Nonintegrability by discrete quadratures

Guy Casale Julien Roques

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Abstract

We give a necessary condition for integrability by discrete quadratures of systems of difference equations : the discrete variational equations along algebraic solutions must have virtually solvable Galois groups. This necessary condition à la Morales and Ramis is used in order to prove that q-analogues of Painlevé I and Painlevé III equations are not integrable by discrete quadratures.

Résumé

Nous nous intéressons à la notion d'intégrabilité par quadratures discrètes pour les systèmes d'équations aux différences algébriques. Nous montrons que les groupes de Galois des équations variationnelles discrètes le long des solutions algébriques d'un système aux différences intégrable sont virtuellement résolubles. Cette condition nécessaire à l'intégrabilité, dans la veine de la théorie de Morales et Ramis, est ensuite utilisée pour montrer que deux types d'équations de Painlevé discrètes ne sont pas intégrables par des quadratures discrètes.

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Introduction

There is no consensus on a general notion of integrability for dynamical systems. G. D. Birkhoff wrote in [5] "It is a well-known fact that for certain problems, auxiliary analytic relations can be deduced by means of which the solutions of the system of differential equations can be satisfactorily treated, in which case the system may be said integrable. When, however, one attempts to formulate a precise definition of integrability, many possibility appear, each with a certain intrinsic interest."

In this paper, we are interested in the integrability by *discrete quadratures* of rational systems of nonlinear σ-difference equations of the form

$$\begin{cases} \sigma y_1 & = & E_1(t, y_1, y_2, \ldots, y_m) \\ \vdots & & \\ \sigma y_m & = & E_m(t, y_1, y_2, \ldots, y_m) \end{cases} \tag{1}$$

where σ is, for instance, the shift $t \mapsto t+1$ or the q-dilatation operator $t \mapsto qt$. The system (1) is integrable by discrete quadratures if it admits a general σ-Liouvillian solution (in the sense of C. H. Franke in [20]); see Definition 3.11 for details. Roughly speaking, this ensures that there is a formula involving only "elementary operations" to get sufficiently transcendental solutions of (1).

In [9], we studied the integrability in the sense of Mishchenko-Fomenko (Definition 3.3 in *loc. cit.*) in the symplectic context. This includes the classical integrability in the sense of Liouville (meaning the existence of sufficiently many linearly independant first integrals in involution; see for instance section 3 in *loc. cit.*). There is no inclusion between these notions of integrability and the integrability by discrete quadratures considered in the present paper.

For more informations about the numerous notions of integrability, we refer the reader to the expository articles [69, 70, 29, 24].

One of our main results is :

Theorem. *If* (1) *is integrable by discrete quadratures then the Galois groups of its discrete variational equations (of any order) along particular algebraic solutions are virtually solvable.*

This result is inspired by the so-called Morales-Ramis theory [43, 44, 45] (for its ramifications and its innumerable applications, see [38, 48] for instance).

The statement of the above theorem involves the Galois groups of *linear* difference equations (namely, the discrete variational equations) but its proof heavily relies on the Galois theory of *nonlinear* difference equations.

Galois' ideas were first transposed to linear differential equations by E. Picard [51] and E. Vessiot [71]. Later, they were extended to linear difference equations. A number of authors have contributed to the development of Galois theories for linear difference equations over the past years, among whom C. H. Franke [19, 20, 21, 22], P. Etingof [18], M. van der Put and M. Singer [54], M. van der Put and M. Reversat [53], Z. Chatzidakis and E. Hrushovski [12], J. Sauloy [62], Y. André [2], C. Hardouin and M. Singer [32], etc. The relations between the various Galois theories for difference equations are partially understood. For this question, we refer the reader to [13, 30, 31].

The extension of the Galois theory of linear differential equations to nonlinear differential equations was first considered by E. Vessiot in [72] and J. Drach in [16]. In the recent years, the subject has known a renewal of interest with independent articles by B. Malgrange [40] and H. Umemura [67]. The ideas of these authors have been extended to nonlinear difference equations; we refer to the work of S. Morikawa [46], S. Morikawa and H. Umemura [47] and A. Granier [25, 26]. For comparison results between the various linear and nonlinear Galois theories for differential equations we refer to B. Malgrange [40] and H. Umemura [67, 68]. For comparison between the linear and nonlinear Galois theories for difference equations we refer to C. Hardouin and L. Di Vizio [30, 31] and S. Morikawa and H. Umemura [47]. Almost all these definitions are proved to be compatible. Comparison between A. Granier and S. Morikawa definitions is still missing and is not used here. In this paper, we recall and develop A. Granier's approach based on ideas of B. Malgrange.

Any generalization of differential or difference Galois theory to nonlinear equation uses pseudogroups instead of groups. The definition of algebraic pseudogroups seems to be different in each approach : $\mathcal{D}$-groupoids in [40] or Lie-Ritt functors in [67]. As we follow B. Malgrange's point of view, our main reference for algebraic pseudogroups will be [40] but most of the results used in this paper were already known by differential geometers. See the books of V. Guillemin and S. Sternberg [28] or J.-F. Pommaret [52].

The basic strategy for the proof of the above main Theorem is the following. We prove, using Artin's approximation theorem, that the pseudogroup of an algebraic σ-difference equation controls the Galois group of its linearization along any particular solution in the following sense : the Lie algebra of the Galois group of the linearized equation is a quotient of a Lie subalgebra of the pseudogroup of the nonlinear equation. Then the Theorem is deduced from the fact that the pseudogroup of a nonlinear σ-difference equation integrable by discrete quadratures is infinitesimally solvable.

Note that, from a Galoisian point of view, the Liouville integrability implies the commutativity of Galois groupoid and that the integrability by discrete quadratures implies the resolubility of this groupoid but the converse implications are false.

This paper also provides applications of our main Theorem to q-Painlevé equations.

Corollary. *The q-analogue of Painlevé I equation namely $q - A_7^1$ from H. Sakai's classification [61] is not integrable by q-quadratures.*

Corollary. *The q-analogue of Painlevé III equation namely $q - P_{III}$ from K. Kajiwara and K. Kimura [34, 35] is not integrable by q-quadratures.*

The discrete Painlevé equations are usually qualified as integrable discrete dynamical systems. This is not in contradiction with our results: in the previous sentence "integrable" roughly means that the discrete Painlevé equations have a "non chaotic behavior" whereas the integrability investigated in this paper means solvability by a controlled formula. By the way many discrete Painlevé equations have been found by using "integrability detectors" such as singularity confinement, zero algebraic entropy, diophantine integrability, isomonodromic transformations, etc. Algebraic entropy as defined in [4] (see also [17]) has a special interest in the study of integrability. It is easy to prove that the algebraic entropy of discrete systems integrable by discrete quadratures vanishes *i.e.* the sequences of degrees of the iterations has a polynomial growth. In [66], T. Takenawa proved that discrete Painlevé equations have zero algebraic entropy but questions about integrability by discrete quadratures or more generally reducibility of these equations was left open. Recently S. Nishioka investigated the reducibility of discrete Painlevé equations in [49, 50]. The method proposed in the present paper is more general and systematic than S. Nishioka's calculations but the conclusion is weaker since integrability by discrete quadratures implies reducibility.

To the best of our knowledge, at the moment, there is no general theory for the discrete Painlevé equations, except Sakai's geometric approach in [61] based on previous work by M.-H. Saito et H. Umemura [60]. For detailed informations about discrete Painlevé equations and integrability, we refer to the expository articles [29, 24].

This paper is organized as follows. In sections 1, 2 and 3 we recall the definition of Malgrange pseudogroup and we prove its infinitesimal solvability in the integrable case. In section 4 we recall useful results regarding the Galois theory of linear difference equations. In section 5 we prove our main theorem. In section 6 and in section 7 we prove that q-analogues of Painlevé I and Painlevé III equations are not integrable by discrete quadratures.

1 Frame bundles and prolongations

Let M be a smooth irreducible affine algebraic variety over $\mathbb{C}$ of dimension d whose coordinate ring is denoted by $\mathbb{C}[M]$.

Following B. Malgrange [40], some classical constructions from differential geometry will be presented algebraically by means of their functors of points. We refer the reader to § 2 and § 3 of V. Guillemin and S. Sternberg's booklet [28] for the point of view of differential geometry. Their notations differ from ours (frame bundles are denoted by $D(O, M)$ and pseudogroups by Γ).

The formal frame bundle of M is the complex proalgebraic variety RM of all formal invertible maps $r : (\mathbb{C}^d, 0) \to M$. More precisely, for any $\mathbb{C}$-algebra A, the A-points of RM, called A-frames on M, are given by

$$RM(A) = \{\text{locally invertible } \mathbb{C}\text{-algebra morphisms } f : \mathbb{C}[M] \to A[[t]]\}$$

where $A[[t]] := A[[t_1, \ldots, t_d]]$ is the ring of formal power series in the d indeterminates $t_1, \ldots, t_d$ (t is a shorthand notation for $t_1, \ldots, t_d$). A $\mathbb{C}$-algebra morphism $f : \mathbb{C}[M] \to A[[t]]$ is said to be locally invertible if its scalar extension $f_A : A[M] \to A[[t]]$ induces an isomorphism $df_A : J/J^2 \to (t)/(t)^2$ where $(t) := (t_1, \ldots, t_d)$ is the ideal of $A[[t]]$ generated by $t_1, \ldots, t_d$ and where $J = f_A^{-1}(t)$; it is equivalent to require that f_A induces an isomorphism $\widehat{f}_A : \widehat{A[M], J} \to A[[t]]$ between the formal completion of $A[M] = A \otimes \mathbb{C}[M]$ along J and the ring of formal power series $A[[t]]$ (see [42, chap.29]).

The algebra $\mathbb{C}[RM]$ of this variety is built up from the differential algebra $\mathbb{C}\{M\}$ generated by $\mathbb{C}[M]$ with d commuting derivations $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_d}$ by inverting the Jacobian determinant $\det\left(\frac{\partial}{\partial t_i} x_j\right)_{1 \le i,j \le d}$ of a transcendence basis $x_1, \ldots, x_d$ of $\mathbb{C}[M]$ over $\mathbb{C}$. This ring does not depend on the choice of the transcendence basis.

The projection of RM on M is given on the A-points by $f \mapsto f/(t)$.

