DIFFERENTIAL TRANSCENDENCE CRITERIA FOR SECOND-ORDER LINEAR DIFFERENCE EQUATIONS AND ELLIPTIC HYPERGEOMETRIC FUNCTIONS

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Abstract. We develop general criteria that ensure that any non-zero solution of a given second-order difference equation is differentially transcendental, which apply uniformly in particular cases of interest, such as shift difference equations, q-dilation difference equations, Mahler difference equations, and elliptic difference equations. These criteria are obtained as an application of differential Galois theory for difference equations. We apply our criteria to prove a new result to the effect that most elliptic hypergeometric functions are differentially transcendental.

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1. Introduction

The differential Galois theory for difference equations developed in [HS08] provides a theoretical tool to understand the differential-algebraic properties of solutions of linear difference equations. Given a $\sigma\delta$-field K (i.e., K is equipped with an automorphism σ and a derivation δ such that $\sigma\circ\delta = \delta\circ\sigma$), one considers an n^{th} order linear difference equation of the form

$$a_n\sigma^n(y) + a_{n-1}\sigma^{n-1}(y) + \cdots + a_1\sigma(y) + a_0 y = 0, \tag{1.1}$$

where $a_i \in K$ for $i = 0, \ldots, n$, $a_n a_0 \neq 0$, and y is an indeterminate. The theory of [HS08] associates with (1.1) a geometric object G, called the differential Galois group, that encodes the polynomial differential equations satisfied by the solutions of (1.1). Traditionally, the following special cases have attracted special attention: $K = \mathbb{C}(z)$, and σ is one of the following: a shift operator $\sigma : z \mapsto z + r$, where $0 \neq r \in \mathbb{C}$; the q-dilation operator $\sigma : z \mapsto qz$, where $q \in \mathbb{C}^*$ is not a root of unity; and the Mahler operator $\sigma : z \mapsto z^p$, where $p \in \mathbb{N}_{\geq 2}$.[1] More recently, the elliptic case has also attracted a lot of interest: here $K = \mathcal{M}er(E)$ is the field of meromorphic functions on the elliptic curve $E = \mathbb{C}^*/p^{\mathbb{Z}}$, where $p \in \mathbb{C}^*$ is such that $|p| \neq 1$ — or equivalently, K is the field of (multiplicatively) p-periodic meromorphic functions $f(z)$ on $\mathbb{C}^*$ such that $f(pz) = f(z)$ — and $\sigma : f(z) \mapsto f(qz)$, where $q \in \mathbb{C}^*$ is such that $p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}$ (or equivalently, q represents a non-torsion point of E). In each one of these four cases of interest there is a corresponding choice of derivation δ that makes K into a $\sigma\delta$-field.

The main contribution of this work (Theorem 3.4) is the development of a new set of criteria for second-order equations

$$\sigma^2(y) + a\sigma(y) + by = 0, \tag{1.2}$$

which guarantee that any non-zero solution y of (1.2) must be differentially transcendental over K, i.e., for any $m \in \mathbb{N}$ there is no non-zero polynomial $P \in K[y_0, y_1, \ldots, y_m]$ such that $P(y, \delta(y), \ldots, \delta^m(y)) = 0$. These criteria apply uniformly under mild conditions on the base $\sigma\delta$-field K (see Definition 2.1), which are satisfied in the four particular cases mentioned above: shift, q-dilation, Mahler, and elliptic. Moreover, the verification of the criteria only requires one to check whether the following auxiliary equations associated with (1.2) admit any solutions in K: if there is no $u \in K$ such that

$$u\sigma(u) + au + b = 0, \tag{1.3}$$

and there are no $g \in K$ and linear differential operator $\mathcal{L} \in \mathbb{C}[\delta]$ such that

$$\mathcal{L}\left(\frac{\delta(b)}{b}\right) = \sigma(g) - g, \tag{1.4}$$

then every non-zero solution of (1.2) must be differentially transcendental over K. Therefore, although we do apply the differential Galois theory for difference equations [HS08] in the proof that our criteria are correct, the actual verification of the criteria does not involve any prior knowledge of

1. In the Mahler case the base field must be taken to be $K = \mathbb{C}(\{z^{1/\ell}\}_{\ell\in\mathbb{N}})$ with $\sigma(z^{1/\ell}) = z^{p/\ell}$ in order for σ to be an automorphism of K and not merely an (injective) endomorphism.

this theory at all. Moreover, in each of the four cases of interest mentioned above there are effective algorithms to decide whether the Riccati equation (1.3) and the telescoping problem (1.4) admit solutions, for which we provide case-by-case references below. Hence these user-friendly criteria are of practical import to non-experts seeking to decide differential transcendence of solutions of second-order difference equations in many settings that arise in applications.

Indeed, we illustrate the practical applicability of our criteria in the elliptic case by proving differential transcendence of "most" elliptic hypergeometric functions. The elliptic hypergeometric functions form a common analogue of classical hypergeometric functions and q-hypergeometric functions, which have been a focus of intense study in the last 200 years within the theory of special functions and are ubiquitous in physics and mathematics. The general theory of these elliptic hypergeometric functions was initiated by Spiridonov in [Spi16] and has been a dynamic field of research, see for instance [vdB$^+$07, FR09, M$^+$09, Rai10, Ros02]. In the intervening years a number of remarkable analogues of known properties and applications of classical and q-hypergeometric functions have been discovered for the elliptic hypergeometric functions; see [Spi16] for more details.

The theoretical part of our strategy to prove differential transcendence for elliptic hypergeometric functions is in the tradition of other applications of the differential Galois theory for difference equations of [HS08] to questions about shift difference equations [Arr17], q-difference equations, [DHR16], deterministic finite automata and Mahler functions [DHR18], lattice walks in the quarter plane [DHRS18, DR17, DHRS17], and shift, q-dilation, and Mahler difference equations in general [AS17]. The Galois correspondence of [HS08] implies in particular that if the differential Galois group G is "large" then there are "few" differential-algebraic relations among the solutions of (1.1). However, this theoretical strategy is only practical in the presence of algorithmic decision procedures that ensure that G is indeed large enough to force any solution of (1.1) to be differentially transcendental. The criteria developed here in Theorem 3.4 serve to fulfill precisely this purpose. To put the novelty and usefulness of these criteria in context, let us briefly recall the state of the art in each of the four particular cases of interest mentioned above.

In the shift case, a complete algorithm to compute the differential Galois group G for (1.2) is developed in [Arr17], based on the earlier algorithm of [Hen98] to compute the non-differential Galois group H of (1.2) [vdPS97]. Even in this case, it is still useful to have the isolated criteria of Theorem 3.4 to decide differential transcendence only, without having to compute the whole Galois group G of (1.2). An algorithm for deciding whether the Riccati equation (1.3) admits a solution in K has been developed in [Hen98], and to decide whether there is a telescoper (1.4) one can apply [HS08, Cor. 3.4].

The situation in the q-dilation and Mahler cases is similar. One knows how to compute the differential Galois group G for first-order equations (1.1) with $n = 1$ by solving an associated telescoping problem (see for example [HS08, Corollary 3.4] in the q-dilation case and [DHR18, Prop. 3.1] in the Mahler case), but there is no general algorithm to compute G for higher-order equations (1.1) with $n \geq 2$. The general criteria developed in [DHR16, DHR18] for differential transcendence of solutions of (1.1) are valid for arbitrary n, but these criteria require prior knowledge of the (non-differential) Galois group H of (1.1) [vdPS97]. At present this group H can only be computed in general when $n \leq 2$ by [Hen97] in the q-dilation case and [Roq18] in the Mahler case. Even when $n = 2$, the criteria given here in Theorem 3.4 strictly generalize those of [DHR16, DHR18], and require no knowledge of (differential) Galois theory of difference equations for their application. Algorithms for deciding whether the Riccati equation (1.3) admits solutions in K have been developed in the the q-dilation [Hen97] and Mahler [Roq18] cases. Algorithms for deciding whether the telescoping problem (1.4) can be solved have also been developed in [HS08, Cor. 3.4] in the q-dilation case and in [DHR18, Prop. 3.1] in the Mahler case.

In the elliptic case, the recent algorithm developed in [DR15] computes the (non-differential) Galois group H of (1.2) associated by the theory of [vdPS97], but there are no general algorithms to compute the differential Galois group G for (1.1) for any order n. In spite of the relative dearth of algorithms in this case, the authors of [DHRS18] were still successful in proving differential transcendence of some first-order (inhomogeneous) elliptic difference equations arising in connection with generating series for walks in the quarter plane. The criteria of Theorem 3.4 are the first to provide a test for differential transcendence that applies to second-order difference equations in the elliptic case. An algorithm for deciding whether the Riccati equation (1.3) admits solutions in K has been developed in [DR15], and an algorithm to decide whether the telescoping problem (1.4) can be solved has also been developed in [DHRS18, Prop. B.8].

