ON SOME ARITHMETIC PROPERTIES OF MAHLER FUNCTIONS

SARA CHECCOLI AND JULIEN ROQUES

Abstract. Mahler functions are power series $f(x)$ with complex coefficients for which there exist a natural number n and an integer $\ell \geq 2$ such that $f(x), f(x^{\ell}), \ldots, f(x^{\ell^{n-1}}), f(x^{\ell^n})$ are linearly dependent over $\mathbb{C}(x)$. The study of the transcendence of their values at algebraic points was initiated by Mahler around the 30's and then developed by many authors. This paper is concerned with some arithmetic aspects of these functions. In particular, if $f(x)$ satisfies $f(x) = p(x)f(x^{\ell})$ with $p(x)$ a polynomial with integer coefficients, we show how the behaviour of $f(x)$ mirrors on the polynomial $p(x)$. We also prove some general results on Mahler functions in analogy with G-functions and E-functions.

1. Introduction

Throughout this article, by *Mahler function* we will mean a solution $f(x)$ of an equation of the form

$$a_n(x)f(x^{\ell^n}) + a_{n-1}(x)f(x^{\ell^{n-1}}) + \ldots + a_0(x)f(x) = 0 \tag{1}$$

where $\ell \geq 2$ is an integer, $a_0(x), \ldots, a_n(x) \in \mathbb{C}(x)$ are rational functions and $a_0(x)a_n(x) \neq 0$. These functions have been extensively studied, starting with the seminal work of Mahler [Mah29], [Mah30a] and [Mah30b], investigating the algebraic relations between the values of these functions at algebraic points. This new approach in transcendence theory, also known as *Mahler's method*, was further explored and developed by many authors, such as Becker, Kubota, Loxton, van der Poorten, Masser, Nishioka, Töpfer (we refer Nishioka's book [Nis97] for an overview on the subject and a complete list of references).

The first results of Mahler concerned functions of order $n = 1$: for instance, he proved that the Thue-Morse number $f(1/2)$ is transcendental, where $f(x)$ satisfies the equation $f(x) = (1-x)f(x^2)$.

In this paper we are also mainly interested in Mahler functions $f(x)$ of order 1 satisfying the following special case of equation (1)

$$f(x) = p(x)f(x^{\ell}) \tag{2}$$

where $p(x) \in \mathbb{Z}[x]$ is a polynomial such that $p(0) = 1$.

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Up to a multiplicative constant, equation (2) has a unique solution in $\mathbb{Q}((x))$, namely

$$f(x) = \sum_{n=0}^{\infty} f_n x^n = \prod_{i\geq 0} p(x^{\ell^i}) \in \mathbb{Z}[[x]].$$

This is the Taylor expansion at 0 of an analytic function over the unit disc $D(0,1) \subset \mathbb{C}$.

One aim of this article is to study how the behaviour of a solution $f(x)$ of (2) and, in particular, of the sequence of its coefficients $\{f_n\}_n$, mirrors on the polynomial $p(x)$.

Duke and Nguyen [DN15] studied an aspect of this question in the case where ℓ is a prime and $p(x) = \Phi_m(x)$ is the m-th cyclotomic polynomial. For instance, the case $\ell = 2$ and $p(x) = \Phi_1(x) = -x+1$ corresponds to the Thue-Morse sequence $\{f_n\}_n$ where $f_n \in \{0,1\}$ is the sum of the binary digits of n modulo 2. Another classical example is obtained when $\ell = 2$ and $p(x) = \Phi_3(x) = x^2+x+1$. The corresponding sequence is the Stern diatomic sequence $\{f_n\}_n$, where f_n counts the way of writing n as a sum of powers of 2 using each power of 2 at most twice (more details on the many properties of this sequence can be found for instance in [SW10]). One of the main results of [DN15] states that $f(x)$ is rational if and only if ℓ divides m, in which case $f(x) = 1/\Phi_d(x^{\ell^{r-1}})$ where $m = \ell^r d$, $r \geq 1$ and $\ell \nmid d$, and that $f(x)$ has the unit circle as a natural boundary otherwise (see [DN15, Theorem 1]). The proof of their result partly bases on a modified version of Mahler's approach to the case $m = 1$, which consists in studying the behaviour of $f(x)$ as x approaches certain roots of unity. More precisely, they determine (see [DN15, Theorem 2]) the asymptotic behaviour of $f(x)$ as x tends radially to a root of unity of order prime to ℓ, by using the arithmetic properties of the Dirichlet series attached to $f(x)$. We mention that, in this setting, the asymptotic of the coefficients of $f(x)^{-1}$ was precisely described by Dumas and Flajolet [DF96] for any integer $\ell \geq 2$ prime to m.

A first natural question is whether the above result can be generalized or, more in general:

Question 1. *If $f(x)$ is a solution of (2) which is bounded as x tends radially to "sufficiently many" roots of unity, what can be said about $p(x)$?*

In this respect, we obtain the following generalization of [DN15, Theorem 1].

Theorem 1.1. *Let $\ell \geq 2$ be an integer, $p(x) \in \mathbb{Z}[x]$ a polynomial with $p(0) = 1$ and let $f(x)$ be a solution of $f(x) = p(x) f(x^\ell)$.*

Suppose that $p(x)$ is monic and that there exists infinitely many integers m prime to ℓ such that $f(x)$ is bounded as x tends radially to any m-th primitive root of unity. Then $p(x)$ is the product of cyclotomic polynomials.

Moreover, if ℓ is a prime, the following conditions are equivalent:

1. *$p(x)$ is monic and, for almost all roots of unity ζ of order prime to ℓ, $f(x)$ is bounded as x tends to ζ radially;*
2. *$p(x)$ is a product of cyclotomic polynomials of order divisible by ℓ;*
3. *$f(x)$ is rational.*

The proof of Theorem 1.1, done in Section 3, relies on a simplified but more general version of [DN15, Theorem 2] given in Proposition 2.1. This describes the asymptotic behaviour of $f(x)$ as x approaches radially a root of unity ζ of order $\ell^n - 1$ and shows, in particular, that this depends on the value $\prod_{k=0}^{n-1} p(\zeta^{\ell^k})$. This description is then used to deduce some strong bound for the absolute value of the norm of $p(x)$ evaluated at roots of unity, which implies that $p(x)$ is a product of cyclotomic polynomials. The proof of Proposition 2.1 is different in nature than the one of [DN15, Theorem 2]: we first consider a certain functional equation (a q-difference equation) satisfied by $f(\zeta e^t)$, then we construct another suitable solution to this equation and use it to study the behaviour of $f(\zeta e^t)$ as t tends to 0.

In the last decades, the interest for Mahler functions was enhanced by their link to the theory of automata. We briefly recall that, if $\mathcal{S}$ is a set, a sequence $\{f_n\}_n \in \mathcal{S}^{\mathbb{N}}$ is called *k-automatic* if, for every n, f_n can be computed by a finite state machine (or *automaton*). This machine, starting on a certain state, takes in input the expansion of n in base k and associates each digit with a state transition. Each state comes with an associated output value and the result of the computation is the output attached to the last reached state. For more informations about the theory of automata, we refer to [AS03].

The link to Mahler functions comes from the fact that if $\{f_n\}_n$ is a k-automatic sequence (or more generally a k-regular sequence) then its generating function $f(x) = \sum_{i=0}^{\infty} f_n x^n$ satisfies some Mahler equation (see for instance [Ran92], [Dum93] and [Bec94]). On the other hand, Becker [Bec94, Theorem 2] proved that the coefficients of any solution of (1) with $a_i(x)$ polynomials and $a_0(x) = 1$ form a k-regular sequence and, from a classical result of Allouche and Shallit [AS03, Theorem 16.1.5], a k-regular sequence takes only finitely many values if and only if it is k-automatic.

The Thue-Morse sequence recalled above is one of the most classical and simplest examples of 2-automatic sequence, while the Stern diatomic sequence is an example of non automatic sequence (indeed the sequence is unbounded since, for instance, it contains the Fibonacci sequence as a subsequence). Another example is the function $f(x)$ introduced by Dilcher and Stolarsky [DS09] which satisfies the order 2 equation $f(x) = (1 + x + x^2)f(x^2) - x^4 f(x^{16})$ and is shown to have all coefficients in $\{0, 1\}$.

A second natural problem to consider is how to characterize Mahler equations giving rise to automatic sequences, a problem which seems highly non trivial even for simple equations of the form (2). We have the following:

Question 2. *What constraints are imposed on $p(x)$ by the fact that $f(x)$ is the generating function of a k-automatic sequence? Is it possible to classify the polynomials $p(x)$ such that $f(x)$ is the generating function of a k-automatic sequence?*

As remarked above, asking that $f(x)$ is the generating function of a k-automatic sequence is equivalent to asking for which polynomials $p(x)$ the set $\mathcal{S}_f$ of values taken by the coefficients of $f(x)$ is finite.

This is easily seen to be the case when $p(x)$ is a product of certain cyclotomic polynomials of order divisible by ℓ (see Proposition 5.1), and the question is whether there are other examples.

The second result of the paper answers Question 2 when $\ell > \deg p(x)$ and when $\ell \leq 3$ and $p(x)$ is monic of degree at most 3.

