ON THE ALGEBRAIC DEPENDENCE OF HOLONOMIC FUNCTIONS

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ABSTRACT. We study the form of possible algebraic relations between functions satisfying linear differential equations. In particular, if f and g satisfy linear differential equations and are algebraically dependent, we give conditions on the differential Galois group associated to f guaranteeing that g is a polynomial in f. We apply this to hypergeometric functions and iterated integrals.

SUR LA DÉPENDANCE ALGÉBRIQUE DES FONCTIONS HOLONOMES

RÉSUMÉ. Nous nous intéressons aux relations algébriques vérifiées par des solutions d'équations différentielles linéaires. En particulier, si f et g satisfont des équations différentielles linéaires et sont algébriquement dépendantes, nous donnons des conditions sur le groupe de Galois différentiel de f garantissant que g est un polynôme en f. Nous appliquons cela aux fonctions hypergéométriques et aux intégrales itérées.

CONTENTS

1. INTRODUCTION

Let K be a differential field of characteristic zero and let k be an algebraically closed differential subfield of K with the same field of constants C :

$$C = \{f \in K \mid f' = 0\} = \{f \in k \mid f' = 0\}.$$

A typical example is the field of Puiseux series $K = \overline{C((x))}$ over the algebraically closed field C endowed with the usual derivation d/dx and $k = \overline{C(x)}$ the field of algebraic functions over $C(x)$.

The present paper is concerned with the following general question.

Question. *We let f and g be elements of $K^{\times}$ satisfying some nontrivial homogeneous linear differential equations over k, say*

$$L(f) = 0 \ and \ M(g) = 0,$$

and we assume that f and g are algebraically dependent over k. What can be said about f and g? about the algebraic relations between f and g over k?

Differential Galois theory has been an efffective tool in understanding the possible algebraic relations among solutions of linear differential equations (see, for example, [Kol68, HS85, HS86, Spe86, Sin86, BBH88, Roq14]). A celebrated illustration of this is the following result of Kolchin answering Question 1 when L and M are first order equations :

Theorem 1 (Kolchin [Kol68]). *If L and M have order 1, then f and g are algebraically dependent over k if and only if there exists $(m, n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that*

$$f^m g^n \in k.$$

The present work started with the following question : what can be said if we assume that L has order 1, but do not make any assumption on the order of M ? Our answer reads as follows :

Theorem 2 (Theorem 12 in Section 3). *If L has order 1 and if $f \notin k$, then f and g are algebraically dependent over k if and only if there exists θ in an algebraic extension of K and a positive integer n such that*

$$f = \theta^n \ and \ g \in k[\theta, \theta^{-1}].$$

The rest of the paper is devoted to the case when the operator L has order ≥ 2. For any $f \in K^{\times}$ as above, we let $\mathcal{A}(f)$ be the k-algebra made of the elements g of K holonomic and algebraically dependent on f over k. In general, the obvious inclusion

$$k[f] \subset \mathcal{A}(f)$$

is not an equality. The principal aim of Sections 4, 5 and 6 is to give criteria on the differential Galois group of L over k ensuring that $\mathcal{A}(f) = k[f]$. In this introduction, we only state the following concrete consequence of these criteria and refer to Sections 4, 5 and 6 for more general statements.

Theorem 3 (Consequence of Proposition 17 in Section 4 and Proposition 22 in Section 5). *Assume that L has order $n \geq 2$ and that its differential Galois group G_L over k is either $\mathrm{SL}_n(C)$ or $\mathrm{GL}_n(C)$. Then, we have*

$$\mathcal{A}(f) = k[f].$$

Remark 4. *1. When $n = 2$, the fact that the differential Galois group over k of L is either $\mathrm{SL}_2(C)$ or $\mathrm{GL}_2(C)$ is equivalent to the irreducibility of L over k [NvdPT08]. We thus have the following statement : If L has order 2 and is irreducible over k then $\mathcal{A}(f) = k[f]$.*

2. Another simple consequence of Theorem 3 is: Assume that L has order $n \geq 2$ and that its differential Galois group G_L over k is either $\mathrm{SL}_n(C)$ or $\mathrm{GL}_n(C)$. If $f \notin k$, then, for $m \geq 2$, $f^{1/m}$ does not satisfy any nontrivial linear differential equation over k. This follows easily form the fact that we cannot have $f^{1/m} \in k[f]$ unless f is algebraic over k and, hence, belongs to k. For related results, see [HS86, Spe86, Sin86].

Theorem 3 applies, for example, to many generalized hypergeometric series

$$_pF_q\left[\boldsymbol{\alpha};\boldsymbol{\beta};x\right] = \sum_{k=0}^{+\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{x^k}{k!} \in \mathbb{C}((x))$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$ for some $p, q \in \mathbb{Z}_{\geq 0}$ and where the Pochhammer symbols $(t)_k$ are defined by $(t)_0 = 1$ and, for $k \in \mathbb{Z}_{\geq 1}$, $(t)_k = t(t+1) \cdots (t+k-1)$. Indeed, this series satisfies the generalized hypergeometric differential equation

$$_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right]\left(_pF_q\left[\boldsymbol{\alpha};\boldsymbol{\beta};x\right]\right) = 0$$

where

$$_p\mathscr{H}_q\left[\boldsymbol{\alpha};\boldsymbol{\beta}\right] = \delta \prod_{k=1}^{q}(\delta + \beta_k - 1) - x \prod_{k=1}^{p}(\delta + \alpha_k)$$

with $\delta = x\frac{d}{dx}$ and the work of Beukers, Brownawell and Heckman [BBH88], Beukers and Heckman [BH89], Duval and Mitschi [DM89], Katz [Kat87, Kat90] and Mitschi [Mit96] give explicit conditions (holding for generic parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$) ensuring that the differential Galois group over $\overline{\mathbb{C}(x)}$ of $_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ is either $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$ (in these references, the differential Galois groups are computed over $\mathbb{C}(x)$, not

$\overline{\mathbb{C}(x)}$; this is not a problem since if the differential Galois group over $\mathbb{C}(x)$ is connected then it does not change when one replaces the base field with $\overline{\mathbb{C}(x)}$ [Mag94, Proposition 5.28]. For instance, it follows from [Kat90, Theorem 3.6] that, for any $\beta \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, the differential Galois group over $\overline{\mathbb{C}(x)}$ of $_0\mathscr{H}_1\left[-; \beta\right]$ is $\mathrm{SL}_2(\mathbb{C})$; so, for $K = \overline{\mathbb{C}((x))}$ endowed with the derivation d/dx and $k = \overline{\mathbb{C}(x)}$, Theorem 3 ensures that :

Corollary 5. *For any $\beta \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we have*

$$\mathcal{A}(_0F_1\left[-; \beta; x\right]) = \overline{\mathbb{C}(x)}[_0F_1\left[-; \beta; x\right]].$$

Moreover, we show in Section 7.1 that our methods (combined with the local formal theory of differential equations) can be used in order to examine in details the algebraic dependence relations between hypergeometric series. For instance, Corollary 5 ensures that if some generalized hypergeometric series $_pF_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right]$ is algebraically dependent on $_0F_1\left[-; \beta; x\right]$, then it is a polynomial in $_0F_1\left[-; \beta; x\right]$; in Section 7.1, we establish a much more precise statement :

Theorem 6. *Consider $\beta \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Consider $\boldsymbol{\gamma} \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^p$ and $\boldsymbol{\delta} \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$. If $_0F_1\left[-; \beta; x\right]$ and $_pF_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right]$ are algebraically dependent over $\overline{\mathbb{C}(x)}$, then*

- *either*
$$q = p + 1 \ and \ {_pF_q}\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right] \in \mathbb{C}(x)_0F_1\left[-; \beta; x\right] + \mathbb{C}(x);$$

- *or*
$$q = p - 1 \ and \ {_pF_q}\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right] \in \overline{\mathbb{C}(x)}.$$

Remark 7. *Whether or not the second condition holds true can be decided by using Beukers and Heckman's [BH89] provided that $\gamma_i - \delta_j \notin \mathbb{Z}$ for all $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$.*

As already mentioned, Theorem 3 is a consequence of results contained in Sections 4, 5 and 6, where the hypothesis that the differential Galois group G_L of L over k is either $\mathrm{SL}_n(C)$ or $\mathrm{GL}_n(C)$ is replaced by the hypothesis that G_L is either simply connected or reductive. These generalizations (more precisely, the generalization to the simply connected case) allow us to describe $\mathcal{A}(f)$ when f is an iterated integral (*i.e.*, when $f^{(n)} \in k$ for some positive integer n, see [RS10]) and to study the algebraic relations among iterated integrals. Indeed, in Section 7.2 we prove the following results.

Theorem 8 (Theorem 28 in Section 7.2)**.** *Let $f \in K^\times \setminus k$ be an iterated integral over k and let $g \in K$ satisfy a nonzero homogeneous linear*

differential equation over k. Assume that k contains an element x with $x' = 1$. Then, we have

$$\mathcal{A}(f) = k[f].$$

Proposition 9 (Proposition 34 in Section 7.2). *Assume that there is an element $x \in k$ such that $x' = 1$. Let $f_1, \ldots, f_n$ be iterated integrals over k. If $f_1, \ldots, f_n$ are algebraically dependent over k then there exist $u_1, \ldots, u_n \in k$, not all zero, such that*

$$u_1 f_1 + \cdots + u_n f_n \in k.$$

Remark 10. *Our proof of Proposition 9 does not use the fact that k is algebraically closed, it only requires that C be algebraically closed.*

This paper is organized as follows. Section 3 is devoted to the case where L has order 1 and contains a proof of Theorem 2 above. In Section 4 we give a result when the differential Galois group G_L of L over k is simply connected, apply this to the case when $G_L = \mathrm{SL}_n(C)$ (proving the $\mathrm{SL}_n(C)$-case of Theorem 3 above). In Section 5 we consider the case when G_L is $\mathrm{GL}_n(C)$ (proving the $\mathrm{GL}_n(C)$-case of Theorem 3 above) and in Section 6 the case when G_L is a general reductive group. We have included a full treatment of the $\mathrm{GL}_n(C)$-case before considering the general reductive case because the proof is less technical in the $\mathrm{GL}_n(C)$-case but already contains most of the ideas used in the general reductive case. In Section 7.1 we apply the results of the previous sections to generalized hypergeometric series and prove Theorem 6 above. In Section 7.2 we study the algebraic relations among iterated integrals (proving Proposition 8 and Proposition 9 above). The Appendix gives an informal introduction to the tannakian approach to the differential Galois theory - an approach that appears in several of our proofs.

2. Notations and conventions

In the whole paper, except where noted to the contrary (*e.g.*, in Section 7.2):

- K is a differential field of characteristic zero;
- k is an algebraically closed differential subfield of K;
- we assume that K and k have the same field of constants C :

$$C = \{f \in K \mid f' = 0\} = \{f \in k \mid f' = 0\}.$$

Note that C has characteristic zero and is algebraically closed because it is the field of constants of k which itself has characteristic zero and is algebraically closed (see [dPS03, Exercise 1.5.2]). Also note that

the assumption that k is algebraically closed implies that the differential Galois group of any Picard-Vessiot extension is connected [dPS03, Proposition 1.34].

We will assume that the reader is familiar with some basic notions concerning linear algebraic groups (a general reference is [Hum75]) and the basic differential Galois theory (a general reference is [Mag94] or [dPS03]). In our proofs we will also sometimes use the tannakian correspondence. Briefly this states that for a differential equation $Y' = AY$, there is a correspondence between representations of the associated differential Galois group and equations gotten from the original equations via applying the constructions of linear algebra (direct sums, tensor products, duals, submodules and quotients) to the differential module associated with $Y' = AY$. In the Appendix we give a fuller but still informal description of this. Formally this is given as an equivalence of categories; for precise details we refer to [DM81, Kat72, Kat87, dPS03]).

Except with explicit mention to the contrary, by "algebraic group", we will mean "linear algebraic group over C".

3. The first order case

We start this Section with a preliminary result concerning algebraic groups. This result will be used several times in this paper.

Lemma 11. *Let $\varphi : G_1 \to G_2$ be an isogeny between two algebraic groups G_1 and G_2. If G_1 is connected and if G_2 is a torus, then G_1 is a torus as well and G_1 and G_2 have the same dimension.*

Proof. To show that the connected group G_1 is a torus it is enough to show that it consists of semisimple elements (c.f. [Hum75, Exercise 2 in Section 21.4]). It is equivalent to showing that e_{G_1} is the unique unipotent element of G_1 (because any linear algebraic group over C contains the semisimple and the unipotent parts of the Jordan decompositions of its elements). Since the image by φ of a unipotent element of G_1 is unipotent and since the only unipotent element of G_2 is e_{G_2}, the unipotent elements of G_1 are all in $\ker \varphi$. Since $\ker \varphi$ is finite and unipotent elements have infinite order or are the iedentity, G_1 has a unique unipotent element, namely e_{G_1}.

It remains to prove that G_1 and G_2 have the same dimension. This follows from the fact that $\dim G_1 = \dim \ker \phi + \dim \operatorname{im} \phi$, [Hum75, CH. 7.3, Proposition B] ∎

Theorem 12. *Let $f \in K^\times \setminus k$ be such that*
$$f' = af$$

for some $a \in k$. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over k. We then have that f and g are algebraically dependent over k if and only if there exists θ in an algebraic extension of K and an integer n such that

$$\text{(1)} \qquad f = \theta^n \text{ and } g \in k[\theta, \theta^{-1}].$$

Proof. If f and g satisfy (1), they are clearly algebraically dependent over k.

