[\frac{-1+i\sqrt{3}}{2}:1]$

In the four situations, we see that $|\alpha_j| < 1$ implies $\alpha_j = 0$, proving our claim.

Recall that we are interested in the poles of xy with $|x| < 1$. With the claim we have to determine whether $([0:1],[1:0])$ is a pole of xy. We have seen in the proof of the claim that in the 9 cases under consideration, $Q_1 \neq Q_2$ since their x-coordinates are different. So y has at most simple pole at $([0:1],[1:0])$. This shows that $([0:1],[1:0])$ is not a pole of xy. Combined with the claim, this shows that xy has no poles for $|x| < 1$, proving the lemma.

□

Lemma 7.3. *The function g has at least one pole in $\Delta_x \cup \Delta_y$, that is, one pole at a point on the elliptic cuve in $\{|x| < 1\} \cup \{|y| < 1\}$.*

Proof of Lemma 7.3. According to Figures 4 and 5, there exist a pair (Q,n) with $n>0$ such that Q and $\tau^n(Q)$ are poles of b_2, and for all $\ell \in]-\infty,-1]\cup[n+1,\infty[$, $\tau^\ell(Q)$ is not a pole of b_2. Since Condition 6.1 is satisfied, one sees that $\tau^n(Q)$ and $\tau(Q)$ should be pole of g. Using Figures 4 and 5, we see that for 8 over 9 cases (all except IIB.7), this implies that g should have a pole for an $\omega\in\mathbb{C}$ that corresponds to $\{|x|<1\}\cup\{|y|<1\}$ in the elliptic curve, proving the lemma for those case.

It remains to treat IIB.7. In this case, b_2 has two double poles $Q,\tau^4(Q)$ and two simples poles $\tau(Q),\tau^3(Q)$ with $Q=([1:0],[-1:1])$. The operator L in Condition 6.1 have coefficients in $\mathbb{C}$. Let n be its order. With Lemma 3.2, we find that $L(b_2)$ have only two poles of order $n+2$, that are $Q,\tau^4(Q)$, and no poles of higher order. With $L(b_2)=\tau(g)-g$, we find that g should have a pole of order $n+2$ at $\tau^4(Q)$. So $\tau(g)$ have a pole of order $n+2$ at $\tau^3(Q)$. Since $L(b_2)=\tau(g)-g$ and $L(b_2)$ have no poles of order $n+2$ or higher at $\tau^3(Q)$, g should have a pole of order $n+2$ at $\tau^3(Q)$. But the y-coordinates of $\tau^3(Q)$ is $[0:1]$, proving the lemma in this case. □

Let us complete the proof of the theorem. From Lemma 7.2 and Lemma 7.3, we see that there exists $\omega_0\in\mathbb{C}$, such that g has a pole at ω_0 and such that r_y is analytic at ω_0. Since L has coefficients in $\mathbb{C}$, $L(r_y)$ is analytic at ω_0. This contradicts $L(r_y)-g\in\mathbb{C}$. □

8. Further works

8.1. **Singular walks.** If the walk is singular, then $\overline{E_t}$ is a rational curve, *i.e.*, is birational to $\mathbb{P}^1(\mathbb{C})$. The difference equations on elliptic curves involved in the nonsingular case should be replaced by finite difference or q-difference equations on a rational curve. It seems plausible that a suitable version of the assumptions 2.9 is satisfied and that our Galoisian methods can be used in order to study the generating series of the singular walks as well.

8.2. **Weighted walks.** We consider a walk with small steps in the quarter plane $\mathbb{Z}^2_{\geq 0}$. Following [FIM99, KY15], the step $(i,j)\in\{0,\pm1\}^2$ is weighted by some $d_{i,j}\in\mathbb{Q}_{\geq 0}$. Let $\mathcal{D}$ be the set $\{d_{i,j}\mid (i,j)\in\{0,\pm1\}^2\}$. For $i,j,k\in\mathbb{Z}_{\geq 0}$, we let $q_{\mathcal{D},i,j,k}$ be the number of walks in $\mathbb{Z}^2_{\geq 0}$ with weighted steps in $\mathcal{D}$ starting at $(0,0)$ and ending at (i,j) in k steps and we consider the corresponding trivariate generating series

$$Q_{\mathcal{D}}(x,y,t):=\sum_{i,j,k\geq 0} q_{\mathcal{D},i,j,k}x^iy^jt^k.$$

We can then ask for these weighted walks the same questions as for the unweighted walks considered in the rest of this paper. It turns out that in the weighted context, we have generalizations of the basic tools used in the unweighted case (generalizations of the kernel, of the functional equation (2.1), *etc*). Moreover, explicit conditions can be deduced from [FIM99], Section 2.3.2, Corollary 4.2.11 and Theorem 3.2.1 in order to have the assumptions 2.9 satisfied. This should be the starting point to apply our technics in this context.

Appendix A.