The ring $\mathbb{C}[RM]$ is filtered by the subrings $\mathbb{C}[RM]_k$ of differential polynomials of order less than or equal to k. The ring $\mathbb{C}[RM]_k$ is the coordinate ring of the order k frame bundle R_kM of M. For any $\mathbb{C}$-algebra A, the A-points of R_kM are given by

$$R_kM(A) = \{\text{locally invertible } \mathbb{C}\text{-algebra morphisms } f : \mathbb{C}[M] \to A[[t]]/(t)^{k+1}\}.$$

A derivation or an endomorphism of $\mathbb{C}[RM]$ is said to be of degree less than or equal to n if it maps $\mathbb{C}[RM]_k$ in $\mathbb{C}[RM]_{k+n}$ for all $k \in \mathbb{N}$.

Example 1.1. If $M = \mathbb{A}^d$ then $\mathbb{C}[M] = \mathbb{C}[x_1, \ldots, x_d]$. We have

$$\mathbb{C}[RM] = \mathbb{C}\left[x_i^\alpha \mid 1 \le i \le d, \alpha \in \mathbb{N}^d\right]\left[\delta^{-1}\right]$$

where the x_i^α's are indeterminates and $\delta = \det\left(x_i^{\epsilon(j)}\right)_{1 \le i,j \le d}$ with $\epsilon(j) = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^d$ where the 1 is at the jth coordinate. We have

$$\mathbb{C}[R_kM] = \mathbb{C}\left[x_i^\alpha \mid 1 \le i \le d, \alpha \in \mathbb{N}^d \text{ s.t. } |\alpha| \le k\right]\left[\delta^{-1}\right]$$

where $|\alpha|$ is the sum of the coordinates of α. The order k frame bundle is

$$R_kM = \mathbb{A}^d \times GL_d \times \left(\bigtimes_{\ell=2}^{k} \mathbb{A}^{\binom{\ell+d}{d}}\right).$$

The projection on M is given by the identification of x_i with x_i^0.

1.1 RM has a canonical parallelism

There is a morphism L from the $\mathbb{C}$-Lie algebra $\widehat{\chi}$ of formal vector fields on $(\mathbb{C}^d, 0)$ to the $\mathbb{C}$-Lie algebra of vector fields on RM of degree less than or equal to 1. Indeed, the Lie algebra $\widehat{\chi}$ is the left $\mathbb{C}[[t]]$-module $\bigoplus_i \mathbb{C}[[t]]\frac{\partial}{\partial t_i}$ of (t)-continuous derivations of $\mathbb{C}[[t]]$. This Lie algebra $\widehat{\chi}$ is the projective limit $\lim\limits_{\leftarrow} \chi_n$ of the finite dimensional $\mathbb{C}$-vector spaces $\chi_n = \widehat{\chi} \underset{\mathbb{C}[[t]]}{\otimes} \mathbb{C}[[t]]/(t)^{n+1}$. For any $\mathbb{C}$-algebra A, for any A-point $Z \in \widehat{\chi}\widehat{\otimes}A = \lim\limits_{\leftarrow} \chi_n \underset{\mathbb{C}}{\otimes} A$ of $\widehat{\chi}$, the map

$$\begin{array}{rcl} RM(A) & \to & RM(A[\epsilon]/(\epsilon^2)) \\ f & \mapsto & (\mathbb{I} + \epsilon Z) \circ f, \end{array}$$

where $\mathbb{I}$ is the identity, defines a vector field LZ on RM by means of dual numbers (see [33, p. 80]). This vector field is called the canonical lift of Z. Note that L is compatible with the Lie bracket : $L[Z_1, Z_2] = [LZ_1, LZ_2]$.

One can check that the lifts of non vanishing vector fields at $0 \in (\mathbb{C}^d, 0)$ have degree 1 whereas the lifts of vanishing vector fields at $0 \in (\mathbb{C}^d, 0)$ have degree 0. The Lie subalgebra $(t)\widehat{\chi} = (t_1, ..., t_d)\widehat{\chi}$ of vanishing vector fields at $0 \in (\mathbb{C}^d, 0)$ is denoted by $\widehat{\chi}^0$. This Lie algebra $\widehat{\chi}^0$ is the projective limit $\lim\limits_{\leftarrow} \chi_n^0$ of the finite dimensional $\mathbb{C}$-vector spaces $\chi_n^0 = \widehat{\chi}^0 \underset{\mathbb{C}[[t]]}{\otimes} \mathbb{C}[[t]]/(t)^{n+1}$.

This action of $\widehat{\chi}$ gives an isomorphism $\widehat{\chi} \times RM \to TRM$ above RM. The inverse morphism sends the tangent vector $f + \epsilon g : \mathbb{C}[M] \to A[[t]][\epsilon]/(\epsilon^2)$ in $T_f RM$ on $(\widehat{g}_A \circ \widehat{f}_A^{-1}, f)$ in $\widehat{\chi} \times RM$. Such a trivialization of the tangent vector bundle is called a parallelism. At each point f of RM, the image of $\widehat{\chi}$ spans $T_f RM$ and its Lie subalgebra $\widehat{\chi}^0$ spans the relative tangent $T_f(RM/M)$.

1.2 RM is a principal bundle

Let Γ be the complex proalgebraic group whose A-points are :

$$\Gamma(A) = \{\text{invertible } (t)\text{-continuous } A\text{-algebra morphisms } g : A[[t]] \to A[[t]]\}.$$

The proalgebraic structure is given by the projective limit $\Gamma = \lim\limits_{\leftarrow}\Gamma_k$ of the complex linear algebraic groups Γ_k whose A-points are given by

$$\Gamma_k(A) = \{\text{invertible } A\text{-algebra morphisms } g : A[[t]]/(t)^{k+1} \to A[[t]]/(t)^{k+1}\}.$$

We have an isomorphism

$$\begin{array}{rcl} L(\Gamma)(A) : \Gamma(A) \times RM(A) & \to & RM(A) \underset{M(A)}{\times} RM(A) \\ (g, f) & \mapsto & (f \circ g, f) \end{array}$$

so that RM is a Γ-principal bundle over M. The Lie algebra of Γ is $\widehat{\chi}^0$. The tangent of $L(\Gamma)$ gives a map from the tangent at the identity of Γ to the relative tangent of the projection of $RM \underset{M}{\times} RM \to RM$ onto the first factor along the diagonal :

$$T_{id}L(\Gamma) : T_{id}\Gamma \times RM \to T_{diag}(RM \underset{M}{\times} RM)/RM \sim TRM.$$

Evaluating on $Z \in \widehat{\chi}^0\widehat{\otimes}A = \lim\limits_{\leftarrow}\chi_n^0 \otimes A$, one gets LZ.

1.3 RM is a 'natural' bundle

Let us consider two A-points p, p' of M. The formal completion of M along p is denoted by $\widehat{M,p}$. We refer to [33, pp. 194-195] for formal schemes. Let φ be an isomorphism from be the formal scheme $\widehat{M,p}$ with ring $\widehat{\mathbb{C}[M],p}$ to the formal scheme $\widehat{M,p'}$ with ring $\widehat{\mathbb{C}[M],p'}$. We have a natural lift $R\varphi$ of φ :

$$R\varphi : \widehat{M,p} \underset{M}{\times} RM = \widehat{RM,p} \to \widehat{RM,p'} = \widehat{M,p'} \underset{M}{\times} RM$$

defined, for any $f : \widehat{\mathbb{C}[M],p'} \to A[[t]]$, by $R\varphi(f) = f \circ \varphi$. This isomorphism has degree 0. This lift is called the prolongation of φ. Note that $R(\varphi_1 \circ \varphi_2) = R\varphi_1 \circ R\varphi_2$.

Let p be a A-point of M and let f be a A-frame at p. The frame f is a morphism $f : \mathbb{C}[M] \to A[[t]]$ such that $f/(t) = p$ thus one can take the formal completion $\widehat{f} : \widehat{\mathbb{C}[M],p} \to A[[t]]$. Let X be a formal vector field at p i.e. $\mathbb{I}+\epsilon X$ is a morphism from $\widehat{\mathbb{C}[M],p}$ to $(\widehat{\mathbb{C}[M],p})[\epsilon]/(\epsilon^2)$. We have a natural lift RX of X:

$$\begin{array}{ccc} \widehat{RM,p}(A) & \to & \widehat{RM,p}(A[\epsilon]/(\epsilon^2)) \\ \widehat{f} & \mapsto & \widehat{f}_\epsilon \circ (\mathbb{I}+\epsilon X) \end{array}$$

where $\widehat{f}_\epsilon$ is the algebra morphism from $(\widehat{\mathbb{C}[M],p})[\epsilon]/(\epsilon^2)$ to $A[\epsilon]/(\epsilon^2)$ defined by $\widehat{f}$ on $(\widehat{\mathbb{C}[M],p})$ and by $\widehat{f}_\epsilon(\epsilon)=\epsilon$. This lift RX is called the prolongation of X on RM. One has $[RX_1, RX_2] = R[X_1, X_2]$. These prolongations commute with the actions of Γ and $\widehat{\chi}$.

1.4 $\mathbb{C}[RM]$ is a differential ring

The fact that $\mathbb{C}[RM]$ is a differential ring is clear from its definition but the differential structure is not canonical. Let $\mathfrak{h}$ be a Lie algebra such that $\widehat{\chi}=\mathfrak{h}\oplus\widehat{\chi}^0$. Then $\mathbb{C}[RM]$ is the $\mathfrak{h}$-differential ring generated by $\mathbb{C}[M]$ where the Jacobian determinant of a transcendence basis of $\mathbb{C}[M]$ over $\mathbb{C}$ is invertible. This ring does not depend on the choice of the transcendence basis.

By duality on gets a differential $d:\mathfrak{h}^*\to\wedge^2\mathfrak{h}^*$. The action of $\mathfrak{h}$ gives a total derivative $D:\mathbb{C}[RM]\to\mathbb{C}[RM]\otimes\mathfrak{h}^*$ describing the differential structure as it is done in [2, **2.1.2.1.**]. It can be extended into $D:\mathbb{C}[RM]\otimes\wedge^n\mathfrak{h}^*\to\mathbb{C}[RM]\otimes\wedge^{n+1}\mathfrak{h}^*$ by $D(f\otimes h^*)=Df\wedge h^*+f\otimes dh^*$ and, by Jacobi's identity, one has $DD=0$.

Example 1.2. If M is a subvariety of $\mathbb{A}^q$ of dimension d defined by an ideal $I\subset\mathbb{C}[x_1,\ldots,x_q]$ and if $t_1,\ldots,t_d$ are coordinates on $(\mathbb{C}^d,0)$ then one can choose $\mathfrak{h}=\oplus_i\mathbb{C}\frac{\partial}{\partial t_i}$. Let R be the ring

$$\mathbb{C}\left[x_i^\alpha, |1\le i\le q, \alpha\in\mathbb{N}^d\right]/DI$$

where DI is the differential ideal generated by I for the following differential structure : the $(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_d})$-differential structure in the sense of E. R. Kolchin [37] of $\mathbb{C}\left[x_i^\alpha|1\le i\le q,\alpha\in\mathbb{N}^d\right]$ defined by $\frac{\partial}{\partial t_j}x_i^\alpha = x_i^{\alpha+\epsilon(j)}$. Using previous notations, we have $D:R\to R\otimes\mathfrak{h}^*$ defined, for $a\in R$, by $Da=\sum_{i=1}^d\frac{\partial a}{\partial t_i}dt_i$ where dt_i is the dual bases of $\mathfrak{h}^*$.