Let us prove the converse implication. Let L be a nonzero linear differential operator with coefficients in k annihilating g and of minimal order r for this property. Let $Y' = A_L Y$ be the corresponding linear differential system.

We let E be a Picard-Vessiot extension over k for the differential system

$$\text{(2)} \qquad Y' = \begin{pmatrix} a & 0 \\ 0 & A_L \end{pmatrix} Y$$

containing f and g. Let G_E be the corresponding differential Galois group. The differential system (2) has a fundamental solution of the form

$$\mathfrak{Y} = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & g_1 & \cdots & g_r \\ 0 & g_1' & \cdots & g_r' \\ 0 & \vdots & \ddots & \vdots \\ 0 & g_1^{(r-1)} & \cdots & g_r^{(r-1)} \end{pmatrix} \in GL_{r+1}(E)$$

with $g_1 = g$ (thus, E is the field extension of k generated by the entries of $\mathfrak{Y}$). The field extension E_1 (resp. E_2) of k generated by f (resp. by the $g_i^{(j)}$) is a Picard-Vessiot extension over k for $y' = ay$ (resp. $Y' = A_L Y$); we let G_1 (resp. G_2) be the corresponding differential Galois group over k.

By minimality of L, we have

$$\operatorname{span}_C Gg = \operatorname{span}_C G_2 g = \operatorname{span}_C \{g_1, \ldots, g_r\}$$

(note that $Gg = G_2 g$ because the restriction morphisms $G \to G_2$ is surjective). Moreover, the C-line generated by f is left invariant by G. These facts, together with the fact that g is algebraic over $k(f)$, imply that the $g_i^{(j)}$ are algebraic over $k(f)$ as well, *i.e.*, E is a finite extension of E_1.

The differential Galois theory implies that the restriction morphism $G \to G_1$ is surjective and hence induces a short exact sequence

$$0 \to H \to G \to G_1 \to 0.$$

The kernel H (which is the group of differential field automorphisms of E over E_1) is finite because E is a finite extension of E_1. Moreover, G_1 is a one dimensional torus (it is $C^\times$ in the representation given by f). Lemma 11 implies that G itself is a one dimensional torus.

Therefore, there exists a basis of solutions $\widetilde{g}_1, \ldots, \widetilde{g}_r$ of L in E such that the image of the representation of the Galois group G with respect to

$$
\widetilde{\mathfrak{Y}} = \begin{pmatrix}
f & 0 & \cdots & 0 \\
0 & \widetilde{g}_1 & \cdots & \widetilde{g}_r \\
0 & \widetilde{g}_1' & \cdots & \widetilde{g}_r' \\
0 & \vdots & \ddots & \vdots \\
0 & \widetilde{g}_1^{(r-1)} & \cdots & \widetilde{g}_r^{(r-1)}
\end{pmatrix} \in \mathrm{GL}_n(E)
$$

is given by

$$
\{\mathrm{diag}(t^{n_0}, \ldots, t^{n_m}) \mid t \in C^\times\}
$$

for some $n_0, \ldots, n_m \in \mathbb{Z}$.

Therefore, if θ is such that $f = \theta^{n_0}$, then $\widetilde{g}_i = a_i \theta^{n_i}$ for some $a_i \in k^\times$ (because $(\widetilde{g}_i/\theta^{n_i})^{n_0}$ is fixed by G). This concludes the proof since g is in the C-span of the $\widetilde{g}_i$. ∎

Corollary 13. *Let $f \in K^\times$ be such that*

$$
f' = af
$$

for some $a \in k$. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over k. Assume that

$$
f = P(g)
$$

for some nonconstant polynomial P with coefficients in k. Then, there exists θ in K such that $\theta'/\theta \in k$

$$
f = \theta^n \ \text{and} \ g = c + d\theta
$$

for some $n \in \mathbb{Z}_{\geq 0}$ and some $c, d \in k$.

Proof. If f is algebraic over k, g is algebraic over k as well and, hence, belongs to k.

So, we can and will assume that f is transcendental over k. Theorem 12 ensures that there exists θ and an integer $n \in \mathbb{Z}$ such that

$$
f = \theta^n \ \text{and} \ g \in k[\theta, \theta^{-1}].
$$

Since k is algebraically closed, we may factor $P(Y)$ as a product of linear factors and, hence, the equation $P(g) = f$ can be rewritten as

$$
e_1 \prod_{i=1}^{\ell} (g - d_i) = \theta^n
$$

for some e_1 and d_i in k. Since θ is transcendental over k, the unique factorization property of $k[\theta, \theta^{-1}]$ implies that each factor in the above equality belong to $k\theta^{\mathbb{Z}}$; in particular, we have for each i

$$g - d_i = e_i \theta^{r_i}$$

for some $e_i \in k^\times$ and some $r_i \in \mathbb{Z}$. For any pair $r_i, r_j, i \neq j$, we have $d_i - d_j = e_i \theta^{r_i} - e_j \theta^{r_j}$. Since θ is transcendental, we must have $d_i = d_j := d, e_i = e_j := e$ and $r_i = r_j := r$. So $g = e\theta^r + d$ and $e, d \in k$. In particular we have that $n = rn'$ for some $n' \in \mathbb{Z}_{\geq 0}$. Whence the desired result (the θ in the statement of the Corollary is the θ^r of the proof, and the n of the statement is the n' of the proof). ∎

Corollary 14. *Let $f, g \in K$ satisfy some nonzero homogeneous linear differential equations over k. Assume that*

$$f \in k(g).$$

Then,

- *either $f \in k[g]$,*
- *or there exists θ in K such that $\theta'/\theta \in k$ and*

$$g = c + d\theta \text{ and } f \in k[\theta, \theta^{-1}]$$

for some $c, d \in k$.

Proof. Consider $R \in k(X)$ such that $f = R(g)$. Let $A, B \in k[X]$ be such that $\gcd(A, B) = 1$ and $R = A/B$. Bézout's Identity states that there exist $U, V \in k[X]$ such that $AU + BV = 1$, so $RU + V = 1/B$. It follows that $fU(g) + V(g) = R(g)U(g) + V(g) = 1/B(g)$ satisfies a nonzero homogeneous linear differential equations over k. By [HS85] (see also [Spe86], [Sin86]), we have $B(g)'/B(g) \in k$ and hence we can apply Corollary 13: if B is nonconstant then there exists θ such that

$$B(g) = \theta^n \text{ and } g = c + d\theta$$

for some $n \in \mathbb{Z}_{\geq 0}$ and some $c, d \in k$.

Of course, if B is constant, then $f \in k[g]$. ∎

4. The simply connected case

We begin with some facts concerning simply connected[1] algebraic groups.

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[1]Here, simply connected means : a connected algebraic group G not admitting any non-trivial isogeny $\phi : H \to G$ where H is also a connected algebraic group. For semisimple algebraic groups over the field of complex numbers this definition is equivalent to the topological one.

Lemma 15. *A simply connected algebraic group G has no torus quotient, that is, there is no nontrivial homomorphism $\phi : G \to T = (C^*)^n$. In particular, any multiplicative character of G is trivial.*

Proof. From [Hoc81, Chapter XVIII, Proposition 3.2], we know that the radical $R(G)$ is unipotent. Let $\phi : G \to T = (C^*)^n$ be a homomorphism. Since homomorpisms preserve unipotent elements, we must have that $R(G) \subset \text{Ker}(\phi)$. Therefore ϕ factors through $H := G/R(G)$. Since H is semisimple[2], it coincides with its derived subgroup H'. Therefore, ϕ takes its values in the derived subgroup T' of T. Since T' is trivial, we have that ϕ is trivial. $\square$

In the proof of the following proposition, we give two arguments - one based on the Picard-Vessiot approach to the Galois theory and the other based on the tannakian approach. The tannakian approach plays the dominant role in what follows and this allows the reader to compare the two approaches in this simpler situation.

Proposition 16. *Let $f \in K^\times \setminus k$ satisfy a nonzero homogeneous linear differential equation $L(f) = 0$ over k of order $n \geq 1$ and assume that the differential Galois group G_L over k of the latter equation is simply connected. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over k. Assume that f and g are algebraically dependent over k.*

1. *If F is a differential field extension of $k(f,g)$ with field of constants C containing a basis of solutions $f_1 = f, f_2, \ldots, f_n$ of L, then*
$$g \in k[W(f_1, \ldots, f_n)]$$
   where $W(f_1, \ldots, f_n)$ denotes the wronskian matrix associated to $f_1, \ldots, f_n$.
2. *Assume*

   (*) $k(f)$ is relatively algebraically closed in $k(W(f_1, \ldots, f_n))$.

   Then,
$$g \in k[f].$$

Proof.   1. Let $Y' = A_L Y$ be the linear differential system corresponding to L.

Let M be a nonzero linear differential operator with coefficients in k annihilating g and of minimal order m for this property. Let $Y' = A_M Y$ be the corresponding linear differential system.

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[2]$R(G)$ is the smallest closed normal subgroup K of G such that G/K is semisimple, cf. [Hum75, Section 19.5 and Theorem 27.5]

We let E be a Picard-Vessiot extension over k of the differential system

$$(3) \qquad Y' = \begin{pmatrix} A_L & 0 \\ 0 & A_M \end{pmatrix} Y$$

containing f and g. Let G be the corresponding differential Galois group over k. The differential system (3) has a fundamental solution of the form

$$\mathfrak{Y} = \begin{pmatrix} \mathfrak{Y}_L & 0 \\ 0 & \mathfrak{Y}_M \end{pmatrix} \in \mathrm{GL}_{m+n}(E)$$

where

$$\mathfrak{Y}_L = \begin{pmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} \in \mathrm{GL}_n(E)$$

and

$$\mathfrak{Y}_M = \begin{pmatrix} g_1 & \cdots & g_m \\ g_1' & \cdots & g_m' \\ \vdots & \ddots & \vdots \\ g_1^{(m-1)} & \cdots & g_m^{(m-1)} \end{pmatrix} \in \mathrm{GL}_m(E)$$

with $f_1 = f$ and $g_1 = g$ (thus, E is the field extension of k generated by the entries of $\mathfrak{Y}_L$ and $\mathfrak{Y}_M$). The field extension E_L (resp. E_M) of k generated by the $f_i^{(j)}$ (resp. by the $g_i^{(j)}$) is a Picard-Vessiot extension over k for $Y' = A_L Y$ (resp. $Y' = A_M Y$); we let G_L (resp. G_M) be the corresponding differential Galois group over k.

By minimality of M, we have

$$\mathrm{span}_C \, Gg = \mathrm{span}_C \, G_M g = \mathrm{span}_C \{g_1, \ldots, g_m\}$$

(note that $Gg = G_M g$ because the restriction morphism $\pi_M : G \to G_M$ is surjective). Moreover, E_L is left invariant by G. These facts, together with the fact that g is algebraic over E_L (because f and g are algebraically dependent over k and f is transcendental over k), imply that the $g_i^{(j)}$ are algebraic over E_L as well, *i.e.*, E is a finite extension of E_L.

Again, the differential Galois theory implies that the restriction morphism $\pi_L : G \to G_L$ is surjective and hence induces a short exact sequence

$$0 \to H \to G \xrightarrow{\pi_L} G_L \to 0.$$

The kernel H (which is the group of differential fields automorphisms of E over E_L) is finite because E is a finite extension of E_L.

Since G_L is a connected and simply connected algebraic group, the kernel H is trivial and $\pi_L : G \to G_L$ is an isomorphism. In particular, since H is trivial, the Galois correspondence implies that $E = E_L$. The C-span of the entries of $\mathfrak{Y}_M$ form a $G = G_L$-invariant finite dimensional C-vector space. It follows from [dPS03, Corollary 1.38] that all of these entries, and in particular $g = g_1$, lie in the Picard-Vessiot ring $k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}]$. Note that G_L leaves the C-line spanned by $\det(\mathfrak{Y}_L)$ invariant and, since all characters of G_L are trivial (see Lemma 15), it must leave $\det(\mathfrak{Y}_L)$ fixed. Therefore $\det(\mathfrak{Y}_L) \in k$ and so $k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}] = k[\mathfrak{Y}_L]$. This proves the first claim of the Proposition.

We now turn to a tannakian proof of the first claim. As already noted the fact that G_L is a connected and simply connected algebraic group implies that the kernel H is trivial and $\pi_L : G \to G_L$ is an isomorphism. It follows that there exists an algebraic group morphism $u : G_L \to G_M$ such that

$$G = \mathrm{im}(\pi_L \oplus \pi_M) = \mathrm{im}(\mathrm{id}_{G_L} \oplus u).$$

We now adopt the tannakian point of view (see Appendix A): the category of rational linear representations of G is equivalent to the tannakian category generated by the differential system (3). In this equivalence, the differential system $Y' = A_L Y$ (resp. $Y' = A_M Y$) corresponds to the restriction morphism $\pi_L : G \to G_L$ (resp. $\pi_M : G \to G_M$). The factorization

$$\pi_M = u \circ \pi_L$$

ensures that $Y' = A_M Y$ belongs to the tannakian category generated by $Y' = A_L Y$. It follows that the entries of the solutions of $Y' = A_M Y$ belong to the Picard-Vessiot ring $R_L = k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}]$ of $Y' = A_L Y$. As before G_L leaves the line spanned by $\det(\mathfrak{Y}_L)$ invariant and, since all characters of G_L are trivial (see Lemma 15), it must leave $\det(\mathfrak{Y}_L)$ fixed. Therefore $\det(\mathfrak{Y}_L) \in k$ and so $k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}] = k[\mathfrak{Y}_L]$. This yields the second proof of the first claim of the Proposition.