The paper is organized as follows. In Section 2, we recall some facts about the difference Galois theory developed in [vdPS97]. To a difference equation (1.1) is associated an algebraic group. The larger the group, the fewer the algebraic relations that exist among the solutions of the difference equation. In Section 3, we recall some facts about the differential Galois theory for difference equations of [HS08]. Here the Galois group is a linear differential algebraic group, that is, a group of matrices defined by a system of algebraic differential equations in the matrix entries. This group encodes the polynomial differential relations among the solutions of the difference equation. In this section we prove our differential transcendence criteria for second-order difference equations (1.2) in Theorem 3.4. In Section 4 we restrict ourselves to the situation where the coefficients of the difference equation are elliptic functions. We recall some results from [DR15], where the authors explain how to compute the difference Galois group of [vdPS97] for order two equations with elliptic coefficients. This computation was inspired by Hendricks' algorithm, see [Hen97]. In Section 5, we follow [Spi16] in defining the elliptic analogue of the hypergeometric equation (5.4) and, under a certain genericity assumption, we prove that its nonzero solutions are differentially transcendental, see Theorem 5.7.

2. Difference Galois theory

For details on what follows, we refer to [vdPS97, Chapter 1]. Unless otherwise stated, all rings are commutative with identity and contain the field of rational numbers. In particular, all fields are of characteristic zero.

A σ-ring (or difference ring) (R,σ) is a ring R together with a ring automorphism $\sigma : R \to R$. If R is a field then (R,σ) is called a σ-field. When there is no possibility of confusion the σ-ring (R,σ) will be simply denoted by R. There are natural notions of σ-ideals, σ-ring extensions, σ-algebras, σ-morphisms, *etc.* We refer to [vdPS97, Chapter 1] for the definitions.

The ring of σ-constants R^σ of the σ-ring (R,σ) is defined by

$$R^\sigma := \{f \in R \mid \sigma(f) = f\}.$$

We now let $(\mathbf{K},\sigma)$ be a σ-field. We assume that the field of constants $\mathcal{C} := \mathbf{K}^\sigma$ is algebraically closed and that the characteristic of $\mathbf{K}$ is 0.

We consider a difference equation of order two with coefficients in $\mathbf{K}$:

$$\sigma^2(y) + a\sigma(y) + by = 0 \text{ with } a \in \mathbf{K} \text{ and } b \in \mathbf{K}^* \tag{2.1}$$

and the associated difference system:

$$\sigma Y = AY \text{ with } A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \in \mathrm{GL}_2(\mathbf{K}). \tag{2.2}$$

By [vdPS97, §1.1], there exists a σ-ring extension (R,σ) of $(\mathbf{K},\sigma)$ such that

1) there exists $U \in \mathrm{GL}_2(R)$ such that $\sigma(U) = AU$ (such a U is called a fundamental matrix of solutions of (2.2));
2) R is generated, as a $\mathbf{K}$-algebra, by the entries of U and $\det(U)^{-1}$;
3) the only σ-ideals of (R,σ) are $\{0\}$ and R.

Note that the last assumption implies $R^\phi = \mathcal{C}$. Such an R is called a σ-Picard-Vessiot ring, or σ-PV ring for short, for (2.2) over $(\mathbf{K},\sigma)$. It is unique up to isomorphism of $(\mathbf{K},\sigma)$-algebras. Note that a σ-PV ring is not always an integral domain, but it is a direct sum of integral domains transitively permuted by σ.

The corresponding σ-Galois group $\mathrm{Gal}(R/\mathbf{K})$ of (2.2) over $(\mathbf{K},\sigma)$, or σ-Galois group for short, is the group of $(\mathbf{K},\sigma)$-automorphisms of R:

$$\mathrm{Gal}(R/\mathbf{K}) := \{\phi \in \mathrm{Aut}(R/\mathbf{K}) \mid \sigma \circ \phi = \phi \circ \sigma\}.$$

A straightforward computation shows that, for any $\phi \in \mathrm{Gal}(R/\mathbf{K})$, there exists a unique $C(\phi) \in \mathrm{GL}_2(\mathcal{C})$ such that $\phi(U) = UC(\phi)$. According to [vdPS97, Theorem 1.13], one can identify $\mathrm{Gal}(R/\mathbf{K})$ with an *algebraic* subgroup G of $\mathrm{GL}_2(\mathcal{C})$ via the faithful representation

$$\begin{array}{rccc} \rho: & \mathrm{Gal}(R/\mathbf{K}) & \to & \mathrm{GL}_2(\mathcal{C}) \\ & \phi & \mapsto & C(\phi). \end{array}$$

If we choose another fundamental matrix of solutions U, we find a conjugate representation. In what follows, by "σ-Galois group of the difference equation (2.1)", we mean "σ-Galois group of the difference system (2.2)".

We shall now introduce a property relative to the base σ-field $(\mathbf{K}, \sigma)$, which appears in [vdPS97, Lemma 1.19].

Definition 2.1. We say that the σ-field $(\mathbf{K}, \sigma)$ satisfies the property $(\mathcal{P})$ if:
— the field $\mathbf{K}$ is a $\mathcal{C}^1$-field [2];
— and the only finite field extension $\mathbf{L}$ of $\mathbf{K}$ such that σ extends to a field endomorphism of $\mathbf{L}$ is $\mathbf{L} = \mathbf{K}$.

Example 2.2. The following are natural examples of difference fields that satisfy property $(\mathcal{P})$:
S: Shift case with $\mathbf{K} = \mathbb{C}(z)$, $\sigma : f(z) \mapsto f(z+h)$, $h \in \mathbb{C}^*$. See [Hen97].
Q: q-difference case. $\mathbf{K} = \mathbb{C}(z^{1/*}) = \bigcup_{\ell \in \mathbb{N}^*} \mathbb{C}(z^{1/\ell})$, $\sigma : f(z) \mapsto f(qz)$, $q \in \mathbb{C}^*$, $|q| \neq 1$. See [Hen98].
M: Mahler case. $\mathbf{K} = \mathbb{C}(z^{1/*})$, $\sigma : f(z) \mapsto f(z^p)$, $p \in \mathbb{N}_{\geq 2}$. See [Roq18].
E: Elliptic case. See Section 4, and [DR15].

The following result is due to van der Put and Singer. We recall that two difference systems $\sigma Y = AY$ and $\sigma Y = BY$ with $A, B \in \mathrm{GL}_2(\mathbf{K})$ are isomorphic over $\mathbf{K}$ if and only if there exists $T \in \mathrm{GL}_2(\mathbf{K})$ such that $\sigma(T)A = BT$. Note that $\sigma(Y) = AY$ if and only if $\sigma(TY) = BTY$.

Theorem 2.3. *Assume that $(\mathbf{K}, \sigma)$ satisfies property $(\mathcal{P})$. Then the following properties relative to $G = \rho(\mathrm{Gal}(R/\mathbf{K}))$ hold:*
— *G/G° is cyclic, where G° is the identity component of G;*
— *there exists $B \in G(\mathbf{K})$ such that (2.2) is isomorphic to $\sigma Y = BY$ over $\mathbf{K}$.*

Let $\widetilde{G}$ be an algebraic subgroup of $\mathrm{GL}_2(\mathcal{C})$ such that $A \in \widetilde{G}(\mathbf{K})$. The following properties hold:
— *G is conjugate to a subgroup of $\widetilde{G}$;*
— *any minimal element (with respect to inclusion) in the set of algebraic subgroups $\widetilde{H}$ of $\widetilde{G}$ for which there exists $T \in \mathrm{GL}_2(\mathbf{K})$ such that $\sigma(T)AT^{-1} \in \widetilde{H}(\mathbf{K})$ is conjugate to G;*
— *G is conjugate to $\widetilde{G}$ if and only if, for any $T \in \widetilde{G}(\mathbf{K})$ and for any proper algebraic subgroup $\widetilde{H}$ of $\widetilde{G}$, one has that $\sigma(T)AT^{-1} \notin \widetilde{H}(\mathbf{K})$.*

Proof. The proof of [vdPS97, Propositions 1.20 and 1.21] in the special case where $\mathbf{K} := \mathbb{C}(z)$ and σ is the shift $\sigma(f(z)) := f(z+h)$ with $h \in \mathbb{C}^*$, extends *mutatis mutandis* to the present case. □

This theorem is at the heart of many algorithms to compute σ-Galois groups, see for example [Hen97, Hen98, DR15, Roq18].

3. Parametrized Difference Galois theory

3.1. **General facts.** A (σ, δ)-ring (R, σ, δ) is a ring R endowed with a ring automorphism σ and a derivation $\delta : R \to R$ (this means that δ is additive and satisfies the Leibniz rule $\delta(ab) = a\delta(b) + \delta(a)b$) such that $\sigma \circ \delta = \delta \circ \sigma$. If R is a field, then (R, σ, δ) is called a (σ, δ)-field. When there is no possibility of confusion, we write R instead of (R, σ, δ). There are natural notions

2. Recall that $\mathbf{K}$ is a $\mathcal{C}^1$-field if every non-constant homogeneous polynomial P over $\mathbf{K}$ has a non-trivial zero provided that the number of its variables is more than its degree.

of (σ,δ)-ideals, (σ,δ)-ring extensions, (σ,δ)-algebras, (σ,δ)-morphisms, *etc.* We refer to [HS08, Section 6.2] for the definitions.