We have seen in Section 3.3 that the study of the hypertranscendance of $F^1(x,t)$ and $F^2(y,t)$ is intimately related to the study of equations of the form $L(b)=\tau(g)-g$ for some nonzero linear differential operator L with coefficients in $\mathbb{C}$ and some $b,g\in\mathbb{C}(\overline{E_t})$. In other contexts (cf. [CS12]), L is referred to as a *telescoper for* b and g as a *witness*. The aim of this appendix is to study in more details these equations.

A.1. **Telescopers and orbit residues.** Let E be an elliptic curve defined over an algebraically closed field k of characteristic zero and $k(E)$ be its function field. Let P be a non-torsion point on E and let $\tau : k(E) \to k(E)$ denote map corresponding to $Q \mapsto Q \oplus P$ on E, where $\oplus$ denotes the group law on E. We let Ω be a non zero regular differential form on E. A straightforward generalization of Lemma 3.1 shows that the derivation δ of $k(E)$ such that $d(f) = \delta(f)\Omega$ commutes with τ.

We will prove the following

Proposition A.1. *Let $b \in k(E)$. The following are equivalent.*

(1) *There exist $g \in k(E)$ and a nonzero operator $L \in k[\delta]$ such that $L(b) = \tau(g) - g$.*
(2) *For all poles Q_0 of b, we have that*

$$h(X) = \sum_{i=1}^{t} b(X \oplus n_i P)$$

is regular at $X = Q_0$ where $Q_0 \oplus n_1 P, \dots, Q_0 \oplus n_t P$ are the poles of b that belong to $Q_0 \oplus \mathbb{Z}P$.

This proposition allows one to give the following useful criteria guaranteeing when Condition 6.1 does or does not hold.

Corollary A.2. *Let $b \in k(E)$ and assume that there exists $Q_0 \in E$ such that*

(1) *b has a pole of order $m > 0$ at Q_0, and*
(2) *b has no other pole of order $\geq m$ in $Q_0 \oplus \mathbb{Z}P$.*

Then there is no nonzero $L \in k[\delta]$ and $g \in k(E)$ such that $L(b) = \tau(g) - g$.

Proof. This follows easily from Proposition A.1 since the pole of $b(X)$ at Q_0 cannot be cancelled by any pole of any $b(X \oplus nP)$ and so $h(X)$ is not regular at $X = Q_0$. □

Corollary A.3. *Let $b \in k(E)$ and assume that there exists $Q_0 \in E$ such that*

(1) *all poles of b occur in $Q_0 \oplus \mathbb{Z}P$, and*
(2) *all poles of b are simple.*

Then there exist $g \in k(E)$ and a nonzero operator $L \in k[\delta]$ such that $L(b) = \tau(g) - g$.

Proof. Using Lemma A.14 below, one can show that the hypotheses of Corollary A.3 imply condition 2 of Proposition A.4 and therefore that the conclusion holds (see the remark following Lemma A.14). □

To prove Proposition A.1 we shall prove two ancillary results, Propositions A.4 and A.7. These results give conditions equivalent to the conditions in Proposition A.1.

Before proceeding, we recall the following standard notation. If D is a divisor of E, we will denote by $\mathcal{L}(D)$ the finite dimensional k-space $\{f \in k(E) \mid (f) + D \geq 0\}$, where (f) is the divisor of f. In Section A.1.1 we will prove

Proposition A.4. *Let $b \in k(E)$. The following are equivalent.*

(1) *There exist $g \in k(E)$ and a nonzero operator $L \in k[\delta]$ such that $L(b) = \tau(g) - g$.*
(2) *There exists $Q \in E$, $e \in k(E)$ and $h \in \mathcal{L}(Q + (Q \ominus P))$[3] such that*

$$b = \tau(e) - e + h.$$

[3] The symbol "+" represents the formal sum of divisors. We will use $\oplus$ and $\ominus$ for addition and subtraction of points on the curve.

To state the next equivalence, we need two definitions. Corresponding to each point $Q \in E$ there exists a valuation ring $\mathfrak{O}_Q \subset k(E)$. A generator u_Q of the maximal ideal of $\mathfrak{O}_Q$ is called a local parameter at Q. Local parameters are unique up to multiplication by a unit of $\mathfrak{O}_Q$.

Definition A.5. *Let $\mathcal{S} = \{u_Q \mid Q \in E\}$ be a set of local parameters at the points of E. We say S is a* coherent set of local parameters *if for any $Q \in E$,*

$$u_{Q\ominus P} = \tau(u_Q).$$

We fix, once and for all, a coherent set of local parameters. All local parameters mentioned henceforth will be from this set.

Let u_Q be a local parameter at a point $Q \in E$ and let v_Q be the valuation corresponding to the valuation ring at Q. If $f \in k(E)$ has a pole at Q or order n, we may write

$$f = \frac{c_{Q,n}}{u_Q^n} + \ldots + \frac{c_{Q,2}}{u_Q^2} + \frac{c_{Q,1}}{u_Q} + \tilde{f}$$

where $v_Q(\tilde{f}) \geq 0$. The following definition is similar to Definition 2.3 of [CS12].