Now, we get $\mathbb{C}[RM]$ from R by extension by the inverse of the Jacobian determinant of a transcendental basis of $\mathbb{C}[M]$ over $\mathbb{C}$. This kind of ring extension has a unique structure of differential ring extension.

From now on such a $\mathfrak{h}$ is fixed.

2 Groupoids and Algebroids

References for general definitions and elementary properties of groupoids and algebroids are [23] for an algebraic geometry point of view and [39] for a differential geometry point of view. In this section, we will review specific examples needed in the rest of the paper.

2.1 The groupoid EM

The product $EM=RM\times RM$ has a structure of groupoid on RM given by

- two projections s and t onto the first and the second factor respectively called source and target,
- a composition $c: EM \underset{{}^tRM^s}{\times} EM\to EM$: the projection on the first and third factors,
- an identity $id:RM\to EM$: the diagonal,
- an inverse $inv:EM\to EM$ exchanging the two factors.

These maps have order 0 and satisfy some commutative diagrams [23]. This product situation is the prototype of groupoid. Subgroupoids of EM are algebraic equivalence relations on RM [39, Example 1.5].

This space is endowed with two commuting prolongation procedures for formal vector fields on M. The first one, the source prolongation of a formal vector field X on M, denoted by R^sX, is defined by RX on $s^*\mathbb{C}[RM]$ and 0 on $t^*\mathbb{C}[RM]$. The target prolongation R^tX is defined similarly (*mutatis mutandis*).

One can also prolong an isomorphism φ from the formal scheme $\widehat{M,p}$ to the formal scheme $\widehat{M,p'}$ in two ways. The source prolongation $R^s\varphi$ is defined by $R\varphi$ on $s^*\mathbb{C}[RM]$ and Id on $t^*\mathbb{C}[RM]$. The target prolongation is defined similarly (*mutatis mutandis*).

These prolongations are also called right and left translations by analogy with groups in [39, Chap. II, Definition 1.2].

2.2 The groupoid $AutM$

The group Γ acts diagonally on the product $RM \times RM = EM$ by $g \cdot (f_1, f_2) = (f_1 \circ g, f_2 \circ g)$. The quotient EM/Γ has two projections on $RM/\Gamma = M$ inherited from s and t and still denoted by s and t. The groupoid structure of the direct product $RM \times RM = EM$ induces a groupoid structure on the quotient EM/Γ still denoted by c, id and inv. This quotient groupoid will be denoted by $AutM$. Points of this space can be identified with formal maps φ between formal neighborhoods of points in M : let f_1, f_2 be two A-frames on M, p and q the A-points defined by $f_1^{-1}(t)$ and $f_2^{-1}(t)$ respectively and $\widehat{f}_{1A} : \widehat{A[M],p} \to A[[t]]$, $\widehat{f}_{2A} : \widehat{A[M],q} \to A[[t]]$ be the corresponding formal completions, then

$$\begin{array}{ccc} RM(A) \times RM(A) & \to & AutM(A) \\ (f_1, f_2) & \mapsto & (\widehat{f}_{1A})^{-1} \circ \widehat{f}_{2A} \end{array}$$

is the quotient map. Because source and target prolongations R^s and R^t defined in section 2.1 commute with the action of Γ, they induce prolongations on $AutM$ still denoted by R^s and R^t.

One can find RM from $AutM$ by choosing a closed point $p \in M$ and a formal $\mathbb{C}$-frame $r : (\mathbb{C}^d, 0) \to (M, p)$ at p. The part $Aut_{(p,M)}$ of $AutM$ above $\{p\} \times M$ for the (source,target) projection can be identified with RM by means of r. Two such isomorphisms are related by the action of Γ.

The group Γ_k acts on R_kM in the same way and one can define Aut_kM to be the quotient of $R_kM \times R_kM$ by the diagonal action of Γ_k. Let us denote by the same symbol π_k^{k+1} the projection of Γ_{k+1} onto Γ_k and the projection of $R_{k+1}M$ onto R_kM. One has $(\pi_k^{k+1} \times \pi_k^{k+1})(g_{k+1} \cdot (f_{1,k+1}, f_{2,k+1})) = (\pi_k^{k+1}(g_{k+1}) \cdot (\pi_k^{k+1}(f_{1,k+1}), \pi_k^{k+1}(f_{2,k+1})))$. The groupoid $AutM$ inherits a proalgebraic structure given by $\varprojlim Aut_kM$.

This groupoid has a differential structure coming from the differential structure of RM. The structural group Γ of the principal bundle $RM \to M$ is a solvable extension of GL_d as it is described in [52] (see also subsection 5.2). Because of this, it is a special group (see [64, **4.4**]) and there exist rational sections $r : M \dashrightarrow RM$ of the projection of RM on M, *i.e.* rational moving frames on M, which can be used in order to trivialize some bundles :

- $TM \overset{\sim}{\dashrightarrow} M \times \mathfrak{h}$ by $(p, v) \mapsto (p, \widehat{T_p r(p)}^{-1} v)$,
- $M \times RM \overset{\sim}{\dashrightarrow} AutM$ by $(p, f) \mapsto f^{-1} \circ r(p)$.

The group $\Gamma(\mathbb{C}(M))$ of gauge transformations acts transitively on rational moving frames and on the trivializations obtained. Using these identifications one can define the differential structure of $AutM$ by the connection obtained by the following compositions :

$$\mathbb{C}[AutM] \to \mathbb{C}[RM] \otimes \mathbb{C}[M] \overset{D\otimes 1}{\to} \mathbb{C}[RM] \otimes \mathfrak{h}^* \otimes \mathbb{C}[M] \to \mathbb{C}[RM] \otimes \Omega^1(M) \to \mathbb{C}[AutM] \underset{s^*\mathbb{C}[M]}{\otimes} \Omega^1(M)$$

Because of the action of $\Gamma(\mathbb{C}(M))$, the composition above is independent of r and is well defined. It gives the total derivation of the differential ring $D : \mathbb{C}[AutM] \to \mathbb{C}[AutM] \underset{s^*\mathbb{C}[M]}{\otimes} \Omega^1(M)$.

2.3 Subgroupoids and pseudogroups

There are two possibilities to define subgroupoids of $Aut(M)$. The first one is to consider algebraic subvarieties of $Aut(M)$ (projective limits of subvarieties of $Aut_k(M)$) such that restrictions of source, target, composition, identity and inverse maps induce a groupoid structure. The second one is to take limit of groupoids i.e. $G = \lim\limits_{\leftarrow} G_k$ with G_k a subvariety of $Aut_k(M)$ and a subgroupoid for k large enough. These objects are too smooth for our purpose and we have to introduce singular subgroupoids and pseudogroups following [40, Definition 4.1.1.].

Definition 2.1. A singular (sub)groupoid of $AutM$ is an algebraic subvariety $G = \lim\limits_{\leftarrow} G_k$ of $AutM$ with G_k a subvariety of $Aut_k(M)$ and a subgroupoid of $Aut_k(M - S)$ for k large enough and S a closed subvariety of M independent of k. Such a S must be smaller than M. The smallest S is called the singular locus of G.

From the differential structure one gets the notion of differential subvarieties : a D-variety (for short) is an algebraic subvariety defined by a differential ideal I (i.e. such that $DI \subset I \otimes \Omega^1(M)$).

Definition 2.2. An algebraic subpseudogroup G of $AutM$ is a singular algebraic groupoid given by a differential ideal (they are called D-groupoids in [40]) for the differential structure previously defined.

2.4 The Lie algebroid eM

The Lie algebroid of $RM \times RM$ is the relative tangent bundle for the source projection above the diagonal. It is a vector bundle over RM canonically isomorphic to its tangent TRM.

The pull back of the usual Lie bracket of vector fields on the sections of TRM defines a Lie bracket on the sections of eM. It is compatible with the graduation i.e. it induces a bracket on the vector fields on R_kM. It is obvious that such objects satisfy the definition of Lie algebroid from [39].

A Lie subalgebroid of eM is a linear subspace of eM in the sense of [27] stable by the bracket. The relative tangent bundle for the source projection of a singular subgroupoid of EM is a Lie algebroid.

Example 2.3 ([40])**.** A foliation on M defined by a linear subspace $F \subset TM$ gives by prolongation of sections a linear subspace $R^tF|_{diag} \subset eM$ stable under Lie bracket. It is a Lie subalgebroid of eM.

Example 2.4. [39, 40] An equivalence relation $E \subset M \times M$ gives by prolongation of sections an equivalence relation on EM *i.e.* a subgroupoid. Its Lie algebroid is the Lie algebroid defined by the foliation of M by equivalence classes of E.

Remark 2.5. If a foliation F has no first integrals, it cannot be the foliation by equivalence classes of an equivalence relation. From [23], it is not the Lie algebroid of an algebraic groupoid (even singular).

2.5 The Lie algebroid $autM$

The Lie algebroid $autM$ of the pseudogroup $AutM$ is the relative tangent of the source projection along the identity : id^*TAutM/s.

Since the identity $id : M \to AutM$ is a differential morphism for the differential structures given by the exterior derivative d on $\mathbb{C}[M]$ and by D on $\mathbb{C}[AutM]$, the tangent bundle along id inherits a differential structure. Furthermore these structures are compatible with source projection because $D|_{s^*\mathbb{C}[M]} = d$, thus the relative tangent bundle along the identity $autM$ has also a differential structure i.e. it is a $\mathbb{C}[M]$-module with connection.

On the other hand, the tangent of the quotient map $RM \times RM \to AutM$ by the diagonal action of Γ gives an isomorphism between $(T(RM/M))/\Gamma$ and $autM$. An element of the fiber $(autM)_p$ of $autM$ at p is identified with a Γ-invariant vector field on the fiber RM_p of RM at p.

Now RM_p is a principal homogeneous space for two groups whose actions on RM_p commute.

Let A be a $\mathbb{C}$-algebra. The first group is Γ and the action of $g \in \Gamma(A)$ on $f \in RM_p(A)$ is the composition $g \cdot f : \mathbb{C}[M] \xrightarrow{f} A[[t]] \xrightarrow{g} A[[t]]$.

