ON THE KERNEL CURVES ASSOCIATED WITH WALKS IN THE QUARTER PLANE

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Abstract. The kernel method is an essential tool for the study of generating series of walks in the quarter plane. This method involves equating to zero a certain polynomial - the kernel polynomial - and using properties of the curve - the kernel curve - this defines. In the present paper, we investigate the basic properties of the kernel curve (irreducibility, singularties, genus, parametrization, *etc*).

Contents

Introduction

We consider a weighted walk in the quarter plane $\mathbb{Z}^2_{\geq 0}$ satisfying the following properties:

- it starts at $(0,0)$;
- it takes steps in a certain subset of the set of cardinal directions.

The weights of such a walk are certain elements $d_{i,j}$ of $\mathbb{Q} \cap [0,1]$ indexed by $(i,j) \in \{0,\pm 1\}^2$ such that $\sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j} = 1$. For $(i,j) \in \{0,\pm1\}^2 \backslash \{(0,0)\}$ (resp. $(0,0)$), $d_{i,j}$ is a weight on the step (i,j) which can be viewed as the probability for the

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walk to go in the direction (i,j) (resp. to stay at the same position)*. The step set or the *model* of the walk is the set of directions with nonzero weights, that is

$$\{(i,j)\in\{0,\pm1\}^2\backslash\{(0,0)\} \mid d_{i,j}\neq 0\}.$$

If $d_{0,0}=0$ and if the nonzero $d_{i,j}$ all have the same value, we say that the model is unweighted.

The *weight of a given walk* is defined to be the product of the weights of its component steps. For any $(i,j)\in\mathbb{Z}_{\geq 0}^2$ and any $k\in\mathbb{Z}_{\geq 0}$, we let $q_{i,j,k}$ be the sum of the weights of all walks reaching the position (i,j) from the initial position $(0,0)$ after k steps. We introduce the corresponding trivariate generating series†

$$Q(x,y,t):=\sum_{i,j,k\geq 0} q_{i,j,k}x^iy^jt^k.$$

The study of the nature of this generating series has attracted the attention of many authors, see for instance [BvHK10, BRS14, BBMR15, BBMR17, BMM10, DHRS18, DHRS20, DR19, DH19, KR12, Mis09, MR09, MM14, Ras12]. The typical questions are: is $Q(x,y,t)$ rational, algebraic, holonomic, *etc*? The starting point of most of these works is the following functional equation (see for instance [DHRS20, Lemma 1.1], and [BMM10] for the unweighted case)

$$K(x,y,t)Q(x,y,t)=xy+K(x,0,t)Q(x,0,t)+K(0,y,t)Q(0,y,t)+td_{-1,-1}Q(0,0,t)$$

where

$$K(x,y,t)=xy(1-tS(x,y))$$

with

$$S(x,y)=\sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j}x^iy^j.$$

The polynomial $K(x,y,t)$ is called the kernel polynomial and is the main character of the kernel method.

Roughly speaking, the first step of the kernel method consists in "eliminating" the left hand side of the above functional equation by restricting our attention to the (x,y) such that $K(x,y,t)=0$. The set E_t made of the (x,y) such that $K(x,y,t)=0$ is called the kernel curve:

$$E_t=\{(x,y)\in\mathbb{C}\times\mathbb{C} \mid K(x,y,t)=0\}.$$

Thus, for $(x,y)\in E_t$, one has

$$0=xy+K(x,0,t)Q(x,0,t)+K(0,y,t)Q(0,y,t)+td_{-1,-1}Q(0,0,t), \tag{1}$$

provided that the various series can be evaluated at the given points.

The second step of the kernel method is to exploit certain involutive birational transformations ι_1,ι_2 of the kernel curve E_t of the form

$$\iota_1(x,y)=(x,y') \text{ and } \iota_2(x,y)=(x',y)$$

in order to deduce from the latter equation some functional equations for $Q(x,0,t)$ and $Q(0,y,t)$. Hence ι_1 and ι_2 switch the roots of the degree two polynomials $y\mapsto K(x,y,t)$

*The eight cardinal directions will be identified with the elements of $\{0,\pm1\}^2\setminus\{(0,0)\}$.

†In several papers it is not assumed that $\sum_{i,j} d_{i,j}=1$. But after a rescaling of the t variable, we may always reduce to the case $\sum_{i,j} d_{i,j}=1$.

and $x \mapsto K(x,y,t)$ respectively. Concretely, the birational transformations ι_1, ι_2 are induced by restriction to the curve of the involutive birational transformations i_1, i_2 of $\mathbb{C}^2$ given by

$$i_1(x,y) = \left(x, \frac{A_{-1}(x)}{A_1(x)y}\right) \text{ and } i_2(x,y) = \left(\frac{B_{-1}(y)}{B_1(y)x}, y\right)$$

where the $A_i(x) \in x^{-1}\mathbb{Q}[x]$ and the $B_i(y) \in y^{-1}\mathbb{Q}[y]$ are defined by

$$S(x,y) = A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x)y = B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y)x$$

(see [BMM10, Section 3], [KY15, Section 3] or [FIM17]). These i_1 and i_2 are the generators of the group of the walk; see [BMM10] for details.