Definition A.6. *Let $f \in k(E)$ and $S = \{u_Q \mid Q \in E\}$ be a coherent set of local parameters and $Q \in E$. For each $j \in \mathbb{N}_{>0}$ we define the* orbit residue of order j at *Q to be*

$$\mathrm{ores}_{Q,j}(f) = \sum_{i\in\mathbb{Z}} c_{Q\oplus iP,j}.$$

Note that if $Q' = Q + tP$ for some $t \in \mathbb{Z}$, then $\mathrm{ores}_{Q',j}(f) = \mathrm{ores}_{Q,j}(f)$ for any $j \in \mathbb{N}_{>0}$. Furthermore $\mathrm{ores}_{Q,j}(f) = \mathrm{ores}_{Q,j}(\tau(f))$. We shall prove the next result in Section A.1.2.

Proposition A.7. *Let $b \in k(E)$ and $S = \{u_Q \mid Q \in E\}$ be a coherent set of local parameters. The following are equivalent.*

(1) *There exists $Q \in E$, $e \in k(E)$ and $g \in \mathcal{L}(Q + (Q \ominus P))$ such that*
$$b = \tau(e) - e + g.$$
(2) *For any $Q \in E$ and $j \in \mathbb{N}_{>0}$*
$$\mathrm{ores}_{Q,j}(b) = 0.$$

Proposition A.4, Proposition A.7 and the following lemma immediately imply Proposition A.1.

Lemma A.8. *Let $b \in k(E)$. The following are equivalent*

(1) *For all poles Q_0 of f, we have that*
$$h(X) = \sum_{i=1}^{t} b(X \oplus n_i P)$$
is regular at $X = Q_0$ where $Q_0 \oplus n_1 P, \ldots, Q_0 \oplus n_t P$ are the poles of b that belong to $Q_0 \oplus \mathbb{Z}P$.
(2) *For any $Q \in E$ and $j \in \mathbb{N}_{>0}$*
$$\mathrm{ores}_{Q,j}(b) = 0.$$

Proof. If u is the local parameter at Q_0, we may write

$$h = \frac{c_n}{u^n} + \ldots + \frac{c_1}{u} + h'$$

where $v_{Q_0}(h') \geq 0$. One easily sees that

$$c_j = \mathrm{ores}_{Q_0,j}(b).$$

The conclusion now follows. ☐

Remark A.9. *Assume that $k = \mathbb{C}$. Then, one can consider the analytification E^{an} of E. Instead of considering algebraic local parameters on E, one can consider analytic local parameters $\{u_Q \mid Q \in E\}$, i.e., for any $Q \in E$, u_Q is a biholomorphism between a neighborhood of Q in E^{an} and a neighborhood of 0 in $\mathbb{C}$. There is an obvious notion of coherent analytic local parameters, extending the notion introduced in Definition A.5, and a corresponding notion of $\mathrm{ores}_{Q,j}$. Lemma A.8 remains true in this context, with the same proof.*

Remark A.10. *The proof that (2) implies (1) in Proposition A.7 is constructive. One only needs a constructive method for finding the bases of certain $\mathcal{L}$ spaces (e.g. [Hes02]). The proof that (2) implies (1) in Proposition A.4 is also constructive. Therefore given $b \in k(E)$ one can decide if there exist $g \in k(E)$ and a nonzero operator $L \in k[\delta]$ such that $L(b) = \tau(g) - g$.*

A.1.1. *Proof of Proposition A.4.* In the following lemma, we will collect some facts concerning the local behavior of functions under the actions of τ. Its proof is a straightforward generalization of the proof of Lemma 3.2.

Lemma A.11. *Let u be a local parameter of $k(E)$ and let v_u be the associated valuation. Then $v_u(\delta(u)) = 0$ and, for any $f \in k(E)$ such that $v_u(f) \neq 0$, we have*

1. *if $v_u(f) \geq 0$ then $v_u(\delta(f)) \geq 0$;*
2. *if $v_u(f) < 0$ then $v_u(\delta(u)) = v_u(f) - 1$.*

We will also need a consequence of the Riemann-Roch Theorem for elliptic curves: If D is a positive divisor on E and $l(D)$ is the dimension of the space $\mathcal{L}(D)$ then

$$l(D) = \text{degree of } D.$$

This implies that if Q is a point on E, u is a local parameter at Q, $n \geq 2$, and $c_2, \ldots, c_n \in k$, then there exists an $f \in \mathcal{L}(nQ)$ and $c_1 \in k$ such that

$$f = \frac{c_n}{u^n} + \ldots + \frac{c_2}{u^2} + \frac{c_1}{u} + \tilde{f}$$

where $v_u(\tilde{f}) \geq 0$. *A priori*, we have no control of the element c_1.