If $\mathbf{K}$ is a δ-field, and if $y_1,\dots,y_n$ belong to some δ-field extension of $\mathbf{K}$, then $\mathbf{K}\{y_1,\dots,y_n\}_\delta$ denotes the δ-algebra generated over $\mathbf{K}$ by $y_1,\dots,y_n$ and $\mathbf{K}\langle y_1,\dots,y_n\rangle_\delta$ denotes the δ-field generated over $\mathbf{K}$ by $y_1,\dots,y_n$.

We now let $(\mathbf{K},\sigma,\delta)$ be a (σ,δ)-field. We assume that the field of σ-constants $\mathcal{C} := \mathbf{K}^\sigma$ is algebraically closed and that $\mathbf{K}$ is of characteristic 0.

In order to apply the (σ,δ)-Galois theory developed in [HS08], we need to work with a base (σ,δ)-field $\mathbf{L}$ such that $\mathcal{C} = \mathbf{L}^\sigma$ is δ-closed.[3] To this end, the following lemma will be useful.

Lemma 3.1 ([DHR18, Lemma 2.3])**.** *Suppose that $\mathcal{C}$ is algebraically closed and let $\widetilde{\mathcal{C}}$ be a δ-closure of $\mathcal{C}$ (the existence of such a $\widetilde{\mathcal{C}}$ is proved in* [Kol74]*). Then the ring $\widetilde{\mathcal{C}} \otimes_{\mathcal{C}} \mathbf{K}$ is an integral domain whose fraction field $\mathbf{L}$ is a (σ,δ)-field extension of $\mathbf{K}$ such that $\mathbf{L}^\sigma = \widetilde{\mathcal{C}}$.*

We still consider the difference equation (2.1) and the associated difference system (2.2). By [HS08, § 6.2.1], there exists a (σ,δ)-ring extension (S,σ,δ) of $(\mathbf{L},\sigma,\delta)$ such that

1) there exists $U \in \mathrm{GL}_2(S)$ such that $\sigma(U) = AU$;
2) S is generated, as an $\mathbf{L}$-δ-algebra, by the entries of U and $\det(U)^{-1}$;
3) the only (σ,δ)-ideals of S are $\{0\}$ and S.

Such an S is called a (σ,δ)-Picard-Vessiot ring, or (σ,δ)-PV ring for short, for (2.2) over $(\mathbf{L},\sigma,\delta)$. It is unique up to isomorphism of $(\mathbf{L},\sigma,\delta)$-algebras. Note that a (σ,δ)-PV ring is not always an integral domain, but it is the direct sum of integral domains that are transitively permuted by σ.

The corresponding (σ,δ)-Galois group $\mathrm{Gal}^\delta(S/\mathbf{L})$ of (2.2) over $(\mathbf{L},\sigma,\delta)$, or (σ,δ)-Galois group for short, is the group of $(\mathbf{L},\sigma,\delta)$-automorphisms of S:

$$\mathrm{Gal}^\delta(S/\mathbf{L}) = \{\phi \in \mathrm{Aut}(S/\mathbf{L}) \mid \sigma\circ\phi = \phi\circ\sigma \text{ and } \delta\circ\phi = \phi\circ\delta\}.$$

In what follows, by "(σ,δ)-Galois group of the difference equation (2.1)", we mean "(σ,δ)-Galois group of the difference system (2.2)".

A straightforward computation shows that, for any $\phi \in \mathrm{Gal}^\delta(S/\mathbf{L})$, there exists a unique $C(\phi) \in \mathrm{GL}_2(\widetilde{\mathcal{C}})$ such that $\phi(U) = UC(\phi)$. By [HS08, Proposition 6.18], the faithful representation

$$\begin{array}{rccl} \rho^\delta: & \mathrm{Gal}^\delta(S/\mathbf{L}) & \rightarrow & \mathrm{GL}_2(\widetilde{\mathcal{C}}) \\ & \phi & \mapsto & C(\phi) \end{array}$$

identifies $\mathrm{Gal}^\delta(S/\mathbf{L})$ with a linear differential algebraic group G^δ, that is, a subgroup of $\mathrm{GL}_2(\widetilde{\mathcal{C}})$ defined by a system of δ-polynomial equations over $\widetilde{\mathcal{C}}$ in the matrix entries. If we choose another fundamental matrix of solutions U, we find a conjugate representation.

3. The field $\widetilde{\mathcal{C}}$ is called δ-closed if, for every (finite) set of δ-polynomials $\mathcal{F}$ with coefficients in $\widetilde{\mathcal{C}}$, if the system of δ-equations $\mathcal{F}=0$ has a solution with entries in some δ-field extension $\mathbf{L}|\widetilde{\mathcal{C}}$, then it has a solution with entries in $\widetilde{\mathcal{C}}$. Note that when the derivation δ is trivial, *i.e.* $\delta = 0$, then a field is δ-closed if and only if it is algebraically closed.

Let S be a (σ,δ)-PV ring for (2.2) over $\mathbf{L}$ and let $U \in \mathrm{GL}_2(S)$ be a fundamental matrix of solutions. Then the $\mathbf{L}$-σ-algebra R generated by the entries of U and $\det(U)^{-1}$ is a σ-PV ring for (2.2) over $\mathbf{L}$. We can (and will) identify $\mathrm{Gal}^{\delta}(S/\mathbf{L})$ with a subgroup of $\mathrm{Gal}(R/\mathbf{L})$ by restricting the elements of $\mathrm{Gal}^{\delta}(S/\mathbf{L})$ to R.

Proposition 3.2 ([HS08], Proposition 2.8)**.** *The group* $\mathrm{Gal}^{\delta}(S/\mathbf{L})$ *is a Zariski-dense subgroup of* $\mathrm{Gal}(R/\mathbf{L})$.

3.2. **Differential transcendence criteria.** The aim of this section is to develop a galoisian criterion for the differential transcendence of the nonzero solutions of (2.1).

Definition 3.3. Let $\mathbf{F}/\mathbf{K}$ be a (σ,δ)-field extension. We say that $f \in \mathbf{F}$ is differentially algebraic over $\mathbf{K}$ if there exists $n \in \mathbb{N}$ such that $f, \dots, \delta^n(f)$ are algebraically dependent over $\mathbf{K}$. Otherwise, we say that f is differentially transcendental over $\mathbf{K}$.

Recall that $\mathbf{K}$ *be a* (σ,δ)*-field satisfying property* $(\mathcal{P})$ *such that* $\mathcal{C} = \mathbf{K}^{\sigma}$ *is algebraically closed and such that* $\mathbf{K}$ *has characteristic* 0.

Let $\widetilde{\mathcal{C}}$ be a δ-closure of $\mathcal{C}$. According to Lemma 3.1, $\widetilde{\mathcal{C}} \otimes_{\mathcal{C}} \mathbf{K}$ is an integral domain and $\mathbf{L} := \mathrm{Frac}(\widetilde{\mathcal{C}} \otimes_{\mathcal{C}} \mathbf{K})$ is a (σ,δ)-field extension of $\mathbf{K}$ such that $\mathbf{L}^{\sigma} = \widetilde{\mathcal{C}}$. Let S be a (σ,δ)-PV ring for (2.2) over $\mathbf{L}$ and let $R \subset S$ be a σ-PV ring for (2.2) over $\mathbf{L}$. We also consider a σ-PV ring $\widetilde{R}$ for (2.2) over $\mathbf{K}$.

Our differential transcendence criteria are given in our main result below.

Theorem 3.4. *Consider the second-order difference equation* (2.1)*:*

$$\sigma^2(y) + a(y) + by = 0,$$

where $a \in \mathbf{K}$ *and* $b \in \mathbf{K}^*$ *and* $\mathbf{K}$ *satisfies property* $(\mathcal{P})$*. Assume the following:*

(1) there is no $u \in \mathbf{K}$ *such that* $u\sigma(u) + au + b = 0$*; and*

(2) there are no $g \in \mathbf{K}$ *and non-zero linear differential operator* $\mathcal{L} \in \mathcal{C}[\delta]$ *such that*

$$\mathcal{L}\left(\frac{\delta(b)}{b}\right) = \sigma(g) - g.$$

Then any non-zero solution of (2.1) *in any* (σ,δ)*-field extension* $\mathbf{F}$ *of* $\mathbf{K}$ *is differentially transcendental over* $\mathbf{K}$*.*

Note that the first criterion of Theorem 3.4 is equivalent to the irreducibility of $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$, and may be tested algorithmically in many contexts, see [Hen97, Hen98, DR15, Roq18]. The following lemma similarly relates the second criterion to a different largeness condition on $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$.

Lemma 3.5 (Proposition 2.6, [DHR18])**.** *The* (σ,δ)*-Galois group of* $\sigma y = by$ *over* $\mathbf{L}$ *is a proper subgroup of* $\mathrm{GL}_1(\widetilde{\mathcal{C}})$ *if and only if there exist a nonzero linear differential operator* $\mathcal{L}$ *with coefficients in* $\mathcal{C}$ *and* $g \in \mathbf{K}$ *such that*

$$\mathcal{L}\left(\frac{\delta(b)}{b}\right) = \sigma(g) - g.$$

The following lemma will be used in the proof of Theorem 3.4.