The third step of the kernel method is to use the above mentioned functional equations of $Q(x,0,t)$ and $Q(0,y,t)$ to continue these series as multivalued meromorphic functions on the kernel curve. To perform this step, we need an explicit parametrization of the kernel curve.

The aim of the present paper is to study the kernel curve E_t and the birational transformations ι_1, ι_2. Note that a similar study has been done in the case $t = 1$ in [FIM17] and in the unweighted case in [KR12]. The goal of the present paper is to extend these works to the weighted case.

The paper is organized as follows. In Section 1, we describe the nondegenerate models of walks. In Section 2, we determine the singularities and the genus of the kernel curve. In Section 3, we establish the basic properties of ι_1 and ι_2. Finally, in Section 4, we give an explicit parametrization of the kernel curve.

1. Nondegenerate walks

From now on, we let t be a transcendental number in $]0,1[$. We start by recalling the notion of *degenerate walks introduced in* [FIM17].

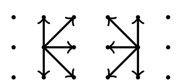
Definition 1.1. A model of walk is called *degenerate* if one of the following holds:

- $K(x,y,t)$ is reducible as an element of the polynomial ring $\mathbb{C}[x,y]$,
- $K(x,y,t)$ has x-degree less than or equal to 1,
- $K(x,y,t)$ has y-degree less than or equal to 1.

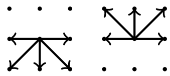
The following result is the analog of [FIM17, Lemma 2.3.2], that focuses on the case $t = 1$.

Proposition 1.2. *A model of walk is degenerate if and only if at least one of the following holds:*

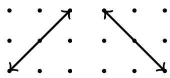
(1) *There exists* $i \in \{-1,1\}$ *such that* $d_{i,-1} = d_{i,0} = d_{i,1} = 0$. *This corresponds to models of walks with steps supported in one of the following configurations*



(2) *There exists* $j \in \{-1, 1\}$ *such that* $d_{-1,j} = d_{0,j} = d_{1,j} = 0$. *This corresponds to models of walks with steps supported in one of the following configurations*



(3) *All the weights are* 0 *except maybe* $\{d_{1,1}, d_{0,0}, d_{-1,-1}\}$ *or* $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$. *This corresponds to models of walks with steps supported in one of the following configurations*



Proof. This proof is organized as follows. We begin by showing that (1) (resp. (2)) corresponds to $K(x, y, t)$ having x-degree ≤ 1 or x-valuation ≥ 1 (resp. y-degree ≤ 1 or y-valuation ≥ 1). In these cases, the model of the walk is clearly degenerate. Assuming (1) and (2) do not hold, we then show that (3) holds if and only if $K(x, y, t)$ is reducible.

Cases (1) and (2). It is clear that $K(x, y, t)$ has x-degree ≤ 1 if and only if $d_{1,-1} = d_{1,0} = d_{1,1} = 0$. Similarly, $K(x, y, t)$ has y-degree ≤ 1 if and only if we have $d_{-1,1} = d_{0,1} = d_{1,1} = 0$. Furthermore, $d_{-1,-1} = d_{-1,0} = d_{-1,1} = 0$ if and only if $K(x, y, t)$ has x-valuation ≥ 1. Similarly, $d_{-1,-1} = d_{0,-1} = d_{1,-1} = 0$ if and only if $K(x, y, t)$ has y-valuation ≥ 1. In these cases, the model of the walk is clearly degenerate.

Case (3). We now assume that cases (1) and (2) do not hold.

If the model of the walk has steps supported in $\{\swarrow, \nearrow\}$ (note that this implies that $d_{1,1} \neq 0$), then the kernel

$$K(x, y, t) = -d_{-1,-1}t + xy - d_{0,0}txy - d_{1,1}tx^2y^2 \in \mathbb{C}[xy]$$

is a degree two polynomial in xy. Thus it may be factorized in the following form $K(x, y, t) = -d_{1,1}t(xy - \alpha)(xy - \beta)$ for some $\alpha, \beta \in \mathbb{C}$. If the model of the walk has steps supported in $\{\nwarrow, \searrow\}$, then

$$K(x, y, t) = -d_{-1,1}ty^2 + xy - d_{0,0}txy - d_{1,-1}tx^2.$$

In this situation, $K(x, y, t)y^{-2} \in \mathbb{C}[x/y]$ may be factorized in the ring $\mathbb{C}[x/y]$, proving that $K(x, y, t)$ may be factorized in $\mathbb{C}[x, y]$ as well.

Conversely, let us assume that the model of the walk is degenerate. Recall that we have assumed that cases (1) and (2) do not hold, so $K(x, y, t)$ has x- and y-degree two, x- and y-valuation 0, and is reducible. We have to prove that the model of the walk has steps supported by $\{\searrow, \nwarrow\}$ or $\{\swarrow, \nearrow\}$. Let us write a factorization

$$K(x, y, t) = -f_1(x, y)f_2(x, y),$$

with $f_1(x, y), f_2(x, y) \in \mathbb{C}[x, y]$ not constant.