2. If (*) is satisfied, then g belongs to $k(f)$ (because g is algebraic over $k(f)$). Corollary 14 implies that either $g \in k[f]$, or there exists θ in K such that $\theta'/\theta \in k$ and $f = c + d\theta$ and $g \in k[\theta, \theta^{-1}]$ for some $c, d \in k$. We claim that the latter case is impossible. Indeed, otherwise the line $C\theta$ spanned by θ would be left invariant by G_L. But Lemma 15 states that any character of G_L is trivial. So G_L would act as the identity on θ. By Galois correspondance, this would imply that $\theta \in k$ and, hence, that $f \in k$. This is excluded by the assumption. ∎

In order to draw the conclusion of Proposition 17.2, one needs to verify condition (*). We will give two examples of this below but first mention a general approach.

Since we are assuming k is algebraically closed, the Galois group G of any Picard-Vessiot extension of k is connected and the Picard-Vessiot ring of any linear differential equation over k is k-isomorphic to the coordinate ring $k[G \otimes k]$ of the group G over k ([Mag94, Corollary 5.16], [dPS03, Proposition 1.31, Corollary 1.32]). In particular, this implies that the Picard-Vessiot field K is isomorphic to the function field of G. This latter field is known to be a purely transcendental extension of k ([Bor91, Remark 14.14]). Therefore, if one can find algebraically independent elements $z_1 = f, z_2, \ldots, z_m$ such that $K = k(z_1, \ldots, z_m)$ then $k(f)$ will be relatively algebraically closed in K. Note that in Proposition 17 the field $k(W(f_1, \ldots, f_n))$ is the Picard-Vessiot field of the equation $L(y) = 0$.

4.1. When $G_L = \mathrm{SL}_n(C)$.

In this section, we will prove

Proposition 17. *Let $f \in K^\times \setminus k$ satisfy an nth order nonzero homogeneous linear differential equation $L(f) = 0$ over k and assume that the differential Galois group G_L over k of this equation is $\mathrm{SL}_n(C)$ seen as a subgroup of $\mathrm{GL}_n(C)$. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over k. Assume that f and g are algebraically dependent over k. Then,*

$$g \in k[f].$$

The proof will depend on the following Lemma.

Lemma 18. *Let K, f, L, G_L, g be as in Proposition 16. Assume that G_L, seen as a subgroup of $\mathrm{GL}_n(C)$ via the basis of solutions $f_1, \ldots, f_n$, also satisfies the following property : if we let*

$$k[G_L \otimes k] = k[(X_{i,j})_{1 \le i,j \le n}, \det(X_{i,j})_{1 \le i \le n}^{-1}]/I$$

be the coordinate ring of $G_L \otimes k$, then, for any $a \in \mathrm{span}_k\{X_{i,1} \mid 1 \le i,j \le n\} \mod I$, $k(a)$ is relatively algebraically closed in $k(G_L \otimes k)$. Then,

$$g \in k[f].$$

Furthermore, this conclusion holds if its hypothesis is replaced by the following: assume that G_L, seen as a subgroup of $\mathrm{GL}_n(C)$ via some basis of solutions of L, satisfies the following property : for any $a \in \mathrm{span}_k\{X_{i,j} \mid 1 \le i,j \le n\} \mod I$, $k(a)$ is relatively algebraically closed in $k(G_L \otimes k)$.

Proof. We know that the Picard-Vessiot ring R_L of L is isomorphic, as a k-algebra, to $k[G_L \otimes k]$, more precisely (c.f. [Mag94, Corollary 5.16], [dPS03, Proposition 1.31, Corollary 1.32]) there exists $v \in \mathrm{GL}_n(k)$ such that the k-algebra morphism

$$k[(X_{i,j})_{1 \leq i,j \leq n}, \det((X_{i,j})_{1 \leq i,j \leq n})^{-1}] \to R_L$$

defined by $X \mapsto v\mathfrak{Y}_L$ induces an isomorphism

$$k[G_L \otimes k] = k[(X_{i,j})_{1 \leq i,j \leq n}, \det(X_{i,j})^{-1}_{1 \leq i,j \leq n}]/I \xrightarrow{\sim} R_L.$$

In this isomorphism, f corresponds to an element of $\mathrm{span}_k \{X_{i,1} \mid 1 \leq i \leq n\} \mod I$, so, the hypothesis on $k(G_L \otimes k)$ implies $k(f)$ is relatively algebraically closed in $k(W(f_1, \ldots, f_n))$. Proposition 16 implies the conclusion.

The final assertion follows from this. Indeed, the differential Galois group of L seen as a group of matrices via some basis of solutions of L is conjugate via some element of $\mathrm{GL}_n(C)$ to the same differential Galois group seen as a group of matrices via the basis of solutions $f_1, \ldots, f_n$. It follows that if the final hypothesis is satisfied then the previous hypothesis is satisfied as well. ∎

Remark 19. *The difference between the two parts of Lemma 18 is that, in the first part, G_L is seen as a group of matrices via the basis of solutions $f_1, \ldots, f_n$ which has $f_1 = f$ as first element, whereas, in the second, f is not required to be an element of the basis used to see G_L as a group of matrices.*

Remark 20. *The hypotheses of Lemma 18 concerning the coordinate ring of $G_L \otimes k$ are not hypotheses on the differential Galois group of L as an "abstract" algebraic group but hypotheses on the incarnation G_L of this differential Galois group as a subgroup of $\mathrm{GL}_n(C)$ via the linear representation given by the choice of a basis of solutions of L. In other words, these hypotheses are not necessarily invariant by isomorphism of algebraic groups. For instance, this hypothesis is satisfied by $G_L = \mathrm{SL}_2(C) \subset \mathrm{GL}_2(C)$, but not by $\mathrm{Sym}^3(\mathrm{SL}_2(C)) \subset \mathrm{GL}_4(C)$, despite the fact that these two algebraic groups are isomorphic. The fact that the hypothesis is satisfied when $G_L = \mathrm{SL}_2(C)$ is shown in the proof of Proposition 17 below. Let us prove that it is not satisfied by $G_L = \mathrm{Sym}^3(\mathrm{SL}_2(C))$. We have denoted by $\mathrm{Sym}^3(\mathrm{SL}_2(C))$ the image of $\mathrm{SL}_2(C)$ by its (faithful) 3rd symmetric power representation*

$$\mathrm{Sym}^3 : \mathrm{SL}_2(C) \to \mathrm{GL}_4(C),$$

which has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & * & * & b^2d \\ ac^2 & * & * & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix}.$$

Let

$$k[\mathrm{Sym}^3(\mathrm{SL}_2(C)) \otimes k] = k[(X_{i,j})_{1 \leq i,j \leq 3}, \det(X_{i,j})_{1 \leq i,j \leq 3}^{-1}]/I$$
$$= k[(x_{i,j})_{1 \leq i,j \leq 3}, \det(x_{i,j})_{1 \leq i,j \leq 3}^{-1}]$$

be the coordinate ring of $\mathrm{Sym}^3(\mathrm{SL}_2(C))$. *Then,*

$$f = 3^{-1}x_{1,2}(x_{2,4}/x_{1,4} - x_{2,1}/x_{1,1}) \in k(\mathrm{Sym}^3(\mathrm{SL}_2(C)) \otimes k)$$

satisfies $f^3 = x_{1,1}$. *Therefore,* f *is algebraic over* $k(x_{1,1})$ *but does not belong to* $k(x_{1,1})$ *and, hence,* $k(x_{1,1})$ *is not relatively algebraically closed in* $k(\mathrm{Sym}^3(\mathrm{SL}_2(C)) \otimes k)$.

This is not a surprise: the conclusion of Proposition 17 is not satisfied when $G_L = \mathrm{Sym}^3(\mathrm{SL}_2(C))$. Indeed, let M be a linear differential equation over k of order 2 with differential Galois group $\mathrm{SL}_2(C)$. Consider the third symmetric power $L = M^{\circledS 3}$; this is a linear differential equation over k of order 4 with differential Galois group $G_L = \mathrm{Sym}^3(\mathrm{SL}_2(C))$. If (u,v) is a basis of solutions of M, then (u^3, u^2v, uv^2, v^3) is a basis of solutions of L. Therefore, $f = u^3$ is a solution of L, $g = u$ is a solution of M, $f \notin k$ and f and g are algebraically dependent over k but $g \notin k(f)$.

Proof of Proposition 17. We shall prove that the hypotheses of Lemma 18 are satisfied when $G_L = \mathrm{SL}_n(C)$. The fact that $\mathrm{SL}_n(C)$ is connected and simply connected is well-known (cf. [Hum75, Section 31.1]). It remains to prove that, if we denote by

$$k[\mathrm{SL}_n(C) \otimes k] = k[(X_{i,j})_{1 \leq i,j \leq n}, \det(X_{i,j})_{1 \leq i,j \leq n}^{-1}]/(\det(X_{i,j})_{1 \leq i,j \leq n} - 1)$$
$$= k[(x_{i,j})_{1 \leq i,j \leq n}]$$

the coordinate ring of $\mathrm{SL}_n(C) \otimes k$, we have: for any

$$a = \sum_{1 \leq i,j \leq n} \lambda_{i,j} x_{i,j} \in \mathrm{span}_k\{x_{i,j} \mid 1 \leq i,j \leq n\},$$

$k(a)$ is relatively algebraically closed in $k(\mathrm{SL}_n(C) \otimes k)$.

Let us first assume that at least one of the $\lambda_{i,j}$ is zero, say $\lambda_{i_0,j_0} = 0$. Then, we have

$$k(\mathrm{SL}_n(C) \otimes k) = k((x_{i,j})_{1 \leq i,j \leq n})$$
$$= k((x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0)}) = k(a, (x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0), (i_1,j_1)})$$

where (i_1, j_1) is chosen such that $\lambda_{i_1,j_1} \neq 0$. The second equality above follows from that fact $x_{i_0,j_0} \in k((x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0)})$ which itself follows from the equality $\det(x_{i,j})_{1 \leq i,j \leq n} = 1$. The third equality follows from the fact that x_{i_1,j_1} belongs to the k-span of a and $(x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0), (i_1,j_1)}$. Since

$$\mathrm{tr.}\deg(k(\mathrm{SL}_n(C) \otimes k)/k) = n^2 - 1$$

and

$$\sharp\{a, (x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0), (i_1,j_1)}\} \leq n^2 - 1,$$

we get that the family $(a, (x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0), (i_1,j_1)})$ is algebraically independent over k. In particular, $k(a)$ is algebraically closed in $k(\mathrm{SL}_n(C) \otimes k)$.

Let us now assume that the $\lambda_{i,j}$ are nonzero. We have

$$k(\mathrm{SL}_n(C) \otimes k) = k((x_{i,j})_{1 \leq i,j \leq n}) = k(a, (x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (1,1)}).$$

The latter equality follows from the fact that $x_{1,1}$ belongs to the k-span of a and $(x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (1,1)}$. But, we have

$$x_{2,1} = \lambda_{2,1}^{-1} a - \sum_{\substack{1 \leq i,j \leq n \\ (i,j) \neq (1,1),(2,1)}} \lambda_{2,1}^{-1} \lambda_{i,j} x_{i,j} - \lambda_{2,1}^{-1} \lambda_{1,1} x_{1,1}.$$

The equality $\det(x_{i,j})_{1 \leq i,j \leq n} = 1$ ensures that

$$x_{1,1} = -\frac{\delta_{2,1}}{\delta_{1,1}} x_{2,1} - \sum_{i=3}^{n} \frac{\delta_{i,1}}{\delta_{1,1}} x_{i,1} + \frac{1}{\delta_{1,1}}$$

where $\delta_{i,j}$ is the (i, j) cofactor of the matrix $(x_{i,j})_{1 \leq i,j \leq n}$. It follows that

$$\left(1 - \lambda_{2,1}^{-1} \lambda_{1,1} \frac{\delta_{2,1}}{\delta_{1,1}}\right) x_{2,1} =$$

$$\lambda_{2,1}^{-1} a - \sum_{\substack{1 \leq i,j \leq n \\ (i,j) \neq (1,1),(2,1)}} \lambda_{2,1}^{-1} \lambda_{i,j} x_{i,j} + \lambda_{2,1}^{-1} \lambda_{1,1} \left(\sum_{i=3}^{n} \frac{\delta_{i,1}}{\delta_{1,1}} x_{i,1} - \frac{1}{\delta_{1,1}}\right).$$

This shows that $x_{2,1}$ belongs to $k(a, (x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (1,1),(2,1)})$ and, hence, that

$$k(\mathrm{SL}_n(C) \otimes k) = k(a, (x_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (1,1),(1,2)}).$$

We can now conclude that $k(a)$ is algebraically closed in $k(\mathrm{SL}_n(C) \otimes k)$ by using a transcendence degree argument as we did above. ∎

5. A special case of reductive G_L: $\mathrm{GL}_n(C)$

Lemma 21. *Let $\varphi : G_1 \to G_2$ be an isogeny with G_1 connected. Then:*

- $\varphi^{-1}(Z_{G_2}) = Z_{G_1}$, $\varphi(Z_{G_1}) = Z_{G_2}$ and $\varphi(Z^0_{G_1}) = Z^0_{G_2}$;
- *if $G_2 = Z^0_{G_2} G'_2$, then $G_1 = Z^0_{G_1} G'_1$*[3];
- *if $Z^0_{G_2}$ is a torus of dimension d, then $Z^0_{G_1}$ is a torus of dimension d as well;*
- *if G'_2 is semi-simple, then G'_1 is semi-simple as well;*
- *if G_2 is reductive, then G_1 is reductive as well.*

Proof. We first prove that $\varphi^{-1}(Z_{G_2}) \subset Z_{G_1}$. Consider $g \in \varphi^{-1}(Z_{G_2})$. For any $h \in G_1$, $\varphi([g,h]) = [\varphi(g), \varphi(h)] = e_{G_2}$, so $[g,h] \in \ker \varphi$. Since G_1 is connected, the image the morphism of algebraic varieties

$$G_1 \quad \to \quad \ker \varphi$$
$$h \quad \mapsto \quad [g,h]$$

is contained in $(\ker \varphi)^0 = \{e_{G_1}\}$, so $g \in Z_{G_1}$ as expected.