Lemma 3.6. *Assume that (2.1) has a nonzero differentially algebraic solution in a (σ,δ)-field extension $\mathbf{F}$ of $\mathbf{K}$. Then (2.1) has a nonzero differentially algebraic solution in S.*

Proof of Lemma 3.6. Since any two (σ,δ)-PV rings for (2.1) over $\mathbf{L}$ are isomorphic, it is sufficient to prove the lemma for some (σ,δ)-PV ring, not necessarily for S itself. Let f be a nonzero differentially algebraic solution of (2.1) in $\mathbf{F}$. We consider the localization T of $\mathbf{L}\langle f,\sigma(f)\rangle_\delta\{X_{1,2},X_{2,2}\}_\delta$ at $fX_{2,2}-\sigma(f)X_{1,2}$, where $X_{1,2},X_{2,2}$ are δ-indeterminates over $\mathbf{L}\langle f,\sigma(f)\rangle_\delta$. This ring has a natural structure of $\mathbf{L}$-(σ,δ)-algebra such that $\sigma\begin{pmatrix}X_{1,2}\\X_{2,2}\end{pmatrix}=A\begin{pmatrix}X_{1,2}\\X_{2,2}\end{pmatrix}$ and $\begin{pmatrix}f & X_{1,2}\\ \sigma(f) & X_{2,2}\end{pmatrix}$ is a fundamental matrix of solutions of $\sigma Y=AY$ with coefficients in T. If we let $\mathfrak{M}$ be a maximal (σ,δ)-ideal of T, then the quotient $T/\mathfrak{M}$ is a (σ,δ)-PV ring for $\sigma Y=AY$ over $\mathbf{L}$ and the image of f in this quotient is differentially algebraic. Let us prove that it is nonzero. Otherwise the image of the fundamental solution in the (σ,δ)-PV ring $T/\mathfrak{M}$ would have a zero first column and therefore would not be inversible, leading to a contradiction. This concludes the proof. □

Proof of Theorem 3.4. Assume to the contrary that (2.1) has a nonzero differentially algebraic solution in a (σ,δ)-field extension $\mathbf{F}$ of $\mathbf{K}$. According to Lemma 3.6, there exists a nonzero differentially algebraic solution f of (2.1) in S.

By [Hen97, Lemma 4.1] combined with Theorem 2.3, one of the following three cases holds

— $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$ is reducible.

— $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$ is irreducible and imprimitive.

— $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$ contains $\mathrm{SL}_2(\mathcal{C})$.

By [DR15, Lemma 13], the assumption that there is no solution in $\mathbf{K}$ for the Riccati equation $u\sigma(u)+au+b=0$ is equivalent to the irreducibility of $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$. Hence only the last two cases may occur. Then we split our study in two cases depending on whether $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$ is imprimitive or not.

Let us first assume that $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$ is imprimitive. It follows from Theorem 2.3 and [Hen97, Section 4.3] that (2.1) is equivalent over $\mathbf{K}$ to

$$\sigma^2(y)+ry=0 \tag{3.1}$$

for some $r\in\mathbf{K}^*$. More precisely, let

$$\sigma Y=BY \text{ with } B=\begin{pmatrix}0 & 1\\ -r & 0\end{pmatrix}\in\mathrm{GL}_2(\mathbf{K}),$$

be the system associated to (3.1). Then there exists $T\in\mathrm{GL}_2(\mathbf{K})$ such that $\sigma(T)A=BT$. Let $T=(t_{i,j})$. Since $\sigma Y=AY$ if and only if $\sigma(TY)=BTY$, we obtain that $t_{1,1}f+t_{1,2}\sigma(f)$ satisfies (3.1) with $(t_{1,1},t_{1,2})\neq(0,0)$. Let us prove that $t_{1,1}f+t_{1,2}\sigma(f)$ is non zero. If $t_{1,1}f+t_{1,2}\sigma(f)=0$, then $f\neq 0$ implies $t_{1,1}t_{1,2}\neq 0$ and then $\sigma(f)/f$ is solution of the Riccati equation $u\sigma(u)+au+b=0$, which contradicts the first assumption of Theorem 3.4.

Since f is differentially algebraic over $\mathbf{K}$, we have that $\sigma(f)$, and hence also $t_{1,1}f+t_{1,2}\sigma(f)$, are differentially algebraic over $\mathbf{L}$. By [HS08, Proposition 6.26], this implies that the (σ^2,δ)-Galois group of (3.1) over $\mathbf{L}$ is a proper subgroup of $\mathrm{GL}_1(\widetilde{\mathcal{C}})$. By Lemma 3.5 there exist a nonzero $\mathcal{D} \in \mathcal{C}[\delta]$ and $h \in \mathbf{K}$ such that

$$\mathcal{D}(\tfrac{\delta(r)}{r}) = \sigma^2(h) - h = \sigma(\sigma(h)+h) - (\sigma(h)+h). \tag{3.2}$$

Taking the determinant in $\sigma(T)A = BT$ allows us to deduce the existence of $p \in \mathbf{K}^*$ such that $b = \frac{\sigma(p)}{p} r$, and therefore the (σ,δ)-Galois groups for $\sigma(y) = ry$ and $\sigma(y) = by$ are the same. Consequently, by Lemma 3.5 and the assumption on the (σ,δ)-Galois group of $\sigma y = by$ over $\mathbf{L}$, for any nonzero $\mathcal{D} \in \mathcal{C}[\delta]$ and any $g \in \mathbf{K}$, we have $\mathcal{D}(\frac{\delta(r)}{r}) \neq \sigma(g) - g$. This is in contradiction with (3.2).

Assume now that $\mathrm{Gal}(\widetilde{R}/\mathbf{K})$ is not imprimitive, so it contains $\mathrm{SL}_2(\mathcal{C})$. By [DHR18, Proposition 2.10], we deduce that

$$G_m := \left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \;\middle|\; c \in \widetilde{\mathcal{C}}^* \right\} \subset \mathrm{Gal}^\delta(S/\mathbf{L}).$$

Let $n \in \mathbb{N}$ be as small as possible such that there exists $0 \neq P \in \mathbf{L}[X_0,\dots,X_n]$ with $P(f,\delta(f),\dots,\delta^n(f)) = 0$, and suppose that this P has smallest possible total degree $d \in \mathbb{N}$. For $c \in \widetilde{\mathcal{C}}^*$, let $\phi_c \in G_m$ with corresponding matrix $\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)$. For all $c \in \widetilde{\mathcal{C}}^*$, we find

$$\begin{aligned} \phi_c P(f,\delta(f),\dots,\delta^n(f)) &= P(\phi_c(f),\phi_c(\delta(f)),\dots,\phi_c(\delta^n(f))) \\ &= P(cf,\delta(cf),\dots,\delta^n(cf)) = 0. \end{aligned}$$

Since $\widetilde{\mathcal{C}}$ is differentially closed, there exists $c \in \widetilde{\mathcal{C}}^*$ such that $\delta(c) = 0$ and $c^d \neq 1$. Since $\delta^i(cf) = c\delta^i(f)$ for such a c, we have that

$$c^d P(f,\delta(f),\dots,\delta^n(f)) - P(cf,c\delta(f),\dots,c\delta^n(f)) = 0,$$

and we find that P must be homogeneous of degree d, for otherwise the total degree d would not be minimal. We may further assume that the degree d_n of X_n in P is as small as possible. Again since $\widetilde{\mathcal{C}}$ is differentially closed, there exists $c \in \widetilde{\mathcal{C}}$ such that $\delta^2(c) = 0$ but $\delta(c) \neq 0$. But then

$$\begin{aligned} 0 = P(cf,\delta(cf),\dots,\delta^n(cf)) &= P(cf,c\delta(f)+\delta(c)f,\dots,c\delta^n(f)+\delta(c)\delta^{n-1}(f)) \\ = c^d P(f,\delta(f),\dots,\delta^n(f)) &+ Q(f,\delta(f),\dots,\delta^n(f)) = Q(f,\delta(f),\dots,\delta^n(f)) \end{aligned}$$

for some nonzero homogeneous polynomial $Q \in \mathbf{L}[X_0,\dots,X_n]$ of total degree d in which the degree of X_n is strictly smaller than d_n. This contradiction concludes the proof. □

4. Difference equations over elliptic curves

In this section we will be mainly interested in difference equations

$$\sigma^2(y) + a\sigma(y) + by = 0, \tag{4.1}$$

with $a, b \in \mathrm{M}_p$, where

— M_p denotes the field of meromorphic functions over the elliptic curve $\mathbb{C}^*/p^{\mathbb{Z}}$ for some $p \in \mathbb{C}^*$ such that $|p| < 1$, *i.e.* the field of meromorphic functions on $\mathbb{C}^*$ satisfying $f(z) = f(pz)$;

— σ is the automorphism of M_p defined by

$$\sigma(f)(z) := f(qz)$$

for some $q \in \mathbb{C}^*$ such that $|q| \neq 1$ and $p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}$.

Note that this choice ensures that σ is non cyclic.