Finally we need some definitions:

Definition A.12. *Let $f \in k(E)$ and $Q \in E$.*

1. *If Q is a pole of f, the* polar dispersion of f at Q, $\mathrm{pdisp}(f, Q)$ *is the largest nonnegative integer ℓ such that $Q \oplus \ell P$ is also a pole of f.*
2. *The* polar dispersion of f, pdisp(f), *is* $\max\{\mathrm{pdisp}(f, Q) \mid Q \text{ a pole of } f\}$.
3. *The* weak polar dispersion of f, wpdisp(f), *is* $\max\{\ell \mid \exists Q \in E \text{ s.t. } f \text{ has a pole of order at least } 2 \text{ at } Q \text{ and } Q \oplus \ell P\}$.

The following is an analogue of [HS08, Lemma 6.2].

Lemma A.13. *Let $f \in k(E)$. There exist $f^*, g \in k(E)$ such that $\mathrm{pdisp}(f^*) \leq 1$, $wpdisp(f^*) = 0$ and $f = f^* + \tau(g) - g$.*

Proof. We begin by showing that there exist $f^*, g \in k(E)$ such that $\mathrm{wpdisp}(f^*) = 0$ and $f = f^* + \tau(g) - g$. We will then further refine f^* so that $\mathrm{pdisp}(f^*) \leq 1$ as well.

Let $N = \mathrm{wpdisp}(f) \geq 1$ and $n_f =$ the number of points $Q \in E$ such that f has poles of order at least two at Q and $Q \ominus NP$. Fix such a point Q and let u be a local parameter at Q. We may write

$$f = \sum_{i=1}^{m} \frac{a_i}{u^i} + h_f$$

where $m \geq 2$ and $v_Q(h_f) \geq 0$. The Riemann-Roch Theorem implies that there exists a $\tilde{g} \in \mathcal{L}(mQ)$ such that

$$\tilde{g} = \sum_{i=1}^{m} \frac{b_i}{u^i} + h_{\tilde{g}}$$

where $b_i = -a_i$ for $i = 2, \dots, m$ and $v_Q(h_{\tilde{g}}) \geq 0$. Note that $\tau(\tilde{g})$ has a pole of order m at $Q \ominus P$. Letting $\tilde{f} = f - (\tau(\tilde{g}) - \tilde{g})$, one sees that $\tilde{f}$ has a pole of order at most 1 at Q. Therefore either the $\mathrm{wpdisp}(f) = \mathrm{wpdisp}(\tilde{f})$ and $n_{\tilde{f}} < n_f$ or $\mathrm{wpdisp}(f) > \mathrm{wpdisp}(\tilde{f})$. An induction allows us to conclude that there exist $f^*, g \in k(E)$ such that $\mathrm{wpdisp}(f^*) = 0$ and $f = f^* + \tau(g) - g$.

We may now assume that $\mathrm{wpdisp}(f) = 0$ and let $\mathrm{pdisp}(f) = N \geq 2$. Let f have poles at both Q and $Q \oplus NP$. Since $\mathrm{wpdisp}(f) = 0$, f has a pole of order greater than one at no more than one of these two points. We deal with the two cases separately.

<u>f has a pole of order 1 at $Q \oplus NP$.</u> The Riemann-Roch Theorem implies that there exists a nonconstant $\tilde{g} \in \mathcal{L}((Q \oplus (N-1)P) + (Q \oplus NP))$. Note that $\tau(\tilde{g}) \in \mathcal{L}((Q \oplus (N-2)P) + (Q \oplus (N-1)P))$. For some $a \in k$, $\tilde{f} - (\tau(a\tilde{g}) - a\tilde{g})$ has no pole at $Q \oplus NP$ and so $\mathrm{pdisp}(f, Q) < N$. An induction finishes the proof.

<u>f has a pole of order 1 at Q.</u> The Riemann-Roch Theorem implies that there exists a nonconstant $\tilde{g} \in \mathcal{L}((Q \oplus P) + (Q \oplus 2P))$. Note that $\tau(\tilde{g}) \in \mathcal{L}((Q) + (Q \oplus P))$. For some $a \in k$, $\tilde{f} - (\tau(a\tilde{g}) - a\tilde{g})$ has no pole at Q and so $\mathrm{pdisp}(f, Q) < N$. An induction again finishes the proof. □

We now turn to the

Proof that 1. implies 2. in Proposition A.4. Applying Lemma A.13, we may assume that $\mathrm{pdisp}(b) \leq 1$ and $\mathrm{wpdisp}(b) = 0$. We will first show that for any $Q \in E$, if b has a pole at Q then this pole must be simple and it has another pole of the same order at $Q \oplus P$ or at $Q \ominus P$. To see this note that if b has a pole at Q, then either g or $\tau(g)$ has a pole at Q. Assume that g has a pole at Q (the argument assuming $\tau(g)$ has a pole at Q is similar). Let r be the largest integer such that $Q \oplus rP$ is a pole of g and s be the largest integer such that $Q \ominus sP$ is a pole of g. We then have that $Q \oplus rP$ and $Q \ominus (s+1)P$ are both poles of $\tau(g) - g$ and therefore of $L(b)$. Using Lemma 3.2.2 above, one sees that they must also be poles of b. Since $\mathrm{pdisp}(b) \leq 1$, we have $r = s = 0$. In particular, the only pole of g in $Q \oplus \mathbb{Z}P$ is at Q, the only pole of $\tau(g)$ in $Q \oplus \mathbb{Z}P$ is at $Q \ominus P$ and they must have the same orders. Once again, Lemma 3.2.2 above implies that b has poles at these points of equal orders. Since $\mathrm{wpdisp}(b) = 0$, the orders of these poles must be 1.