We claim that both $f_1(x, y)$ *and* $f_2(x, y)$ *have bidegree* $(1, 1)$. Suppose to the contrary that $f_1(x, y)$ or $f_2(x, y)$ does not have bidegree $(1, 1)$. Since K is of bidegree at most $(2, 2)$ then at least one of the f_i's has degree 0 in x or y. Up to interchange of x and y and f_1 and f_2, we may assume that $f_1(x, y)$ has y-degree 0 and we denote it by $f_1(x)$. Since $K(x, y, t) = -f_1(x)f_2(x, y)$, we find in particular that $f_1(x)$ is a common factor of the nonzero polynomials $d_{-1,-1}t + d_{0,-1}tx + d_{1,-1}tx^2$ and

$d_{-1,0}t + (d_{0,0}t - 1)x + d_{1,0}tx^2$ (these polynomials are nonzero because we are not in Cases (1) and (2) of Proposition 1.2). Since t is transcendental and the $d_{i,j}$ are algebraic, we find that the roots of $d_{-1,-1}t + d_{0,-1}tx + d_{1,-1}tx^2 = 0$ are algebraic, while the roots of $d_{-1,0}t + (d_{0,0}t - 1)x + d_{1,0}tx^2 = 0$ are transcendental. Therefore, they are polynomials with no common roots, and must be relatively prime, showing that $f_1(x)$ has degree 0, *i.e.* $f_1(x) \in \mathbb{C}$. This contradicts $f_1(x,y)$ not constant and shows the claim.

We claim that $f_1(x,y)$ and $f_2(x,y)$ are irreducible in the ring $\mathbb{C}[x,y]$. If not, then we find $f_1(x,y) = (ax-b)(cy-d)$ for some $a,b,c,d \in \mathbb{C}$. Since $f_1(x,y)$ has bidegree $(1,1)$, we have $ac \neq 0$. We then have that

$$0 = K(b/a, y, t) = \frac{b}{a}y - t(\tilde{A}_{-1}(\frac{b}{a}) + \tilde{A}_0(\frac{b}{a})y + \tilde{A}_1(\frac{b}{a})y^2)$$

where $\tilde{A}_i = xA_i \in \mathbb{Q}[x]$. Equating the y^2-terms we find that $\tilde{A}_1(\frac{b}{a}) = 0$ so $\frac{b}{a} \in \overline{\mathbb{Q}}$ (note that $\tilde{A}_1(x)$ is nonzero because $K(x,y,t)$ has bidegree $(2,2)$). Equating the y-terms, we obtain that $\frac{b}{a} - t\tilde{A}_0(\frac{b}{a}) = 0$. Using $t \notin \overline{\mathbb{Q}}$[‡] and $\frac{b}{a} \in \overline{\mathbb{Q}}$ we deduce $\frac{b}{a} = 0$. Therefore $b = 0$. This contradicts the fact that K has x-valuation 0. A similar argument shows that $f_2(x,y)$ is irreducible.

Let $\overline{f}_i(x,y)$ denote the polynomial whose coefficients are the complex conjugates of those of $f_i(x,y)$. Unique factorization of polynomials implies that since $-K(x,y,t) = f_1(x,y)f_2(x,y) = \overline{f_1}(x,y)\overline{f_2}(x,y)$, there exists $\lambda \in \mathbb{C}^*$ such that

- either $\overline{f_1}(x,y) = \lambda f_2(x,y)$ and $\overline{f_2}(x,y) = \lambda^{-1} f_1(x,y)$;
- or $\overline{f_1}(x,y) = \lambda f_1(x,y)$ and $\overline{f}_2(x,y) = \lambda^{-1} f_2(x,y)$.

In the former case, we have $f_1(x,y) = \overline{\lambda}\,\overline{f_2}(x,y) = \overline{\lambda}\lambda^{-1} f_1(x,y)$ and so $\overline{\lambda}\lambda^{-1} = 1$. This implies that λ is real and replacing $f_1(x,y)$ by $|\lambda|^{-1/2} f_1(x,y)$ and $f_2(x,y)$ by $|\lambda|^{1/2} f_2(x,y)$, we can assume that either $\overline{f_1}(x,y) = f_2(x,y)$ and $\overline{f_2}(x,y) = f_1(x,y)$ or $\overline{f_1}(x,y) = -f_2(x,y)$ and $\overline{f_2}(x,y) = -f_1(x,y)$.

A similar computation in the latter case shows that $|\lambda| = 1$. Letting μ be a square root of λ we have $\mu^{-1} = \overline{\mu}$ so $\lambda = \mu/\overline{\mu}$. Replacing $f_1(x,y)$ by $\mu f_1(x,y)$ and $f_2(x,y)$ by $\overline{\mu} f_2(x,y)$, we can assume that $\overline{f_1}(x,y) = f_1(x,y)$ and $\overline{f_2}(x,y) = f_2(x,y)$.

To summarize, we have two possibilities:

- there exists $\epsilon \in \{\pm 1\}$ such that $\overline{f_1}(x,y) = \epsilon f_2(x,y)$, or
- $\overline{f_1}(x,y) = f_1(x,y) \in \mathbb{R}[x,y]$ and $\overline{f_2}(x,y) = f_2(x,y) \in \mathbb{R}[x,y]$.