4.1. **The base field.** The difference Galois groups of linear difference equations over elliptic curves have been studied in [DR15]. In *loc. cit.* the elliptic curves are given by quotients of the form $\mathbb{C}/\Lambda$ for some lattice Λ. However, in the present work, we are mainly interested in difference equations on elliptic curves given by quotients of the form $\mathbb{C}^*/p^{\mathbb{Z}}$ for some $p \in \mathbb{C}^*$ such that $|p| < 1$. The translation between elliptic curves of the form $\mathbb{C}/\Lambda$ and elliptic curves of the form $\mathbb{C}^*/p^{\mathbb{Z}}$ is standard, namely by using the fact that if $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ with $\Im(\tau) > 0$ and $p = e^{2\pi i\tau}$ then the map $\mathbb{C} \to \mathbb{C}^* : w \mapsto e^{2\pi i w}$ induces an isomorphism $\mathbb{C}/\Lambda \simeq \mathbb{C}^*/p^{\mathbb{Z}}$.

We shall now recall some constructions and results from [DR15], restated in the "$\mathbb{C}^*/p^{\mathbb{Z}}$ context" *via* the above identification between $\mathbb{C}/\Lambda$ and $\mathbb{C}^*/p^{\mathbb{Z}}$. For $k \in \mathbb{N}^*$ we denote by $\mathbb{C}_k^*$ the Riemann surface of $z^{1/k}$, and we let z_k be a coordinate function on each $\mathbb{C}_k^*$ such that $z_{dk}^d = z_k$ for every $d \in \mathbb{N}^*$. We will write $\mathbb{C}_1^* = \mathbb{C}^*$ and $z_1 = z$.

We let $\mathrm{M}_{p,k}$ denote the field of meromorphic functions on $\mathbb{C}_k^*$ satisfying $f(pz_k) = f(z_k)$, or equivalently the field of meromorphic functions on the elliptic curve $\mathbb{C}_k^*/p^{\mathbb{Z}}$. The d-power map $\mathbb{C}_{dk}^* \to \mathbb{C}_k^* : \xi \mapsto \xi^d$ induces an inclusion of function fields $\mathrm{M}_{p,k} \hookrightarrow \mathrm{M}_{p,dk}$ for each $k, d \in \mathbb{N}^*$. We denote by $\mathbf{K}$ the field defined by

$$\mathbf{K} := \bigcup_{k \geq 1} \mathrm{M}_{p,k}\,.$$

We endow $\mathbf{K}$ with the non-cyclic field automorphism σ defined by

$$\sigma(f)(z_k) := f(q_k z_k) \tag{4.2}$$

where $q_1 = q \in \mathbb{C}^*$ is such that $|q| \neq 1$ and $p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}$, and $q_k \in \mathbb{C}_k^*$ defines a compatible system of k-th roots of $q_1 = q$ such that $q_{dk}^d = q_k$ for every $d \in \mathbb{N}^*$ (cf. [Hen98, Section 2]). Then $(\mathbf{K}, \sigma)$ is a difference field and we have the following properties.

Proposition 4.1 ([DR15], Proposition 5)**.** *The field of constants of* $(\mathbf{K}, \sigma)$ *is* $\mathbf{K}^\sigma = \mathbb{C}$.

Proposition 4.2 ([DR15], Proposition 6)**.** *The difference field* $(\mathbf{K}, \sigma)$ *satisfies property* $(\mathcal{P})$ *(see Definition 2.1).*

Remark 4.3. The field $\mathrm{M}_p = \mathrm{M}_{p,1}$ equipped with the automorphism σ does not satisfy property $(\mathcal{P})$. This is why we work over $(\mathbf{K}, \sigma)$ instead of (M_p, σ).

Corollary 4.4. *The conclusions of Theorem 2.3 are valid for* $(\mathbf{K}, \sigma)$.

4.2. **Theta functions.** We shall now recall some basic facts and notations about theta functions extracted from [DR15, Section 3] (but stated in the "$\mathbb{C}^*/p^{\mathbb{Z}}$ context", see the beginning of the previous section). For the proofs, we refer to [Mum07, Chapter I]. We still consider $p \in \mathbb{C}^*$ such that $|p| < 1$. We consider the infinite product

$$(z;p)_\infty = \prod_{j\geq 0}(1-zp^j).$$

The theta function defined by

$$\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty \tag{4.3}$$

satisfies

$$\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p). \tag{4.4}$$

Let Θ_k be the set of holomorphic functions on $\mathbb{C}_k^*$ of the form

$$c \prod_{\xi\in\mathbb{C}_k^*} \theta(\xi z_k)^{n_\xi}$$

with $c \in \mathbb{C}^*$ and $(n_\xi)_{\xi\in\mathbb{C}_k^*} \in \mathbb{N}^{(\mathbb{C}_k^*)}$ with finite support. We denote by Θ_k^{quot} the set of meromorphic functions on $\mathbb{C}_k^*$ that can be written as a quotient of two elements of Θ_k. We have

$$\mathrm{M}_{p,k} \subset \Theta_k^{quot}.$$

We define the divisor $\mathrm{div}_k(f)$ of $f \in \Theta_k^{quot}$ as the following formal sum of points of $\mathbb{C}_k^*/p^{\mathbb{Z}}$:

$$\mathrm{div}_k(f) := \sum_{\lambda\in\mathbb{C}_k^*/p^{\mathbb{Z}}} \mathrm{ord}_\lambda(f)[\lambda],$$

where $\mathrm{ord}_\lambda(f)$ is the $(z_k-\xi)$-adic valuation of f, for an arbitrary $\xi \in \lambda$ (it follows from (4.4) that this valuation does not depend on the chosen $\xi \in \lambda$). For any $\lambda \in \mathbb{C}_k^*/p^{\mathbb{Z}}$ and any $\xi \in \lambda$, we set

$$[\xi]_k := [\lambda].$$

Moreover, we will write

$$\sum_{\lambda\in\mathbb{C}_k^*/p^{\mathbb{Z}}} n_\lambda[\lambda] \;\leq\; \sum_{\lambda\in\mathbb{C}_k^*/p^{\mathbb{Z}}} m_\lambda[\lambda]$$

if $n_\lambda \leq m_\lambda$ for all $\lambda \in \mathbb{C}_k^*/p^{\mathbb{Z}}$. We also introduce the weight $\omega_k(f)$ of f defined by

$$\omega_k(f) := \prod_{\lambda\in\mathbb{C}_k^*/p^{\mathbb{Z}}} \lambda^{\mathrm{ord}_\lambda(f)} \in \mathbb{C}_k^*/p^{\mathbb{Z}}$$

and its degree $\deg_k(f)$ given by

$$\deg_k(f) := \sum_{\lambda\in\mathbb{C}_k^*/p^{\mathbb{Z}}} \mathrm{ord}_\lambda(f) \in \mathbb{Z}.$$

Example 4.5. Consider $\theta = \theta(z;p)$ defined above. Then it follows from (4.3) that $\mathrm{div}_1(\theta) = [1]$, since $\theta(z;p)$ has a zero of multiplicity one at each point of the subgroup $p^{\mathbb{Z}} \subset \mathbb{C}^*$. However, since $z = z_k^k$, we have that

$$\mathrm{div}_k(\theta) = \sum_{i,j=0}^{k-1} \left[\zeta_k^i \sqrt[k]{p^j}\right],$$

where $\zeta_k \in \mathbb{C}_k^*$ denotes a primitive k-th root of unity and $\sqrt[k]{p^j}$ is the j-th power of an arbitrary choice $\sqrt[k]{p}$ of k-th root of p.

Similarly, for any $f(z) \in \mathrm{M}_p = \mathrm{M}_{p,1}$ we have that $\mathrm{div}_k(f) = \varphi_k^*(\mathrm{div}_1(f))$, where $\varphi_k : \mathbb{C}_k^*/p^{\mathbb{Z}} \to \mathbb{C}^*/p^{\mathbb{Z}}$ denotes the k-power map and φ_k^* denotes the induced pull-back map on divisors.

4.3. **Irreducibility of the σ-Galois groups.** One of the criteria of Theorem 3.4 concerns the non-existence of a solution in $\mathbf{K}$ of a difference Riccati equation. The main tool used in this paper to address this is the following result.

Theorem 4.6 (Proposition 17 in [DR15])**.** *Let G be the σ-Galois group of (4.1) over $\mathbf{K}$. The following statements are equivalent:*
— *the group G is reducible;*
— *the following Riccati equation has a solution in $\mathrm{M}_{p,2}$:*

$$u\sigma(u) + au + b = 0. \tag{4.5}$$

Moreover, if $p_1 \in \Theta_2 \cup \{0\}$ and $p_2, p_3 \in \Theta_2$ are such that

$$a = \frac{p_1}{p_3} \text{ and } b = \frac{p_2}{p_3},$$

then any solution $u \in \mathrm{M}_{p,2}$ of (4.5) is of the form

$$u = \frac{\sigma(r_0)}{r_0}\frac{r_1}{r_2}$$

for some $r_0, r_1, r_2 \in \Theta_2$ such that
(i) $\mathrm{div}_2(r_1) \leq \mathrm{div}_2(p_2)$,
(ii) $\mathrm{div}_2(r_2) \leq \mathrm{div}_2(\sigma^{-1}(p_3))$,
(iii) $\deg_2(r_1) = \deg_2(r_2)$,
(iv) $\omega_2(r_1/r_2) = q_2^{\deg_2(r_0)} \mod p^{\mathbb{Z}}$.