We can therefore conclude that b has only poles of order 1 and the poles of b occur in pairs $\{Q_1, Q_1 \ominus P\}, \dots, \{Q_r, Q_r \ominus P\}$ where $Q_i \oplus \mathbb{Z}P \cap Q_j \oplus \mathbb{Z}P = \emptyset$ for $i \neq j$.

We will now show how one can construct an element e such that $b - (\tau(e) - e)$ has exactly one pair of poles $\{Q, Q \ominus P\}$. This will yield the conclusion of the Proposition. Assume $r > 1$ and that b has simple poles at the pairs $\{Q_1, Q_1 \ominus P\}$ and $\{Q_2, Q_2 \ominus P\}$. Let $h \in k(E)$ be a nonconstant element of $\mathcal{L}(Q_1 + Q_2)$. There exists an $a \in k$ such that $\tilde{b} = b - (\tau(ah) - ah)$ has no pole at Q_1. The element $\tilde{b}$ has only simple poles and $\mathrm{pdisp}(b) \leq 1$. Therefore its poles occur at possibly $Q_1 \ominus P, \{Q_2, Q_2 \ominus P\}, \dots, \{Q_r, Q_r \ominus P\}$. Since $\tilde{b}$ satisfies an equation of the form $L(\tilde{b}) = \tau(\tilde{g}) - \tilde{g}$ for some $\tilde{g} \in k(E)$ the poles of such an $\tilde{b}$ must occur in pairs. Therefore we have that $\tilde{b}$ has no pole at $Q_1 \ominus P$. Continuing in this way we eventually find an $e \in k(E)$ such that $b - (\tau(e) - e)$ has exactly one pair of poles $\{Q, Q \ominus P\}$. □

In the proof that 2. implies 1. in Proposition A.7 we will need the following technical lemma. Let u be a local parameter at Q. Note that $\tau(u)$ is a local parameter at $Q \ominus P$.

Lemma A.14. *If $g \in \mathcal{L}(Q+(Q\ominus m_1P)+\ldots+(Q\ominus m_tP))$ where $m_i, \ldots, m_t \in \mathbb{Z}\backslash\{0\}$ then*

$$\mathrm{ores}_{Q,1}(g) = 0.$$

Proof. This result will follow from the fact that the sum of the residues of a differential form must be zero. We start by noting that Lemma 3.2 states that $v_{u_Q}(\delta(u_Q)) = 0$ so we may write $\delta(u_Q)^{-1} = \alpha + \bar{u}$ where $0 \neq \alpha \in k$ and $\bar{u}$ is regular and zero at Q. For each $i \in \mathbb{Z}$ we write

$$g = \frac{c_{Q\ominus iP,-1}}{u_{Q\ominus iP}} + g_{Q\ominus iP}$$

where $g_{Q\ominus iP}$ is regular and zero at $Q \ominus iP$. Now consider the differential $g\Omega$. Since for any $i \in \mathbb{Z}$, $\Omega = \delta(u_{Q\ominus iP})^{-1} du_{Q\ominus iP}$, we have

$$\begin{aligned} g\Omega &= (\frac{c_{Q\ominus iP,-1}}{u_{Q\ominus iP}} + g_{Q\ominus iP})(\delta(u_{Q\ominus iP})^{-1} du_{Q\ominus iP}) \\ &= (\frac{c_{Q\ominus iP,-1}}{u_{Q\ominus iP}} + g_{Q\ominus iP})(\tau^i(\delta(u_Q)^{-1}) du_{Q\ominus iP}) \\ &= (\frac{c_{Q\ominus iP,-1}}{u_{Q\ominus iP}} + g_{Q\ominus iP})((\alpha + \tau^i(\bar{u})) du_{Q\ominus iP}) \end{aligned}$$

where the second equality follows from the fact that $u_{Q\ominus iP} = \tau^i(u_Q)$ and $\tau\delta = \delta\tau$. Therefore the residue of $g\Omega$ at $Q \ominus iP$ is $\alpha c_{Q\ominus iP,-1}$. Since $\alpha \neq 0$ and the sum of the residues of a differential form is 0 we have $\mathrm{ores}_{Q,1}(g) = 0$. $\square$

Remark A.15. *The proof of Lemma A.14 shows that if the poles of $g \in k(E)$ are simple and belong to $Q \oplus \mathbb{Z}P$, then there exists $0 \neq \alpha \in \mathbb{C}$ such that $\mathrm{ores}_{Q,1}(g) = \alpha \sum_{i\in\mathbb{Z}} \mathrm{Res}_{Q\oplus iP}(g\Omega)$. Therefore, $\mathrm{ores}_{Q,1}(g) = 0$ if and only if $\sum_{i\in\mathbb{Z}} \mathrm{Res}_{Q\oplus iP}(g\Omega) = 0$.*