For $i = 1, 2$, let us write

$$f_i(x,y) = (\alpha_{i,4}x + \alpha_{i,3})y + (\alpha_{i,2}x + \alpha_{i,1}),$$

[‡] We have denoted by $\overline{\mathbb{Q}}$ the field of algebraic numbers.

with $\alpha_{i,j} \in \mathbb{C}$. Equating the terms in $x^i y^j$ with $-1 \le i,j \le 1$, in $f_1(x,y)f_2(x,y) = -K(x,y,t)$, we find (recall that $d_{i,j} \in [0,1]$, $t \in]0,1[$)

term	coefficient in $f_1(x,y)f_2(x,y)$	coefficient in $-K(x,y,t)$
1	$\alpha_{1,1}\alpha_{2,1}$	$d_{-1,-1}t \ge 0$
x	$\alpha_{1,2}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,2}$	$d_{0,-1}t \ge 0$
x^2	$\alpha_{1,2}\alpha_{2,2}$	$d_{1,-1}t \ge 0$
y	$\alpha_{1,3}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,3}$	$d_{-1,0}t \ge 0$
xy	$\alpha_{1,4}\alpha_{2,1} + \alpha_{1,3}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,3} + \alpha_{1,1}\alpha_{2,4}$	$d_{0,0}t - 1 < 0$
x^2y	$\alpha_{1,4}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,4}$	$d_{1,0}t \ge 0$
y^2	$\alpha_{1,3}\alpha_{2,3}$	$d_{-1,1}t \ge 0$
xy^2	$\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4}$	$d_{0,1}t \ge 0$
x^2y^2	$\alpha_{1,4}\alpha_{2,4}$	$d_{1,1}t \ge 0$

Let us treat separately two cases.

Case 1: $f_1(x,y), f_2(x,y) \notin \mathbb{R}[x,y]$**.** So, in this case we have either $\overline{f_1}(x,y) = f_2(x,y)$ or $\overline{f_1}(x,y) = -f_2(x,y)$.

Let us first assume that $\overline{f_1}(x,y) = f_2(x,y)$. Then, evaluating the equality $K(x,y,t) = -f_1(x,y)f_2(x,y)$ at $x = y = 1$, we get the following equality $K(1,1,t) = -f_1(1,1)f_2(1,1) = -|f_1(1,1)|^2$. But this is impossible because the left-hand term $K(1,1,t) = 1 - t\sum_{i,j\in\{-1,0,1\}^2} d_{i,j} = 1-t$ is >0 whereas the right-hand term $-|f_1(1,1)|^2$ is ≤ 0.

Let us now assume that $\overline{f_1}(x,y) = -f_2(x,y)$. Equating the constant terms in the equality $f_1(x,y)f_2(x,y) = -K(x,y,t)$, we get $-|\alpha_{1,1}|^2 = d_{-1,-1}t$, so $\alpha_{1,1} = \alpha_{2,1} = d_{-1,-1} = 0$. Equating the coefficients of x^2 in the equality $f_1(x,y)f_2(x,y) = -K(x,y,t)$, we get $-|\alpha_{1,2}|^2 = d_{1,-1}t$, so $\alpha_{1,2} = \alpha_{2,2} = d_{1,-1} = 0$. It follows that the y-valuation of $f_1(x,y)f_2(x,y) = -K(x,y,t)$ is ≥ 2, whence a contradiction.

Case 2: $f_1(x,y), f_2(x,y) \in \mathbb{R}[x,y]$**.** We first claim that, after possibly replacing $f_1(x,y)$ by $-f_1(x,y)$ and $f_2(x,y)$ by $-f_2(x,y)$, we may assume that $\alpha_{1,4}, \alpha_{2,4}, \alpha_{1,3}, \alpha_{2,3} \ge 0$.

Let us first assume that $\alpha_{1,4}\alpha_{2,4} \ne 0$. Since $\alpha_{1,4}\alpha_{2,4} = d_{1,1}t \ge 0$, we find that $\alpha_{1,4}, \alpha_{2,4}$ belong simultaneously to $\mathbb{R}_{>0}$ or $\mathbb{R}_{<0}$. After possibly replacing $f_1(x,y)$ by $-f_1(x,y)$ and $f_2(x,y)$ by $-f_2(x,y)$, we may assume that $\alpha_{1,4}, \alpha_{2,4} > 0$. Since $\alpha_{1,3}\alpha_{2,3} = d_{-1,1}t \ge 0$, we have that $\alpha_{1,3}, \alpha_{2,3}$ belong simultaneously to $\mathbb{R}_{\ge 0}$ or $\mathbb{R}_{\le 0}$. Then, the equality $\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} = d_{0,1}t \ge 0$ implies that $\alpha_{1,3}, \alpha_{2,3} \ge 0$.

We can argue similarly in the case $\alpha_{1,3}\alpha_{2,3} \ne 0$.

It remains to consider the case $\alpha_{1,4}\alpha_{2,4} = \alpha_{1,3}\alpha_{2,3} = 0$. After possibly replacing $f_1(x,y)$ by $-f_1(x,y)$ and $f_2(x,y)$ by $-f_2(x,y)$, we may assume that $\alpha_{1,4}, \alpha_{2,4} \ge 0$. The case $\alpha_{1,4} = \alpha_{1,3} = 0$ is impossible because, otherwise, we would have $d_{1,1} = d_{-1,1} = d_{0,1} = 0$, which is excluded. Similarly, the case $\alpha_{2,4} = \alpha_{2,3} = 0$ is impossible. So, we are left with the cases $\alpha_{1,4} = \alpha_{2,3} = 0$ or $\alpha_{2,4} = \alpha_{1,3} = 0$. In both cases, the equality $\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} = d_{0,1}t \ge 0$ implies that $\alpha_{1,4}, \alpha_{2,4}, \alpha_{1,3}, \alpha_{2,3} \ge 0$.