5. Application to the elliptic hypergeometric functions

5.1. **The elliptic hypergeometric functions.** We shall now introduce the elliptic hypergeometric functions following [Spi16]. Consider $p, q \in \mathbb{C}^*$ such that $|p| < 1$, $|q| < 1$, and $q^{\mathbb{Z}} \cap p^{\mathbb{Z}} = \{1\}$. Define

$$(z; p, q)_\infty = \prod_{j,k \geq 0} (1 - zp^j q^k) \quad \text{and} \quad \Gamma(z; p, q) = \frac{(pq/z; p, q)_\infty}{(z; p, q)_\infty}.$$

We have

$$\Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q) \quad \text{and} \quad \Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q).$$

For $t_1, \ldots, t_8 \in \mathbb{C}^*$ such that $|t_j| < 1$ for each $1 \leq j \leq 8$ and satisfying the balancing condition $\prod_{j=1}^{8} t_j = p^2 q^2$, we set

$$V(t_1, \ldots, t_8; p, q) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^{8} \Gamma(t_j z; p, q)\Gamma(t_j/z; p, q)}{\Gamma(z^2; p, q)\Gamma(z^{-2}; p, q)} \frac{dz}{z}, \tag{5.1}$$

where $\mathbb{T}$ denotes the positively oriented unit circle and $\kappa = \frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}}$. For $z \in \mathbb{C}^*$, we follow [Spi16] by setting $t_6 = cz$, $t_7 = c/z$, and introducing new parameters

$$\varepsilon_j = \frac{q}{ct_j} \text{ for } j = 1, \dots, 5, \quad \varepsilon_8 = \frac{c}{t_8}, \quad \varepsilon_7 = \frac{\varepsilon_8}{q}, \quad c = \frac{\sqrt{\varepsilon_6 \varepsilon_8}}{p^2}. \tag{5.2}$$

We denote $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_8)$. Note that we still have the balancing condition

$$\prod_{j=1}^{8} \varepsilon_j = p^2 q^2. \tag{5.3}$$

Definition 5.1. The elliptic hypergeometric function $f_{\boldsymbol{\varepsilon}}(z)$ is defined by the following formula

$$f_{\boldsymbol{\varepsilon}}(z) := \frac{V(q/c\varepsilon_1, \dots, q/c\varepsilon_5, cz, c/z, c\varepsilon_8; p, q)}{\Gamma(c^2 z/\varepsilon_8; p, q)\Gamma(z/\varepsilon_8; p, q)\Gamma(c^2/z\varepsilon_8; p, q)\Gamma(1/z\varepsilon_8; p, q)}.$$

Remark 5.2. As explained in [Spi16], the function $V(\underline{t}; p, q)$ defined in (5.1) can be extended by analytic continuation, so that $\prod_{1 \le j < k \le 8} (t_j t_k; p, q)_\infty V(\underline{t}; p, q)$ is holomorphic for $t_1, \dots, t_8 \in \mathbb{C}^*$. We should also mention for completeness that, as explained in [Spi16], in Definition 5.1 it is initially necessary to impose the constraints (expressed in terms of the old parametrization) $\sqrt{|pq|} < |t_j| < 1$ for $j = 1, \dots, 5$ and $\sqrt{|pq|} < |q^{\pm 1} t_j| < 1$ for $j = 6, 7, 8$, which can then be relaxed by analytic continuation. These important but subtle considerations will not play a role in what follows.

5.2. **The elliptic hypergeometric equation.** The elliptic hypergeometric function $f_{\boldsymbol{\varepsilon}}(z)$ satisfies the following equation

$$A(z)(y(qz) - y(z)) + A(z^{-1})(y(q^{-1}z) - y(z)) + \nu y(z) = 0, \tag{5.4}$$

where

$$A(z) = \frac{1}{\theta(z^2; p)\theta(qz^2; p)} \prod_{j=1}^{8} \theta(\varepsilon_j z; p) \quad \text{and} \quad \nu = \prod_{j=1}^{6} \theta(\varepsilon_j \varepsilon_8/q; p). \tag{5.5}$$

It is easily seen that $A(pz) = A(z)$, so that the previous equation has coefficients in $\mathrm{M}_{p,1}$.

Replacing z by qz in (5.4), we obtain the following equation:

$$\sigma^2(y) + a\sigma(y) + by = 0, \tag{5.6}$$

with $a = \frac{\nu - A(qz) - A(q^{-1}z^{-1})}{A(qz)}, b = \frac{A(q^{-1}z^{-1})}{A(qz)} \in \mathrm{M}_{p,1}$.

Remark 5.3. Note that the new parameters $\varepsilon_1, \dots, \varepsilon_8$ used in the definition of $f_{\boldsymbol{\varepsilon}}(z)$ are not defined to be free independent parameters, since they are defined in terms of the old parameters $t_1, \dots, t_8$ (which are free parameters save for the balancing condition $\prod_{j=1}^{8} t_j = p^2 q^2$), and in fact one of the equations in the reparametrization (5.2) is equivalent to $\varepsilon_8 = \varepsilon_7 q$.

On the other hand, the elliptic hypergeometric equation (5.4) is defined for arbitrary parameters $\varepsilon_1, \dots, \varepsilon_8 \in \mathbb{C}^*$, subject only to the balancing condition (5.3), which is equivalent to imposing that the coefficients $A(z)$ and $A(z^{-1})$ actually belong to the field of elliptic functions $M_{p,1}$.

For this reason, we prove two related but distinct results on differential transcendence: (A) differential transcendence of solutions of the elliptic hypergeometric equation (5.4), where we think of the ε_j as free parameters subject only to the balancing condition (5.3) and without imposing the additional constraint $\varepsilon_8 = \varepsilon_7 q$; and (B) differential transcendence of the elliptic hypergeometric functions $f_{\boldsymbol{\varepsilon}}(z)$ where the ε_j are defined in terms of the t_j as in (5.2), and where in particular we do impose the additional constraint $\varepsilon_8 = \varepsilon_7 q$.

Note that in case (B) above the balancing condition (5.3) for the remaining independent parameters $\varepsilon_1, \dots, \varepsilon_7$ becomes

$$\left(\prod_{j=1}^{6} \varepsilon_j\right) \varepsilon_7^2 = p^2 q. \tag{5.7}$$

In the next lemma we show that in case (B) there are no universal relations among the parameters $\varepsilon_1, \dots, \varepsilon_7$ induced from the reparametrization (5.2), save for formal algebraic consequences of the balancing condition (5.7). This result ensures that the hypothesis in case (B) of Theorem 5.6 and Theorem 5.7 below are not vacuous.

Lemma 5.4. *Assume that case (B) holds. Every multiplicative relation among the $\varepsilon_1, \dots, \varepsilon_7, p, q$ is induced by (5.7), in the sense that if there are integers $\alpha_1, \dots, \alpha_7, m, n$ such that*

$$\prod_{j=1}^{7} \varepsilon_j^{\alpha_j} = p^m q^n,$$

then $\alpha_1 = \cdots = \alpha_6 = \alpha = n$ and $m = \alpha_7 = 2\alpha$ for some $\alpha \in \mathbb{Z}$.

Proof. Let us begin to write c and the ε_j in terms of the t_j. We have $c = \sqrt{t_6 t_7}$, and

$$\varepsilon_6 = \frac{c^2 p^4}{\varepsilon_8} = c p^4 t_8 = p^4 \sqrt{t_6 t_7}\, t_8, \quad \varepsilon_7 = \frac{\varepsilon_8}{q} = \frac{c}{q t_8} = \frac{\sqrt{t_6 t_7}}{q t_8}.$$

Assume now that there are integers $\alpha_1, \dots, \alpha_8, m, n$ such that

$$\prod_{j=1}^{7} \varepsilon_j^{\alpha_j} = p^m q^n.$$

Let us write this equality in term of t_j. The relation $\prod_{j=1}^{7} \varepsilon_j^{\alpha_j} = p^m q^n$ gives

$$\left(\prod_{j=1}^{5} \frac{q^{\alpha_j}}{(t_6 t_7)^{\alpha_j/2} t_j^{\alpha_j}}\right) p^{4\alpha_6} (t_6 t_7)^{(\alpha_6+\alpha_7)/2} t_8^{\alpha_6-\alpha_7} q^{-\alpha_7} = p^m q^n. \tag{5.8}$$

Using the balancing condition $\prod_{j=1}^{8} t_j = p^2 q^2$, we obtain the existence of an integer α such that

$$\alpha_1 = \cdots = \alpha_5 = \alpha.$$

Furthermore, regarding the terms in q, p, t_j, $j=6,7$, and t_8 respectively, we find

$$n-5\alpha+\alpha_7=-2\alpha,\quad m-4\alpha_6=-2\alpha,\\ -5\alpha/2+\alpha_6/2+\alpha_7/2=-\alpha,\quad \alpha_6-\alpha_7=-\alpha. \tag{5.9}$$

If we put the equality of the fourth relation $\alpha_6=\alpha_7-\alpha$ into the third, we obtain $2\alpha=\alpha_7$. With $\alpha_6=\alpha_7-\alpha$, we find $\alpha=\alpha_6$. Finally from the first and the second equality, we deduce $n=\alpha$ and $m=2\alpha$. ☐

Remark 5.5. Assume that case (B) holds. With Lemma 5.4 and the relation $\varepsilon_8=q\varepsilon_7$ it follows that if there are integers $\alpha_1,\dots,\alpha_8,m,n$ such that

$$\prod_{j=1}^{8}\varepsilon_j^{\alpha_j}=p^m q^n$$

then $\alpha_1=\dots=\alpha_6=\alpha=n-\alpha_8$ and $\alpha_7+\alpha_8=2\alpha=m$ for some $\alpha\in\mathbb{Z}$.