The inclusion $Z_{G_1} \subset \varphi^{-1}(Z_{G_2})$ follows from the fact that φ is onto and therefore $\varphi^{-1}(Z_{G_2}) = Z_{G_1}$.

The equality $\varphi(Z_{G_1}) = Z_{G_2}$ follows form this and from the fact that φ is onto.

It follows that $\varphi(Z^0_{G_1}) = \varphi(Z_{G_1})^0 = Z^0_{G_2}$.

Since φ is onto, we have $\varphi(G'_1) = G'_2$. We have seen that $\varphi(Z^0_{G_1}) = Z^0_{G_2}$. Therefore, $\varphi(Z^0_{G_1} G'_1) = Z^0_{G_2} G'_2 = G_2$. Since $\ker \varphi$ is central, we get $G_1 \subset (\ker \varphi) Z^0_{G_1} G'_1 \subset Z_{G_1} G'_1$, so $G_1 = Z_{G_1} G'_1$. But $Z^0_{G_1} G'_1$ is a closed subgroup of $G_1 = Z_{G_1} G'_1$ of finite index, so $G^0_1 = G_1 \subset Z^0_{G_1} G'_1$, whence the equality $G_1 = Z^0_{G_1} G'_1$.

Assume that $Z^0_{G_2}$ is a torus of dimension d. We have seen that $\varphi(Z^0_{G_1}) = Z^0_{G_2}$, so φ induces an isogeny between $Z^0_{G_1}$ and $Z^0_{G_2}$. Lemma 11 implies that $Z^0_{G_1}$ is a torus of dimension d.

Since $\varphi(G'_1) = G'_2$, φ induces an isogeny between G'_1 and G'_2. Therefore, G'_1 is semi-simple if and only if G'_2 is semi-simple.

The last assertion follows from the previous one. ∎

In the proof of the next result we will use the tannakian correspondence. Appendix A contains a review of this tool.

Proposition 22. *Let $f \in K^\times$ satisfy a nonzero homogeneous linear differential equation over k of order n and assume that the differential*

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[3]Recall that G' is the derived subgroup of the algebraic group G.

Galois group over k of the latter equation is $\mathrm{GL}_n(C)$. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over k. Assume that f and g are algebraically dependent over k. We have:

(1) *if $n = 1$, then there exist θ and a relative integer n such that*

$$f = \theta^n \ and \ g \in k[\theta, \theta^{-1}];$$

(2) *if $n > 1$, then*

$$g \in k[f].$$

Proof.   Let L be a nonzero linear differential operator of order n with coefficients in k and with differential Galois group over k equal to $\mathrm{GL}_n(C)$ annihilating f. Let $Y' = AY$ be the corresponding linear differential system.

Let M be a nonzero linear differential operator with coefficients in k annihilating g and of minimal order m for this property. Let $Y' = BY$ be the corresponding linear differential system.

We let E be a Picard-Vessiot extension over k of the differential system

$$(4) \qquad\qquad Y' = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Y$$

containing f and g. Let G be the corresponding differential Galois group over k. The differential system (4) has a fundamental solution of the form

$$\mathfrak{Y} = \begin{pmatrix} \mathfrak{Y}_A & 0 \\ 0 & \mathfrak{Y}_B \end{pmatrix} \in \mathrm{GL}_{m+n}(E)$$

where

$$\mathfrak{Y}_A = \begin{pmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} \in \mathrm{GL}_n(E)$$

and

$$\mathfrak{Y}_B = \begin{pmatrix} g_1 & \cdots & g_m \\ g_1' & \cdots & g_m' \\ \vdots & \ddots & \vdots \\ g_1^{(m-1)} & \cdots & g_m^{(m-1)} \end{pmatrix} \in \mathrm{GL}_m(E)$$

with $f_1 = f$ and $g_1 = g$ (thus, E is the field extension of k generated by the entries of $\mathfrak{Y}_A$ and $\mathfrak{Y}_B$). The field extension E_A (resp. E_B) of k generated by the $f_i^{(j)}$ (resp. by the $g_i^{(j)}$) is a Picard-Vessiot extension over k for $Y' = AY$ (resp. $Y' = BY$); we let G_A (resp. G_B) be the corresponding differential Galois group over k.

By minimality of M, we have

$$\mathrm{span}_C \, Gg = \mathrm{span}_C \, G_B g = \mathrm{span}_C \{g_1, \ldots, g_m\}$$

(note that $Gg = G_B g$ because the restriction morphism $G \to G_B$ is surjective). Moreover, E_A is left invariant by G. These facts, together with the fact that g is algebraic over E_A (because f and g are algebraically dependent over k and f is transcendental over k because G_A is $\mathrm{GL}_n(C)$) imply that the $g_i^{(j)}$ are algebraic over E_A as well, *i.e.*, E is a finite extension of E_A.

It is a general fact that the restriction morphism $\pi_A : G \to G_A$ is surjective and hence induces a short exact sequence

$$0 \to H \to G \xrightarrow{\pi_A} G_A \to 0.$$

The kernel H (which is the group of differential field automorphisms of E over E_A) is finite because E is a finite extension of E_A.

Since k is algebraically closed, G is connected. Applying Lemma 21 to $\pi_A : G \to G_A$ (with $G_A = \mathrm{GL}_n(C) = Z^0_{G_A} G'_A$ with $Z^0_{G_A} = C^\times I_n$ and $G'_A = \mathrm{SL}_n(C)$), we get

$$G = Z_G G' = Z^0_G G'$$

and also that Z^0_G is a one dimensional torus. There exists $P \in \mathrm{diag}(I_n, \mathrm{GL}_m(C)) \subset \mathrm{GL}_{n+m}(C)$ such that

$$PZ^0_G P^{-1} = \{\mathrm{diag}(t^{\alpha_0} I_n, t^{\alpha_1} I_{m_1}, \ldots, t^{\alpha_\ell} I_{m_\ell}) \mid t \in C^\times\}$$

for some $m_1, \ldots, m_\ell \in \mathbb{Z}_{\geq 1}$ and some $\alpha_0, \ldots, \alpha_\ell \in \mathbb{Z}$ with $\alpha_1, \ldots, \alpha_\ell$ two by two distinct. Therefore, PGP^{-1} is diagonal by block:

$$PGP^{-1} \subset \mathrm{diag}(\mathrm{GL}_n(C), \mathrm{GL}_{m_1}(C), \ldots, \mathrm{GL}_{m_\ell}(C)).$$

It follows that there exist $B_1 \in M_{m_1}(k), \ldots, B_\ell \in M_{m_\ell}(k)$ such that $Y' = BY$ is equivalent over k to

$$Y' = \mathrm{diag}(B_1, \ldots, B_\ell)Y.$$

So, up to changing $\mathfrak{Y}_B$ by $R\mathfrak{Y}_B$ for some $R \in \mathrm{GL}_m(k)$, we can and will assume that $B = \mathrm{diag}(B_1, \ldots, B_\ell)$ and that $P = I_n$.

Let $\widetilde{A} = A - \mathrm{tr}(A)/n I_n$ and consider the differential system

(5) $$Y' = \mathrm{diag}(\mathrm{tr}(A)/n I_n, \widetilde{A}, B_1, \ldots, B_\ell)Y.$$

Since $\mathrm{tr}(\widetilde{A}) = 0$, the differential Galois group of $\widetilde{A}$ is unimodular ([dPS03, Exercise 1.35.5]). We let H be the differential Galois group of (5). In the tannakian correspondance (cf., Example 35), the equation

$$Y' = \mathrm{diag}(A, B_1, \ldots, B_\ell)Y$$

corresponds to the representation $\rho : H \to G$ given by

$$\rho : \mathrm{diag}(t, \widetilde{h}, h_1, \ldots, h_l) \mapsto \mathrm{diag}(t\widetilde{h}, h_1, \ldots, h_\ell).$$

Applying Lemma 21 to $\rho : H \to G$ (again the differential Galois theory implies that ρ is onto, and that it has finite kernel), we have

$$H = Z_H^0 H'$$

and $Z_H^0 = \rho^{-1}(Z_G^0)^0$ is a one dimensional torus. The restriction

$$\rho_{|Z_H^0} : Z_H^0 = \rho^{-1}(Z_G^0)^0 \to Z_G^0$$

is given by

$$h \mapsto (\chi_0(h)I_n, \chi_1(h)I_{m_1}, \ldots, \chi_\ell(h)I_{m_\ell})$$

for some characters $\chi_i : Z_H^0 \to C^\times$. Another morphism $Z_H^0 \to Z_G^0$ is given by

$$h = \mathrm{diag}(t, \widetilde{h}, h_1, \ldots, h_\ell) \mapsto \mathrm{diag}(t^{\alpha_0} I_n, t^{\alpha_1} I_{m_1}, \ldots, t^{\alpha_\ell} I_{m_\ell}).$$

Therefore (since Z_K^0 and Z_G^0 are one dimensional tori: see Lemma 24), there exist nonzero relatively prime integers N, M such that, for all i,

$$\chi_i(h)^N = t^{\alpha_i M}.$$

Consider the differential system

$$Y' = \mathrm{diag}(\mathrm{tr}(A)/nN, \widetilde{A}, B_1, \ldots, B_\ell)Y.$$

Let H_N be the differential Galois group of this system. In the tannakian correspondance, the equation

$$Y' = \mathrm{diag}(A, B_1, \ldots, B_\ell)Y$$

corresponds to

$$\varrho := \rho \circ p : H_N \to G$$

where $p : H_N \to H$ (which corresponds to $Y' = \mathrm{diag}(\mathrm{tr}(A)/n, \widetilde{A}, B_1, \ldots, B_\ell)Y$ in the tannakian correspondence) is given by

$$p : \mathrm{diag}(z, \widetilde{h}, h_1, \ldots, h_l) \mapsto \mathrm{diag}(z^N, \widetilde{h}, h_1, \ldots, h_\ell).$$

Applying Lemma 21 to $p : H_N \to H$, we get

$$H_N = Z_{H_N}^0 H_N'$$

and $Z_{H_N}^0 = p^{-1}(Z_H^0)^0$ is a one dimensional torus.

If $h = \mathrm{diag}(z, \widetilde{h}, h_1, \ldots, h_\ell) \in Z_{H_N}^0 = p^{-1}(Z_H^0)^0$, then

$$\chi_i(p(h))^N = z^{\alpha_i NM}$$

so

$$\chi_i(p(h)) = z^{\alpha_i M}.$$

So, we have

$$\varrho_{|Z_{H_N}^0} : (z, \widetilde{h}, h_1, \ldots, h_\ell) \mapsto (z^N I_n = z^{\alpha_0 M} I_n, z^{\alpha_1 M} I_{m_1}, \ldots, z^{\alpha_\ell M} I_{m_\ell}).$$

On the other hand, the representation corresponding to $Y' = AY$ in the tannakian correspondence is

$$\varpi_A = \pi_A \circ \varrho : H_N \to G_A.$$

It is surjective (differential Galois theory) and has finite kernel (because ϱ and π_A have finite kernels), so it is an isogeny. Therefore, it induces an isogeny

$$\varpi_{A|H'_N} = \pi_A \circ \varrho : H'_N \to G'_A.$$

Since $G'_A = \mathrm{SL}_n(C)$ is a connected and simply connected algebraic group,

$$\varpi_{A|H'_N} = \pi_A \circ \varrho : H'_N \to G'_A.$$

is actually an isomorphism. Since $G = \mathrm{im}(\pi_A \circ \varrho \oplus \pi_B \circ \varrho)$, it follows that there exists an algebraic group morphism $u = (u_1, \ldots, u_\ell) : G'_A \to G_B$ such that

$$\pi_B \circ \varrho = u \circ (\pi_A \circ \varrho).$$

So,

$$\varrho_{|H'_N} : (1, \widetilde{h}, h_1, \ldots, h_\ell) \mapsto (\widetilde{h}, u_1(\widetilde{h}), \ldots, u_\ell(\widetilde{h})).$$

Finally, we see that the representation $\varpi_B = \pi_B \circ \varrho$ corresponding to $Y' = BY$ in the tannakian correspondence is given by

$$\varpi_B = \pi_B \circ \varrho : (z, \widetilde{h}, h_1, \ldots, h_l) \mapsto (z^{\alpha_1 M} u_1(\widetilde{h}), \ldots, z^{\alpha_\ell M} u_\ell(\widetilde{h})).$$

This ensures, by the tannakian correspondance, that $Y' = B_i Y$ is equivalent to a system of the form $Y' = (\mathrm{Constr}_i(\widetilde{A}) + \alpha_i M (\mathrm{tr}(A)/nN) I_{m_i}) Y$ where $\mathrm{Constr}_i(\widetilde{A})$ is obtained from $\widetilde{A}$ by some construction of linear algebra. It follows that the entires of the solutions of $Y' = BY$ belong to $k[\mathfrak{Y}_L, \sqrt[q]{\det(\mathfrak{Y}_L)}^{-1}]$ for some positive integer q; in particular, g belongs to $k[\mathfrak{Y}_L, \sqrt[q]{\det(\mathfrak{Y}_L)}^{-1}]$.