Theorem 1.2. *Let $\ell \geq 2$ be an integer and let $f(x) = \sum_{i=0}^{\infty} f_n x^n$ be a solution of the equation $f(x) = p(x)f(x^\ell)$, with $p(x) \in \mathbb{Z}[x]$ such that $p(0) = 1$. Then:*

- *(i) If $\ell > \deg p(x)$, the sequence $\{f_n\}_n$ is automatic if and only if the coefficients of $p(x)$ are in $\{0, \pm 1\}$.*
- *(ii) If $\ell = 2$ and $p(x) = x^2 + bx + 1$, the sequence $\{f_n\}_n$ is automatic if and only if $b \in \{0, -1\}$.*
- *(iii) If $\ell \leq 3$ and $p(x) = x^3 + bx^2 + cx + 1$, the sequence $\{f_n\}_n$ is automatic if and only if:*
  - *— $\ell = 2$ and $b = c = 0$;*
  - *— $\ell = 3$ and $(b, c) \in \{(0,0), (-1,0), (0,-1)\}$.*

Moreover, in all the above cases $\{f_n\}_n$ can be generated by a 3-state automaton.

This theorem is proved in Section 4. It is easy to deduce from (2) that the coefficients of $f(x)$ satisfy a divide-and-conquer type recurrence (see [Dum93] for more details). When $\ell > \deg p(x)$ the recurrence is particularly simple and is enough to easily answer Question 2 (see Proposition 5.2). If $\ell \leq \deg p(x)$, things gets more involved and the result is proved using, in addition, some norm estimates for the values of $p(x)$ at roots of unity (see Section 4.1) which are deduced again from Proposition 2.1. The criterion we use is not subtle enough to treat the case $\deg p(x) \geq 4$ and $\ell \leq \deg p(x)$ (see Remark 5.5) but it is strong enough to obtain the following information (see Section 6.2, Proposition 6.6) on the vanishing of $p(x)$ at 1.

Proposition 1.3. *Assume that the set of coefficients of $f(x)$ is bounded. Then, for every integer $\alpha > 0$ and every prime number ℓ' prime to ℓ, if $p(x)$ has no root which is a primitive $(\ell')^\alpha$-th root of unity, the order of 1 as a root of $p(x)$ is less than or equal to $\log_{\ell'}(\ell^{(\ell')^{\alpha-1}(\ell'-1)})$.*

A reason *a parte* to study Mahler functions with bounded coefficients is given in Section 6 in relation to G-functions and E-functions, which are power series satisfying a homogeneous linear differential equation and whose coefficients fulfill some special growth conditions. Being a G-function or an E-function is a quite strong property. For instance, in the E-function case, André [And00] proved that the minimal nonzero differential operator L annihilating an E-function has at most two non trivial singularities, namely 0 and ∞, the former being regular singular, the latter being in general irregular but the slopes of the Newton polygon attached to L at ∞ are included in $\{0, 1\}$ (see Section 6.2 for the definition of these slopes). In Proposition 6.4, we prove that the coefficients of any Mahler function satisfy automatically the conditions defining the G-functions, except that they are not solutions of linear homogeneous differential equations in general. Actually, they even have some stronger properties characteristic of globally bounded functions, see Definition 6.3. It is therefore quite natural to seek whether more restrictive assumptions on the coefficients of Mahler functions give rise to interesting classes of functions. In particular, in view of the previous discussion, it is natural to wonder whether the fact that a Mahler function $f(x)$ has bounded integer coefficients implies some restrictions on the Newton polygon attached at $x = 1$ to the minimal Mahler operator of $f(x)$. It turns out that when $f(x)$ is a solution of (2), this Newton polygon has only one slope given by the order of vanishing of $p(x)$ at 1, so Proposition 1.3 can be interpreted as an evidence to an affirmative answer to this question in the case of equation (2). Further evidences are given in Section 6.2.

2. The asymptotic behaviour of $f(x)$ at roots of unity

Throughout this section we assume that $f(x)$ is a Mahler function satisfying equation (2) *i.e.* $f(x) = p(x)f(x^\ell)$ with $\ell \geq 2$ an integer, $p(x) \in \mathbb{Z}[x]$ and $p(0) = 1$. The aim of this section is to prove a general asymptotic formula for $f(x)$ as x approaches radially a root of unity ζ of order $\ell^n - 1$. We prove in particular that this behaviour depends on the integer $\prod_{k=0}^{n-1} p(\zeta^{\ell^k})$, providing a generalization of [DN15, Theorem 2] (see Remark 2.2 below). We use the standard symbol $\sim$ to denote asymptotic equivalence.

Proposition 2.1. *Suppose that $f(x)$ is a Mahler function satisfying equation (2). Let n be a positive integer and let ζ be an $(\ell^n - 1)$-th complex root of unity such that $\prod_{k=0}^{n-1} p(\zeta^{\ell^k}) \neq 0$. Then*

$$f(\zeta e^t) \sim m_\zeta(t) t^{-\frac{\log(\prod_{k=0}^{n-1} p(\zeta^{\ell^k}))}{\log(\ell^n)}} \quad \text{as } t \to 0^-$$

where $m_\zeta(t)$ is some non zero meromorphic function on the left half-plane $\{t \in \mathbb{C} \mid \Re(t) < 0\}$ such that $m_\zeta(\ell^n t) = m_\zeta(t)$.

Proof. Consider the function

$$g(z) = f(\zeta z) \in \overline{\mathbb{Q}}[[x]].$$

Then $g(z)$ is analytic on the unit disc $D(0,1)$ and satisfies the functional equation

$$g(z) = q(z)g(z^{\ell^n}) \text{ with } q(z) = \prod_{k=0}^{n-1} p(\zeta^{\ell^k} z^{\ell^k}).$$

Therefore, the function

$$h(t) = g(e^t) = f(\zeta e^t),$$

which is analytic on the left half-plane $\{t \in \mathbb{C} \mid \Re(t) < 0\}$, satisfies the functional equation

$$h(t) = r(t)h(\ell^n t) \text{ with } r(t) = q(e^t). \tag{3}$$

We shall now construct another solution $k(t)$ of equation (3) and use it to study the behaviour of $h(t)$ has t tends to 0^-.

The infinite product

$$l(t) = \prod_{j \geq 1} (r(0)^{-1} r(\ell^{-nj} t))^{-1}$$

defines a meromorphic function over $\mathbb{C}$ (notice that $r(0)^{-1}r(q^{-nj}t)$ equals 1 at $t=0$), is analytic at $t=0$ with $l(0)=1$, and satisfies

$$l(t) = r(0)^{-1}r(t)l(\ell^n t).$$

Let $c = -\frac{\log(r(0))}{\log(\ell^n)} = -\frac{\log(\prod_{k=0}^{n-1} p(\zeta^{\ell^k}))}{\log(\ell^n)}$ for some choice of $\log(r(0))$, and consider the function

$$k(t) = t^c l(t).$$

Clearly $k(t)$ also satisfies equation (3). It follows that the function $m(t) = h(t)/k(t)$ is meromorphic on $\{t \in \mathbb{C} \mid \Re(t) < 0\}$ (we fix a branch of t^c on this half-plane) and satisfies

$$m(\ell^n t) = m(t).$$

Now, the result follows from the facts that $h(t) = m(t)k(t)$ and that $k(t) \sim t^c$ as $t \to 0^-$. □

Remark 2.2. *Clearly* [DN15, Theorem 2] *follows from Proposition 2.1 by noticing that if ζ is an m-th root of unity such that $\gcd(\ell, m) = 1$ and if κ is the order of ℓ in $(\mathbb{Z}/m\mathbb{Z})^*$, then ζ is also an $(\ell^\kappa - 1)$-th root of unity.*

3. Radial asymptotic boundedness and rationality: proof of Theorem 1.1

In this section we prove Theorem 1.1 from the Introduction. We assume again that $f(x)$ is a Mahler function satisfying (2). The proof is split in two parts: in Proposition 3.2 we show that if $p(x)$ is monic and $f(x)$ is bounded as x tends radially to 'many' roots of unity, then $p(x)$ must be a product of cyclotomic polynomials. Then in Proposition 3.3 we show how, when ℓ is prime, this 'radial boundedness' condition is equivalent to the rationality of $f(x)$ and to the fact that $p(x)$ is a product of cyclotomic polyomials of order divisible by ℓ. We start with a simple lemma on the norm of polynomials evaluated at roots of unity (if K is a number field, we denote by $N_{K/\mathbb{Q}}$ the norm function from K to $\mathbb{Q}$).

Lemma 3.1. *Let $\ell \geq 2$ be an integer and $p(x) \in \mathbb{Z}[x]$ a polynomial. Let m be an integer prime to ℓ and κ be the order of ℓ in $(\mathbb{Z}/m\mathbb{Z})^\times$. Let ζ be a primitive m-th root of unity. There exist some primitive m-th roots of unity $\zeta_1, \ldots, \zeta_r$ such that*

$$N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta)) = \prod_{i=1}^{r} \prod_{k=0}^{\kappa-1} p(\zeta_i^{\ell^k}).$$

Proof. Let μ'_m be the set of m-th primitive roots of unity. Since $\gcd(m, \ell) = 1$, we have

$$\mu'_m = \bigcup_{\zeta \in \mu'_m} \{\zeta^{\ell^j} \mid j \geq 0\}.$$

Assume that $\{\zeta_1^{\ell^j} \mid j \geq 0\} \cap \{\zeta_2^{\ell^j} \mid j \geq 0\} \neq \emptyset$ for some $\zeta_1, \zeta_2 \in \mu'_m$. Then, we have $\zeta_1^{\ell^i} = \zeta_2^{\ell^j}$ for some $i, j \geq 0$. Up to renumbering, we can assume that $j \geq i$. Then, $(\zeta_1^{-1}\zeta_2^{\ell^{j-i}})^{\ell^i} = 1$. Since, $\zeta_1^{-1}\zeta_2^{\ell^{j-i}}$ is a m-th root of unity and $\gcd(m, \ell) = 1$, we get $\zeta_1^{-1}\zeta_2^{\ell^{j-i}} = 1$ *i.e.* $\zeta_1 = \zeta_2^{\ell^{j-i}}$ and, hence,

$\{\zeta_1^{\ell^j} \mid j \geq 0\} \subset \{\zeta_2^{\ell^j} \mid j \geq 0\}$. Therefore, one can find $\zeta_1, \ldots, \zeta_r \in \mu'_m$ such that μ'_m is a disjoint union of the r sets $\{\zeta_i^{\ell^j} \mid j \geq 0\}$. Therefore,

$$N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta)) = \prod_{\xi \in \mu'_m} p(\xi) = \prod_{i=1}^{r} \prod_{k=0}^{\kappa-1} p(\zeta_i^{\ell^k}).$$

□

Proposition 3.2. *Let $f(x)$ be a Mahler function satisfying equation (2). Suppose that $p(x)$ is monic and that, for infinitely many integers m prime to ℓ and for all primitive m-th roots of unity ζ, the function $f(x)$ is bounded as x tends to ζ radially. Then $p(x)$ is a product of cyclotomic polynomials.*

Proof. Let E be the infinite set of roots of unity mentioned in the hypotheses of the proposition. We can assume that, for all $\zeta \in E$, for all $j \in \mathbb{Z}$, $p(\zeta^{\ell^j}) \neq 0$ (this condition excludes at most finitely many elements of E).