5.3. **Irreducibility of the σ-Galois group of the elliptic hypergeometric function.** From now on, we denote by G the σ-Galois group of (5.6) over $\mathbf{K}$ (with respect to some σ-PV ring).

Theorem 5.6. *Assume one of the two hypotheses (A) or (B) below.*

(A) Every multiplicative relation among the $\varepsilon_1,\dots,\varepsilon_8,p,q$ is induced by (5.3), in the sense that if there are integers $\alpha_1,\dots,\alpha_8,m,n$ such that

$$\prod_{j=1}^{8}\varepsilon_j^{\alpha_j}=p^m q^n$$

then $\alpha_1=\dots=\alpha_8=:\alpha$ and $m=n=2\alpha$ for some $\alpha\in\mathbb{Z}$.

(B) $\varepsilon_8=\varepsilon_7 q$ and every multiplicative relation among the $\varepsilon_1,\dots,\varepsilon_7,p,q$ is induced by (5.7), in the sense that if there are integers $\alpha_1,\dots,\alpha_8,m,n$ such that

$$\prod_{j=1}^{8}\varepsilon_j^{\alpha_j}=p^m q^n$$

then $\alpha_1=\dots=\alpha_6=\alpha=n-\alpha_8$ and $\alpha_7+\alpha_8=2\alpha=m$ for some $\alpha\in\mathbb{Z}$.

Then G is irreducible.

Proof. To the contrary, assume that G is reducible. According to Theorem 4.6, the following Riccati equation has a solution $u\in \mathrm{M}_{p,2}$:

$$u\sigma(u)+au+b=0. \tag{5.10}$$

First, note that $u\in\mathrm{M}_{p,2}$ is a solution of (5.10) if and only if $v(\sigma(v)+\sigma^{-1}(a))+\sigma^{-1}(b)=0$ with $v=\sigma^{-1}(u)\in\mathbf{K}$. Then to simplify the expression of the divisors of a and b, we may replace them by $\sigma^{-1}(a)=\frac{\nu-A(z)-A(z^{-1})}{A(z)}$, $\sigma^{-1}(b)=\frac{A(z^{-1})}{A(z)}$, and consider the Riccati equation satisfied by v. Consider $p_1\in\Theta_2\cup\{0\}$ and $p_2,p_3\in\Theta_2$ such that

$$\sigma^{-1}(a)=\frac{p_1}{p_3}\text{ and }\sigma^{-1}(b)=\frac{p_2}{p_3}.$$

In view of the explicit expressions for $\sigma^{-1}(a)$ and $\sigma^{-1}(b)$, we see that we may take p_2 and p_3 such that

$$\begin{aligned}\operatorname{div}_2(p_2) =& \sum_{j=1}^{8} \left[\sqrt{\varepsilon_j}\right] + \left[-\sqrt{\varepsilon_j}\right] + \left[\sqrt{p\varepsilon_j}\right] + \left[-\sqrt{p\varepsilon_j}\right] \\ &+ \sum_{j=0}^{3} \left[\sqrt[4]{p^j/q}\right] + \left[-\sqrt[4]{p^j/q}\right] + \left[\mathrm{i}\sqrt[4]{p^j/q}\right] + \left[-\mathrm{i}\sqrt[4]{p^j/q}\right]\end{aligned}$$

and

$$\begin{aligned}\operatorname{div}_2(p_3) =& \sum_{j=1}^{8} \left[\sqrt{1/\varepsilon_j}\right] + \left[-\sqrt{1/\varepsilon_j}\right] + \left[\sqrt{p/\varepsilon_j}\right] + \left[-\sqrt{p/\varepsilon_j}\right] \\ &+ \sum_{j=0}^{3} \left[\sqrt[4]{qp^j}\right] + \left[-\sqrt[4]{qp^j}\right] + \left[\mathrm{i}\sqrt[4]{qp^j}\right] + \left[-\mathrm{i}\sqrt[4]{qp^j}\right].\end{aligned}$$

We note for convenience that

$$\begin{aligned}\operatorname{div}_2(\sigma^{-1}(p_3)) =& \sum_{j=1}^{8} \left[\sqrt{q/\varepsilon_j}\right] + \left[-\sqrt{q/\varepsilon_j}\right] + \left[\sqrt{qp/\varepsilon_j}\right] + \left[-\sqrt{qp/\varepsilon_j}\right] \\ &+ \sum_{j=0}^{3} \left[\sqrt{q}\sqrt[4]{qp^j}\right] + \left[-\sqrt{q}\sqrt[4]{qp^j}\right] + \left[\mathrm{i}\sqrt{q}\sqrt[4]{qp^j}\right] + \left[-\mathrm{i}\sqrt{q}\sqrt[4]{qp^j}\right].\end{aligned}$$

We now consider $r_0, r_1, r_2 \in \Theta_2$ as in Theorem 4.6. For $i = 1, 2$, let

$$\mathcal{S}_i := \{\lambda \in \mathbb{C}_2^*/p^{\mathbb{Z}} \mid \operatorname{ord}_\lambda(r_i) \neq 0\}$$

denote the support of $\operatorname{div}_2(r_i)$. For each $j \in \{1, \ldots, 8\}$ we let $\alpha_j \in \mathbb{N}$ denote the number of points in $\mathcal{S}_1$ of the form $\pm\sqrt{\varepsilon_j}$ or $\pm\sqrt{p\varepsilon_j}$. Similarly, for each $j \in \{1, \ldots, 8\}$ we let $\alpha'_j \in \mathbb{N}$ denote the number of points in $\mathcal{S}_2$ of the form $\pm\sqrt{q/\varepsilon_j}$ or $\pm\sqrt{qp/\varepsilon_j}$. We find that there exist $\ell_1, \ell_2 \in \{0, 1, 2, 3\}$ and $\gamma \in \mathbb{N}$ such that

$$\omega_2(r_1/r_2) = \mathrm{i}^{\ell_1} \sqrt[4]{p}^{\ell_2} \prod_{j=1}^{8} \sqrt{\varepsilon_j}^{\alpha_j+\alpha'_j} \sqrt{q}^{-\deg_2(r_2)} \sqrt[4]{q}^{-\gamma} = \sqrt{q}^{\deg_2(r_0)} \mod p^{\mathbb{Z}},$$

where the second equality is obtained from property (iv) of Theorem 4.6. After taking fourth powers we see that

$$\prod_{j=1}^{8} \varepsilon_j^{2\alpha_j+2\alpha'_j} = p^m q^{2\deg_2(r_2)+\gamma+2\deg_2(r_0)} \tag{5.11}$$

for some $m \in \mathbb{Z}$. We now claim that v is constant.

Suppose first that we are in case (A). Since every multiplicative relation among the $\varepsilon_1, \ldots, \varepsilon_8, p, q$ is induced by (5.3), it follows from (5.11) that there exists $\alpha \in \mathbb{N}$ such that $2\alpha_j + 2\alpha'_j = \alpha$ for every $j \in \{1, \ldots, 8\}$ and $m = 2\deg_2(r_2)+\gamma+2\deg_2(r_0) = 2\alpha$. In particular, we have that $2\deg_2(r_2) \leq 2\alpha$. On the other hand, it follows from properties (i) and (ii) of Theorem 4.6, respectively, that $\alpha_1 + \cdots + \alpha_8 \leq \deg_2(r_1)$ and $\alpha'_1 + \cdots + \alpha'_8 \leq \deg_2(r_2)$. We note that by property (iii) of Theorem 4.6 $2\deg_2(r_2) = \deg_2(r_1) + \deg_2(r_2)$. Putting together these inequalities we obtain

$$4\alpha = \sum_{j=1}^{8} \alpha_j + \alpha'_j \leq \deg_2(r_1) + \deg_2(r_2) = 2\deg_2(r_2) \leq 2\alpha.$$

It follows from this that $\alpha = \deg_2(r_1) = \deg_2(r_2) = 0$. Hence, r_1/r_2 is constant and

$$\omega_2(r_1/r_2) = 1 = \sqrt{q}^{\deg_2(r_0)} \mod p^{\mathbb{Z}}$$

by property (iv) of Theorem 4.6. Since $p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}$, we see that $\deg_2(r_0) = 0$ also.