For $n = 1$, this gives the desired result (which, of course, already follows from Corollary 13).

Let us now assume $n > 1$. We have that $R_L = k[\mathfrak{Y}_L, \det(\mathfrak{Y}_L)^{-1}]$ is isomorphic, as a k-algebra, to $k[\mathrm{GL}_n(k)]$. More precisely there exists $v \in \mathrm{GL}_n(k)$ such that the k-algebra morphism

$$k[(X_{i,j})_{1 \leq i,j \leq n}, \det((X_{i,j})_{1 \leq i,j \leq n})^{-1}] \to R_L$$

defined by $X \mapsto v\mathfrak{Y}_L$ is an isomorphism; it induces an isomorphism

$$k[(X_{i,j})_{1 \leq i,j \leq n}, \sqrt[q]{\det((X_{i,j})_{1 \leq i,j \leq n})}^{-1}] \to k[\mathfrak{Y}_L, \sqrt[q]{\det(\mathfrak{Y}_L)}^{-1}]$$

In this isomorphism, f corresponds to an element a of $\mathrm{span}_k\{X_{i,1} \mid 1 \leq i \leq n\}$ mod I. The fact that $k(a)$ is relatively algebraically closed in $k((X_{i,j})_{1 \leq i,j \leq n}, \sqrt[q]{\det((X_{i,j})_{1 \leq i,j \leq n})}^{-1})$ (see Lemma 23 below) implies that

$$g \in k(f).$$

Corollary 14 ensures that:

- either $g \in k[f]$,
- or there exists θ such that $\theta'/\theta \in k$ and

$$f = c + d\theta \text{ and } g \in k[\theta, \theta^{-1}]$$

for some $c, d \in k$.

The latter possibility is incompatible with the fact that $G_L = \mathrm{GL}_n(C)$ acts irreducibly on the solution space of L. ∎

Lemma 23. *For any $a \in \mathrm{span}_k\{X_{i,j} \mid 1 \leq i, j \leq n\}$, $k(a)$ is relatively algebraically closed in*

$$k((X_{i,j})_{1 \leq i,j \leq n}, \Delta^{-1}) \text{ where } \Delta = \sqrt[q]{\det((X_{i,j})_{1 \leq i,j \leq n})}.$$

Proof. Let

$$L = k((X_{i,j})_{1 \leq i,j \leq n}, \Delta).$$

and let $a = \sum \lambda_{i,j} X_{i,j}$ be in the k-span of the $X_{i,j}$.

Let us first assume at least one of the $\lambda_{i,j}$ is zero, say $\lambda_{i_0,j_0} = 0$. Then, we have

$$L = k((X_{i,j})_{1 \leq i,j \leq n}, \Delta) = k((X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0)}, \Delta)$$
$$= k(a, (X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0), (i_1,j_1)}, \Delta)$$

where (i_1, j_1) is such that $\lambda_{i_1,j_1} \neq 0$. The second equality follows from the fact that $X_{i_0,j_0} \in k((X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0)}, \Delta)$ which itself follows from the equality

$$\det(X_{i,j})_{1 \leq i,j \leq n} = \sum_{j=1}^{n} X_{i_0,j} \delta_{i_0,j} = \Delta^q$$

where $\delta_{i,j}$ is the (i,j) cofactor of $(X_{i,j})_{1 \leq i,j \leq n}$. The third equality follows from the fact that X_{i_1,j_1} belongs to the k-span of a and $(X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0), (i_1,j_1)}$. Since

$$\mathrm{tr.\,deg}(L/k) = n^2 \text{ and } \sharp\{a, (X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0), (i_1,j_1)}, \Delta\} \leq n^2,$$

we get that the family $(a, (X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (i_0,j_0), (i_1,j_1)}, \Delta)$ is algebraically independent over k. In particular, $k(a)$ is algebraically closed in L.

Let us now assume that the $\lambda_{i,j}$ are nonzero. We have

$$L = k((X_{i,j})_{1 \leq i,j \leq n}, \Delta) = k(a, (X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (1,1)}, \Delta)$$

The latter equality follows from the fact that $X_{1,1}$ belongs to the k-span of a and $(X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (1,1)}$. We have

$$X_{2,1} = \lambda_{2,1}^{-1} a - \sum_{\substack{1 \leq i,j \leq n \\ (i,j) \neq (1,1),(2,1)}} \lambda_{2,1}^{-1} \lambda_{i,j} X_{i,j} - \lambda_{2,1}^{-1} \lambda_{1,1} X_{1,1}.$$

Furthermore the equality $\det(X_{i,j})_{1 \leq i,j \leq n} = \Delta^q$ ensures that

$$X_{1,1} = -\frac{\delta_{2,1}}{\delta_{1,1}} X_{2,1} - \sum_{i=3}^{n} \frac{\delta_{i,1}}{\delta_{1,1}} X_{i,1} + \frac{\delta^q}{\delta_{1,1}}$$

where $\delta_{i,j}$ is the (i,j) cofactor of $(X_{i,j})_{1 \leq i,j \leq n}$. It follows that

$$\left(1 + \lambda_{2,1}^{-1} \lambda_{1,1} \frac{\delta_{2,1}}{\delta_{1,1}} \right) X_{2,1} =$$

$$\lambda_{2,1}^{-1} a - \sum_{\substack{1 \leq i,j \leq n \\ (i,j) \neq (1,1),(2,1)}} \lambda_{2,1}^{-1} \lambda_{i,j} X_{i,j} + \lambda_{2,1}^{-1} \lambda_{1,1} \left(\sum_{i=3}^{n} \frac{\delta_{i,1}}{\delta_{1,1}} X_{i,1} - \frac{\delta^q}{\delta_{1,1}} \right).$$

This shows that $X_{2,1}$ belongs to $k((X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (1,1),(2,1)}, \Delta)$ and, hence, that

$$L = k(a, (X_{i,j})_{1 \leq i,j \leq n, (i,j) \neq (1,1),(1,2)}, \Delta).$$

We can now conclude as above by a transcendence degree argument. ∎

6. The general reductive case

Lemma 24. *Let T_1 and T_2 be tori of dimension d. Let $\phi, \varphi : T_1 \to T_2$ be two isogenies. Then, there exist $N \in \mathbb{Z}_{\geq 1}$ and an isogeny $\psi : T_2 \to T_2$ such that $\varphi^N = \psi \circ \phi$.*

Proof. Using the fact that T_1 and T_2 are isomorphic to $(C^\times)^d$, we can assume that $T = T_1 = T_2 = (C^\times)^d$. We let $X(T)$ be the character group of T; it is a free abelian group of rank d. We set $\phi = (\phi_1, \ldots, \phi_d)$ and $\varphi = (\varphi_1, \ldots, \varphi_d)$; the ϕ_i and φ_i are thus elements of $X(T)$. Since $\phi : T \to T$ is an isogeny, $\phi_1, \ldots, \phi_d$ are multiplicatively independent elements of $X(T)$ and, hence, they generate a free subgroup $\langle \phi_1, \ldots, \phi_d \rangle$ of rank d of $X(T)$. Thus, every element of $X(T)$ has a positive power belonging to $\langle \phi_1, \ldots, \phi_d \rangle$. In particular, there exist a positive integer N such that the φ_i^N belong to $\langle \phi_1, \ldots, \phi_d \rangle$. Therefore, there exists $\psi_i \in X(T)$ such that $\varphi_i^N = \psi_i \circ \phi$. Thus, the morphism $\psi = (\psi_1, \ldots, \psi_d) : T \to T$ is such that $\varphi^N = \psi \circ \phi$. Since φ^N is an isogeny, ψ is an isogeny as well. ∎

Proposition 25. *Let $f \in K^\times \setminus k$ satisfy a nonzero homogeneous linear differential equation $L(f) = 0$ over k of order n and assume that the differential Galois group over k of the latter equation is reductive and has a simply connected derived subgroup. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over k. Assume that f and g are algebraically dependent over k.*

(1) *If F is a differential field extension of $k(f, g)$ with field of constants C containing a basis of solutions $f_1 = f, f_2, \ldots, f_n$ of L, then*

$$g \in k[W(f_1, \ldots, f_n), y_1^{-1/q}, \ldots, y_s^{-1/q}]$$

where $W(f_1, \ldots, f_n)$ denotes the wronskian matrix associated to $f_1, \ldots, f_n$, the $y_i \in k[W(f_1, \ldots, f_n)]$ are such that $y_i'/y_i \in k$ and q is a positive integer.

(2) *Assume*

> *(*) $k(f)$ is relatively algebraically closed in $k(W(f_1, \ldots, f_n), y_1^{-1/q}, \ldots, y_s^{-1/q})$.*

Then,

$$g \in k[f].$$

Remark 26. *Actually, the proof of Proposition 25 gives the following more precise informations. We let $L = L_1 \cdots L_r$ be a factorization of L over k into irreducible factors. The proof of Proposition 25 shows that $s \leq r$ and that the y_i can be chosen such that y_i are solutions of $y' = \mathrm{tr}(A_i)y$ where A_i is a certain matrix gauge equivalent to the companion matrix of $L_i(Y) = 0$. In particular, with these refinements in mind, we recover the main ingredients of the proof of Proposition 22 (the only additional ingredient of the proof of Proposition 22 is the fact that the hypothesis (*) is satisfied under the assumptions of Proposition 22).*

Proof. Let $Y' = AY$ be the linear differential system corresponding to L. Its differential Galois group over k (the same as L) being reductive, we have that $Y' = AY$ is equivalent over k to

$$Y' = \overline{A}Y, \quad \overline{A} = \mathrm{diag}(A_1, \ldots, A_s)$$

for some $A_i \in M_{n_i}(k)$ such that $Y' = A_iY$ is irreducible over k.

Let M be a nonzero linear differential operator with coefficients in k annihilating g and of minimal order m for this property. Let $Y' = BY$ be the corresponding linear differential system.

We let E be a Picard-Vessiot extension over k of the differential system

(6)
$$Y' = \begin{pmatrix} \overline{A} & 0 \\ 0 & B \end{pmatrix} Y$$

containing f and g. Let G be the corresponding differential Galois group over k. The differential system (6) has a fundamental solution of the form

$$\mathfrak{Y} = \begin{pmatrix} \mathfrak{Y}_{\overline{A}} & 0 \\ 0 & \mathfrak{Y}_B \end{pmatrix} \in \mathrm{GL}_{m+n}(E)$$

where $\mathfrak{Y}_{\overline{A}}$ is a fundamental solution of $Y' = \overline{A}Y$ and $\mathfrak{Y}_B$ is a fundamental solution of $Y' = BY$ (thus, E is the field extension of k generated by the entries of $\mathfrak{Y}_{\overline{A}}$ and $\mathfrak{Y}_B$). The field extension $E_{\overline{A}}$ (resp. E_B) of k generated by the entries of $\mathfrak{Y}_{\overline{A}}$ (resp. by the $\mathfrak{Y}_B$) is a Picard-Vessiot extension over k for $Y' = \overline{A}Y$ (resp. $Y' = BY$); we have $f \in E_{\overline{A}}$ and $g \in E_B$. We let $G_{\overline{A}}$ (resp. G_B) be the corresponding differential Galois group over k. We recall that $G_{\overline{A}}$ is reductive with simply connected derived subgroup $G'_{\overline{A}}$; we will denote by d the dimension of the torus $Z^0_{G_{\overline{A}}}$.

By minimality of M, we have that E_B is generated, as a differential field extension of k, by $Gg = G_Bg$ (we recall that $Gg = G_Bg$ because the restriction morphism $G \to G_B$ is surjective). Moreover, $E_{\overline{A}}$ is left invariant by G. These facts, together with the fact that g is algebraic over E_A, imply that E is a finite extension of $E_{\overline{A}}$.

We again have that the restriction morphism $\pi_{\overline{A}} : G \to G_{\overline{A}}$ is surjective and hence induces a short exact sequence

$$0 \to J \to G \xrightarrow{\pi_{\overline{A}}} G_{\overline{A}} \to 0.$$

The kernel J (which is the group of differential fields automorphisms of E over $E_{\overline{A}}$) is finite because E is a finite extension of $E_{\overline{A}}$.