Fix $\zeta \in E$. Let m be the order of ζ as a root of unity and let κ be the order of ℓ in $(\mathbb{Z}/m\mathbb{Z})^\times$. Then ζ is an $(\ell^\kappa - 1)$-th root of unity and, by Proposition 2.1, the behaviour of $f(x)$ as x tends radially to ζ is given by

$$f(\zeta e^t) \sim_{t \to 0^-} m_\zeta(t) t^{-\frac{\log(\prod_{k=0}^{\kappa-1} p(\zeta^{\ell^k}))}{\log(\ell^\kappa)}}$$

where $m_\zeta(t)$ is some non identically zero meromorphic function on the left half-plane $\{t \in \mathbb{C} \mid \Re(t) < 0\}$ such that $m_\zeta(\ell^\kappa t) = m_\zeta(t)$.

The fact that $f(x)$ is bounded as x tends to ζ radially ensures that $\Re(-\frac{\log(\prod_{k=0}^{\kappa-1} p(\zeta^{\ell^k}))}{\log(\ell^\kappa)}) \geq 0$ *i.e.* $|\prod_{k=0}^{\kappa-1} p(\zeta^{\ell^k})| \leq 1$. Using Lemma 3.1, we get $|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta))| \leq 1$ and, hence (we remind that $p(\zeta) \neq 0$),

$$|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta))| = 1.$$

Hence, for infinitely many roots of unity ζ, we see that $p(\zeta)$ is a unit of $\mathbb{Z}[\zeta]$. If $p_1(x), \ldots, p_r(x)$ denote the irreducible factors of $p(x)$ in $\mathbb{Z}[x]$, we get that, for infinitely many roots of unity ζ, $p_1(\zeta), \ldots, p_r(\zeta)$ are units of $\mathbb{Z}[\zeta]$. It follows from [Kam88, Theorem 2] that $p_1(x), \ldots, p_r(x)$ are cyclotomic polynomials. □

We now conclude the proof of Theorem 1.1. In what follows, we will need to compute $N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\Phi_n(\zeta))$ where $\Phi_n(x)$ is the n-th cyclotomic polynomial and where ζ is some primitive q-th root of unity. We remind that (see for instance Section 3.3.6 of [Pras04] or [Apo70]) we have

$$N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\Phi_n(\zeta)) = \begin{cases} p^{\varphi(n)} & \text{if } q/n \text{ is in } p^{\mathbb{Z}^*} \text{ for some prime } p; \\ 0 & \text{if } n = q; \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

Proposition 3.3. *Let $f(x)$ be a Mahler function satisfying equation (2). Assume that ℓ is a prime. Then the following conditions are equivalent:*

(1) $p(x)$ is monic and, for almost all roots of unity ζ of order prime to ℓ, $f(x)$ is bounded as x tends to ζ radially;

(2) $p(x)$ is a product of cyclotomic polynomials of order divisible by ℓ;

(3) $f(x)$ is rational.

Proof. We first prove that condition (1) implies condition (2).

Proposition 3.2 ensures that $p(x)$ is a product of cyclotomic polynomials. Let $\Phi(x) = \Phi_k(x)$ be a cyclotomic factor of $p(x)$. Arguing as in the proof of Proposition 3.2, we see that for all root of unity ζ of sufficiently large order prime to ℓ, we have $|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\Phi(\zeta))| = 1$. Now, suppose that ℓ does not divide k. Let $p \neq \ell$ be a prime and $r \geq 1$ an integer. Then if ζ is a $p^r k$-th root of unity, we have $|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\Phi(\zeta))| = p \neq 1$, which is a contradiction. So, k is a multiple of ℓ.

Suppose that condition (2) holds. In order to prove that $f(x)$ is rational, it is sufficient to prove that this is true when $p(x) = \Phi_k(x)$ is the k-th cyclotomic polynomial with k multiple of ℓ. This is a direct consequence of the fact that $\Phi_k(x) = \frac{\Phi_{k'}(x^{\ell^v})}{\Phi_{k'}(x^{\ell^{v-1}})}$ where $k = \ell^v k'$ with $v \geq 1$ and $k' \geq 1$ prime to ℓ.

Finally, assume that condition (3) holds. We have $f(x) = x^{v_\infty} g(x)$ for some relative integer v_∞ and some rational function $g(x)$ regular at ∞. It follows that $p(x) = \frac{f(x)}{f(x^\ell)} = x^{v_\infty(1-\ell)} \frac{g(x)}{g(x^\ell)}$. In particular, this implies that $p(x)$ is monic. Moreover, the radial boundedness property of condition (1) is obviously satisfied when $f(x)$ is rational. □

4. The case where the coefficients of $f(x)$ are bounded

In this section we also focus on Mahler functions $f(x)$ for which (2) holds. In particular, we prove, under the assumption that $f(x)$ has bounded coefficients, some results on the norm estimates of the value of $p(x)$ at roots of unity and on the Mahler measure of $p(x)$. These will be used in the proof of Theorem 1.2.

4.1. Norms estimatates for the value of $p(x)$ at roots of unity.

Proposition 4.1. *Let $f(x)$ be a Mahler function satisfying (2). Assume that the set of coefficients of $f(x)$ is bounded. Let $n \geq 1$ be an integer and let ζ be an $(\ell^n - 1)$-th root of unity. Then*

$$\left| \prod_{k=0}^{n-1} p(\zeta^{\ell^k}) \right| \leq \ell^n.$$

Proof. We can and will assume that $\prod_{k=0}^{n-1} p(\zeta^{\ell^k}) \neq 0$.

Since the coefficients of $f(x)$ are bounded, there exists a constant $C > 0$ such that, for all $t < 0$,

$$|f(\zeta e^t)| \leq \frac{C}{1 - e^t}.$$

So $|tf(\zeta e^t)|$ is bounded as t tends to 0^-.

On the other hand, according to Proposition 2.1, we have

$$f(\zeta e^t) \sim_{t \to 0^-} m_\zeta(t) t^{-\frac{\log(\prod_{k=0}^{n-1} p(\zeta^{\ell^k}))}{\log(\ell^n)}}$$

where $m_\zeta(t)$ is some non zero meromorphic function on the left half-plane $\{t \in \mathbb{C} \mid \Re(t) < 0\}$ such that $m_\zeta(\ell^n t) = m_\zeta(t)$.

Therefore, $\left|tm_\zeta(t)t^{-\frac{\log(\prod_{k=0}^{n-1}p(\zeta^{\ell^k}))}{\log(\ell^n)}}\right|$ is bounded as t tends to 0^-. So, $\Re\left(1-\frac{\log(\prod_{k=0}^{n-1}p(\zeta^{\ell^k}))}{\log(\ell^n)}\right)\geq 0$. Whence the result. □

Corollary 4.2. *Assume that $f(x)$ satisfies (2) and that the set of coefficients of $f(x)$ is bounded. Let ζ be a complex root of unity of order prime to ℓ. Then $|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta))|\leq \ell^{[\mathbb{Q}(\zeta):\mathbb{Q}]}$ and $|N_{\mathbb{Q}(p(\zeta))/\mathbb{Q}}(p(\zeta))|\leq \ell^{[\mathbb{Q}(p(\zeta)):\mathbb{Q}]}$.*

Proof. Let m be the order of ζ as a root of unity. Let κ be the order of ℓ in $(\mathbb{Z}/m\mathbb{Z})^\times$. According to Lemma 3.1, there exist some primitive m-th roots of unity $\zeta_1,\ldots,\zeta_r$ such that

$$N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta))=\prod_{i=1}^{r}\prod_{k=0}^{\kappa-1}p(\zeta_i^{\ell^k}).$$

But, for any $i\in\{1,\ldots,r\}$, ζ_i is an $(\ell^\kappa-1)$-th root of unity and by Proposition 4.1, we have

$$|\prod_{k=0}^{\kappa-1}p(\zeta_i^{\ell^k})|\leq \ell^\kappa.$$

Therefore,

$$|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta))|=|\prod_{i=1}^{r}\prod_{k=0}^{\kappa-1}p(\zeta_i^{\ell^k})|\leq\prod_{i=1}^{r}\ell^\kappa=\ell^{r\kappa}=\ell^{\varphi(m)}=\ell^{[\mathbb{Q}(\zeta):\mathbb{Q}]}.$$

Whence the first estimate. The second one follows from the fact that $|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta))|=|N_{\mathbb{Q}(p(\zeta))/\mathbb{Q}}(p(\zeta))|^{[\mathbb{Q}(\zeta):\mathbb{Q}(p(\zeta))]}$. □

4.2. **A bound for the Mahler measure of $p(x)$.** In addition to the constraints relying on $p(x)$ proved in the previous sections, we have the following bound for the Mahler measure of $p(x)$.