The second group is $\widehat{Diff}(M,p)$ whose A-points are

$$\widehat{Diff}(M,p)(A) = \{\text{invertible } (p)\text{-continuous } A\text{-algebra morphisms } \varphi : \widehat{A[M],p} \to \widehat{A[M],p}\}$$

and the action of $\varphi \in \widehat{Diff}(M,p)(A)$ on $f \in RM_p(A)$ is the composition

$$\varphi \cdot f : \mathbb{C}[M] \hookrightarrow \widehat{A[M],p} \xrightarrow{\varphi} \widehat{A[M],p} \xrightarrow{\widehat{f}_A} A[[t]]$$

Since these two actions commute, the Γ-invariant vector fields on RM_p are exactly the infinitesimal generators of the action of $\widehat{Diff}(M,p)$. The Lie algebra of $\widehat{Diff}(M,p)$ is the Lie algebra $\widehat{\chi}_{M,p}$ of formal vector fields at p. So, an element of $(autM)_p$ can be canonically identified with a formal vector field on M at p.

Let $G \subset AutM$ be a pseudogroup and let p be a closed point of M not in the singular locus S of G. The Lie algebroid of G is the relative tangent bundle $\mathfrak{g} = id^*TG/s$ of the source projection along id. Its Lie algebra at p is $\mathfrak{g}(p) = p^*\mathfrak{g} \subset \widehat{\chi}_{M,p}$ under the identification described above.

2.6 Some properties of pseudogroups

A pseudogroup G is usually an algebraic groupoid only out of a singular locus. Nevertheless it defines a set theoretical groupoid.

Theorem 2.6. *Let φ_1, φ_2 be two composable formal maps in neighborhood of two $\mathbb{C}$-points and let G be a pseudogroup. If φ_1 and φ_2 are in G so is $\varphi_1 \circ \varphi_2$.*

The proof of this theorem can be found in [11, Theorem 1.5]. The main ingredient is Artin's approximation theorem (already used in a special case in [45]).

A consequence of this theorem is that for any closed point p of M, $G_{(p,p)} = s^{-1}(p) \cap t^{-1}(p)$ is a group and $G_{(p,M)} = s^{-1}(p)$ is a $G_{(p,p)}$-principal bundle over $t(G_{(p,M)}) \subset M$.

The proof of the previous theorem has also the following consequence proved in [11, Lemma 3.3].

Lemma 2.7. *If the Lie algebra of a pseudogroup G at a closed point of $M - S$ is solvable with a sequence of derived algebras of length N then, at any closed point $p \in M$, $\mathfrak{g}(p)$ is a Lie algebra with a sequence of derived algebras of length less than or equal to N.*

3 Malgrange's pseudogroup

3.1 Definition

A set theoretical pseudogroup on M is a set of analytic invertible maps $\varphi : U \to V$ between analytic open sets of M stable by composition (when defined), inversion, restriction, analytic continuation (under the invertibility condition) [28, Definition 1.2].

Let $\Phi : M \dashrightarrow M$ be a rational dominant map. This map is locally invertible. So, its restrictions to suitable open subsets are invertible and thus generate a set theoretical pseudogroup denoted by $PG\Phi$. Such a set of invertible maps describes a subset of $AutM$ by taking all Taylor expansions of every map at every point where it is defined. This subset of $AutM$ is a set theoretical groupoid and is still denoted by $PG\Phi$.

Example 3.1. Let $n \in \mathbb{N}$ and let $\Phi : \mathbb{C} \to \mathbb{C}$ be the map defined by $\Phi(x) = x^n$. The pseudogroup generated by Φ is given by the set restrictions of determinations of the following maps on any open set where it is invertible : $x \mapsto \zeta x^{n^a}$ with $a \in \mathbb{Z}$ and ζ is 1 if $a \geq 0$ otherwise ζ is a n^{-a}th root of unity.

Example 3.2 (Granier [26]). Let $Y(qt) = A(t)Y(t)$ be a linear q-difference equation with $A(t) \in GL_n(\mathbb{C}(t))$. The map $\Phi : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$ defined by $\Phi(t, Y) = (qt, A(t)Y)$ is dominant. Graphs of solutions of this equations are Φ-invariant curves. The pseudogroup generated by Φ is the set of determinations of $(t, Y) \mapsto (q^n, A_n(t)Y)$ where $A_n(t)$ is $\prod_{i=0}^{n-1} A(q^i t)$ if $n \in \mathbb{N}$ and $\prod_{i=n}^{-1} A(q^i t)$ if $n \in \mathbb{Z} - \mathbb{N}$.

Definition 3.3. Let M be a smooth complex algebraic variety and let $\Phi : M \dashrightarrow M$ be a rational dominant map. The Malgrange pseudogroup $Mal\Phi$ of Φ is the Zariski closure in $AutM$ of the set theoretical pseudogroup $PG\Phi$ generated by Φ.

We have to prove that such an object is an algebraic pseudogroup.

Proof. – Let Z be the Zariski closure of $PG\Phi$ in $Aut(M)$. Let $\varphi \in PG\Phi$ be a map defined on an open set U. The pointwise Taylor extension

$$\begin{array}{cccc} j\varphi : & U & \longrightarrow & AutM \\ & x_0 & \mapsto & \sum \frac{\partial^\alpha \varphi}{\partial x^\alpha}(x_0)\frac{(x-x_0)^\alpha}{\alpha!} \end{array}$$

of φ gives an analytic section of the source projection $AutM \to M$.

This section is differential for the differential structure d of $\mathcal{O}(U)$ and D of $\mathbb{C}[AutM]$ i.e. the morphism $\varphi^\# : \mathbb{C}[AutM] \to \mathcal{O}(U)$ is differential. Let I be the ideal of Z. Then $E \in I$ can be written as $\varphi^\# E = 0$ for all $\varphi \in PG\Phi$. This implies that $\varphi^\# DE = 0$ for all $\varphi \in PG\Phi$ and I is differential.

$Aut(M)$ acts on itself by target composition and one can consider the stabilizer of Z. This is the largest subvariety $G \subset Aut(M)$ such that there is an hypersurface $S \subset M$ and an inclusion $c(Z \underset{M}{\times} G) \subset Z$ above $(M - S) \times (M - S)$. One can prove the existence of G above the generic point $\mathbb{C}(M \times M)$ by elimination. This is proved in [8, pp. 773-774] following classical argument from algebraic group theory [65, Proposition 2.2.4]. Then G is a D-subvariety of $Aut(M)$ above the generic point of $M \times M$, by Ritt-Raudenbush basis theorem ([57]) one gets existence of G above $M \times M$.

Stability of G by composition out of an hypersurface follows from the associativity of the composition and thus $G \cap i(G)$ is a singular subgroupoid of $Aut(M)$ included in Z. By [40] $\widetilde{S}$ may be chosen to be some $(S \times M) \cup (M \times S)$ with S a hypersurface of M. Let U and V be two Zariski dense open subset of M such that $\Phi|_U : U \to V$ is a covering. Because $R^t\Phi|_U(PG\Phi)|_U = (PG\Phi)|_V$, one has $\overline{(PG\Phi)|_V} \subset R^t\Phi|_U\left(\overline{(PG\Phi)|_U}\right) \subset \overline{R^t\Phi|_U(PG\Phi)|_U} \subset \overline{(PG\Phi)|_V}$.

Reading equations of G, we get that, for any $p \in U$, the Taylor expansion $j_{x_0}\Phi$ of Φ at x_0 as well as $(j_{x_0}\Phi)^{-1}$ belong to G. So $PG\Phi \subset G \cap i(G)$ thus $Z \subset G \cap i(G)$ and Z is a singular groupoid. □

Remark 3.4. In the original definition of A. Granier, $Mal(\Phi)$ was the smallest algebraic pseudogroup containing $PG\Phi$. This definition gives a simplification as the smallest algebraic variety containing $PG\Phi$ is an algebraic pseudogroup.

Example 3.5 ([6, 7])**.** Let $n \in \mathbb{N}$ and let $\Phi : \mathbb{C} \to \mathbb{C}$ be the map defined by $\Phi(x) = x^n$. We get easily the inclusion

$$Mal(\Phi) \subset \{\varphi \mid \varphi(x) = \alpha x^\beta, \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}^*, x \in s(\varphi)\}.$$

The right hand side is an algebraic pseudogroup in $Aut(\mathbb{C})$ whose ideal is generated as a differential ideal by $x\varphi\varphi'' - x(\varphi')^2 + \varphi\varphi'$.

Example 3.6 (Granier [26])**.** Let $Y(qt) = A(t)Y(t)$ be a linear q-difference system with $A(t) \in GL_n(\mathbb{C}(t))$ and $|q| > 1$. Consider $\Phi : \mathbb{C}^{n+1} \dashrightarrow \mathbb{C}^{n+1}$ defined by $\Phi(t, Y) = (qt, A(t)Y)$. We have

$$Mal(\Phi) \subset \{(t, Y) \mapsto (\alpha t, B(t)Y) \mid \alpha \in \mathbb{C}^*, B \in GL_n(\mathcal{O}(U)), U \text{ any open set of } \mathbb{C}\}.$$

This inclusion may be strict. Determination of cases of strict inclusion is the aim of the differential difference Galois theory developed by C. Hardouin and M. Singer in [32].

Theorem 3.7. *Let M and N be two smooth complex algebraic varieties endowed with rational dominant maps $\Phi : M \dashrightarrow M$ and $\Psi : N \dashrightarrow N$. A rational dominant morphism $\pi : N \dashrightarrow M$ is a difference morphism if $\pi \circ \Psi = \Phi \circ \pi$. Assume that such a rational dominant difference morphism exists. Then, there exists a rational dominant groupoid morphism*

$$\pi_\star : Mal\Psi \dashrightarrow Mal\Phi.$$

Proof. – One defines the algebraic pseudogroup $Aut(\mathcal{F}_\pi)$ following [40, p 492], as the pseudogroup whose points are the formal invertible π-projectable maps :

$$\begin{array}{rcl} Aut(\mathcal{F}_\pi) & = & \{\varphi \in Aut(N) \mid \varphi_*(\pi^*\Omega^1_M) \subset \pi^*\Omega^1_M\} \\ & = & \{\varphi \in Aut(N) \mid \exists \pi_\star\varphi \in Aut(M) \ s.t. \ \pi \circ \varphi = (\pi_\star\varphi) \circ \pi\}. \end{array}$$

The projection of such a map induces a map

$$\pi_\star : Aut(\mathcal{F}_\pi) \dashrightarrow AutM.$$

By assumption $\pi \circ \Psi = \Phi \circ \pi$ so $PG\Psi \subset Aut(\mathcal{F}_\pi)$ and $Mal\Psi \subset Aut(\mathcal{F}_\pi)$. Let $\overline{\pi_\star(Mal\Psi)}$ be the Zariski closure of the image of $Mal\Psi$ in $\underline{Aut(\mathcal{F}_\pi)}$. Because Ψ is π-projectable on Φ, this is also true for $PG\Psi$ on $PG\Phi$ and $Mal\Phi \subset \overline{\pi_\star(Mal\Psi)}$. But $\pi_\star^{-1}(Mal\Phi)$ is an algebraic pseudogroup containing any map π-projectable on a map in $Mal\Phi$ thus $Mal\Psi \subset \pi_\star^{-1}(Mal\Phi)$. The theorem follows from this inclusion by applying $\pi_\star$. □

Remark 3.8. In usual Galois theory this theorem just states that if $\mathbb{Q} \subset L \subset M$ is a tower of number fields such that $M/\mathbb{Q}$ and $L/\mathbb{Q}$ are Galoisian extensions then the small Galois group is a quotient of the big one.