Arguing as above, we see that $\alpha_{1,2}, \alpha_{2,2}, \alpha_{1,1}, \alpha_{2,1}$ all belong to $\mathbb{R}_{\ge 0}$ or $\mathbb{R}_{\le 0}$. Using the equation of the xy-coefficients, we find that $\alpha_{1,2}, \alpha_{2,2}, \alpha_{1,1}, \alpha_{2,1}$ are all in $\mathbb{R}_{\le 0}$.
Now, equating the coefficients of x^2y in the equality $f_1(x,y)f_2(x,y) = -K(x,y,t)$ we get

$\alpha_{1,4}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,4} = d_{1,0}t$. Using the fact that $\alpha_{1,4}\alpha_{2,2}, \alpha_{1,2}\alpha_{2,4} \leq 0$ and that $d_{1,0}t \geq 0$, we get $\alpha_{1,4}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,4} = d_{1,0} = 0$. Similarly, using the coefficients of y, we get $\alpha_{1,3}\alpha_{2,1} = \alpha_{1,1}\alpha_{2,3} = d_{-1,0} = 0$.
So, we have

$$\alpha_{1,4}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,4} = \alpha_{1,3}\alpha_{2,1} = \alpha_{1,1}\alpha_{2,3} = 0.$$

The fact that $K(x, y, t)$ has x- and y-degree two and x- and y-valuation 0 implies that, for any $i \in \{1, 2\}$, none of the vectors $(\alpha_{i,4}, \alpha_{i,3})$, $(\alpha_{i,2}, \alpha_{i,1})$, $(\alpha_{i,4}, \alpha_{i,2})$ and $(\alpha_{i,3}, \alpha_{i,1})$ is $(0, 0)$. Since $\alpha_{1,4}\alpha_{2,2} = 0$, we have $\alpha_{1,4} = 0$ or $\alpha_{2,2} = 0$. If $\alpha_{1,4} = 0$, from what precedes, we find

$$\alpha_{1,4} = \alpha_{2,4} = \alpha_{2,1} = \alpha_{1,1} = 0.$$

If $\alpha_{2,2} = 0$ we obtain

$$\alpha_{2,2} = \alpha_{1,2} = \alpha_{1,3} = \alpha_{2,3} = 0.$$

In the first case, the model of the walk has steps supported by $\left\{\searrow, \nwarrow\right\}$. In the second case, we find that the model of the walk has steps supported by $\left\{\swarrow, \nearrow\right\}$. This completes the proof. □

Remark 1.3. The "degenerate models of walks" are called "singular" by certain authors, *e.g.*, in [FIM99, FIM17]. Note also that, in [KR12], "singular walks" has a different meaning and refers to models of walks such that the associated Kernel defines a genus zero curve.

Remark 1.4. In [DR19, Proposition 3], the authors show that Proposition 1.2 extends *mutatis mutandis* to the case when $t \in]0, 1[$ is algebraic. Their proof relies on Proposition 1.2 and its proof.

From now on, we will only consider nondegenerate models of walks. In terms of models of walks, this only discards one dimensional problems and models of walks in the half-plane restricted to the quarter plane that are easier to study, as explained in [BMM10, Section 2.1].

2. Singularities and genus of the kernel curve

The Kernel curve E_t is the complex affine algebraic curve defined by

$$E_t = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid K(x, y, t) = 0\}.$$

We shall now consider a compactification of this curve. We let $\mathbb{P}^1(\mathbb{C})$ be the complex projective line, which is the quotient of $\mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ by the equivalence relation $\sim$ defined by

$$(x_0, x_1) \sim (x'_0, x'_1) \Leftrightarrow \exists \lambda \in \mathbb{C}^*, (x'_0, x'_1) = \lambda(x_0, x_1).$$

The equivalence class of $(x_0, x_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ is denoted by $[x_0 : x_1] \in \mathbb{P}^1(\mathbb{C})$. The map $x \mapsto [x : 1]$ embeds $\mathbb{C}$ inside $\mathbb{P}^1(\mathbb{C})$. The latter map is not surjective: its image is $\mathbb{P}^1(\mathbb{C}) \setminus \{[1 : 0]\}$; the missing point $[1 : 0]$ is usually denoted by ∞. Now, we embed E_t inside $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ via $(x, y) \mapsto ([x : 1], [y : 1])$. The kernel curve $\overline{E_t}$ is the closure of this embedding of E_t. In other words, the kernel curve $\overline{E_t}$ is the algebraic curve defined by

$$\overline{E_t} = \{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid \overline{K}(x_0, x_1, y_0, y_1, t) = 0\}$$

where $\overline{K}(x_0,x_1,y_0,y_1,t)$ is the following bihomogeneous polynomial

$$\overline{K}(x_0,x_1,y_0,y_1,t)=x_1^2y_1^2K(\frac{x_0}{x_1},\frac{y_0}{y_1},t)=x_0x_1y_0y_1-t\sum_{i,j=0}^{2}d_{i-1,j-1}x_0^ix_1^{2-i}y_0^jy_1^{2-j}. \tag{2.1}$$