Proposition 4.3. *Let $f(x)$ be a solution of (2) with $p(x)=a_nx^n+\ldots+a_1x+1$ and assume that the set of coefficients of $f(x)$ is bounded. Then the Mahler measure $M(p)$ of $p(x)$ is bounded as*

$$M(p)\leq|a_n|\ell.$$

Proof. By definition

$$M(p)=\frac{|a_n|}{|\alpha_1\cdots\alpha_m|}$$

where $\alpha_1,\ldots,\alpha_m$ are the roots of $p(x)$ in $D(0,1)$ counted with multiplicities. We want to show that

$$|\alpha_1\cdots\alpha_m|\geq\ell^{-1}.$$

Up to renumbering, we can assume that $|\alpha_1|\leq\cdots\leq|\alpha_m|$. Applying Jensen's formula to $f(x)$ in the disk $\overline{D}(0,r)$, we get

$$\sum_{k=1}^{m}\sum_{j=0}^{J_{k,r}}\ell^j\log\left(\frac{r}{|\alpha_k|^{\ell^{-j}}}\right)=\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|$$

where $J_{k,r}=\max\{j\geq 0\mid |\alpha_k|^{\ell^{-j}}\leq r\}$.

On the one hand, letting C be an upper bound for the absolute values of the coefficients of $f(x)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \leq \log\left(\frac{C}{1-r}\right).$$

On the other hand, since $J_{m,r} \subset \cdots \subset J_{1,r}$, we have

$$\sum_{k=1}^{m}\sum_{j=0}^{J_{m,r}} \ell^j \log\left(\frac{r}{|\alpha_k|^{\ell^{-j}}}\right) \leq \sum_{k=1}^{m}\sum_{j=0}^{J_{k,r}} \ell^j \log\left(\frac{r}{|\alpha_k|^{\ell^{-j}}}\right).$$

Now, for any positive integer N and for $r = |\alpha_m|^{\ell^{-N}}$, we have $J_{m,r} = N$, hence

$$\sum_{k=1}^{m}\sum_{j=0}^{N} \ell^j \log\left(\frac{|\alpha_m|^{\ell^{-N}}}{|\alpha_k|^{\ell^{-j}}}\right) = \sum_{k=1}^{m}\sum_{j=0}^{J_{m,r}} \ell^j \log\left(\frac{r}{|\alpha_k|^{\ell^{-j}}}\right).$$

Since $|\alpha_k|^{\ell^{-N}} \leq |\alpha_m|^{\ell^{-N}}$, this gives

$$\sum_{k=1}^{m}\sum_{j=0}^{N} \ell^j \log\left(\frac{|\alpha_k|^{\ell^{-N}}}{|\alpha_k|^{\ell^{-j}}}\right) \leq \sum_{k=1}^{m}\sum_{j=0}^{N} \ell^j \log\left(\frac{|\alpha_m|^{\ell^{-N}}}{|\alpha_k|^{\ell^{-j}}}\right).$$

Finally, we get

$$\sum_{k=1}^{m}\sum_{j=0}^{N} \ell^j \log\left(\frac{|\alpha_k|^{\ell^{-N}}}{|\alpha_k|^{\ell^{-j}}}\right) \leq \log\left(\frac{C}{1-|\alpha_m|^{\ell^{-N}}}\right).$$

But

$$\begin{aligned}\sum_{k=1}^{m}\sum_{j=0}^{N} \ell^j \log\left(\frac{|\alpha_k|^{q^{-N}}}{|\alpha_k|^{\ell^{-j}}}\right) &= \sum_{k=1}^{m}\sum_{j=0}^{N} \ell^j(\ell^{-j}-\ell^{-N})\log(|\alpha_k|^{-1}) \\ &= \sum_{k=1}^{m}\left(N - \frac{\ell^{N+1}-1}{\ell-1}\ell^{-N}\right)\log(|\alpha_k|^{-1})\end{aligned}$$

which is equivalent to $N\log(|\alpha_1\cdots\alpha_m|^{-1})$. Moreover, $\log\left(\frac{C}{1-|\alpha_m|^{\ell^{-N}}}\right)$ is equivalent to $N\log(\ell)$. So $\log(|\alpha_1\cdots\alpha_m|^{-1}) \leq \log(\ell)$ *i.e.* $|\alpha_1\cdots\alpha_m| \geq \ell^{-1}$, as wanted. □

5. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2 from the Introduction. Again, we suppose that $f(x) = \sum_{n\geq 0} f_n x^n$ is a Mahler function satisfying $f(x) = p(x)f(x^\ell)$, with $p(x) \in \mathbb{Z}[x]$ such that $p(0) = 1$, and we denote by $\mathcal{S}_f$ the set of coefficients of $f(x)$. We want to study for which polynomials $p(x)$ the set $\mathcal{S}_f$ is finite.

A first simple class of examples for which $\mathcal{S}_f$ is finite is the following.

Proposition 5.1. *Let $f(x)$ be a Mahler function satisfying (2). Assume that ℓ is a prime number and that $p(x) = \Phi_{n_1}(x)\cdots\Phi_{n_r}(x)$ is the product of cyclotomic polynomials $\Phi_{n_1}(x),\ldots,\Phi_{n_r}(x)$ of order divisible by ℓ. We set $n_i = n_i'\ell^{v_i}$ with $\gcd(n_i',\ell) = 1$. If $n_1',\cdots,n_r'$ are pairwise distinct, then $\mathcal{S}_f$ is finite.*

Proof. It is sufficient to prove the result for $f(x) = \prod_{i\geq 0} p(x^{\ell^i})$. We have $\Phi_{n_i}(x) = \frac{\Phi_{n'_i}(x^{\ell^{v_i}})}{\Phi_{n'_i}(x^{\ell^{v_i-1}})}$. It follows that $f(x) = \frac{1}{\Phi_{n'_1}(x^{\ell^{v_1-1}})\cdots\Phi_{n'_r}(x^{\ell^{v_r-1}})}$. Since the polynomials $\Phi_{n'_1}(x^{\ell^{v_1-1}}),\ldots,\Phi_{n'_r}(x^{\ell^{v_r-1}})$ are pairwise coprime, we get that $f(x)$ is a $\mathbb{Q}[x]$-linear combination of the $\frac{1}{\Phi_{n'_i}(x^{\ell^{v_i-1}})}$. So, it is sufficient to prove that the Taylor expansion at 0 of $\frac{1}{\Phi_{n'_i}(x^{\ell^{v_i-1}})}$ has finitely many distinct coefficients. This follows immediately from the fact that $\Phi_{n'_i}(x^{\ell^{v_i-1}})$ divides $x^{n_i}-1$. □

A natural question is whether there are any more examples and if it is possible to classify them.

The situation is very clear when $\ell > \deg p(x)$, indeed we have the following:

Proposition 5.2. *Let $f(x)$ be a Mahler function satisfying (2) where $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + 1$ with $a_i \in \mathbb{Z}$ and $\ell > n$. Then $\mathcal{S}_f$ is finite if and only if $|a_i| \leq 1$ for all i, in which case $\mathcal{S}_f \subseteq \{0, 1, -1\}$.*

Proof. We remark that it is sufficient to prove the result for $f(x) = \prod_{i\geq 0} p(x^{\ell^i})$. Equation $f(x) = p(x)f(x^\ell)$ implies that the coefficients $\{f_i\}_i$ satisfy the recurrence relation

$$\begin{aligned} &f_0 = 1, \\ &f_{m\ell+i} = a_i f_m \text{ for } i = 0, 1, \ldots, n, \\ &f_k = 0 \text{ otherwise.} \end{aligned} \tag{5}$$

So, f_i is either 0 or it has the form $\prod_{j=1}^n a_j^{r_j}$ for some integers r_i. In particular we have that

$$f_{i(\ell^k+\cdots+\ell+1)} = a_i^{k+1}$$

which implies that $\mathcal{S}_f$ is finite if and only if $|a_i| \leq 1$ for all i, in which case $\mathcal{S}_f \subseteq \{0, 1, -1\}$. □

This result covers the case where $\deg p(x) = 1$, as ℓ is supposed to be at least 2. Thus we are left with the case $2 \leq \ell \leq \deg p(x)$.

The following result treats the case where $p(x)$ is monic and $\deg p(x) = 2$.

Proposition 5.3. *Let $f(x)$ be a Mahler function satisfying (2) where $p(x) = x^2 + bx + 1$ with $b \in \mathbb{Z}$ and $\ell = 2$. Then $\mathcal{S}_f$ is finite if and only if $b \in \{0, -1\}$.*

Proof. It is sufficient to prove the result for $f(x) = \prod_{i\geq 0} p(x^{\ell^i})$. First notice that the equation $f(x) = (x^2+bx+1)f(x^2)$ implies that the sequence $\{f_n\}_n$ of the coefficients of $f(x) = \sum_{i\geq 0} f_n x^n$ satisfies the recurrence relation

$$f_0 = 1, f_1 = b, f_{2n+1} = bf_n, f_{2n} = f_{n-1} + f_n. \tag{6}$$

Suppose now that $\mathcal{S}_f$ is finite. Then by (6) $b \in \{-1, 0, 1\}$: indeed we have $f_{2^n-1} = b^n$. This is true for $n = 1$ as $f_0 = 1$. Suppose it true for n. Then, by (6), $f_{2^{n+1}-1} = f_{2(2^n-1)+1} = bf_{2^n-1} = b^{n+1}$.