Remark A.16. *Lemma A.14 and Proposition A.1 imply Corollary A.3. To see this note that for f as in Corollary A.3 we have that $f \in \mathcal{L}(Q + (Q \ominus m_1P) + \ldots + (Q \ominus m_tP))$ where $m_i = -n_i$. The residue of $h(X) = \sum_{i=1}^{t} f(X \oplus n_iP)$ at $X = P$ is $\mathrm{ores}_{Q,1}(f)$, so $h(X)$ is regular at Q_0. Applying Proposition A.1 yields the conclusion of Corollary A.3.*

Proof that 2. implies 1. in Proposition A.4. Let us assume that condition 2. holds. We claim that it is enough to find an element $\tilde{g}$ and a nonzero operator L such that $L(h) = \tau(\tilde{g}) - \tilde{g}$. Assume that we have done this. Then

$$L(b) = L(\tau(e) - e + h) = \tau(L(e)) - L(e) + \tau(\tilde{g}) - \tilde{g} = \tau(g) - g$$

where $g = L(e) + \tilde{g}$.

If h is constant, then the result is obvious (take $L = \delta$ and $\tilde{g} = 0$). We shall now assume that h is not constant.

To simplify notation, we write u for u_Q and let $\delta(u) = u_0 + \bar{u}$, where $0 \neq u_0 \in k$ and $\bar{u}$ is regular and zero at Q, and so $\delta(\tau(u)) = u_0 + \tau(\bar{u})$. Using Lemma A.14, one sees that

$$\begin{aligned} \delta(h) &= \frac{-u_0 a}{u^2}(1 + h_u) \\ &= \frac{u_0 a}{\tau(u)^2}(1 + h_{\tau(u)}) \end{aligned}$$

where $v_u(h_u) > 0$ and $v_{\tau(u)}(h_{\tau(u)}) > 0$. Therefore there exists an element $f \in \mathcal{L}(2Q)$ such that

$$\delta(h) - (\tau(f) - f) \in \mathcal{L}(Q + (Q \ominus P)).$$

Since $\{1, h\}$ forms a basis of $\mathcal{L}(Q + (Q \ominus P))$ (recall that h is not constant), there exist elements $c, d \in k$ such that

$$\delta(h) - (\tau(f) - f) - ch - d = 0.$$

Therefore

$$\delta^2(h) - c\delta(h) = \tau(\delta(f)) - \delta(f)$$

and conclusion 2. holds for $L = \delta^2 - c\delta$ and $\tilde{g} = \delta(f)$. □

We note that 2. cannot be changed to conclude that $b = \tau(e) - e + c$ for some constant $c \in k$. To see this, let b be a nonconstant element of $\mathcal{L}(Q + (Q \ominus P))$. Note that $\mathrm{pdisp}(g, Q \ominus P) = 1$. We have just shown that b satisfies 1. of Proposition A.7. Now assume $b = \tau(e) - e + c$ for some $e \in k(E), c \in k$. Since $\mathrm{pdisp}(\tau(e) - e) = \mathrm{pdisp}(e) + 1$ if $e \notin k$, we have $\mathrm{pdisp}(e) = 0$. Since b has no poles outside of $\{Q, Q \ominus P\}$, we would have that e has at most one pole and this pole would be simple. Therefore e must be constant. Since b is not a constant, we have that $b \neq \tau(e) - e + c$ for any $e \in k(E), c \in k$.

A.1.2. *Proof of Proposition A.7.*

Proof that 1. implies 2. in Proposition A.7. For any $Q \in E$ and $j \in \mathbb{N}_{>0}$, we have $\mathrm{ores}_{Q,j}(e) = \mathrm{ores}_{Q,j}(\tau(e))$. Furthermore Lemma A.14 implies that $\mathrm{ores}_{Q,j}(g) = 0$. Therefore $\mathrm{ores}_{Q,j}(f) = \mathrm{ores}_{Q,j}(\tau(e) - e + g) = \mathrm{ores}_{Q,j}(\tau(e)) - \mathrm{ores}_{Q,j}(e) + \mathrm{ores}_{Q,j}(g) = 0$. □

Proof that 2. implies 1. in Proposition A.7. The proof of this implication is similar to the proof that 1. implies 2. in Proposition A.4. Lemma A.13 implies that we may assume that $\mathrm{pdisp}(f) \leq 1$ and wdisp(f) = 0. Therefore if f has a pole of order $j \geq 2$ at some $Q \in E$, then Q is the only point in $Q + \mathbb{Z}P$ at which f has a pole. Since $\mathrm{ores}_{Q,j}(f) = 0$, we have that f has no poles of order greater than 1. Since we also have $\mathrm{pdisp}(f) \leq 1$, we can conclude that that f has only poles of order 1 and the poles of f occur in pairs $\{Q_1, Q_1 \ominus P\}, \ldots, \{Q_r, Q_r \ominus P\}$ where $Q_i \oplus \mathbb{Z}P \cap Q_j \oplus \mathbb{Z}P = \emptyset$ for $i \neq j$.