Applying Lemma 21 to $\pi_{\overline{A}} : G \to G_{\overline{A}}$, we get that G is reductive and that Z^0_G is a torus of dimension d.

Let

$$\widetilde{A} = \overline{A} - \mathrm{diag}(\mathrm{tr}(A_1)/n_1 I_{n_1}, \dots, \mathrm{tr}(A_s)/n_s I_{n_s})$$

and consider the differential system

$$Y' = \mathrm{diag}(\mathrm{tr}(A_1)/n_1, \dots \mathrm{tr}(A_s)/n_s, \widetilde{A}, B)Y.$$

We let H be the differential Galois group over k of this equation. In the tannakian correspondance, the equation (6) corresponds to the representation $\rho : H \to G$ given by

$$\rho : \mathrm{diag}(c_1, \dots, c_s, \widetilde{g}_1, \dots, \widetilde{g}_s, g) \mapsto \mathrm{diag}(c_1\widetilde{g}_1, \dots, c_s\widetilde{g}_s, g).$$

Applying Lemma 21 to the isogeny $\rho : H \to G$ (as before, ρ is onto, and it is clear that it has finite kernel because the $\widetilde{g}_i$ have determinant 1), we have that H is reductive and that $Z_H^0 = \rho^{-1}(Z_G^0)^0$ is a torus of dimension d. We claim that there exists an isogeny $\Xi : Z_H^0 \to Z_G^0$ of the form

$$\text{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \Xi(c_1, \ldots, c_s),$$

an isogeny $\psi : Z_G^0 \to Z_G^0$ and a positive integer N such that

$$\rho_{|Z_H^0}^N = \psi \circ \Xi.$$

Indeed, the image of Z_H^0 by

$$\text{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \text{diag}(c_1, \ldots, c_s)$$

is a torus of dimension d (this is because $\pi_{\overline{A}} \circ \rho : H \to G_{\overline{A}}$, which is given by

$$\text{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \text{diag}(c_1 \widetilde{g}_1, \ldots, c_s \widetilde{g}_s)$$

is an isogeny and, hence, induces an isogeny $Z_H \to Z_{G_{\overline{A}}}$, but $Z_{G_{\overline{A}}}$ is made of diagonal matrices of the form

$$\text{diag}(*I_{n_1}, \ldots, *I_{n_s}),$$

and the $\widetilde{g}_i$ have determinant 1, so $(\pi_{\overline{A}} \circ \rho_{|Z_H})^{n_1 \cdots n_s}$ can be described as follows :

$$(\pi_{\overline{A}} \circ \rho_{|Z_H})^{n_1 \cdots n_s} : \text{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \text{diag}(c_1 I_{n_1}, \ldots, c_s I_{n_s})^{n_1 \cdots n_s}$$

and hence the image of Z_H by

$$\text{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \text{diag}(c_1 I_{n_1}, \ldots, c_s I_{n_s})$$

has the same dimension as the image of $\pi_{\overline{A}} \circ \rho_{|Z_H}$, which is equal to the dimension of $Z_{G_{\overline{A}}}$), so there exists an isogeny $\Xi : Z_H^0 \to Z_G^0$ of the form

$$\text{diag}(c_1, \ldots, c_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \Xi(c_1, \ldots, c_s).$$

But $\rho_{|Z_H^0} : Z_H^0 \to Z_G^0$ is also an isogeny, so Lemma 24 ensures that there exist a positive integer N and an isogeny $\psi : Z_G^0 \to Z_G^0$ such that

$$\psi \circ \Xi = \rho_{|Z_H^0}^N.$$

This proves our claim.

Consider the differential system

$$Y' = \text{diag}(\text{tr}(A_1)/(Nn_1), \ldots, \text{tr}(A_s)/(Nn_s), \widetilde{A}, B)Y.$$

Let H_N be the differential Galois group over k of this system. In the tannakian correspondance, the equation

$$Y' = \text{diag}(A, B)Y$$

corresponds to

$$\varrho := \rho \circ p : H_N \to G$$

where $p : H_N \to H$ (which corresponds to $Y' = \operatorname{diag}(\operatorname{tr}(A_1)/n_1, \ldots \operatorname{tr}(A_s)/n_s, \widetilde{A}, B)Y$ in the tannakian correspondence) is given by

$$p : \operatorname{diag}(e_1, \ldots, e_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \operatorname{diag}(e_1^N, \ldots, e_s^N, \widetilde{g}_1, \ldots, \widetilde{g}_s, g).$$

Applying Lemma 21 to $p : H_N \to H$, we get that H_N is reductive and that $Z_{H_N}^0 = p^{-1}(Z_H^0)^0$ is torus of dimension d.

If $\operatorname{diag}(e_1, \ldots, e_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \in Z_{H_N}^0$, then

$$\varrho(\operatorname{diag}(e_1, \ldots, e_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g))^N = \rho_{|Z_H^0}(\operatorname{diag}(e_1^N, \ldots, e_s^N, \widetilde{g}_1, \ldots, \widetilde{g}_s, g))^N$$
$$= \psi \circ \Xi(e_1^N, \ldots, e_s^N) = \psi \circ \Xi(e_1, \ldots, e_s)^N$$

so

$$\varrho_{|Z_{H_N}^0} = \psi \circ \Xi.$$

Moreover, the representation of H_N corresponding to $Y' = \overline{A}Y$ in the tannakian correspondence is

$$\varpi_{\overline{A}} = \pi_{\overline{A}} \circ \varrho : H_N \to G_{\overline{A}}.$$

It is surjective (general fact) and has finite kernel (because ϱ and $\pi_{\overline{A}}$ have finite kernels), so it is an isogeny. Therefore, it induces an isogeny

$$\varpi_{\overline{A}|H_N'} = \pi_{\overline{A}} \circ \varrho : H_N' \to G_{\overline{A}}'.$$

Since $G_{\overline{A}}'$ is a connected and simply connected algebraic group,

$$\varpi_{\overline{A}|H_N'} = \pi_{\overline{A}} \circ \varrho : H_N' \to G_{\overline{A}}'.$$

is actually an isomorphism. Since $G = \operatorname{im}(\pi_{\overline{A}} \circ \varrho \oplus \pi_B \circ \varrho)$, it follows that there exists an algebraic group morphism $u : G_{\overline{A}}' \to G_B'$ such that

$$\pi_B \circ \varrho = u \circ (\pi_{\overline{A}} \circ \varrho).$$

So, the restriction of ϱ to the derived subgroup H_N' of H_N is given by

$$\varrho_{|H_N'} : (1, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \operatorname{diag}(\widetilde{g}_1, \ldots, \widetilde{g}_s, u(\widetilde{g}_1, \ldots, \widetilde{g}_s)).$$

Finally, we see that the representation $\varpi_B = \pi_B \circ \varrho$ of H_N corresponding to $Y' = BY$ in the tannakian correspondence is given by

$$\varpi_B = \pi_B \circ \varrho : \operatorname{diag}(e_1, \ldots, e_s, \widetilde{g}_1, \ldots, \widetilde{g}_s, g) \mapsto \psi \circ \Xi(e_1, \ldots, e_s) u(\widetilde{g}_1, \ldots, \widetilde{g}_s).$$

It follows that the differential system $Y' = BY$ is equivalent over k to

$$Y' = \operatorname{diag}(B_1, \ldots, B_\ell)Y$$

where each $B_i \in M_{m_i}(k)$ has the form

$$\mathrm{Constr}_i(\widetilde{A}) + \sum_{j=1}^{s} \alpha_{i,j} \, \mathrm{tr}(A_j)/(n_j N) I_{m_j}$$

for some $\alpha_{i,j} \in \mathbb{Z}$ and some $\mathrm{Constr}_i(\widetilde{A})$ obtained from $\widetilde{A}$ by some construction of linear algebra. It follows that the entires of the solutions of $Y' = BY$ belong to $k[\mathfrak{Y}_{\overline{A}}, \sqrt[q]{y_1}^{-1}, \ldots, \sqrt[q]{y_s}^{-1}]$ for some positive integer q, where each y_j satisfies $y_j' = \mathrm{tr}(A_j)y_j$. ∎

7. Applications

In this section we will apply the previous results to generalized hypergeometric series and to iterated integrals.

7.1. Generalized hypergeometric series.
The aim of this Section is to prove Theorem 6 stated in Section 1.

7.1.1. The generalized hypergeometric series and equations. We recall that, for any $p, q \in \mathbb{Z}_{\geq 0}$, the generalized hypergeometric series with parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$ is given by

$$_pF_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}; x\right] = \sum_{k=0}^{+\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{x^k}{k!} \in \mathbb{C}((x))$$

where the Pochhammer symbols $(t)_k$ are defined by $(t)_0 = 1$ and, for $k \in \mathbb{Z}_{\geq 1}$, $(t)_k = t(t+1) \cdots (t+k-1)$.

This series satisfies the generalized hypergeometric differential equation

$$_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right] \left(_pF_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}; x\right]\right) = 0$$

where

$$_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right] = \delta \prod_{k=1}^{q}(\delta + \beta_k - 1) - x \prod_{k=1}^{p}(\delta + \alpha_k)$$

with $\delta = x\frac{d}{dx}$.

In that case, $_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]$ is a linear differential operator of order $q+1$, and

- if $q + 1 = p$, then $_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]$ has order $q + 1 = p$ and has at most three singularities, namely 0, 1 and ∞, all regular;
- if $q + 1 > p$, then $_p\mathscr{H}_q\left[\boldsymbol{\alpha}; \boldsymbol{\beta}\right]$ has order $q + 1$ and has at most two singularities, namely 0 and ∞; 0 is regular whereas ∞ is irregular.

- if $q + 1 < p$, then $_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ has order p and has at most two singularities, namely 0 and ∞; 0 is irregular whereas ∞ is regular.

7.1.2. *Formal structure at ∞ of the generalized hypergeometric equations when $q + 1 > p$.* We shall now recall some basic facts concerning the formal structure at ∞ of $_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ when $q + 1 > p$. Consider the uniformiser at ∞ given by $t = x^{-1}$. We have

$$_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}] = -t^{-1}\left((-1)^q t\delta_t \prod_{k=1}^q (\delta_t - \beta_k + 1) - (-1)^p \prod_{k=1}^p (\delta_t - \alpha_k)\right)$$

with $\delta_t = t\frac{d}{dt}$. By definition, the Newton polygon $\mathcal{N}_\infty(_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}])$ of $_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ at ∞ is the convex hull of

$$\{(x,y) \in \mathbb{R}^2 \mid \text{there is a monomial } t^m\delta_t^n$$
$$\text{in } _p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}] \text{ with } (x,y) \geq (n,m)\}$$

where $(x_1,y_1) \geq (x_2,y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \geq y_2$. This polygon has three extremal points, namely $(0,0)$, $(p,0)$ and $(q+1,1)$. Thus, the slopes of $_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ at ∞ are

$$\lambda_1 = 0 \text{ with multiplicity } p \text{ and } \lambda_2 = 1/\sigma \text{ with multiplicity } \sigma,$$

where

$$\sigma = q - p + 1.$$

It follows that the differential module $\mathcal{M}$ over $\mathbb{C}(x)$ associated to $_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ satisfies

$$\mathbb{C}((t)) \otimes_{\mathbb{C}(x)} \mathcal{M} = \widehat{\mathcal{M}}_{\lambda_1} \oplus \widehat{\mathcal{M}}_{\lambda_2}$$

where $\widehat{\mathcal{M}}_{\lambda_1}$ is regular singular of rank p and $\widehat{\mathcal{M}}_{\lambda_2}$ is irregular of rank σ, with only one slope, namely λ_2. According to [dPS03, Remark 3.34], we have

$$\mathbb{C}((t^{1/\sigma})) \otimes_{\mathbb{C}((x))} \widehat{\mathcal{M}}_{\lambda_2} = \oplus_{j=1}^\sigma \widehat{\mathcal{R}} \otimes_{\mathbb{C}((t^{1/\sigma}))} \widehat{\mathcal{E}}(q_j(t^{-1/\sigma}))$$

where:
- $\widehat{\mathcal{R}}$ is a regular singular differential module over $\mathbb{C}((t^{1/\sigma}))$ of rank one,
- for all $j \in \{1,\ldots,\sigma\}$, $q_j(t^{-1/\sigma}) = q(\zeta_j t^{-1/\sigma})$ with $\zeta_j = e^{\frac{2\pi i j}{\sigma}}$ for some $q(X) \in X\mathbb{C}[X]$ of X-adic valuation 1,
- $\widehat{\mathcal{E}}(q_j(t^{-1/\sigma}))$ is the rank one differential module over $\mathbb{C}((t^{1/\sigma}))$ defined by $\widehat{\mathcal{E}}(q_j(t^{-1/\sigma})) = \mathbb{C}((t^{1/\sigma}))e_j$ with $\partial(e_j) = q_j(t^{-1/\sigma})e_j$.