We shall now study the singularities and compute the genus of $\overline{E_t}$. We recall that $P=([a:b],[c:d])\in\overline{E_t}$ is called a singularity of $\overline{E_t}$ if

$$\frac{\partial\overline{K}(a,b,c,d,t)}{\partial x_0}=\frac{\partial\overline{K}(a,b,c,d,t)}{\partial x_1}=\frac{\partial\overline{K}(a,b,c,d,t)}{\partial y_0}=\frac{\partial\overline{K}(a,b,c,d,t)}{\partial y_1}=0$$

Here, we have used the fact that $K(x,y,t)$ is irreducible in $\mathbb{C}[x,y]$, the model of walk under consideration being nondegenerate by hypothesis. Actually, two amongst the above equalities are automatically satisfied, depending on the affine chart containing P. For instance, if $b,d\neq 0$, then P is a singularity of $\overline{E_t}$ if and only if

$$\frac{\partial\overline{K}(a,b,c,d,t)}{\partial x_0}=\frac{\partial\overline{K}(a,b,c,d,t)}{\partial y_0}=0.$$

If $P=([a:b],[c:d])\in\overline{E_t}$ is not a singularity of $\overline{E_t}$, then it is called a smooth point of $\overline{E_t}$.

We also recall that $\overline{E_t}$ is called singular if it has at least one singular point. Otherwise, we say that $\overline{E_t}$ is nonsingular or smooth.

Proposition 2.1 bellow shows that the smothness of $\overline{E_t}$ is intimately related to the value of the genus of $\overline{E_t}$. We will freely use the fact that the genus $g(C)$ of any irreducible curve $C\subset\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ of bidegree (d_1,d_2) is given by

$$g(C)=1+d_1d_2-d_1-d_2-\sum_{P\in\mathrm{Sing}}\sum_{i}\frac{m_i(P)(m_i(P)-1)}{2}, \tag{2.2}$$

where $m_i(P)$ is a positive integer standing for the multiplicity of a point P[§]. This follows for instance from [Har77, Exercise 5.6, Page 231-232 and Example 3.9.2, Page 393]. We define the genus of the weighted model of walk, as the genus of its kernel curve $\overline{E_t}$.

For any $[x_0:x_1]$ and $[y_0:y_1]$ in $\mathbb{P}^1(\mathbb{C})$, we denote by $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ the discriminants of the degree two homogeneous polynomials given by $y\mapsto\overline{K}(x_0,x_1,y,t)$ and $x\mapsto\overline{K}(x,y_0,y_1,t)$ respectively, *i.e.*

$$\begin{aligned}\Delta^x_{[x_0:x_1]}=t^2\Big(&(d_{-1,0}x_1^2-\frac{1}{t}x_0x_1+d_{0,0}x_0x_1+d_{1,0}x_0^2)^2\\&-4(d_{-1,1}x_1^2+d_{0,1}x_0x_1+d_{1,1}x_0^2)(d_{-1,-1}x_1^2+d_{0,-1}x_0x_1+d_{1,-1}x_0^2)\Big)\end{aligned}$$

and

$$\begin{aligned}\Delta^y_{[y_0:y_1]}=t^2\Big(&(d_{0,-1}y_1^2-\frac{1}{t}y_0y_1+d_{0,0}y_0y_1+d_{0,1}y_0^2)^2\\&-4(d_{1,-1}y_1^2+d_{1,0}y_0y_1+d_{1,1}y_0^2)(d_{-1,-1}y_1^2+d_{-1,0}y_0y_1+d_{-1,1}y_0^2)\Big).\end{aligned}$$

[§] that is, some partial derivative of order $m_i(P)$ does not vanish while for every $\ell<m_i(P)$, the partial derivatives of order ℓ vanish at P.

Proposition 2.1. *The following facts are equivalent:*

1. *the curve $\overline{E_t}$ is a genus zero curve;*
2. *the curve $\overline{E_t}$ is singular;*
3. *the curve $\overline{E_t}$ has exactly one singularity $\Omega \in \overline{E_t}$;*
4. *there exists $([a:b],[c:d]) \in \overline{E_t}$ such that the discriminants $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ have a root $[a:b] \in \mathbb{P}^1(\mathbb{C})$ and $[c:d] \in \mathbb{P}^1(\mathbb{C})$ respectively;*
5. *there exists $([a:b],[c:d]) \in \overline{E_t}$ such that the discriminants $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ have a double root $[a:b] \in \mathbb{P}^1(\mathbb{C})$ and $[c:d] \in \mathbb{P}^1(\mathbb{C})$ respectively.*

If these properties are satisfied, then the singular point is $\Omega = ([a:b],[c:d])$ where $[a:b] \in \mathbb{P}^1(\mathbb{C})$ is a double root of $\Delta^x_{[x_0:x_1]}$ and $[c:d] \in \mathbb{P}^1(\mathbb{C})$ is a double root of $\Delta^y_{[y_0:y_1]}$. If the previous properties are not satisfied, then $\overline{E_t}$ is a smooth curve of genus one.