Now suppose we are in case (B). Since every multiplicative relation among the $\varepsilon_1, \dots, \varepsilon_8, p, q$ is induced by (5.7), it follows from (5.11) that there exists $\alpha \in \mathbb{N}$ such that $2\alpha_j + 2\alpha'_j = \alpha = 2\deg_2(r_2) + \gamma + 2\deg_2(r_0) - (2\alpha_8 + 2\alpha'_8)$ for every $j \in \{1, \dots, 6\}$, and $2\alpha_7 + 2\alpha'_7 + 2\alpha_8 + 2\alpha'_8 = 2\alpha = m$. It follows from the second set of equations that $2\alpha_8 + 2\alpha'_8 \leq 2\alpha$. From this and the first set of equations it then follows that $2\deg_2(r_2) \leq 3\alpha$. On the other hand, it follows from properties (i) and (ii) of Theorem 4.6, respectively, that $\alpha_1 + \cdots + \alpha_8 \leq \deg_2(r_1)$ and $\alpha'_1 + \cdots + \alpha'_8 \leq \deg_2(r_2)$. We note that by property (iii) of Theorem 4.6, $2\deg_2(r_2) = \deg_2(r_1) + \deg_2(r_2)$. Putting together these inequalities we obtain

$$4\alpha = \sum_{j=1}^{8} \alpha_j + \alpha'_j \leq \deg_2(r_1) + \deg_2(r_2) = 2\deg_2(r_2) \leq 3\alpha.$$

It follows from this that $\alpha = \deg_2(r_1) = \deg_2(r_2) = 0$. Hence, r_1/r_2 is constant and

$$\omega_2(r_1/r_2) = 1 = \sqrt{q}^{\deg_2(r_0)} \mod p^{\mathbb{Z}}$$

by property (iv) of Theorem 4.6. Since $p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}$, we see that $\deg_2(r_0) = 0$ also.

It follows from the above in either of the cases (A) or (B) that $v \in \mathbb{C}^*$ is constant. Therefore (5.10) can be rewritten as

$$v^2 A(z) + v(\nu - A(z) - A(z^{-1})) + A(z^{-1}) = 0, \tag{5.12}$$

i.e.

$$(v^2 - v)A(z) + v\nu = (v - 1)A(z^{-1}). \tag{5.13}$$

But since $\sqrt{q}^{-1}$ is a pole of $A(z)$ but not of $A(z^{-1})$ and, on the other hand, $\sqrt{q}$ is a pole of $A(z^{-1})$ but not of $A(z)$, we obtain that $v^2 - v = v - 1 = v\nu = 0$. So we must have $\nu = 0$. On the other hand, we see from the definition of ν in (5.5) that $\nu = 0$ if and only if $\varepsilon_j \varepsilon_8 = qp^\ell$ for some $\ell \in \mathbb{Z}$ and $j = 1, \dots, 6$, which is ruled out by our hypotheses in both cases (A) and (B). This contradiction concludes the proof that G is irreducible. □

5.4. **Differential transcendence of the elliptic hypergeometric functions.** We may equip $(\mathbf{K}, \sigma)$ with the classical derivation $\delta := z\frac{d}{dz}$ as in [DHRS18, Section 3.1]. Note that δ commutes with σ. Let $\widetilde{\mathbb{C}}$ be the δ-closure of $\mathbb{C}$. Following Lemma 3.1, we may consider $\mathbf{L} := \mathrm{Frac}(\mathbf{K} \otimes_{\mathbb{C}} \widetilde{\mathbb{C}})$ and we have $\mathbf{L}^\sigma = \widetilde{\mathbb{C}}$. Recall that $f_{\varepsilon}(z)$ is meromorphic on $\mathbb{C}^*$, and note that the field of meromorphic functions on $\mathbb{C}^*$ is a (σ, δ)-extension of $\mathbf{K}$.

Theorem 5.7. *Assume one of the two hypotheses (A) or (B) below.*

(A) Every multiplicative relation among the $\varepsilon_1, \dots, \varepsilon_8, p, q$ *is induced by* (5.3)*, in the sense that if there are integers* $\alpha_1, \dots, \alpha_8, m, n$ *such that*

$$\prod_{j=1}^{8} \varepsilon_j^{\alpha_j} = p^m q^n$$

then $\alpha_1 = \dots = \alpha_8 =: \alpha$ *and* $m = n = 2\alpha$ *for some* $\alpha \in \mathbb{Z}$*.*

(B) $\varepsilon_8 = \varepsilon_7 q$ *and every multiplicative relation among the* $\varepsilon_1, \dots, \varepsilon_7, p, q$ *is induced by* (5.7)*, in the sense that if there are integers* $\alpha_1, \dots, \alpha_8, m, n$ *such that*

$$\prod_{j=1}^{8} \varepsilon_j^{\alpha_j} = p^m q^n$$

then $\alpha_1 = \dots = \alpha_6 = \alpha = n - \alpha_8$ *and* $\alpha_7 + \alpha_8 = 2\alpha = m$ *for some* $\alpha \in \mathbb{Z}$*.*

Then any non-zero solution to (5.4) *is differentially transcendental over* $\mathbf{K}$*.*

Proof. We apply the criteria of Theorem 3.4. We proved in Theorem 5.6 that G is irreducible, which by [DR15, Lemma 13] is equivalent to the non-existence of a solution $u \in \mathbf{K}$ to the Riccati equation

$$u\sigma(u) + au + b = 0.$$

It remains to show that there is no nonzero linear differential operator $\mathcal{L}$ in δ with coefficients in $\mathbb{C}$ and $g \in \mathbf{K}$ such that

$$\mathcal{L}\left(\frac{\delta b}{b}\right) = \sigma(g) - g.$$

Let $k \in \mathbb{N}^*$ such that $g \in \mathrm{M}_{p,k}$ and consider b as an element of $\mathrm{M}_{p,k}$. Let $\omega \in \mathbb{C}_k^*/p^{\mathbb{Z}}$ be a zero or a pole of b. Then it is a pole of $\frac{\delta b}{b}$. Since $\mathcal{L}$ has constant coefficients, we get that ω is also a pole of $\mathcal{L}\left(\frac{\delta b}{b}\right)$. Therefore, ω is a pole of $\sigma(g) - g$ and hence also a pole of $\sigma(g)$ or of g. Furthermore, $\sigma(g) - g$ has at least two distinct poles $\omega', \omega'' \in \mathbb{C}_k^*/p^{\mathbb{Z}}$ such that $\omega \equiv \omega' \equiv \omega''$ mod $q_k^{\mathbb{Z}}$, where $q_k \in \mathbb{C}_k^*$ is as in (4.2). These ω' and ω'' are poles of $\frac{\delta b}{b}$, and hence zeros or poles of b has well. We have proved that, for every $\omega \in \mathbb{C}_k^*/p^{\mathbb{Z}}$ that is a pole or zero of b, there exists $\ell \in \mathbb{Z}_{\neq 0}$ such that ωq_k^{ℓ} is a pole or zero of b.

Let us now consider b as an element of $\mathrm{M}_{p,1}$. From the preceding, we deduce that for every $\omega \in \mathbb{C}^*/p^{\mathbb{Z}}$, pole or zero of b, there exists $\ell \in \mathbb{Z}_{\neq 0}$ such that ωq^{ℓ} is a pole or zero of b. We will use this to find a contradiction. Note that the set of zeros or poles of $b = \frac{\theta(q^2z^2;p)\theta(q^3z^2;p)}{\theta(q^{-2}z^{-2};p)\theta(q^{-1}z^{-2};p)} \times \prod_{j=1}^{8} \frac{\theta(\varepsilon_j q^{-1} z^{-1}; p)}{\theta(\varepsilon_j q z; p)}$, seen as an element of $\mathrm{M}_{p,1}$, is included in

$$\mathcal{S} = \{q^{-1}\varepsilon_1^{\pm 1}, \dots, q^{-1}\varepsilon_8^{\pm 1}, \pm q^{-1/2}, \pm q^{-1/2}\sqrt{p}, \pm q^{-3/2}, \pm q^{-3/2}\sqrt{p}\} \mod p^{\mathbb{Z}}.$$

Let us prove that the elements of $\mathcal{S}$ are all distinct. To see this, note that if any two elements of $\mathcal{S}$ were the same modulo $p^{\mathbb{Z}}$ then we would find a non-trivial multiplicative relation satisfied by at most four elements among $p, q, \varepsilon_1, \dots, \varepsilon_8$. This contradicts the hypothesis in both cases (A) and (B). Therefore, no simplifications occur and $\mathcal{S}$ is exactly the set of zeros or poles of b. It suffices to show that for all $\ell \in \mathbb{Z}_{\neq 0}$, we have $\mathcal{S} \cap \{q^{\ell} q^{-1}\varepsilon_1 \mod p^{\mathbb{Z}}\} = \varnothing$. Let $\ell \in \mathbb{Z}$ such that $\mathcal{S} \cap \{q^{\ell} q^{-1}\varepsilon_1 \mod p^{\mathbb{Z}}\} \neq \varnothing$. If $\ell \neq 0$, then we again find a non-trivial multiplicative relation satisfied by at most four elements among $p, q, \varepsilon_1, \ldots, \varepsilon_8$. In either case (A) or case (B) this contradiction to the hypothesis concludes the proof. □

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