We will now show how one can construct an element e such that $f - (\tau(e) - e)$ has at most one pair of poles $\{Q, Q \ominus P\}$. This will yield condition 2. of the Proposition. We can assume that $r > 1$. Let $h \in k(E)$ be a nonconstant element of $\mathcal{L}(Q_1 + Q_2)$. There exists an $a \in k$ such that $\tilde{f} = f - (\tau(ag) - ag)$ has no pole at Q_1. The element $\tilde{f}$ has only simple poles and $\mathrm{pdisp}(f) \leq 1$. Therefore its poles occur at possibly $Q_1 \ominus P, \{Q_2, Q_2 \ominus P\}, \ldots, \{Q_r, Q_r \ominus P\}$. Since $\mathrm{ores}_{Q_1,1}(f) = 0$, f cannot have a singe pole in $Q_1 + \mathbb{Z}P$. Therefore we have that $\tilde{f}$ has no pole at $Q_1 \ominus P$. Continuing in this way we eventually find an $e \in k(E)$ such that $f - (\tau(e) - e)$ has exactly one pair of poles $\{Q, Q \ominus P\}$. □

A.2. **Some computation of orbit residues.** Let E be an elliptic curve defined over an algebraically closed field k of characteristic zero and $k(E)$ be its function field. Let P be a non-torsion point on E and let $\tau : k(E) \to k(E)$ denote map corresponding to $Q \mapsto Q \oplus P$ on E, where $\oplus$ denotes the group law on E. Let ι_1 and ι_2 two involutions of E such that $\tau = \iota_2 \circ \iota_1$. We let Ω be a non zero regular differential form on E and we keep notation as in §A.1.

Lemma A.17. *Let $b \in k(E)$ such that $\iota_1(b) = -b$. Let $Q \in E$ be a simple pole of b such that $Q \neq \iota_1(Q)$. Then, $\iota_1(Q)$ is a simple pole of b and the residue of $b\Omega$ at Q coincides with its residue at $\iota_1(Q)$.*

Proof. The assertion follows from the fact that, since $\iota_1(b) = -b$ and $\iota_1^*(\Omega) = -\Omega$ (see [Dui10, Lemma 2.5.1 and Proposition 2.5.2]), the form $\eta = b\Omega$ satisfies $\eta = \iota_1^*(\eta)$. (Indeed, if u_Q is a local parameter at Q, then we have $\eta = v du_Q$ with $v = \frac{c_Q}{u_Q} + \overline{v}$ where $c_Q \in \mathbb{C}$ is the residue of η at Q and $\overline{v}$ is regular at Q. Hence, $\iota_1(u_Q)$ is a local parameter at $\iota_1(Q)$, and we have $\iota_1^*(\eta) = \iota_1(v)d\iota_1(u_Q)$ with $\iota_1(v) = \frac{c_Q}{\iota_1(u_Q)} + \iota_1(\overline{v})$ where $\iota_1(\overline{v})$ is regular at $\iota_1(Q)$. So, $\mathrm{Res}_{\iota_1(Q)}(\eta) = \mathrm{Res}_{\iota_1(Q)}(\iota_1^*(\eta)) = c_Q = \mathrm{Res}_Q(\eta)$.) □

For the notion of coherent analytic parameters used below, we refer to Section A.1, especially to Remark A.9.

Lemma A.18. *There exists a coherent set of analytic local parameters $\{u_Q \mid Q \in E\}$ on E^{an} such that $\iota_1(u_Q) = -u_{\iota_1(Q)}$. Let $b \in k(E)$ such that $\iota_1(b) = -b$. For such a set of local parameters, if*

$$b = \frac{c_{Q,n}}{u_Q^n} + \ldots + \frac{c_{Q,2}}{u_Q^2} + \frac{c_{Q,1}}{u_Q} + \tilde{f}$$

where $v_Q(\tilde{f}) \geq 0$, then

$$b = \frac{c_{\iota_1(Q),n}}{u_{\iota_1(Q)}^n} + \ldots + \frac{c_{\iota_1(Q),2}}{u_{\iota_1(Q)}^2} + \frac{c_{\iota_1(Q),1}}{u_{\iota_1(Q)}} + \tilde{g}$$

where $v_{\iota_1(Q)}(\tilde{g}) \geq 0$ and $c_{\iota_1(Q),j} = (-1)^{j+1}c_{Q,j}$. If follows that, if all the poles of b belong to the same τ-orbit, then, for any even number j, we have $\mathrm{ores}_{Q,j}(b) = 0$.