3.2 Examples

For basic definition and notation concerning difference algebra, we refer to the book of R.M. Cohn [14] specially chapter 2 section 4 for notations.

σ-Liouvillian functions

Definition 3.9 (Franke [20, 22])**.** Let $(\mathbb{C}(t), \sigma)$ be the difference field of rational functions with difference operator σ a Moebius transformation. A difference extension $(K, \overline{\sigma})$ of $(\mathbb{C}(t), \sigma)$ is said to be σ-Liouvillian if there exists a tower of differential extensions

$$(\mathbb{C}(t), \sigma) = (K_0, \sigma_0) \subset (K_1, \sigma_1) \ldots \subset (K_n, \sigma_n) = (K, \overline{\sigma})$$

such that, for all $i \in \{1, ..., n\}$, the extension $K_{i-1} \subset K_i$ is either

- algebraic,
- or additive in the sense that there exist $n_i \in \mathbb{N}$ and $z_i \in K_i$ such that $\sigma_i^{n_i} z_i - z_i \in K_{i-1}$ and $K_i = K_{i-1}(z_i, \ldots, \sigma_i^{n_i-1} z_i)$,
- or multiplicative in the sense that there exist $n_i \in \mathbb{N}$ and $z_i \in K_i$ such that $\frac{\sigma_i^{n_i} z_i}{z_i} \in K_{i-1}$ and $K_i = K_{i-1}(z_i \ldots, \sigma_i^{n_i-1} z_i)$.

The σ-Liouvillian functions are elements of σ-Liouvillian extensions.

Note that the notion of σ-Liouvillian function is a natural discretization of the usual notion of Liouvillian function. Indeed, assume for instance that $K = \mathbb{C}(t)$ and that $\sigma = \sigma_q$ is the q-dilatation operator $f(t) \mapsto f(qt)$ with $q \in \mathbb{C}^*$. Then, roughly speaking, as q tends to 1, $\frac{\sigma_q - Id}{(q-1)t}$ tends to the usual derivative and $\frac{\sigma_q - Id}{(q-1)t}$ tends to the usual logarithmic derivative.

Assume that K is σ-Liouvillian and that the transcendence degree of K over $\mathbb{C}$ is $d = 1 + \sum_i n_i$. Let N be a model for a field K i.e. $\mathbb{C}(N) = K$. Because this field is a difference field, N is endowed with a rational map Ψ.

Proposition 3.10. *The Lie algebra of the Malgrange pseudogroup of Ψ at $p \in N$ is solvable.*

Proof. – We maintain the notations introduced above. Let m be the smallest common multiple of n_i's and $z_0 = t, z_1, \ldots, z_d$ be a transcendence basis in the partial order given by the definition. We have

$$\overline{\sigma}^m(z_i) = a_i(z_0, \ldots, z_{i-1}) + z_i$$

in the additive case and

$$\overline{\sigma}^m(z_i) = b_i(z_0, \ldots, z_{i-1})z_i$$

in the multiplicative case for some rational fractions a_i and b_i with coefficients in K.

By lemma 2.7, it is enough to prove the proposition at a generic point. There exist some special rational differential 1-forms $\theta_0, \ldots, \theta_d$ on W satisfying, for all $0 \leq i \leq d$, $(\Psi^m)^*\theta_i = \theta_i \mod (\theta_0, \ldots, \theta_{i-1})$. The construction of these forms is direct from the definition of the σ-Liouvillian extensions : we can consider $\theta_0 = dt = dz_0$ and, for all $1 \leq i \leq d$, $\theta_i = dz_i$ in the additive case or $\theta_i = \frac{dz_i}{z_i}$ in the multiplicative case.

The equations $(\Psi^m)^*\theta_i = \theta_i \mod (\theta_0, \ldots, \theta_{i-1})$ for $0 \leq i \leq d$ are a synthetic way of writing infinitely many algebraic equations satisfied by elements of $PG(\Psi^m)$. The Lie algebroid of $Mal(\Psi^m)$ at a point p must be included in the solutions of the linearized equations : the vector fields Y in the Lie algebroid must satisfiy $\mathcal{L}_Y\theta_i = 0 \mod (\theta_0, \ldots, \theta_{i-1})$ where $\mathcal{L}_Y$ denotes the Lie derivative along Y. Let p be a generic point on N and $t_1, ..., t_d$ be local analytic coordinates such that $dt_i = \theta_i \mod (\theta_0, \ldots, \theta_{i-1})$. The vector field Y can be written as

$$Y = c_0\frac{\partial}{\partial t_0} + c_1(t_0)\frac{\partial}{\partial t_1} + c_2(t_0, t_1)\frac{\partial}{\partial t_2} + \cdots + c_d(t_0, \cdots, t_{d-1})\frac{\partial}{\partial t_d}.$$

The $(d+1)$th derived algebra of this type of Lie algebra of formal vector fields is trivial.

The theorem is proved for Ψ^m. One gets inclusions

$$Mal(\Psi^m) \subset Mal(\Psi) \subset \bigcup_{k=0}^{m-1} \overline{R^s\Psi^k\left(Mal(\Psi^m)\right)}.$$

This implies that tangent spaces of $Mal(\Psi^m)$ and $Mal(\Psi)$ coincide at smooth points thus the Lie algebroids are equal. □

Rational systems integrable by σ-quadratures

Definition 3.11. Let

$$\begin{cases} \sigma y_1 &= E_1(t, y_1, y_2, \ldots, y_m) \\ \vdots & \\ \sigma y_m &= E_m(t, y_1, y_2, \ldots, y_m) \end{cases} \quad (1)$$

be a rank m system of nonlinear rational σ-difference equations. This system is said to be integrable by σ-quadratures if it admits a σ-Liouvillian solution $(f_1, \ldots, f_m)$ (the coordinates are σ-Liouvillian) such that $\mathbb{C}(t, f_1, \ldots, f_m)$ is σ-isomorphic to $\mathbb{C}(t)\{y_1, \ldots, y_m\}_\sigma/I$ where $\mathbb{C}(t)\{y_1, ..., y_m\}_\sigma$ is the σ-ring generated by the indeterminates $y_1, ..., y_m$ and where I is the σ-ideal generated by the equations of the system.

In [22] it is proved that if system (1) is linear in y's then integrability by σ-quadratures is equivalent to virtual solvability of the difference Galois group. In the nonlinear case one can prove the analog of one of these implications.

One defines the Malgrange pseudogroup of (1) as that of

$$\Phi : \begin{array}{ccc} \mathbb{C}^{m+1} & \dashrightarrow & \mathbb{C}^{m+1} \\ (t, y_1, \ldots, y_m) & \mapsto & (\sigma(t), E_1, \ldots, E_m) \end{array}.$$

A consequence of proposition 3.10, theorem 3.7 and lemma 2.7 is

Theorem 3.12. *If (1) is integrable by σ-quadratures then the Lie algebra of its Malgrange pseudogroup is solvable.*

As in the differential, case one can find examples of nonlinear systems with solvable Malgrange pseudogroups but which are not integrable by σ-quadratures.

4 Galois theory for linear q-difference equations

In this section we collect some results concerning the Galois theory of linear q-difference equations which will be used in the next sections. We set $\sigma t = \sigma_q t = qt$ with $q \in \mathbb{C}^*$ such that $|q| > 1$. Two rather different approaches for q-difference Galois theory will be used.

4.1 Picard-Vessiot theory

The construction given here of the Galois group follows the first chapter of [54]. We only assume that $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ has a Zariski dense orbit. Let G be a complex linear algebraic group. Let $E \xrightarrow{\pi} \mathbb{P}^1$ be a principal G-bundle i.e. the action of G on E gives an isomorphism $E \underset{\mathbb{P}^1}{\times} E \sim E \times G$ over E for the first projection [64]. For a π-projectable G-equivariant rational dominant map $\Phi : E \dashrightarrow E$ such that $\sigma = \pi_* \Phi$, PV denotes a closed minimal Φ-invariant subvariety of E dominating $\mathbb{P}^1$ and $Gal\Phi$ its stabilizer in G.

- Two such PV are isomorphic under action of G and called Picard-Vessiot varieties of Φ. The ring extension $\mathbb{C}[\pi(PV)] \subset \mathbb{C}[PV]$ is usually called a Picard-Vessiot extension for Φ.
- The group $Gal\Phi$ is well defined up to conjugation in G. It is the Galois group of Φ.
- Common level sets of all invariants of Φ in $\mathbb{C}(E)$, dominating $\mathbb{P}^1$, are Picard-Vessiot varieties.

Up to some modification of the bundle over $\infty \in \mathbb{P}^1$, we can assume that the bundle is trivial : $E = \mathbb{P}^1 \times G$. If $G = GL_n(\mathbb{C})$, one gets $\Phi(t, g) = (\sigma t, A(t)g)$ with $A(t) \in GL_n(\mathbb{C}(t))$. The equations of invariants curves $g = g(t)$ are linear σ-difference systems in fundamental form $g(\sigma t) = A(t)g(t)$. By looking at the first column, one gets the usual vectorial form of linear σ-difference systems : $Y(\sigma t) = A(t)Y(t)$. In this linear situation (E, Φ) will stand either for the system in fundamental form and (V, Φ) for the system in vectorial form.

From a linear σ-difference system on a vector bundle V on $\mathbb{P}^1$ and a point $p \in \mathbb{C}^*$, the fundamental form is given by the action of Φ on the second factor of $E = \left(V_p^\vee \otimes V\right)^*$ where V_p is the fiber of V at p, $V_p^\vee$ is the dual of the trivial bundle $V_p \times \mathbb{P}^1$, $(\ \cdot\ )^*$ denotes the open set of full rank elements and $G = GL(V_p)$. Then $Gal(\Phi, p) \subset GL(V_p)$ is the Galois group defined by the Picard-Vessiot variety containing $Id_p \in E_p$.