Recall that the slopes of $\widehat{\mathcal{M}}_{\lambda_2}$ can be computed using the $q_j(t^{-1/\sigma})$: they are the negative of the t-adic valuations of the $q_j(t^{-1/\sigma})$, which are equal all to $\frac{\deg_X q(X)}{\sigma}$. Since $\widehat{\mathcal{M}}_{\lambda_2}$ has a unique slope $\lambda_2 = 1/\sigma$, we get $\deg_X q(X) = 1$ so $q(X) = aX$ is a monomial and

$$q_j(t^{-1/\sigma}) = a\zeta_j t^{-1/\sigma}.$$

We shall now determine a. Set $w = t^{1/\sigma}$ and $\delta_w = w\frac{d}{dw} = \sigma\delta_t$. We have to find the a such that the Newton polygon (with respect to w) of $e^{-aw^{-1}}{}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]\,e^{aw^{-1}}$ has height $< q+1$ (because the slopes of the latter differential operator are 0 and/or 1). We have

$${}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}] =$$
$$-w^{-\sigma}\left((-1)^q w^\sigma \frac{\delta_w}{\sigma}\prod_{k=1}^{q}\left(\frac{\delta_w}{\sigma} - \beta_k + 1\right) - (-1)^p\prod_{k=1}^{p}\left(\frac{\delta_w}{\sigma} - \alpha_k\right)\right)$$

and

$$e^{-aw^{-1}}{}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]\,e^{aw^{-1}} =$$
$$-w^{-\sigma}\left((-1)^q w^\sigma \left(\frac{\delta_w}{\sigma} - \frac{a}{\sigma}w^{-1}\right)\prod_{k=1}^{q}\left(\frac{\delta_w}{\sigma} - \frac{a}{\sigma}w^{-1} - \beta_k + 1\right)\right.$$
$$\left. -(-1)^p\prod_{k=1}^{p}\left(\frac{\delta_w}{\sigma} - \frac{a}{\sigma}w^{-1} - \alpha_k\right)\right)$$

The coefficient of δ_w^j in $e^{-aw^{-1}}{}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]\,e^{aw^{-1}}$ has w-valuation $\geq -q + j - 1$ and the w-valuation of δ_w^{q+1} is equal to 0; therefore, we have to find the a such that the coefficient of δ_w^0 in $e^{-aw^{-1}}{}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]\,e^{aw^{-1}}$ has valuation $> q+1$. But the latter coefficient is of the form

$$\left(\frac{a}{\sigma}w^{-1}\right)^{q+1} - w^{-\sigma}\left(\frac{a}{\sigma}w^{-1}\right)^{p} + \text{ terms of higher degree in } w;$$

it has valuation $> q + 1$ if and only if $(\frac{a}{\sigma})^{q+1} = (\frac{a}{\sigma})^p$ if and only if $a \in \sigma\mu_\sigma$.

In conclusion, the list of determining polynomials[4] at ∞ of ${}_p\mathscr{H}_q[\boldsymbol{\alpha};\boldsymbol{\beta}]$ is

$$(0 \text{ repeated } p \text{ times}, \zeta_1\sigma t^{-1/\sigma}, \ldots, \zeta_\sigma\sigma t^{-1/\sigma})$$

with $\zeta_j = e^{\frac{2\pi i j}{\sigma}}$.

---

[4]These "determining polynomials" are the "eigenvalues" of [dPS03, Definition 3.26].

7.1.3. *A preliminary remark concerning Proposition 16 in the* $\mathrm{SL}_2(C)$ *case.* The hypothesis of Proposition 16 is:

> Let $f \in K^\times$ satisfy a nonzero homogeneous linear differential equation $L(f) = 0$ over k of order $n \geq 1$ and assume that the differential Galois group G_L over k of the latter equation is simply connected. Let $g \in K$ satisfy a nonzero homogeneous linear differential equation over k. Assume that $f \notin k$ and that f and g are algebraically dependent over k.

With the notations of the proof of Proposition 16, we have

$$\pi_M = u \circ \pi_L.$$

We shall now focus on the case $G_L = \mathrm{SL}_2(\mathbb{C})$. Then, the classification of the representations of $\mathrm{SL}_2(\mathbb{C})$ shows that the representation $u : G_L \to G_M$ is conjugate to the direct sum of symmetric power representations

$$\mathrm{Sym}^{m_i} : G_L = \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_{m_i+1}(\mathbb{C}).$$

Letting $\mathcal{M}$ and $\mathcal{L}$ denote the differential modules associated with $Y' = A_M Y$ and $Y' = A_L Y$, the tannakian correspondence implies that $\mathcal{M}$ is isomorphic to the direct sum of the $\mathrm{Sym}^{m_i}(\mathcal{L})$. We will now use this to study the algebraic relations between generalized hypergeometric series.

7.1.4. *Proof of Theorem 6.* The differential Galois group over $\overline{\mathbb{C}(x)}$ of $L = {}_0\mathscr{H}_1[-;\beta]$ is $\mathrm{SL}_2(\mathbb{C})$; see [Kat90, Theorem 3.6]. As explained in Section 7.1.3, the differential module $\mathcal{M}$ associated to the minimal nonzero differential equation M with coefficients in $\overline{\mathbb{C}(x)}$ annihilating ${}_pF_q[\gamma;\delta;x]$ is isomorphic to a direct sum of symmetric powers of $\mathcal{L}$, say

$$\mathcal{M} \cong \oplus_{i=1}^r \mathrm{Sym}^{m_i}(\mathcal{L}).$$

Let us first assume that $q + 1 > p$. We have seen in Section 7.1.2 that L is irregular at ∞: it has exactly one slope at ∞, namely $1/2$, and its list of determining polynomials at ∞ is $\pm 2x^{1/2}$. Therefore, the list of the determining polynomials of $\mathrm{Sym}^{m_i}(\mathcal{L})$ at ∞ is
(7)
$$-2m_i x^{1/2}, 2(-m_i+2)x^{1/2}, 2(-m_i+4)x^{1/2}, \ldots, 2(m_i-2)x^{1/2}, 2m_i x^{1/2}.$$

So, the list of determining polynomials of $\mathcal{M}$ is the concatenation of the lists (7) for i varying in $\{1, \ldots, r\}$.

On the other hand, M is a factor of ${}_p\mathscr{H}_q[\gamma;\delta]$ so (see Section 7.1.3) the list of determining polynomials of $\mathcal{M}$ is a sublist of

$$0 \text{ with multplicity } p, \zeta_1 \sigma x^{1/\sigma}, \ldots, \zeta_\sigma \sigma x^{1/\sigma}$$

with $\sigma = q - p + 1$ and $\zeta_j = e^{\frac{2\pi i j}{\sigma}}$.

Comparing the two preceding descriptions of the determining factors of $\mathcal{M}$, we find that $\sigma = 2$ and that we have

- either all the m_i are equal to 0;
- or one of the m_i, say m_r, is equal to 1 and the other m_i are equal to 0.

We claim that the first case cannot happen. Indeed, otherwise the entire function $_pF_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right]$ would be algebraic and, hence, polynomial, which is false.

Therefore, we have

$$\mathcal{M} \cong \mathrm{Sym}^0(\mathcal{L}) \oplus \cdots \oplus \mathrm{Sym}^0(\mathcal{L}) \oplus \mathrm{Sym}^1(\mathcal{L}).$$

It follows that $_pF_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right]$ is of the form $a + bg + cg'$ for some $a, b, c \in \overline{\mathbb{C}(x)}$, b or $c \neq 0$, and some nonzero solution g of L. Since the differential Galois group of L over $\overline{\mathbb{C}(x)}$ is $\mathrm{SL}_2(\mathbb{C})$, the only possibility for $a + bg + cg'$ to be algebraically depend with $_0F_1\left[\alpha; \beta; x\right]$ is that $g = d \cdot {_0F_1}\left[-; \beta; x\right]$ for some $d \in \mathbb{C}$ and $c = 0$. Therefore,

$$_pF_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right] \in \overline{\mathbb{C}(x)}_0F_1\left[-; \beta; x\right] + \overline{\mathbb{C}(x)}.$$

The fact that we can descend this linear relation to $\mathbb{C}(x)$ follows from the fact that $_pF_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right]$ and $_0F_1\left[-; \beta; x\right]$ are entire functions and that $_0F_1\left[-; \beta; x\right]$ and 1 are linearly independent over $\overline{\mathbb{C}(x)}$.

Let us now consider the case $q + 1 \leq p$. In that case, we have seen that $_p\mathscr{H}_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}\right]$ is regular at ∞. Since M is a factor of $_p\mathscr{H}_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}\right]$, M and hence $\mathcal{M}$ are regular at ∞ as well. Since $\mathrm{Sym}^{m_i}(\mathcal{L})$ is irregular at ∞ if m_i is nonzero, we infer that all the m_i are equal to 0. Therefore, $_pF_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right]$ belongs to $\overline{\mathbb{C}(x)}$. This excludes the case $q + 1 < p$, because the radius of convergence of $_pF_q\left[\boldsymbol{\gamma}; \boldsymbol{\delta}; x\right]$ is 0 in that case.

7.2. **Iterated integrals.** From [Hoc81, Chapter XVIII, Theorem 3.3] we know that a unipotent group is simply connected. This allows us to make an application of Proposition 17 when the Galois group is unipotent.

Definition 27 (Cf. [RS10])**.** *We say that $f \in K$ is an* iterated integral *over k if for, for some $n \in \mathbb{N}$, $f^{(n)} \in k$.*

Theorem 28. *Let $f \in K^\times$ be a iterated integral over k and let $g \in K$ satisfy a nonzero homogeneous linear differential equation over k. Assume that k contains an element x with $x' = 1$, f is transcendental over k and that f and g are algebraically dependent over k. Then $g \in k[f]$.*

Remark 29. *The condition that k contains an element x such that $x' = 1$ is needed. For example, let $f = \mathbb{C}$ and $K = \mathbb{C}(x), x' = 1$. The element x^2 is an iterated integral over k, x satisfies a linear differential equation over k and these elements are algebraically dependent over k. Nonetheless, $x \notin \mathbb{C}[x^2]$.*

We will use, again and again, the fact that if $E \subset F$ are differential fields with the same algebraically closed field of constants and $y \in F$ with $y' \in E$, then either $y \in E$ or y is transcendental over E. This follows easily from the Galois theory and the fact that the only algebraic subgroups of $\mathbb{G}_a(C)$ are $\{0\}$ and $\mathbb{C}$.

Lemma 30. *Let $E \subset F$ be differential fields with the same algebraically closed subfield C of constants. If $x \in k$ such that $x' = 1$ and $y, z \in F$ such that*

    (1) *$y' \in E$ and y transcendental over E and*
    (2) *$z \in E(y)$ and $z' = y$,*

then $z = Ay + B$ for some $A, B \in E$.

Proof. Let $z = p(y)/q(y)$ where p and q are relatively prime polynomials. As in the proof of Corollary 14, [HS85] implies that $q(y)'/q(y)$ is algebraic over E and therefore in E (since E is algebraically closed in $E(y)$). We then have that $E(q(y))$ is a Picard-Vessiot extension of E whose differential Galois group G is a subgroup of $\mathrm{GL}_1(C)$ and is a quotient of $\mathbb{G}_a(C)$. This implies that G must be trivial so $q(y) \in E$.

Therefore we may write $z = a_m y^m + \ldots + a_0$ with the $a_i \in E$. Differentiating, we have

$$y = z' = a'_m y^m + (m a_m y' + a'_{m-1}) y^{m-1} + \ldots.$$

If $a'_m \neq 0$ we have $m = 1$ and our conclusion follows. If $a'_m = 0$ we must have $m \geq 2$ and we will derive a contradiction. We have that

$$m a_m y' + a'_{m-1} = c \in E$$

where $c = 0$ if $m > 2$ or $c = 1$ if $m = 2$. In either case, $y = \frac{1}{m a_m}(cx - a_{m-1} + d) \in E$ for some constant d, contradicting the fact that y is transcendental over E. $\square$

Lemma 31. *Let $k \subset K$ be differential fields with the same algebraically closed subfield of constants C. Assume that there exists $x \in k$ such that $x' = 1$.*

    1. *If f is a iterated integral over k, then $k\langle f \rangle$ is a Picard-Vessiot extension of k with unipotent Galois group.*

2. *If $f \notin k$, then there exist algebraically independent $y_1, \ldots y_{r-1}, y_r = f$ such that $k\langle f \rangle = k(y_1, \ldots, y_r)$. In particular, $k(f)$ is algebraicaly closed in $k\langle f \rangle$.*

Proof. If $f^{(n)} = h \in k$ then f satisfies

$$(8) \qquad y^{(n+1)} - \frac{h'}{h} y^{(n)} = 0.$$

(1) If $f \in k$, then clearly $k\langle f \rangle$ is a Picard-Vessiot extension of k. If $f \notin k$ then $f, 1, x, \ldots x^{n-1}$ are linearly independent over C and so form a basis of the solution space of (8) and $k\langle f \rangle$ is a Picard-Vessiot extension of k. In addition, if we define $k_i = k_{i-1}(f^{(n-i)})$, we have a tower of differential fields $k = k_0 \subset k_1 \subset \ldots \subset k_n = k\langle f \rangle$ where each k_i is a Picard -Vessiot extension whose differential Galois group is either $\{0\}$ or $\mathbb{G}_a$. Therefore the differential Galois group of $k\langle f \rangle$ over k is unipotent.