Proof. We first prove the existence of analytic local parameters with the desired properties. According to [Dui10, p.35 and Remark 2.3.8], $\iota_1(P) = [-1]P \oplus P_0$ for some $P_0 \in E$. By uniformisation, it is equivalent to prove the following result : *Consider a lattice $\Lambda \subset \mathbb{C}$ and two endomorphisms of the complex torus $\mathbb{C}/\Lambda$ given by $\iota_1 : \overline{z} \mapsto \overline{-z + p_0}$ and $\tau : \overline{z} \mapsto \overline{z + q_0}$ for some $p_0, q_0 \in \mathbb{C}$. Then, there exists a set of analytic local parameters $\{u_{\overline{\omega}} \mid \overline{\omega} \in \mathbb{C}/\Lambda\}$ on the complex torus $\mathbb{C}/\Lambda$ such that $\iota_1(u_{\overline{\omega}}) = -u_{\iota_1(\overline{\omega})}$ and $\tau(u_{\overline{\omega}}) = u_{\tau(\overline{\omega})}$.* Such local parameters are given by $u_{\overline{\omega}} : \overline{z} \mapsto z - \omega$ for z close to ω. The rest of the Lemma is a direct consequence of the following easy computation. Indeed, applying ι_1 to

$$b = \frac{c_{Q,n}}{u_Q^n} + \ldots + \frac{c_{Q,2}}{u_Q^2} + \frac{c_{Q,1}}{u_Q} + \tilde{f},$$

we get

$$\begin{aligned} -b = \iota_1(b) &= \frac{c_{Q,n}}{\iota_1(u_Q)^n} + \ldots + \frac{c_{Q,2}}{\iota_1(u_Q)^2} + \frac{c_{Q,1}}{\iota_1(u_Q)} + \iota_1(\tilde{f}) \\ &= \frac{(-1)^n c_{Q,n}}{u_{\iota_1(Q)}^n} + \ldots + \frac{(-1)^2 c_{Q,2}}{u_{\iota_1(Q)}^2} + \frac{(-1)^1 c_{Q,1}}{u_{\iota_1(Q)}} + \iota_1(\tilde{f}) \end{aligned}$$

where $v_{\iota_1(Q)}(\iota_1(\tilde{f})) \geq 0$, as expected. □

Lemma A.19. *If $g \in \mathcal{L}(2Q + 2(Q \ominus m_1 P) + \ldots + 2(Q \ominus m_s P))$ where $m_1, \ldots, m_s \in \mathbb{Z}\backslash\{0\}$ is such that $\mathrm{ores}_{Q,2}(g) = 0$ then*

$$\mathrm{ores}_{Q,1}(g) = 0.$$

Proof. We may write $\delta(u_Q)^{-1} = \alpha + \beta u_Q + \bar{u}$ where $0 \neq \alpha \in k$, $\beta \in k$ and $\bar{u}$ is regular and has a zero of order 2 at Q. For each $i \in \{0, m_1, \ldots, m_s\}$ we write

$$g = \frac{c_{Q \ominus iP,2}}{u_{Q \ominus iP}^2} + \frac{c_{Q \ominus iP,1}}{u_{Q \ominus iP}} + g_{Q \ominus iP}$$

where $g_{Q\ominus iP}$ is regular and zero at $Q\ominus iP$. Now consider the differential $g\Omega$. Since for any $i\in\mathbb{Z}$, $\omega=\delta(u_{Q\ominus iP})^{-1}du_{Q\ominus iP}$, we have

$$\begin{aligned} g\Omega &= (\frac{c_{Q\ominus iP,2}}{u^2_{Q\ominus iP}}+\frac{c_{Q\ominus iP,1}}{u_{Q\ominus iP}}+g_{Q\ominus iP})(\delta(u_{Q\ominus iP})^{-1}du_{Q\ominus iP})\\ &= (\frac{c_{Q\ominus iP,2}}{u^2_{Q\ominus iP}}+\frac{c_{Q\ominus iP,1}}{u_{Q\ominus iP}}+g_{Q\ominus iP})(\tau^i(\delta(u_Q)^{-1})du_{Q\ominus iP})\\ &= (\frac{c_{Q\ominus iP,2}}{u^2_{Q\ominus iP}}+\frac{c_{Q\ominus iP,1}}{u_{Q\ominus iP}}+g_{Q\ominus iP})((\alpha+\beta u_{Q\ominus iP}+\tau^i(\bar{u}))du_{Q\ominus iP}) \end{aligned}$$

where the second equality follows from the fact that $u_{Q\ominus iP}=\tau^i(u_Q)$ and $\tau\delta=\delta\tau$. Note that $\tau^i(\bar{u})$ is regular and has a zero of order 2 at $Q\ominus iP$. Therefore the residue of $g\omega$ at $Q\ominus iP$ is $\alpha c_{Q\ominus iP,1}+\beta c_{Q\ominus iP,2}$. Since the sum of the residues of a differential form is 0 we get $\alpha\mathrm{ores}_{Q,1}(g)+\beta\mathrm{ores}_{Q,2}(g)=0$. Since $\alpha\neq 0$ and $\mathrm{ores}_{Q,2}(g)=0$, we get $\mathrm{ores}_{Q,1}(g)=0$. □

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