4.2 Tannakian approach

In this section, we will only consider q-difference equations (in view of applications given in sections 6 and 7).

Let $\mathcal{D}_q = \mathbb{C}(t)\langle \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1}\rangle$ be the non commutative algebra of non commutative polynomials with coefficients in $\mathbb{C}(t)$ satisfying to the relation $\boldsymbol{\sigma}_q f = (\sigma_q f)\boldsymbol{\sigma}_q$ for any $f \in \mathbb{C}(t)$. We denote by $\mathcal{F}$ the neutral Tannakian category over $\mathbb{C}$ of q-difference modules over $\mathbb{C}(t)$: it is the full subcategory of the category of left $\mathcal{D}_q$-modules whose objects are the left $\mathcal{D}_q$-modules which are finite dimensional as $\mathbb{C}(t)$-vector spaces. The objects of $\mathcal{F}$ can be interpreted as pairs (V, Φ) where V is a finite dimensional $\mathbb{C}(t)$-vector space V and where Φ is a σ_q-linear automorphism of V; from this point of view, the morphisms from an object (V, Φ) to an object (V', Φ') are the $\mathbb{C}(t)$-linear maps $F : V \to V'$ such that $F\Phi = \Phi' F'$.

We define similar categories $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(\infty)}$ by replacing the field $\mathbb{C}(t)$ by $\mathbb{C}(\{t\})$ and $\mathbb{C}(\{t^{-1}\})$ respectively. We have natural localization functors $\mathcal{F} \rightsquigarrow \mathcal{F}^{(0)}$ and $\mathcal{F} \rightsquigarrow \mathcal{F}^{(\infty)}$.

A q-difference module over $\mathbb{C}(t)$ is regular singular both at 0 and at ∞ if its localization at 0 has a lattice over $\mathbb{C}\{t\}$ and its localization at ∞ has a lattice over $\mathbb{C}\{t^{-1}\}$ which are invariant under the action of $\boldsymbol{\sigma}_q$ and $\boldsymbol{\sigma}_q^{-1}$. We denote by $\mathcal{E}$ the full subcategory of $\mathcal{F}$ made of its regular singular objects; it is a neutral Tannakian subcategory of $\mathcal{F}$. For details on what precedes, we refer to M. van der Put and M. Singer's book [54] and to J. Sauloy's paper [62].

For the general theory of Tannakian categories, we refer to [15]. Let ω be a complex valued fiber functor over $\mathcal{F}$. The Galois group of $\mathcal{F}$ is by definition the complex proalgebraic group $\pi_1^{q-diff} = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ and the Galois group of an object M of $\mathcal{F}$ is the the complex linear algebraic group $Gal(M) = \underline{\mathrm{Aut}}^{\otimes}(\omega|\langle M\rangle)$ where $\langle M\rangle$ denotes the Tannakian subcategory of $\mathcal{F}$ generated by M (it is the full subcategory of $\mathcal{F}$ whose objects are obtained form M by combining the following operations : tensor products $\otimes$, direct sums $\oplus$, duals $\overset{\vee}{\cdot}$, quotients, subobjects).

By Tannakian duality ([15]) ω induces an equivalence of tensor categories between $\mathcal{F}$ and the rational finite dimensional linear representations of π_1^{q-diff}; similarly, for any object M of $\mathcal{F}$, ω induces an equivalence of tensor categories between $\langle M\rangle$ and the rational finite dimensional linear representations of $Gal(M)$. Let M be an object of $\mathcal{F}$ and let $\rho_M : \pi_1^{q-diff} \rightarrow GL(\omega(M))$ be the representation corresponding to M. We can identify $Gal(M)$ with the image of ρ_M ($\subset GL(\omega(M))$).

We will freely use the following easy and classical results.

Proposition 4.1. *An object M of $\mathcal{F}$ is simple if and only if the corresponding rational representation ρ_M is irreducible if and only if $Gal(M)$ acts irreducibly on $\omega(M)$.*

Proposition 4.2. *Let M be an object of $\mathcal{F}$. The determinant of M is trivial if and only if $Gal(M)$ belongs to $SL(\omega(M))$.*

We will also use the following result.

Proposition 4.3. *If an object M of $\mathcal{F}$ has a virtually solvable Galois group then any object of $\langle M\rangle$ has a virtually solvable Galois group; in particular, any subobject of M has a virtually solvable Galois group. We recall that a group if virtually solvable if it has a finite index solvable subgroup. For algebraic groups it is equivalent to have a solvable Lie algebra.*

Proof. – Let N be an object of $\langle M\rangle$ and let ρ_N be the rational linear representation of $Gal(M)$ corresponding to N. Since $Gal(M)^\circ$ is solvable its image by the rational linear representation ρ_N, which is $Gal(N)^\circ$, is also solvable. □

Let us now consider a complex valued fiber functor $\omega^{(0)}$ over $\mathcal{F}^{(0)}$ and take for ω the complex valued fiber functor obtained by composing $\omega^{(0)}$ with the exact, faithful and tensor localization functor $\mathcal{F} \rightsquigarrow \mathcal{F}^{(0)}$. The local Galois group at 0 of an object M of $\mathcal{F}$ is the complex linear algebraic group $G_{loc,0}(M) = \underline{\mathrm{Aut}}^{\otimes}(\omega^{(0)}|\langle M^{(0)}\rangle)$ (where $M^{(0)}$ denotes the localization of M at 0) which can be viewed, as above, as a subgroup of $GL(\omega^{(0)}(M^{(0)})) = GL(\omega(M))$.

The localization functor $\mathcal{F} \rightsquigarrow \mathcal{F}^{(0)}$ induces, for any object of M of $\mathcal{F}$, a closed immersion (this is a consequence of [15, Proposition 2.21.] :

$$G_{loc,0}(M) \quad \hookrightarrow \quad Gal(M).$$

The following result Proposition 12.2 in [54]. Compare with Gabber connectedness criterion [36, Proposition 1.2.5].

Theorem 4.4. *Let M be an object of $\mathcal{F}$. Then we have a natural surjective morphism :*

$$G_{loc,0}(M)/G_{loc,0}(M)^\circ \twoheadrightarrow Gal(M)/Gal(M)^\circ.$$

In particular, if $G_{loc,0}(M)$ is connected then $Gal(M)$ is connected.

The interest of this theorem is that $G_{loc,0}(M)$ is easy to describe if M is an object of $\mathcal{E}$. The following corollary will allow us to greatly simplify the calculations of Galois groups in sections 6 and 7.

Corollary 4.5. *Let M be an object of $\mathcal{E}$. Assume that with respect to some basis the action of $\boldsymbol{\sigma}_q$ on the localization $M^{(0)}$ of M at 0 is given by a matrix $A \in GL_n(\mathbb{C}\{t\})$ such that the eigenvalues of $A(0)$ belong to $q^{\mathbb{Z}}$ then $Gal(M)$ is connected.*

Proof. – Theorem 4.4 ensures that it is sufficient to prove that $G_{loc,0}(M)$ is connected. This is indeed the case because, in virtue of the description of the Galois groups given in Chapter 12 of [54] of Chapter 2.2 of [62], $G_{loc,0}(M)$ is generated, as a complex algebraic group, by a unipotent element. □

In concrete examples, we will work with q-difference systems. Let M be an object of $\mathcal{F}$. Choosing a $\mathbb{C}(t)$-basis of M, we can interpret M as the q-difference system $\sigma_q Y = AY$ where A is the inverse of the matrix representing the action of $\boldsymbol{\sigma}_q$ on M with respect to the given basis. We will also work with associated q-difference operators. Concretely, let Φ_A be the σ_q-linear operator on the n-dimensional $\mathbb{C}(t)$-vector space $V = \mathbb{C}(t)^n$ given by $\Phi_A(X) = A^{-1}\sigma_q X$. We will exhibit $e \in V$ such that $(e, \Phi_A(e), ..., \Phi_A^{n-1}(e))$ is a basis over $\mathbb{C}(t)$ of V and we will work with $P(\boldsymbol{\sigma}_q) \in \mathcal{D}_q$ where $P \in \mathbb{C}(t)[X]$ is the unique unitary polynomial of degree $n-1$ such that $P(\Phi_A)e = 0$; such a e is called a cyclic vector and the theoretical existence of cyclic vector is ensured by the so-called cyclic vector Lemma ([54, 63, 73]).

We shall conclude this section with the remark that the Tannkian approach gives the same Galois groups as the usual Picard-Vessiot theory (because the Picard-Vessiot groups can be seen as the tensor automorphisms of some fibre functor over the algebraically closed field $\mathcal{C}$; see [54, §????].

5 A linearization theorem

5.1 Variational equations

Let $\Phi : M \dashrightarrow M$ be a dominant rational map and let $\mathscr{C}$ be an algebraic rational Φ-invariant curve with $\Phi|_{\mathscr{C}}$ being either $t \mapsto qt$ for some $q \in \mathbb{C}^*$ with $|q| > 1$ or $t \mapsto t+1$. The order k prolongations of Φ are dominant rational maps $R_k\Phi$ on frame bundles R_kM. The restriction of the frame bundles over $\mathscr{C}$ are Γ_k-principal bundles over $\mathscr{C}$ which have a projectable Γ_k-equivariant map given by the restriction of $R_k\Phi$ over $\mathscr{C}$. This is the order k variational equation in fundamental form. The projection $\pi_k^{k+1} : R_{k+1}M \to R_kM$ obtained from the inclusion of $(t)^{k+1} \subset (t)^k$, with notations of the first section, is compatible with $R_k\Phi$ i.e. $R_k\Phi \circ \pi_k^{k+1} = \pi_k^{k+1} \circ R_{k_1}\Phi$. Because $R_{k+1}\Phi$ is π_k^{k+1}-projectable on $R_k\Phi$, this is also true for Galois groups. We have surjective morphisms

$$Gal(R_{k+1}\Phi|_{\mathscr{C}}) \twoheadrightarrow Gal(R_k\Phi|_{\mathscr{C}}).$$

We set

$$Gal(R\Phi|_{\mathscr{C}}) = \lim_{\leftarrow} Gal(R_k\Phi|_{\mathscr{C}}).$$

Theorem 5.1. *Let p be a generic point on $\mathscr{C}$. We have a canonical injective proalgebraic group morphism*

$$Gal(R\Phi|_{\mathscr{C}}, p) \subset Mal\Phi_{(p,p)}.$$

Proof. – By choosing a formal chart at $p \in \mathscr{C}$, the Γ-principal bundle $RM|_{\mathscr{C}}$ is isomorphic to the subspace of $AutM_{(p,\mathscr{C})}$ with source $p \in \mathscr{C}$ and target in $\mathscr{C}$. Under this identification

- $AutM_{(p,p)}$ is Γ,
- $Gal(R\Phi|_{\mathscr{C}}, p)$ is a subgroup acting by left translation.