Proposition 4.1 for $n = 1$ ensures that

$$p(1) = b + 2 \in \{\pm 2, \pm 1, 0\} \tag{7}$$

so $b \in \{-4,-3,-2,-1,0\}$.

Therefore, we have $b \in \{-1, 0\}$.

Viceversa, suppose that:

— $b = 0$: then (6) implies that $f_{2i+1} = 0$ for all $i \geq 0$ while the subsequence $\{f_{2i}\}_i$ satisfies $f_{4i} = f_{2i}$ and $f_{4i+2} = f_{2i}$, which implies that $\mathcal{S}_f$ is bounded.

— $b = -1$: then the sequence $\{f_i\}_i$ satisfies the relation $f_{2i+1} = -f_i$ and $f_{2i} = f_i + f_{i-1}$. It is easy to see, by induction, that the sequence is periodic and satisfies $f_{3i} = 1$, $f_{3i+1} = -1$ and $f_{3i+2} = 0$. In particular $\mathcal{S}_f$ is finite.

□

Next proposition considers the case where $p(x)$ is monic and $\deg p(x) = 3$, completing the proof of Theorem 1.2.

Proposition 5.4. *Let $f(x)$ be a Mahler function satisfying (2) where $p(x) = x^3 + bx^2 + cx + 1$ with $b, c \in \mathbb{Z}$.*

If $\ell = 2$ then $\mathcal{S}_f$ is finite if and only if $b = c = 0$, in which case $\mathcal{S}_f \subseteq \{0, 1, -1\}$.

If $\ell = 3$ then $\mathcal{S}_f$ is finite if and only if $c = 0$ and $b \in \{0, -1\}$ or $c = -1$ and $b = 0$, in which case $\mathcal{S}_f \subseteq \{0, 1, -1\}$.

Proof. We remark that it is sufficient to prove the result for $f(x) = \prod_{i\geq 0} p(x^{\ell^i})$.

Let $\ell = 2$. Suppose that $\mathcal{S}_f$ is finite.

Notice that the coefficients f_n of $f(x)$ satisfy the following recurrence relation:

$$f_0 = 1,\ f_1 = c,\ f_{2n} = f_n + bf_{n-1},\ f_{2n+1} = cf_n + f_{n-1}. \tag{8}$$

Proposition 4.1 for $n = 1$ gives $|b + c + 2| \leq 2$, so that

$$-4 \leq b + c \leq 0, \tag{9}$$

while Proposition 4.1 for $n = 2$ gives $|p(j)|^2 = 4 - bc + b^2 + c^2 - 2(b + c) \leq 4$ which implies

$$bc \geq -2(b + c) + b^2 + c^2. \tag{10}$$

By (9) we must have $b + c \leq 0$ and (10) gives $bc \geq b^2 + c^2 - 2(b + c) \geq 0$ so $bc \geq 0$. In particular b and c are both ≤ 0, so $-4 \leq b, c \leq 0$. We have several cases:

— If $b = 0$ then $c = 0$: indeed, by (10) we have $c^2 - c \leq 0$ (in the same way, if $c = 0$ then $b = 0$).

— If $b = -1$, then (10) gives $c \geq c^2 + 3 > 0$ which cannot hold as $c \leq 0$. This also shows that $c = -1$ cannot occur.

— If $b = -2$ then (10) gives $c^2 + 8 \leq 0$, which is impossible. This also shows that $c = -2$ cannot occur.

— If $b \in \{-3, -4\}$ then $c \in \{0, -1\}$. The case $c = -1$ was discarded already and we saw that $c = 0$ implies $b = 0$, a contradiction.

So the only possible case is $b = c = 0$. Let $f(x) = \sum_{i\geq 0} f_n x^n$ be such that $f(x) = (x^3 + 1)f(x^2)$. Then, by (8) we have $f_0 = 1, f_1 = 0, f_{2i+1} = f_{i-1}, f_{2i} = f_i$, so $\mathcal{S}_f \subseteq \{0, 1\}$.

Assume now that $\ell = 3$. Then, from the equation $f(x) = p(x)f(x^3)$ we see that the coefficients of f satisfy the recurrence $f_0 = 1, f_1 = c, f_2 = b, f_{3n+1} = cf_n$, $f_{3n+2} = bf_n$ and $f_{3n} = f_n + f_{n-1}$. In particular $f_{\frac{3^k-1}{2}} = c^k$ and $f_{3^k-1} = b^k$, which implies that $|b|, |c| \leq 1$. Moreover, one can prove by induction that $f_{\frac{3^{k+1}-3}{2}} = 1 + c + \cdots + c^k$. Indeed, for $k = 1$ we have $f_3 = f_1 + f_0 = c + 1$ and, if it holds for k, then

$$f_{\frac{3^{k+1}-3}{2}} = f_{3\frac{(3^k-1)}{2}} = f_{\frac{(3^k-1)}{2}} + f_{\frac{3^k-3}{2}} = c^k + (1 + c + \cdots + c^{k-1}).$$

So c equals either 0 or -1.

Let us first consider the case $c = 0$. Then, by induction we see that $f_{3^k} = 1 + b + b^2 + \cdots + b^{k-1}$ for every $k \geq 2$, which implies $b = -1$ or $b = 0$.

If $b = 0$ then it is clear that $\mathcal{S}_f$ is finite as $f_n \in \{0, 1, -1\}$ for all n (the only possible non zero elements are the $f_{3n} = f_n + f_{n-1}$ and at least one among f_n and f_{n-1} is zero).

Assume now $b = -1$. We are going to prove that $\mathcal{S}_f$ is finite by showing that, for every $n \geq 3$, the set $\{f_0, \ldots, f_n\}$ consists only of 0, 1 and -1. This is true for $n = 3$. Assume it holds for $n - 1$ and consider the set $\{f_0, \ldots, f_{n-1}, f_n\}$. If n is congruent to 1 modulo 3 then $f_n = 0$. If n is congruent to 2 modulo 3 then $f_n = -f_m$ for some $m \leq n - 1$, so $f_n \in \{0, 1, -1\}$. Thus we may assume $n = 3m$, with $m < n - 1$: in this case $f_n = f_m + f_{m-1}$. In particular, if m is congruent to 1 or 2 modulo 3, then f_n equals either f_{m-1} or f_m, which by assumption are in $\{0, 1, -1\}$. So the only case left is when $m = 3t$ for some $t < m$. In this case $f_n = f_{3t} + f_{3t-1} = f_{3t} + f_{3(t-1)+2} = f_t + f_{t-1} - f_{t-1} = f_t$, as wished.

We now consider the case $c = -1$. Then:

— if $b = 0$ the set $\mathcal{S}_f$ is finite. This follows from the fact that in this case the corresponding polynomial is the reciprocal of the polynomial associated with the choice $c = 0, b = -1$ for which the finiteness of $\mathcal{S}_f$ is proved above.

— if $b = 1$ then the set $\mathcal{S}_f$ is not finite as, for instance, $f_{3^k} = k - 1$ for all $k \geq 1$. This can be proved by induction: for $k = 1$ we have $f_3 = f_1 + f_0 = 0$ and, assuming it true for k, $f_{3^{k+1}} = f_{3^k} + f_{3^k-1} = (k-1) + b^{k+1} = k$.

— if $b = -1$ then $\mathcal{S}_f$ is not finite. Indeed one can prove by induction that $f_{2(3+3^3+\ldots+3^{2k+1})} = (-2)^{k+1}$. This is true for $k = 0$ as $f_6 = f_2 + f_1 = -2$. Now assuming it true for $k - 1$, $k \geq 2$, we have

$$\begin{aligned} f_{2(3+3^3+\ldots+3^{2k+1})} = f_{3\cdot 2(1+3^2+\ldots+3^{2k})} &= f_{2+3\cdot 2(3+\ldots+3^{2k-1})} + f_{1+3\cdot 2(3+\ldots+3^{2k-1})} \\ &= -f_{2(3+\ldots+3^{2k-1})} - f_{2(3+\ldots+3^{2k-1})} = (-2)^{k+1}. \end{aligned}$$

□

Remark 5.5. *For polynomials of higher degree the criterium of Proposition 4.1 is not subtle enough to conclude as above, as the following example shows.*

Let $p(x) = x^4 - x^3 + 1$ *and let* ζ *be a* $2^n - 1$ *root of unity. Then* $|\prod_{k=0}^{n-1} p(\zeta^{2^k})| \leq 2^n$.

Proof. We consider the functions

$$f_1(t) = |p(e^{2i\pi t})|$$

$$f_2(t) = |p(e^{2i\pi t})p(e^{2^2 i\pi t})|$$

$$f_3(t) = |p(e^{2i\pi t})p(e^{2^2 i\pi t})p(e^{2^3 i\pi t})|.$$

We first prove that $f_1(t) \leq 2$ for $t \in [0,1] \setminus (I_1 \cup I_2 \cup I_3)$ where $I_1 = [0.185, 0.276]$, $I_2 = [0.418, 0.581]$ and $I_3 = [0.723, 0.815]$.