(2) Assume that the transcendence degree of $k\langle f \rangle$ over k is r. By assumption $r \geq 1$. Using the construction of the k_i we see that from the set $\{f^{(n-1)}, f^{(n-2)}, \ldots, f', f\}$ we may select a subset of elements $T = \{y_i = f^{(n-n_i)} \mid n_1 < n_2 < \ldots < n_r\}$ that forms a transcendence basis. We wish to show that we can select these so that $n_r = n$. Suppose among all choices of the set T we have selected one with n_r maximal. If $n_r < n$, let $E = k(y_1, \ldots, y_{r-1}), y = y_r = f^{(n-n_r)}, z = f^{(n-(n_r+1))}$. Applying Lemma 30, we have that $y_r = az + b$ for some $a, b \in E$. Therefore

$$\begin{aligned}
k\langle f \rangle &= k(f^{(n-n_1)}, \ldots, f^{(n-n_{r-1})}, f^{(n-n_r)}) \\
&= E(y_r) \\
&= E(z) \\
&= k(f^{(n-n_1)}, \ldots, f^{(n-n_{r-1})}, f^{(n-(n_r+1))}),
\end{aligned}$$

contradicting the maximality of n_r. $\square$

Proof of Theorem 28. This now follows from Proposition 16 ∎

For the rest of Section 7.2 we remove the assumption that k is algebraically closed and only assume it is a differential field of charactersitic zero with an algebraically closed subfield of constants.

A consequence of the Kolchin-Ostrowski Theorem ([Kol68, Section 2]) is that if $k \subset K$ are differential fields with the same constants and $f_1, \ldots, f_n \in K$ with $f_i' \in k$ for $i = 1, \ldots n$ then the f_i are algebraicaly

dependent over k if and only if there exist constants c_i, not all zero, such that $c_1 f_1 + \ldots + c_n f_n \in k$. Proposition 34 below gives a related result for iterated integrals.

Corollary 32. *Let $k \subset K$ have the same field of constants and assume that there is an element $x \in k$ such that $x' = 1$. Let $f, g \in K$ be iterated integrals over k. If f and g are algebraically dependent over k then there exist $u, v \in k$, not both zero, such that $uf + vg \in k$.*

Proof. Let $E = k(f, f', \ldots, f^{(n-1)}, g, g', \ldots, g^{(m-1)})$. Proceeding in a manner similar to the proof of Lemma 31.2, there is an algeraically independent set of elements $\{z_1, \ldots, z_t\} \subset \{f, f', \ldots, f^{(n-1)}, g, g', \ldots, g^{(m-1)}\}$ such that $E = k(z_1, \ldots, z_t)$, that is, E is a purely transcendental extension of k. Therefore if either f or g are algebraic, then both of them will lie in k and the conclusion follows. Therefore we may assume neither f nor g are algebriac over k.

Theorem 28 implies that $f = P(g)$ and $g = Q(f)$ for some polynomials $P, Q \in \bar{k}[Y]$ where $\bar{k}$ is the algebraic closure of k. This implies that P and Q are linear polynomials so f, g and 1 are linearly dependent over $\bar{k}$. Since E is a purely transcendental extension of k, it is linearly disjoint from $\bar{k}$ over k. Therefore f, g and 1 are linearly dependent over k. ∎

Example 33. *Let $k = \mathbb{C}(x), K = \mathbb{C}(x, \log x)$ and let $f = \log x + x, g = x \log x$. We have $f', g'' \in k$ and $xf - g = x^2 \in k$.*

Proposition 34. *Let $k \subset K$ have the same field of constants and assume that there is an element $x \in k$ such that $x' = 1$. Let $f_1, \ldots, f_n$ be iterated integrals over k. If $f_1, \ldots, f_n$ are algebraically dependent over k then there exist $u_1, \ldots, u_n \in k$, not all zero, such that $u_1 f_1 + \ldots + u_n f_n \in k$.*

Proof. We can assume that for each pair i, j, $i \neq j$

- the set $\{f_1, \ldots, \widehat{f}_i, \ldots, \widehat{f}_j, \ldots, f_n\}$ is algebraically independent over k,
- f_i, f_j are transcendental over $F_{i,j} = k(f_1, \ldots, \widehat{f}_i, \ldots, \widehat{f}_j, \ldots, f_n)$, and
- f_i and f_j are algebraically dependent over $F_{i,j}$.

Corollary 32 implies that

$$f_i = A_{i,j} f_j + B_{i,j} \text{ for some } A_{i,j}, B_{i,j} \in F_{i,j}.$$

In particular, $f_1 = A_{1,2} f_2 + B_{1,2}$ for some $A_{1,2}, B_{1,2} \in F_{1,2}$. We shall show that $A_{1,2} \in k$ and that $B_{1,2}$ is a k linear combination of

$f_3, \ldots, f_n, 1$. Note that $k(f_2, \ldots, f_n)$ is a purely transcendental extension of k. We will let ∂_i denote the derivation $\frac{\partial}{\partial f_i}$ on this field.

$A_{1,2} \in k$. Since we also have $f_1 = A_{1,3}f_3 + B_{1,3}$ for some $A_{1,2}, B_{1,2} \in F_{1,3}$, $A_{1,3}f_3 + B_{1,3} = A_{1,2}f_2 + B_{1,2}$. Applying ∂_3, we have $A_{1,3} = \partial_3(A_{1,2})f_2 + \partial_3(B_{1,2})$. Since $A_{1,3}$ is independent of f_3 and $A_{1,2}$ is independent of f_1, f_2 we have $A_{1,2} \in k(y_4, \ldots, y_n)$ so $A_{1,2} = \alpha f_3 + \beta$, where $\alpha, \beta \in k(f_4, \ldots, f_n)$.

We also have $f_2 = A_{2,1}f_1 + B_{2,1}$ for some $A_{2,1}, B_{2,1} \in F_{1,2} = F_{2,1}$ and $f_2 = A_{2,3}f_3 + B_{2,3}$ for some $A_{2,3}, B_{2,3} \in F_{2,3}$. Comparing these two expressions in a manner similar to the above we have that $A_{2,1} = \bar{\alpha}f_3 + \bar{\beta}, \alpha, \beta \in k(f_4, \ldots, f_n)$. Since $A_{2,1} = A_{1,2}^{-1}$, we must have $\alpha = \bar{\alpha} = 0$, that is, $A_{1,2}$ is independent of f_3. Replacing f_3 with each $f_i, i = 4, \ldots, n$ in the above argument yields that $A_{1,2} \in k$.

$B_{1,2}$ is a k-linear combination of $f_3, \ldots, f_n, 1$. Note that the above argument can be modified to work equaly well for $f_1, f_j, \ j > 1$ to conclude that $f_1 = A_{1,j}f_j + B_{1,j}$ where $A_{1,j} \in k$ and $B_{1,j} \in F_{1,j}$. Therefore $\partial_j(f_1) = A_{1,j} \in k$, that is, $f_1 \in k(f_2, \ldots, f_n)$ is linear in each of the $f_i, 2 \leq i \leq n$ and so the conclusion is obtained. ∎

Appendix A. The tannakian correspondence

We give an informal description of this correspondence, enough to understand its use in our proofs. For a formal description see [DM81], [Kat72, Kat87], [dPS03, Ch. 2 and Appendix B].

Let k be a differential field of characterstic zero (not assumed to be algebraically closed) with algebraically closed field of constants C. Let

$$(9) \qquad Y' = AY$$

with $A \in \mathrm{gl}_n(k)$ and let K be its Picard-Vessiot extension. If $\mathfrak{Y} \in \mathrm{GL}_n(K)$ is a funadamental solution matrix of (9), we refer to the ring $k[\mathfrak{Y}, \det(\mathfrak{Y})^{-1}]$ generated by the entries of $\mathfrak{Y}$ and the inverse of its determinant as the Picard-Vessiot ring associated with (9). It is independent of our choice of fundamental solution matrix.

One can associate to (9) a differential module M, that is, a finite dimensional k-vector space M endowed with a map $\partial : M \to M$ such that for all $f \in k$ and $m, n \in M$, $\partial(m + n) = \partial m + \partial n$ and $\partial(fm) = f'm + f\partial m$. This is done in the following way. Let $M = k^n$ and denote the standard basis by $\{e_i\}$. We define ∂ by setting $\partial(e_i) := -\sum_j a_{j,i}e_j$ where $A = (a_{i,j})$. Given any differential module together with a basis $\mathbf{e} = \{e_i\}$, one can reverse this process

and produce a differential equation $Y' = A_{\mathbf{e}}Y$. If $\mathbf{f} = B\mathbf{e}$ is another basis of M with $B \in \mathrm{GL}_n(k)$ then $A_{\mathbf{e}}$ and $A_{\mathbf{f}}$ are related by $A_{\mathbf{e}} = B^{-1}A_{\mathbf{f}}B - B^{-1}B$. In this case we say that the equations are *gauge equivalent* or just *equivalent*.

Given differential modules M_1, M_2 one can define a differential module homomorphism as well as sub and quotient differential modules in the obvious way. One can also form direct sums, tensor products, and duals as in the following table:

Construction	∂	Equation
$M_1 \oplus M_2$	$\partial(m_1 \oplus m_2) = \partial_1 m_1 \oplus \partial_2 m_2$	$Y' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} Y$
$M_1 \otimes M_2$	$\partial(m_1 \otimes m_2) =$ $\partial_1 m_1 \otimes m_2 + m_1 \otimes \partial_2 m_2$	$Y' =$ $(A_1 \otimes I_{m_2} + I_{m_1} \otimes A_2)Y$
M_1^*	$\partial(f)(m) = f(\partial_1(m))$	$Y' = -A^T Y$

The collection of all differential modules forms a category closed under differential module homomorphism and the constructions of submodules, quotients, direct sums, tensor products, and duals, which we refer to as *the constructions of linear algebra*. We denote by $\{\{M\}\}$ the smallest subcategory containing M and closed under these five constructions. Given any $N \in \{\{M\}\}$ and any basis $\mathbf{e}$ of N, we can form the associated differential equation $Y' = A_{N,\mathbf{e}}Y$, $A_{N,\mathbf{e}} \in \mathrm{gl}_m(k)$. It is known ([dPS03, Ch. 2.4]) that this equation has a fundamental matrix $\mathfrak{Y}_{N,\mathbf{e}} \in \mathrm{GL}_m(K)$ and, *a fortiori*, in the Picard-Vessiot ring $k[\mathfrak{Y}, \det(\mathfrak{Y})^{-1}]$ ([dPS03, Corollary 1.38]). The differential Galois group G of K over k acts on the C-space V spanned by the columns of $\mathfrak{Y}$ and so induces a representation of G. The space V depends on our selection of bases of N but it can be shown that the representation of G is independent, up to G-isomorphism, of this choice of bases (see [dPS03, Ch. 2.4] for a basis free way of defining this representation). We will denote this representation of G by $S(N)$.

We denote by Repr_G the category of representations of G, that is the category of finite dimensional C-vector spaces on which G acts. In this category one also has a notion of the five constructions above. The tannakian correspondence [dPS03, Theorem 2.22] says that

> The map $N \mapsto S(N)$ is a bijective map between elements of $\{\{M\}\}$ (modulo isomorphism of differential modules)

and representations of G (modulo conjugacy of representations). This map is compatible with homomorphisms and the five constructions above.

One consequence of this is: if k is algebraically closed, then the Picard-Vessiot ring is naturally isomorphic to the coordinate ring (over k) of the differential Galois group.

Example 35. *Consider $Y' = AY$ with $A \in \mathrm{gl}_n(k)$ and let $\tilde{A} = A - (\mathrm{tr}(A)/n)I_n$. Since $\mathrm{tr}(\tilde{A}) = 0$, the differential Galois group of the equation $Y' = \tilde{A}Y$ is unimodular ([dPS03, Exercise 1.35.5]) and this construction is often used to reduce questions about general linear differential equations to ones that have unimodular Galois groups. We now form the $2n \times 2n$ differential system*

$$(10) \qquad\qquad Y' = \mathrm{diag}((\mathrm{tr}(A)/n)I_n, \tilde{A})Y$$

and let $\tilde{K}$ be its Picard-Vessiot extension with differential Galois group H. Let K be the Picard-Vessiot extension for $Y' = AY$ and let G be its differential Galois group.

If $\mathfrak{Y}$ is the fundamental solution matrix of $Y' = AY$, then $\mathfrak{y} = \sqrt[n]{\det \mathfrak{Y}}$ is a solution of $y' = (\mathrm{tr}\, A/n)y$. Note that $\mathfrak{y}$ need not lie in K but $\mathfrak{y}^n \in K$. A computation shows that if $\tilde{\mathfrak{Y}}$ is a fundamental solution matrix of $y' = \tilde{A}y$, then $\mathfrak{y}I_n \cdot \tilde{\mathfrak{Y}}$ is a fundamental solution matrix of $Y' = AY$. Therefore $K \subset \tilde{K}$. The differential Galois theory implies that restricting H to K gives a short exact sequence

$$0 \to H_0 \to H \xrightarrow{\rho} G \to 0$$

where H_0 is a finite subgroup of $\mathrm{GL}_n(C)$ (since $\tilde{K}$ is a simple radical extension of K) and the map ρ is given by

$$\rho : \mathrm{diag}(t, h) \to th.$$

This defines a homomorphism of $H \subset \mathrm{GL}_{2n}(C)$ to $\mathrm{GL}_n(C)$ and so defines a representation of H.

This example appears in the proof of Proposition 22.

Acknowledgement

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