The closed subvariety $Mal\Phi_{(p,\mathscr{C})}$ with source a and target in $\mathscr{C}$ of $AutM_{(p,\mathscr{C})}$ is

- $R\Phi$-invariant because $Mal\Phi$ and $\mathscr{C}$ are $R\Phi$-invariants,
- dominates $\mathscr{C}$ because its projection contains the orbits of p by Φ which is Zariski dense in $\mathscr{C}$.

This implies that $Gal(R\Phi|_{\mathscr{C}}, p) \subset Stab(Mal\Phi_{(p,\mathscr{C})})$ by source composition thus, by Theorem 2.6, $Gal(R\Phi|_{\mathscr{C}}, p) \subset Mal\Phi_{(p,p)}$. □

5.2 Main theorem

The main theorem is now a consequence of theorem 5.1 and theorem 3.12.

Theorem 5.2. *If Φ is integrable by N σ-quadratures then $Gal(R_k\Phi|_{\mathscr{C}})$ is solvable of length N.*

These groups are algebraic subgroups of Γ_k (defined in subsection 1.2). They inherit special structure of these groups ; see [52] where they are called $GL_k(\mathbb{C}^d)$. Let $\pi^k_{k-1} : \Gamma_k \to \Gamma_{k-1}$ be the projection of order k jets on order $k-1$ jets and let K be its kernel. The elements of K can be written roughly $t \mapsto t + \alpha t^k$ and one gets that K is a vector group. Then the solvability of a subgroup of Γ_k is equivalent to the solvability of its projection in Γ_{k-1} and thus is equivalent to the solvability of its projection in Γ_1. So, if the first variational equation has a virtually solvable Galois group $Gal(R_1\Phi|_{\mathscr{C}})$ then any variational equation has a virtually solvable Galois group $Gal(R_k\Phi|_{\mathscr{C}})$.

6 Nonintegrability of a discrete Painlevé I equation

The rest of this paper is devoted to applications of Theorem 5.2 to concrete discrete Painlevé equations. See the introduction of the present paper for comments and references about these equations.

The following system of non linear q-difference equations (qPA_7' in Sakai's classification [61]) is a q-analogue of Painlevé I equation :

$$\begin{cases} y(qx) &= \dfrac{1-xz(x)}{xy(x)(z(x)-1)}z(x) \\ z(qx) &= \left(\dfrac{1-xz(x)}{xy(x)(z(x)-1)}\right)^2 z(x). \end{cases}$$

The corresponding dynamical system is :

$$\Phi : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} qx \\ \dfrac{1-xz}{xy(z-1)}z \\ \left(\dfrac{1-xz}{xy(z-1)}\right)^2 z \end{bmatrix}.$$

Theorem 6.1. *The qPA_7' Painlevé system is non integrable by q-quadratures.*

Proof. – It is a consequence of sections 6.1 and 6.2 below and of Theorem 5.2. □

Note that a direct proof of nonintegrability of qPA_7' can be found in [49] where the stronger statement of irreducibility of nonalgebraic solutions is proved.

6.1 Invariant curve and discrete variational equations

Let q_4 be a 4th root of q in $\mathbb{C}$. A straightforward calculation shows that Φ leaves globally invariant the curve parameterized by :

$$\varphi : \quad t \quad \mapsto \quad \begin{bmatrix} t^2 \\ q_4/t \\ 1/t \end{bmatrix}$$

and that Φ acts on the variable t as σ_{q_2}, where $q_2 = q_4^2$, in the sense that :

$$\Phi \circ \varphi = \varphi \circ \sigma_{q_2}.$$

This leads to the following discrete variational equation :

$$Y_1(q_2 t) = D\Phi(\varphi(t))(Y_1(t)) = \begin{bmatrix} q & 0 & 0 \\ \dfrac{-1}{q_4 t^3(1-t)} & \dfrac{-1}{q_2} & \frac{-2t}{q_4(1-t)} \\ \dfrac{-2}{q_2 t^3(1-t)} & \dfrac{-2}{q_4^3} & \frac{-3t-1}{q_2(1-t)} \end{bmatrix} Y_1(t) \qquad (2)$$

6.2 Non virtual solvability of the Galois group of the discrete variational equation

In this section we shall prove that the q_2-difference system (2) has a non virtually solvable Galois group. Proposition 4.3 shows that it is sufficient to prove that the following subsystem of (2) has a non virtually solvable Galois group :

$$Y(q_2t) = A(t)Y(t), \quad A(t) = \begin{bmatrix} \frac{-1}{q_2} & \frac{-2t}{q_4(1-t)} \\ \frac{-2}{q_4^3} & \frac{-3t-1}{q_2(1-t)} \end{bmatrix} \in GL_2(\mathbb{C}(t)). \tag{3}$$

This q_2-difference system is actually a basic hypergeometric equation in disguise whose Galois group was computed in [58].

Let us recall that the q_2-hypergeometric operator with parameters $(a,b;c,d) \in (\mathbb{C}^*)^4$ is given by :

$$\begin{aligned} & \left(\frac{c}{q_2}\boldsymbol{\sigma}_{q_2} - 1\right)\left(\frac{d}{q_2}\boldsymbol{\sigma}_{q_2} - 1\right) - t(a\boldsymbol{\sigma}_{q_2} - 1)(b\boldsymbol{\sigma}_{q_2} - 1) \\ = & \left(\frac{cd}{q_2^2} - tab\right)\boldsymbol{\sigma}_{q_2}^2 + \left(-\frac{c+d}{q_2} + t(a+b)\right)\boldsymbol{\sigma}_{q_2} + (1-t). \end{aligned}$$

Coming back to our concrete situation, we claim that $e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{C}^2 \subset \mathbb{C}(t)^2$ is a cyclic vector for (3) and that the corresponding q_2-difference operator is q_2-hypergeometric. We have

$$A^{-1} = \begin{bmatrix} -q_2\frac{3t+1}{(1-t)} & \frac{2q_4^3t}{(1-t)} \\ 2q_4 & -q_2 \end{bmatrix}.$$

Hence

$$\Phi_A(e) = \begin{bmatrix} \frac{2q_4^3t}{(1-t)} \\ -q_2 \end{bmatrix}.$$

Consequently, e is a cyclic vector and we have

$$\begin{aligned} \Phi_A^2(e) &= A^{-1}\begin{bmatrix} \frac{2q_4^5t}{(1-q_2t)} \\ -q_2 \end{bmatrix} \\ &= \begin{bmatrix} -q_2\frac{3t+1}{(1-t)}\frac{2q_4^5t}{(1-q_2t)} + \frac{2q_4^3t}{(1-t)}(-q_2) \\ 2q_4\frac{2q_4^5t}{(1-q_2t)} + (-q_2)(-q_2) \end{bmatrix} \\ &= \begin{bmatrix} \frac{2q_4^3t}{(1-t)}(-\frac{q(3t+1)}{(1-q_2t)} - q_2) \\ \frac{4q_2^3t}{(1-q_2t)} + q \end{bmatrix} \\ &= \begin{bmatrix} \frac{2q_4^3t}{(1-t)}(-\frac{q(3t+1)}{(1-q_2t)} - q_2) \\ -q_2(-\frac{q(3t+1)}{(1-q_2t)} - q_2) \end{bmatrix} + \begin{bmatrix} 0 \\ q_2(-\frac{q(3t+1)}{(1-q_2t)} - q_2) + \frac{4q_2^3t}{(1-q_2t)} + q \end{bmatrix} \\ &= -(\frac{2qt+(q+q_2)}{(1-q_2t)})\Phi_A(e) - \frac{q_2^3(1-t)}{(1-q_2t)}e. \end{aligned}$$

So we get the q_2-difference operator $\boldsymbol{\sigma}_{q_2}^2 + \frac{2qt+(q+q_2)}{(1-q_2t)}\boldsymbol{\sigma}_{q_2} + \frac{q_2^3(1-t)}{(1-q_2t)}$. Using the gauge transformation $y \mapsto ty$, we see that $\boldsymbol{\sigma}_{q_2}^2 + \frac{2qt+(q+q_2)}{(1-q_2t)}\boldsymbol{\sigma}_{q_2} + \frac{q_2^3(1-t)}{(1-q_2t)}$ is equivalent to $\boldsymbol{\sigma}_{q_2}^2 + \frac{2t+(1+q_2^{-1})}{(q_2^{-1}-t)}\boldsymbol{\sigma}_{q_2} + \frac{(1-t)}{(q_2^{-1}-t)}$. This is the q_2-hypergeometric operator with parameters $(a,b;c,d) = (1,1;-q_2,-1)$. It is proved in [58] that its Galois group is $SL_2(\mathbb{C})$ (to be precise, in [58] the Galois group computed is that of the operator obtained by permuting (a,b) and (c,d) in the above operator but this is of course inoffensive).