We can write

$$\begin{aligned} f_1(t)^2 =& |e^{8i\pi t} - e^{6i\pi t} + 1| = (\cos(8\pi t) - \cos(6\pi t) + 1)^2 + (\sin(8\pi t) + \sin(6\pi t))^2 = \\ =& 3 + 2\cos(8\pi t) - 2\cos(6\pi t) - 2\cos(2\pi t). \end{aligned}$$

Setting $\gamma = \cos(2\pi t)$ and using standard properties of the cosinus, we see that $f_1(t)^2$ can be written as a polynomial $r(\gamma)$ in γ, that is

$$f_1(t)^2 = 16\gamma^4 - 8\gamma^3 - 8\gamma^2 + 4\gamma + 5 = r(\gamma).$$

The polynomial $4 - r(\gamma)$ has only 3 real roots $\gamma_1, \gamma_2, \gamma_3$ for $\gamma \in [-1,1]$, which correspond 6 values of t, $0 < t_1 < t_2 < \ldots < t_6 < 1$ such that $f_1(t) = 2$. As $2 - f_1(0)$ and $2 - f_1(1)$ are both positive, this shows that $f_1(t) \leq 2$ for $t \in [0,1] \setminus ([t_1,t_2] \cup [t_2,t_3] \cup [t_5,t_6])$ Notice that the function $4 - r(\cos(2\pi t))$ is positive at all the boundary points of I_1, I_2, I_3 and negative at at least one point inside each interval I_k (e.g. at the points $t = 0.2$, $t = 0.5$ and $t = 0.8$). This shows that $[t_{2k-1}, t_{2k}] \subseteq I_k$ for all $k = 1,2,3$, proving the claim.

We want now to show that $f_2(t) \leq 4$ for $t \in [0,1] \setminus (J_1 \cup J_2)$ where $J_1 = [0.2, 0.287]$ and $J_2 = [0.713, 0.795]$.

Keeping the previous notation, by the duplication formula for the cosinus, we have that $f_2(t)^2 = r(\gamma)r(2\gamma^2 - 1)$. By Sturm's algorithm (and performing the computations with PARI/GP [PARI]) the polynomial $16 - r(\gamma)r(2\gamma^2 - 1)$ has only 2 real roots for $\gamma \in [-1,1]$, which correspond to 4 values of t, $0 < t_1 < t_2 < t_3 < t_4 < 1$. Since the function $4 - f_2(t)$ is positive at $t = 0, 1$, it means that $f_2(t) \leq 4$ for $t \in [0,1] \setminus ([t_1,t_2] \cup [t_3,t_4])$. As before, since the function $4 - f_2(t)$ is positive all the boundary points of J_1 and J_2, and negative at at least on point inside each interval J_k (e.g. at the points $t = 0.22$ and $t = 0.73$), then $[t_1,t_2] \subseteq J_1$ and $[t_3,t_4] \subseteq J_2$, proving the claim.

Finally, we have that

$$f_3(t) \leq 8 \tag{11}$$

for all $t \in [0,1]$. Indeed, using again the duplication formula for the cosinus, we can write $f_3(t)^2 = r(\gamma)r(2\gamma - 1)r(8\gamma^4 - 8\gamma^2 + 1)$ and by Sturm's algorithm one sees that the polynomial $64 - r(\gamma)r(2\gamma - 1)r(8\gamma^4 - 8\gamma^2 + 1)$ has no real roots for $\gamma \in [-1,1]$.

Now, let n be a nonzero integer and let ζ be an $(2^n - 1)$-th complex root of unity.

Assume that n is a multiple of 3. Then the inequality $|\prod_{k=0}^{n-1} p(\zeta^{2^k})| \leq 2^n$ is a direct consequence of (11).

We are going to prove that there exists an index $0 \leq j_0 \leq n-1$ such that $2^{j_0} \notin I_1 \cup I_2 \cup I_3$. Assuming the contrary, we have three cases:

(1) If $2^{j_0} \in I_1$. Then $2^{j_0+1} \in [0.37, 0.552]$. So $2^{j_0+1} \in [0.418, 0.552]$. So $2^{j_0+2} \in [0.836, 1] \cup [0, 0.104] \notin I_1 \cup I_2 \cup I_3$, giving a contradiction.

(2) Assume that $2^{j_0} \in I_2$. Then $2^{j_0+1} \in [0.836, 1] \cup [0, 0.104]$, giving a contradiction as before.

(3) Assume that $2^{j_0} \in I_3$. Then $2^{j_0+1} \in [0.446, 0.63]$. So $2^{j_0+1} \in I_2$ and we are reduced to a previous case.

Moreover, we cannot have $2^j \in J_1 \cup J_2$ for all j because if $2^j \in J_1 \cup J_2$ then $2^{j+1} \notin J_1 \cup J_2$.

Now assume that $n \equiv 1 \mod 3$. Since there exists $0 \leq j_0 \leq n-1$ such that $2^{j_0} \notin I_1 \cup I_2 \cup I_3$, by (11) we see that $|\prod_{k=0}^{n-1} p(\zeta^{2^k})| \leq 2^n$.

Finally, assume that $n \equiv 2 \mod 3$. Let j_0 be such that $2^{j_0} \notin I_1 \cup I_2 \cup I_3$. We are going to prove that either 2^{j_0-1} or 2^{j_0+1} is not in $J_1 \cup J_2$, which, together with (11), concludes the proof.

— If $2^{j_0} \in [0, 0.185[$ then $2^{j_0-1} \in [0, 0.0925] \notin J_1 \cup J_2$.
— If $2^{j_0} \in]0.276, 0.418[$ then $2^{j_0-1} \in]0.138, 0.209[$. If $2^{j_0-1} \in J_1 \cup J_2$ then $2^{j_0-1} \in [0.2, 0.209[$, so $2^{j_0+1} \in [0.8, 0.836[\notin J_1 \cup J_2$.
— If $2^{j_0} \in]0.581, 0.723[$ then $2^{j_0-1} \in]0.2905, 0.3615[\notin J_1 \cup J_2$.
— If $2^{j_0} \in]0.815, 1]$ then $2^{j_0-1} \in]0.4075, 0.5] \notin J_1 \cup J_2$.

□

Although we have been unable to prove it, we believe that for $p(x) = x^4 - x^3 + 1$ *the set* $\mathcal{S}_f$ *is not finite. Indeed, set* $f(x) = \prod_{i \geq 0} p(x^{2^i})$ *and let* $\{f_n\}_n$ *be the sequence of the coefficients of* $f(x)$*. If* $\{f_n\}_n$ *were bounded, then the sequence* $\{d_n\}_n$ *with* $d_n = f_n + f_{n+2}$ *should be also bounded. However, this does not seem to be the case.*

n	*19*	*107*	*359*	*843*	*1703*	*5815*	*6799*	*10983*
d_n	*-2*	*3*	*-4*	*5*	*-6*	*7*	*-8*	*9*

TABLE 1. Some values of the sequence $\{d_n\}_n$

6. Analogies with E-functions and G-functions

In [Sie29] Siegel introduced an important class of functions, which goes nowadays under the name of E-functions. They can be seen as a sort of extensions of the exponential series and were indeed introduced to generalize the Lindemann-Weierstrass theorem.

Formally, an *E-function* is a power series $f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$ which satisfies a nonzero homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ and such that $\{a_n\}_n$ is a sequence of algebraic numbers with the following special 'growth' properties: namely, there exists an absolute constant $C > 0$ such that for all n:

(1) the maximum of the moduli of the conjugates of a_n is at most C^n;

(2) the common denominator of $a_0, \ldots, a_n$ is bounded by C^n (this means that there exists $d_n \in \mathbb{Z}$ such that $|d_n| \leq C^n$ and $d_n a_i$ is an algebraic integer for all $i \leq n$).

As recalled above, Siegel's original motivation was to obtain a Lindemann-Weierstrass type theorem for a more general class of functions. To this aim he initiated a method to study the transcendence and algebraic dependence of values of E-functions at algebraic points, which was later successfully developed by many authors, such as André, Beukers, Nesterenko, Shidlovsky, *etc.*

In [Sie29] Siegel also introduced another class of functions, which this time play the role of generalizations of geometric series, namely G-functions. Similarly to the preceeding definition, a *G-funtion* is a a power series $f(z) = \sum_{n\geq 0} a_n z^n$ which satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ and such that $\{a_n\}_n$ is a sequence of algebraic numbers satisfying conditions (1) and (2) above. However, while the nature of the values of E-functions at algebraic points is well described by Siegel's method, very little is known for those of G-functions.

This section aims to describe some analogies between Mahler functions and E- and G-functions in two respects.

First, we show (see Proposition 6.4) that the coefficients of Mahler functions satisfy automatically the two conditions (1) and (2), and even the stronger condition of being *globally bounded* (see Definition 6.3). This global boundedness property is deduced from a general index theorem for Mahler operators (see Theorem 6.1).

Secondly, in the last section, we study the Newton polygon at 1 of certain Mahler operators. This is motivated by the properties of the Newton polygons of the E- and G-operators recalled at the end of the Introduction. In particular, we show that for Mahler operators whose solutions satisfy equation (2) with $\ell = 2$ and have bounded coefficients, the Newton polygon is pure isoclinic of slope 0 and 1, a behaviour which is reminiscent of a property of the E-functions recalled at the end of the Introduction.

These analogies constitute an additional motivation for our study in Sections 4 and 5 of Mahler functions with bounded coefficients.

6.1. Mahler functions, G-functions and global boundedness.

6.1.1. *Index theorems.* Let E and F be vector spaces over a field K. We recall that a linear map $u : E \to F$ has an *index* if $\ker(u)$ and $\operatorname{coker}(u)$ are both finite dimensional K-vector spaces. In this case, the index $\chi(u)$ of u (also denoted $\chi(u, E, F)$ or $\chi(u, E)$ if $E = F$) is defined as

$$\chi(u) = \dim_K \ker(u) - \dim_K \operatorname{coker}(u).$$

In cohomological terms, $u : E \to F$ has an index if and only if the complex

$$0 \to E \xrightarrow{u} F \to 0$$

has finite dimensional cohomology spaces and, when this happens, $\chi(u)$ is the Euler characteristic of the complex, where E is placed in degree 0.