Proof. Since the curve $\overline{E_t}$ is of bidegree $(2,2)$ in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$, the formula (2.2) ensures that

$$g(\overline{E_t}) = 1 - \sum_{P\in \mathrm{Sing}} \sum_i \frac{m_i(P)(m_i(P)-1)}{2}, \tag{2.3}$$

and, hence, $\overline{E_t}$ is smooth if and only if $g(\overline{E_t}) = 1$. Moreover (2.3) shows that if $\overline{E_t}$ is singular, then there is exactly one singular point that is a double point, and the curve has genus zero. This proves the equivalence between (1), (2) and (3), and the last statement of the Proposition.

Let us prove (4) $\Rightarrow$ (3). Assume that the discriminant $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$) has a root in $[a:b] \in \mathbb{P}^1(\mathbb{C})$ (resp. $[c:d] \in \mathbb{P}^1(\mathbb{C})$). Let us write

$$\begin{array}{lllll} & \overline{K}(x_0,x_1,y_0,y_1,t) & & \\ = & e_{-1,1}(dy_0-cy_1)^2 & +e_{0,1}(bx_0-ax_1)(dy_0-cy_1)^2 & +e_{1,1}(bx_0-ax_1)^2(dy_0-cy_1)^2 \\ + & e_{-1,0}(dy_0-cy_1) & +e_{0,0}(bx_0-ax_1)(dy_0-cy_1) & +e_{1,0}(bx_0-ax_1)^2(dy_0-cy_1) \\ + & e_{-1,-1} & +e_{0,-1}(bx_0-ax_1) & +e_{1,-1}(bx_0-ax_1)^2. \end{array}$$

Since $([a:b],[c:d]) \in \overline{E_t}$, we have by definition that $\overline{K}(a,b,c,d,t)=0$, *i.e.* $e_{-1,-1}=0$. Since $\Delta^x_{[x_0:x_1]}$ has a root in $[a:b] \in \mathbb{P}^1(\mathbb{C})$, $K(a,b,y_0,y_1)$ has a double root at $[c,d]$ and so $e_{-1,0}=0$. Similarly, the fact that $\Delta^y_{[y_0:y_1]}$ has a root in $[c:d] \in \mathbb{P}^1(\mathbb{C})$ implies $e_{0,-1}=0$. This shows that

$$\frac{\partial\overline{K}(a,b,c,d,t)}{\partial x_0} = \frac{\partial\overline{K}(a,b,c,d,t)}{\partial x_1} = \frac{\partial\overline{K}(a,b,c,d,t)}{\partial y_0} = \frac{\partial\overline{K}(a,b,c,d,t)}{\partial y_1} = 0,$$

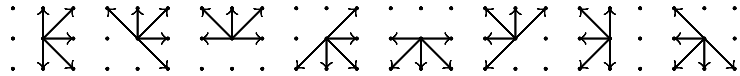
and, hence, $([a:b],[c:d])$ is the singular point of $\overline{E_t}$.

Let us prove (3) $\Rightarrow$ (5). If $\Omega = ([a:b],[c:d])$ is the singular point of $\overline{E_t}$, then $e_{-1,0}=e_{0,-1}=0$, and the discriminants $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ have a double root in $[a:b] \in \mathbb{P}^1(\mathbb{C})$ and $[c:d] \in \mathbb{P}^1(\mathbb{C})$ respectively.

The implication (5) $\Rightarrow$ (4) is obvious. ☐

Our next aim is to describe the genus zero models of walks.

Lemma 2.2. *The discriminant $\Delta^y_{[y_0:y_1]}$ has a double zero if and only if the step set of the model of the walk is supported in one of the following configurations*



Remark 2.3. See also [DR19, Proposition 9] for an extension of Lemma 2.2 to the case when $t \in]0,1[$ is algebraic. Their proof relies on the results of the present section.

Remark 2.4. In the case $t = 1$, it is proved in [FIM17, Lemma 2.3.10] that, besides the models listed in Lemma 2.2, any nondegenerate model such that the drift is zero, i.e.

$$(\sum_i i d_{i,j}, \sum_j j d_{i,j}) = (0,0),$$

has a curve $\overline{E_t}$ of genus 0.

Proof. The computations seem to be too complicated to be performed by hand, so we used MAPLE. Briefly, for a Kernel with indeterminates $d_{i,j}$, one calculates the discriminant of the discriminant $\Delta^y_{[y_0:y_1]}$. This is a polynomial of degree 12 in t with coefficients that are polynomials in the $d_{i,j}$. Since t is transcendental, the polynomial is zero if and only if its t-coefficients are all zero. We set these polynomials equal to zero and solve. This yields 8 solutions corresponding to the above configurations. Note that we may also do this computation by decomposing the radical of an ideal into its prime components.