7 Nonintegrability of a discrete Painlevé III equation

The following system of non linear q-difference equations is a q-analogue of Painlevé III equation :

$$\left\{\begin{array}{rcl} y(qx) & = & \dfrac{1}{y(x)z(x)}\dfrac{1+a_0xz(x)}{a_0x+z(x)} \\ \\ z(q^{-1}x) & = & \dfrac{1}{y(x)z(x)}\dfrac{qa_1x^{-1}+y(x)}{1+qa_1x^{-1}y(x)} \end{array}\right. \tag{4}$$

where $a_0, a_1 \in \mathbb{C}^*$. Note that (4) is equivalent to :

$$\left\{\begin{array}{rcl} y(qx) & = & \dfrac{1}{y(x)z(x)}\dfrac{1+a_0xz(x)}{a_0x+z(x)} \\ \\ z(qx) & = & \dfrac{1}{y(qx)z(x)}\dfrac{a_1x^{-1}+y(qx)}{1+a_1x^{-1}y(qx)} \\ & = & \dfrac{y(x)(a_0x+z(x))(x+a_0x^2z(x)+a_0a_1xy(x)z(x)+a_1y(x)z(x)^2)}{(1+a_0xz(x))(xy(x)z(x)^2+a_0x^2y(x)z(x)+a_0a_1xz(x)+a_1)}. \end{array}\right.$$

The corresponding discrete dynamical system is :

$$\Phi : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} qx \\ \dfrac{1}{yz}\dfrac{1+a_0xz}{a_0x+z} \\ \dfrac{y(a_0x+z)(x+a_0x^2z+a_0a_1xyz+a_1yz^2)}{(1+a_0xz)(xyz^2+a_0x^2yz+a_0a_1xz+a_1)} \end{bmatrix}.$$

Theorem 7.1. *If $a_0a_1 \notin -q^{-\mathbb{N}}$ then the qP_{III} equation (4) is non integrable by q-quadratures.*

Proof. – It is a consequence of sections 7.1 and 7.2 below and of Theorem 5.2. □

7.1 Invariant curve and discrete variational equation

A straightforward calculation shows that Φ leaves globally invariant the curve parameterized by :

$$\varphi : \ t \ \mapsto \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix}$$

and that it acts on the variable t as σ_q in the sense that :

$$\Phi \circ \varphi = \varphi \circ \sigma_q.$$

The discrete variational equation of Φ along φ is easily seen to be given by :

$$Y_1(qt) = D\Phi(\varphi(t))(Y_1(t)) = \begin{bmatrix} q & 0 & 0 \\ 0 & -1 & \dfrac{2a_1}{t+a_1} \\ 0 & -\dfrac{2}{a_0t+1} & -\dfrac{a_0t^2+(1+a_0a_1)t-3a_1}{(a_0t+1)(t+a_1)} \end{bmatrix} Y_1(t) \tag{5}$$

7.2 Non virtual solvability of the Galois group of the discrete variational equation

In this section we shall prove that the q-difference system (5) has a non virtually solvable Galois group. In this purpose, Proposition 4.3 ensures that it is sufficient to prove that the following (regular singular) subsystem of (5) has a non virtually solvable Galois group :

$$Y(qt) = A(t)Y(t), \quad A(t) = \begin{bmatrix} -1 & \frac{2a_1}{t+a_1} \\ -\frac{2}{a_0t+1} & -\frac{a_0t^2+(1+a_0a_1)t-3a_1}{(a_0t+1)(t+a_1)} \end{bmatrix} \in GL_2(\mathbb{C}(t)). \tag{6}$$

We claim that the Galois group of (6) is $SL_2(\mathbb{C})$. This assertion will be a consequence of the following three observations.

First observation: G is connected – Indeed, a simple calculation shows that the complex eigenvalues of $A(0)$ belong to $q^{\mathbb{Z}}$. Corollary 4.5 ensures that G is connected.

Second observation: G acts irreducibly on $\mathbb{C}^2$ – Indeed, G acts irreducibly if and only if (see Proposition 4.1) (6) is irreducible if and only if some q-difference operator associated to (6) is irreducible over $\mathbb{C}(t)$. Let us now determine an explicit q-difference operator associated to (6).

We claim that $e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{C}^2 \subset \mathbb{C}(t)^2$ is a cyclic vector for (6). Indeed, we have:

$$A^{-1} = \begin{bmatrix} -\frac{a_0t^2+(1+a_0a_1)t-3a_1}{(a_0t+1)(t+a_1)} & -\frac{2a_1}{t+a_1} \\ \frac{2}{a_0t+1} & -1 \end{bmatrix},$$

so :

$$\Phi_A(e) = A^{-1}\sigma_q(e) = \begin{bmatrix} -\frac{2a_1}{t+a_1} \\ -1 \end{bmatrix}$$

is non $\mathbb{C}(t)$-colinear with e. Moreover, we have:

$$\begin{aligned} \Phi_A^2(e) &= A^{-1} \begin{bmatrix} -\frac{2a_1}{qt+a_1} \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2a_1(a_0(1+q)t^2+(1+2a_0a_1+q)t-2a_1)}{(t+a_1)(qt+a_1)(a_0t+1)} \\ \frac{a_0qt^2+(a_0a_1+q)t-3a_1}{(qt+a_1)(a_0t+1)} \end{bmatrix} \\ &= -\frac{a_0(1+q)t^2+(1+2a_0a_1+q)t-2a_1}{(qt+a_1)(a_0t+1)}\Phi_A(e) - \frac{t+a_1}{qt+a_1}e. \end{aligned}$$

Hence:

$$\begin{aligned} L &= \boldsymbol{\sigma}_q^2 + \frac{a_0(1+q)t^2+(1+2a_0a_1+q)t-2a_1}{(qt+a_1)(a_0t+1)}\boldsymbol{\sigma}_q + \frac{t+a_1}{qt+a_1} \\ &= \boldsymbol{\sigma}_q^2 + \alpha(t)\boldsymbol{\sigma}_q + \beta(t) \end{aligned}$$

is a q-difference operator associated to (6). In order to prove that L is irreducible over $\mathbb{C}(t)$, we follow closely the method presented in [1]. Assume at the contrary that L is reducible over $\mathbb{C}(t)$ i.e. that there exist r and s in $\mathbb{C}(t)^{\times}$ such that :

$$L = (\sigma_q - s)(\sigma_q - r).$$

Then we have:

$$r(qt)r(t) + \alpha(t)r(t) + \beta(t) = 0. \tag{7}$$

According to [1], r can be decomposed as follows :

$$r = \lambda\frac{u(t)}{v(t)}\frac{c(qt)}{c(t)}$$

for some $\lambda \in \mathbb{C}^*$ and some unitary polynomials $u, v, c \in \mathbb{C}[t] \setminus \{0\}$ such that:

a) $\forall n \in \mathbb{N}$, $u(t) \wedge v(q^nt) = 1$;

b) $u(t) \wedge c(t) = 1$;

c) $v(t) \wedge c(qt) = 1$;

d) $c(0) \neq 0$.

Then equation (7) becomes:

$$\lambda^2 \frac{u(qt)u(t)}{v(qt)v(t)} \frac{c(q^2t)}{c(t)} + \lambda\alpha(t)\frac{u(t)}{v(t)} \frac{c(qt)}{c(t)} + \beta(t) = 0 \tag{8}$$

and, clearing the denominators, we get:

$$\begin{aligned}&\lambda^2(qt + a_1)(a_0t + 1)u(qt)u(t)c(q^2t)\\ &+\lambda(a_0(1 + q)t^2 + (1 + 2a_0a_1 + q)t - 2a_1)u(t)v(qt)c(qt)\\ &+(a_0t + 1)(t + a_1)v(t)v(qt)c(t) = 0.\end{aligned} \tag{9}$$

We see that u is a unitary polynomial dividing $(a_0t+1)(t+a_1)$ and that v is a unitary polynomial dividing $(t+a_1)(a_0q^{-1}t+1)$. Moreover, we claim that we necessary have $\deg(u) = \deg(v)$. Indeed, if $\deg(u) > \deg(v)$ (the case that $\deg(u) < \deg(v)$ is similar), then we would have the following inequalities:

$$\begin{aligned}&\deg(\lambda^2(qt + a_1)(a_0t + 1)u(qt)u(t)c(q^2t))\\ =\;& 2 + 2\deg(u) + \deg(c)\\ >\;& 2 + \deg(u) + \deg(v) + \deg(c)\\ =\;& \max\{\deg(\lambda(a_0(1 + q)t^2 + (1 + 2a_0a_1 + q)t - 2a_1)u(t)v(qt)c(qt)),\\ &\qquad \deg((a_0t + 1)(t + a_1)v(t)v(qt)c(t))\}\\ \geq\;& \deg(-\lambda(a_0(1 + q)t^2 + (1 + 2a_0a_1 + q)t - 2a_1)u(t)v(qt)c(qt)\\ &\qquad -(a_0t + 1)(t + a_1)v(t)v(qt)c(t))\end{aligned}$$

contradicting (9). Hence, the only possibilities for (u, v) are $(1, 1)$, $(t+a_0^{-1}, t+a_1)$, $(t+a_0^{-1}, t+qa_0^{-1})$, $(t+a_1, t+a_1)$, $(t+a_1, t+qa_0^{-1})$ and $((t+a_0^{-1})(t+a_1), (t+a_1)(t+qa_0^{-1}))$. Properties a) - d) listed above allow us to reduce the possibilities for (u, v) to $(1, 1)$, $(t+a_0^{-1}, t+a_1)$ and $(t+a_1, t+qa_0^{-1})$. We now consider each case separately.

- Case $(u, v) = (1, 1)$. Considering equation (8) at $t = \infty$, we obtain $\lambda^2q^{2deg(c)} + \lambda q^{deg(c)}(1 + \frac{1}{q}) + \frac{1}{q} = 0$ i.e. $\lambda = -q^{-deg(c)}$ or $-q^{-deg(c)-1}$. On the other hand, evaluating (8) at $t = 0$ we obtain $\lambda^2 - 2\lambda + 1 = 0$ i.e. $\lambda = 1$. So -1 belongs to the $q^{-\mathbb{N}}$: contradiction.
- Case $(u, v) = (t+a_0^{-1}, t+a_1)$. As above we have $\lambda = -q^{-deg(c)}$ or $-q^{-deg(c)-1}$. On the other hand, evaluating (8) at $t = 0$ we obtain $\lambda^2(\frac{1}{a_0a_1})^2 - 2\lambda\frac{1}{a_0a_1} + 1 = 0$ i.e. $\lambda = a_0a_1$. Hence a_0a_1 belongs to $-q^{-\mathbb{Z}}$: contradiction.
- Case $(u, v) = (t+a_1, t+qa_0^{-1})$. As above we have $\lambda = -q^{-deg(c)}$ or $-q^{-deg(c)-1}$. On the other hand, evaluating (8) at $t = 0$ we obtain $\lambda^2(q^{-1}a_0a_1)^2 - 2\lambda q^{-1}a_0a_1 + 1 = 0$ i.e. $\lambda = \frac{1}{q^{-1}a_0a_1}$. Hence a_0a_1 belongs to $-q^{-\mathbb{Z}}$: contradiction.

Third observation: $G \subset SL_2(\mathbb{C})$. Indeed, this is a direct application of Proposition 4.2 since the determinant A is equal to 1.

Hence, G is a connected algebraic subgroup of $SL_2(\mathbb{C})$ acting irreducibly on $\mathbb{C}^2$. The only possibility is $G = SL_2(\mathbb{C})$.

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Guy Casale
IRMAR UMR 6625 Université de Rennes 1,
Campus de Beaulieu
35042 Rennes Cedex - France
guy.casale@univ-rennes1.fr

Julien Roques
Institut Fourier UMR 5582 Université Grenoble I - CNRS,
100 rue des Maths, BP 74,
38402 St Martin d'Heres Cedex, France
Julien.Roques@ujf-grenoble.fr