Let K be either $\mathbb{C}$ or an algebraically closed field of characteristic 0, complete with respect to an ultrametric norm $|\cdot|$.

For $K = \mathbb{C}$, we let $K\{x\} = \mathbb{C}\{x\}$ be the $\mathbb{C}$-algebra of germs of analytic functions at $0 \in \mathbb{C}$ while, for $K \neq \mathbb{C}$, we consider the K-algebra

$$K\{x\} = \left\{ \sum_{n\geq 0} a_n x^n \in K[[x]] \mid \exists r > 0, \lim_{n\to+\infty} |a_n| r^n = 0 \right\}.$$

In both cases, one can see $K\{x\}$ as the inductive limit as r tends to 0^+ of $K\langle x\rangle_r$ where

$$K\langle x\rangle_r = \left\{\sum_{n\geq 0} a_n x^n \in K[[x]] \mid \lim_{n\to+\infty} |a_n| r^n < \infty\right\}$$

if $K=\mathbb{C}$ and

$$K\langle x\rangle_r = \left\{\sum_{n\geq 0} a_n x^n \in K[[x]] \mid \lim_{n\to+\infty} |a_n| r^n = 0\right\}$$

otherwise.

Let $\ell \geq 2$ be an integer and denote by ϕ_ℓ the operator acting on a function $y(x)$ as follows :

$$\phi_\ell(y(x)) = y(x^\ell).$$

For any $r \in]0,1[$, we consider an operator

$$L = a_n(x)\phi_\ell^n + a_{n-1}(x)\phi_\ell^{n-1} + \cdots + a_0(x) \tag{12}$$

with $a_0(x), ..., a_n(x) \in K\langle x\rangle_r$ and $a_0(x)a_n(x) \neq 0$.

In the following result, we study the index of the K-linear map

$$\begin{aligned} L : E &\to E \\ f &\mapsto L(f) = a_n(x)f(x^{\ell^n}) + a_{n-1}(x)f(x^{\ell^{n-1}}) + \cdots + a_0(x)f(x) \end{aligned} \tag{13}$$

induced by L on various K-vector spaces E, namely $E = K[[x]]$, $K\{x\}$ and $K[[x]]/K\{x\}$.

We will denote by $v_0 : K[[x]] \to \mathbb{N} \cup \{+\infty\}$ the x-adic valuation.

Theorem 6.1. *The map L has an index in E in all the following cases:*

1. *If $E = K[[x]]$ then $\chi(L, K[[x]]) = -v_0(a_0)$ (this holds for every field K and for L with coefficients in $E = K[[x]]$).*
2. *If $E = K\{x\}$, then $\chi(L, K\{x\}) = -v_0(a_0)$.*
3. *If $E = K[[x]]/K\{x\}$ then $\chi(L, K[[x]]/K\{x\}) = 0$. Actually, we have*
   - *(i)* $\operatorname{coker}(L : K[[x]]/K\{x\} \to K[[x]]/K\{x\}) = 0$*;*
   - *(ii)* $\ker(L : K[[x]]/K\{x\} \to K[[x]]/K\{x\}) = 0$.

Proof. The proofs of these properties rely on classical "pertubative methods" in the framework of Banach or ultrametric Banach algebras (see Ramis's [Ram84] and Serre's [Ser62]), and are variants of proofs of similar results by Malgrange [Mal71], Ramis [Ram84], Bézivin [Bez92a, Bez92b], *etc*, concerning differential and q-difference equations. For this reason, we will only give a skeleton of the proofs, except for (1) that we shall now prove in details.

We let d be an integer such that, for $k \geq d$ and $j \in \{1, \ldots, n\}$, $v_0(a_0)+k < v_0(a_j) + k\ell^j$. Then, for all $k \geq d$,

$$L(x^k) = \underbrace{a_{0,v_0(a_0)}x^{v_0(a_0)+k}}_{\neq 0} + \text{ terms of higher valuation}$$

where

$$a_0(x) = a_{0,v_0(a_0)}x^{v_0(a_0)} + \text{ terms of higher valuation.}$$

It follows that, for all $g \in K[[x]]$ with $v_0(g) \geq v_0(a_0) + d$, there exists a unique $f \in K[[x]]$ such that $v_0(f) \geq d$ and $L(f) = g$. In other words, the map

$$L : x^d K[[x]] \to x^{v_0(a_0)+d} K[[x]]$$

is an isomorphism, and, hence, has an index, which is equal to 0. Note that this implies that $L : K[[x]] \to K[[x]]$ has an index. We now consider the map

$$K[[x]] \xrightarrow{x^d} K[[x]] \xrightarrow{L} K[[x]] \tag{14}$$

which is equal to

$$K[[x]] \xrightarrow{x^d} x^d K[[x]] \xrightarrow{L} x^{v_0(a_0)+d} K[[x]] \hookrightarrow K[[x]]. \tag{15}$$

Equating the indices of (14) and (15), which are easily computed using the additivity of the index, we find $\chi(L, K[[x]]) - d = -(v_0(a_0) + d)$. This proves (1). The proof of (2) in the ultrametric case is akin to the proofs of [Bez92b, Propositions 3.1 and 3.2]. We give a sketch of it in this case, the complex case being similar (see also [Bez92a]).

The idea is to consider first the K-algebra $K\langle x\rangle_r$. The Gauss norm defined by

$$|| \sum_{n \geq 0} a_n x^n ||_r = \max_{n \geq 0} |a_n| r^n$$

endows $K\langle x\rangle_r$ with a structure of ultrametric Banach algebra over K. A simple modification of the proof of [Bez92b, Proposition 3.1] shows that, for $r > 0$ small enough, L has an index in $K\langle x\rangle_r$ which is equal to $-v_0(a_0)$. In order to conclude the proof, we note that $K\{x\}$ is the inductive limit as r tends to 0^+ of the $K\langle x\rangle_r$ and we argue as in the proof of [Bez92b, Proposition 3.2].

We shall now prove (3i) *i.e.* that, for all $g \in K[[x]]$, there exist $h \in K\{x\}$ and $f \in K[[x]]$ such that $L(f) = g - h$. To this aim, it is sufficient to choose h as the truncation of g up to the order $v_0(g) \geq v_0(a_0) + d$ for some integer d such that, for $k \geq d$ and $j \in \{1, \ldots, n\}$, $v_0(a_0) + k < v_0(a_j) + k\ell^j$, and then argue as in the proof of (1).

Now (3ii) follows from the previous results and from a general algebraic argument. Indeed, we have the following exact sequence of complexes (the complexes are in column)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0\,. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K\{x\} & \hookrightarrow & K[[x]] & \twoheadrightarrow & K[[x]]/K\{x\} & \longrightarrow & 0 \\
\downarrow & & \downarrow L & & \downarrow L & & \downarrow L & & \downarrow \\
0 & \longrightarrow & K\{x\} & \hookrightarrow & K[[x]] & \twoheadrightarrow & K[[x]]/K\{x\} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Since the first and second complexes have finite dimensional cohomology, the third complex has finite dimensional cohomology as well *i.e.*

$L : K[[x]]/K\{x\} \to K[[x]]/K\{x\}$ has an index. Moreover, by the additivity of the Euler characteristic, $\chi(L, K[[x]]/K\{x\}) = \chi(L, K[[x]]) - \chi(L, K\{x\})$, which is equal to 0 in virtue of (1) and (2). Since $\mathrm{coker}(L : K[[x]]/K\{x\} \to K[[x]]/K\{x\}) = 0$, we get $\ker(L : K[[x]]/K\{x\} \to K[[x]]/K\{x\}) = 0$. □

An immediate corollary of Theorem 6.1(3ii), which will be useful in the next section, is the following:

Corollary 6.2. *Consider $g \in K\{x\}$. Any $f \in K[[x]]$ such that $L(f) = g$ actually belongs to $K\{x\}$. In particular, any solution in $K[[x]]$ of $L(f) = 0$ actually belongs to $K\{x\}$.*

6.1.2. *Application to equations with coefficients in $\overline{\mathbb{Q}}(x)$.* In this section, we will show how Theorem 6.1 implies that the coefficients of Mahler series automatically satisfy some nice properties, which are typical of a certain class of G-functions. According to Christol's terminology, we have the following definition:

Definition 6.3. *A power series $f(x) = \sum_{n \geq -N} a_n x^n \in \overline{\mathbb{Q}}((x))$ is* globally bounded *if:*

- *$f(x)$ defines an analytic function near $0 \in \mathbb{C}$;*
- *there exists a non zero integer C such that the coefficients of $f(Cx)$ are algebraic integers.*

Every globally bounded function $f(x) \in \overline{\mathbb{Q}}[[x]]$ which satisfies a homogeneous linear differential equation is a G-function, but the converse does not hold in general. The following result shows that global boundedness is automatic for Mahler functions. We consider an operator

$$L = a_n(x)\phi_\ell^n + a_{n-1}(x)\phi_\ell^{n-1} + \cdots + a_0(x) \tag{16}$$

with $a_0(x), \ldots, a_n(x) \in \overline{\mathbb{Q}}(x)$ and $a_0(x)a_n(x) \neq 0$.

Proposition 6.4. *Any solution $f(x) \in \overline{\mathbb{Q}}((x))$ of $L(f) = 0$ is globally bounded.*

Proof. By [AB, Lemma 5.1], the coefficients of $f(x)$ belong to a finitely generated $\mathbb{Z}$-algebra and so they all belong to a number field F. In order to conclude the proof, it is sufficient to show that, for any place v of F, there exist $A_v, B_v > 0$ such that $|a_n|_v \leq A_v B_v^n$. This is a direct consequence of Corollary 6.2. □

6.2. **Mahler operators, E-operators and Newton polygons.** In [And00] André defined and studied, amongst others, the structure of E-operators, which are the Fourier-Laplace transforms of G-operators.

We recall that an element $\Phi \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]$ is called a *G-operator* if it satisfies the so-called *Galochkin condition* (see [And00], p.718 for a precise definition). In particular, by a result of Chudnovsky [CC85], this condition is automatic if $\Phi f(x) = 0$ for some G-function $f(x)$ and such that the equation is minimal for $f(x)$.

A differential operator $\Psi \in \overline{\mathbb{Q}}[x, \frac{d}{dx}]$ is an *E-operator* if it is obtained by a G-operator via the formal changes $x \to \frac{d}{dx}$ and $\frac{d}{dx} \to -x$.

André investigated, in particular, the singularities of E-operators and proved that there can be only two of them, namely 0 and ∞ and that the only slopes of the Newton polygon of an E-operator at ∞ are 0 and 1, while 0 is a regular singular point. Let us briefly recall the definition of the slopes of Ψ at ∞. If $\Psi = \sum a_{i,j} x^i (x\frac{d}{dx})^j$, the Newton polygon $\mathcal{N}(L)$ of L at ∞ is the convex hull of the set

$$\{(x,y) \in \mathbb{R}^2 \mid x \leq j, y \leq i, a_{i,j} \neq 0\}.$$

This $\mathcal{N}(L)$ has finitely many extremal points $\{(n_1,m_1),\ldots,(n_{r+1},m_{r+1})\}$ with $0 \leq n_1 < n_2 < \cdots < n_{r+1} = n$. The positive slopes of L are $k_1 < \cdots < k_r$ with $k_i = \frac{m_{i+1}-m_i}{n_{i+1}-n_i}$. If $n_1 > 0$, then one adds a slope $k_0 = 0$. The set of slopes of L at ∞ is either $\{k_1,\ldots,k_{r+1}\}$ or $\{k_0,\ldots,k_{r+1}\}$ depending on whether $n_1 = 0$ or not.

In this section we show that a similar result can be obtained for Mahler operators of the form $L = p(x)\phi_2 - 1$, where $p(x) \in \mathbb{Z}[x]$, assuming that the Mahler function $f(x) = \sum_{n\geq 0} f_n x^n \in \mathbb{Z}[[x]]$ solution of $Lf(x) = 0$ has bounded coefficients and under some conditions on $p(x)$.

We recall the definition of Newton polygon in our context:

Definition 6.5. *The Newton polygon $\mathcal{N}(L)$ of*

$$L = a_n(x)\phi_\ell^n + a_{n-1}(x)\phi_\ell^{n-1} + \cdots + a_0(x)$$

with $a_i(x) \in \mathbb{C}(x)$ is the convex hull in $\mathbb{R}^2$ of $\{(i,j) \mid i \in \mathbb{Z}$ and $j \geq v_{x-1}(a_i)\}$ where v_{x-1} denotes the $(x-1)$-adic valuation on $\mathbb{C}(x)$. This polygon is made of two vertical half lines and of k vectors (r_1,d_1), ..., $(r_k,d_k) \in \mathbb{N}^* \times \mathbb{Z}$ having pairwise distinct slopes, called the slopes L. For any $i \in \{1,...,k\}$, r_i is called the multiplicity of the slope $\frac{d_i}{r_i}$.

For instance, the Newton polygon of $p(x)\phi_\ell - 1$ is the convex subset of $\mathbb{R}^2$ delimited by the vertical half lines $\{0\} \times \mathbb{R}^+$ and $\{1\} \times [v_{x-1}(p), +\infty[$ and by the segment from $(0,0)$ to $(1, v_{x-1}(p))$. So $p(x)\phi_\ell - 1$ is pure isoclinic (*i.e.* its Newton polygon has only one slope) with slope $v_{x-1}(p)$. We now prove Proposition 1.3 from the Introduction, which we recall here for clarity.

Proposition 6.6. *Let $f(x)$ be a Mahler function satisfying (2). Assume that the set of coefficients of $f(x)$ is bounded. Let r be the order of 1 as a root of $p(x)$. For all power $(\ell')^\alpha$ of a prime number ℓ' prime to ℓ and such that $p(x)$ has no root which is a primitive $(\ell')^\alpha$-th root of unity, we have*

$$(\ell')^r \leq \ell^{(\ell')^{\alpha-1}(\ell'-1)}.$$

In particular, all the slopes of the operators $p(x)\phi_\ell - 1$ are bounded by $\log_{\ell'}(\ell^{(\ell')^{\alpha-1}(\ell'-1)})$.

Proof. Let $q(x) \in \mathbb{Z}[x]$ be such that $p(x) = q(x)(x-1)^r$. Let $m \geq 1$ be an integer such that $\gcd(m,\ell) = 1$. Let ζ be a primitive m-th root of unity. According to Corollary 4.2, we have

$$|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta))| \leq \ell^{[\mathbb{Q}(\zeta):\mathbb{Q}]}$$

i.e.

$$|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(q(\zeta))||N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta-1)|^r \leq \ell^{[\mathbb{Q}(\zeta):\mathbb{Q}]}.$$

This gives the expected inequality because, by formula (4), $|N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta-1)| = |N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\Phi_1(\zeta))| = \ell'$ if $m = (\ell')^\alpha$ is the α-th power of a prime ℓ' prime to ℓ. □

For $\ell = 2$ we have the following:

Corollary 6.7. *Let $f(x)$ be a Mahler function satisfying (2). Assume that $\ell = 2$, that the set of coefficients of $f(x)$ is bounded and that one of the following conditions hold:*

(1) $p(j) \neq 0$;

(2) $p(\zeta_5) \neq 0$, where ζ_5 is a primitive 5-th root of unity;

(3) $p(\zeta_9) \neq 0$, where ζ_9 is a primitive 9-th root of unity.

Then, $p(x)\phi_2 - 1$ is pure isoclinic with slope 0 or 1.

Proof. Let r be the order of 1 as a root of $p(x)$. We have to prove that under the above hypothesis $r = 0$ or 1.

In cases (1) and (2) we apply Proposition 6.6 with $\ell' = 3$ and $\ell' = 5$ respectively. We obtain $3^r \leq 4$ and $5^r \leq 16$ respectively. Whence the result.

To prove case (3), we may assume that j is a root of order $r_3 \geq 1$ of $p(x)$, so that $p(x) = q(x)(x-1)^r \Phi_3(x)^{r_3}$ for some $q(x) \in \mathbb{Z}[x]$. By Proposition 4.2, we have

$$|N_{\mathbb{Q}(\zeta_9)/\mathbb{Q}}(p(\zeta_9))| \leq 2^{[\mathbb{Q}(\zeta_9):\mathbb{Q}]}$$

i.e.

$$|N_{\mathbb{Q}(\zeta_9)/\mathbb{Q}}(q(\zeta_9))||N_{\mathbb{Q}(\zeta_9)/\mathbb{Q}}(\zeta_9 - 1)|^r |N_{\mathbb{Q}(\zeta_9)/\mathbb{Q}}\Phi_3(\zeta_9)|^{r_3} \leq 2^{[\mathbb{Q}(\zeta_9):\mathbb{Q}]}.$$

But, by (4), we have $|N_{\mathbb{Q}(\zeta_9)/\mathbb{Q}}(\zeta_9 - 1)| = |N_{\mathbb{Q}(\zeta_9)/\mathbb{Q}}(\Phi_1(\zeta_9))| = 3$ and $|N_{\mathbb{Q}(\zeta_9)/\mathbb{Q}}(\Phi_3(\zeta_9))| = 9$. So we get

$$3^{r+2r_3} \leq 2^6.$$

It follows that, if $r \geq 2$, then $r_3 = 0$ (because $3^4 > 2^6$) and this is a contradiction. Therefore, r is equal to 0 or 1. □

Remark 6.8. *All cases in the previous corollary occur :*

— *if $p(x) = 1$, the slope is equal to 0 and $f(x) = 1$.*

— *if $p(x) = 1 - x$, the slope is equal to 1 and $f(x)$ has coefficients in $\{0, \pm 1\}$.*

More generally, if $\zeta_{\ell'^{k+1}}$ is not a root of $p(x)$, but all the $\zeta_{\ell'^j}$, for $j \leq k$ are, then combining Proposition 4.2 and (4) we have

$$\ell'^{r+(\ell'^k-1)} \leq \ell^{\ell'^k(\ell'-1)}.$$

As we saw, for $\ell = 2$, if $\ell' = 3$ and $k \leq 2$ this implies $r \leq 1$, but already for $k = 3$ it only implies $r \leq 3$.

It is natural to wonder whether Corollary 6.7 remains true without any hypothesis, whence the following question.

Question 3. *Is it true that, if $\ell = 2$ and if the set of coefficients of $f(x)$ is bounded, then $p(x)\phi_2 - 1$ is pure isoclinic with slope 0 or 1 ? More generally, does there exist $C_\ell \in \mathbb{Z}_{\geq 0}$, depending only on ℓ, such that, if the set of coefficients of $f(x)$ is bounded, then $p(x)\phi_2 - 1$ is pure isoclinic with slope $\leq C_\ell$?*

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Université Grenoble Alpes, Institut Fourier, F-38000 Grenoble, France
E-mail address: Sara.Checcoli@univ-grenoble-alpes.fr
E-mail address: Julien.Roques@univ-grenoble-alpes.fr