We begin by calculating the Kernel of the model of the walk

```
>K := expand(x*y*(1-t*(sum(sum(d[i, j]*x^i*y^j, i = -1 .. 1), j = -1 .. 1))));
```

$$K := -x^2y^2td_{1,1} - x^2ytd_{1,0} - xy^2td_{0,1} - x^2td_{1,-1} - xytd_{0,0} - y^2td_{-1,1} - xtd_{0,-1} - ytd_{-1,0} - td_{-1,-1} + xy$$

The discriminant of the Kernel with respect to x is

```
>DX := expand(y[1]^4*subs(y = y[0]/y[1], discrim(K, x)));
```

$$DX := -4\,{y_1}^4t^2d_{-1,-1}d_{1,-1} - 4\,{y_1}^3t^2y_0d_{-1,-1}d_{1,0} - 4\,{y_1}^2t^2{y_0}^2d_{-1,-1}d_{1,1} - 4\,{y_1}^3t^2y_0d_{-1,0}d_{1,-1} - 4\,{y_1}^2t^2{y_0}^2d_{-1,0}d_{1,0} - 4\,y_1t^2{y_0}^3d_{-1,0}d_{1,1} - 4\,{y_1}^2t^2{y_0}^2d_{-1,1}d_{1,-1} - 4\,y_1t^2{y_0}^3d_{-1,1}d_{1,0} - 4\,t^2{y_0}^4d_{-1,1}d_{1,1} + {y_1}^4t^2{d_{0,-1}}^2 + 2\,{y_1}^3t^2y_0d_{0,-1}d_{0,0} + 2\,{y_1}^2t^2{y_0}^2d_{0,-1}d_{0,1} + {y_1}^2t^2{y_0}^2{d_{0,0}}^2 + 2\,y_1t^2{y_0}^3d_{0,0}d_{0,1} + t^2{y_0}^4{d_{0,1}}^2 - 2\,{y_1}^3ty_0d_{0,-1} - 2\,{y_1}^2t{y_0}^2d_{0,0} - 2\,y_1t{y_0}^3d_{0,1} + {y_1}^2{y_0}^2$$

We wish to determine when DX has a double root. We first assume that DX has a double root at (a,b) and that b is not zero. We can then set $y_1 = 1$ and $y_0 = y$ and calculate the discriminant of DX (we suppress the output)

```
>DD := discrim(subs(y[1] = 1, y[0] = y, DX), y);
>degree(DD, t);
```

$$12$$

Since t is transcendental over the $d_{i,j}$, DD is zero if and only if each of the coefficients of powers of t to zero. We now set these equal to zero and solve for the $d_{i,j}$.

```
>S := [solve({coeff(DD, t, 0) = 0, coeff(DD, t, 1) = 0, coeff(DD, t, 2) = 0,
coeff(DD, t, 4) = 0, coeff(DD, t, 5) = 0, coeff(DD, t, 6) = 0, coeff(DD, t, 7) = 0,
coeff(DD, t, 8) = 0, coeff(DD, t, 9) = 0, coeff(DD, t, 10) = 0, coeff(DD, t, 11) = 0,
coeff(DD, t, 12) = 0}, {d[-1, -1], d[-1, 0], d[-1, 1], d[0, -1], d[0, 0], d[0, 1],
d[1, -1], d[1, 0], d[1, 1]})];
>nops(S);
```

$$8$$

The last command indicates that there are 8 systems of equations for the $d_{i,j}$. We now list each of these $S[i]$.

```
>S[1];
```

$$\{d_{-1,-1} = 0, d_{-1,0} = 0, d_{-1,1} = 0, d_{0,-1} = d_{0,-1}, d_{0,0} = d_{0,0}, d_{0,1} = d_{0,1}, d_{1,-1} = d_{1,-1}, d_{1,0} = d_{1,0}, d_{1,1} = d_{1,1}\}$$

```
>S[2];
```

$$\{d_{-1,-1} = 0, d_{-1,0} = 0, d_{-1,1} = d_{-1,1}, d_{0,-1} = 0, d_{0,0} = d_{0,0}, d_{0,1} = d_{0,1}, d_{1,-1} = d_{1,-1}, d_{1,0} = d_{1,0}, d_{1,1} = d_{1,1}\}$$

```
>S[3];
```

$$\{d_{-1,-1} = 0, d_{-1,0} = d_{-1,0}, d_{-1,1} = d_{-1,1}, d_{0,-1} = 0, d_{0,0} = d_{0,0}, d_{0,1} = d_{0,1}, d_{1,-1} = 0, d_{1,0} = d_{1,0}, d_{1,1} = d_{1,1}\}$$

```
>S[4];
```

$$\{d_{-1,-1} = d_{-1,-1}, d_{-1,0} = 0, d_{-1,1} = 0, d_{0,-1} = d_{0,-1}, d_{0,0} = d_{0,0}, d_{0,1} = 0, d_{1,-1} = d_{1,-1}, d_{1,0} = d_{1,0}, d_{1,1} = d_{1,1}\}$$

```
>S[5];
```

$$\{d_{-1,-1} = d_{-1,-1}, d_{-1,0} = d_{-1,0}, d_{-1,1} = 0, d_{0,-1} = d_{0,-1}, d_{0,0} = d_{0,0}, d_{0,1} = 0, d_{1,-1} = d_{1,-1}, d_{1,0} = d_{1,0}, d_{1,1} = 0\}$$

```
>S[6];
```

$$\{d_{-1,-1} = d_{-1,-1}, d_{-1,0} = d_{-1,0}, d_{-1,1} = d_{-1,1}, d_{0,-1} = 0, d_{0,0} = d_{0,0}, d_{0,1} = d_{0,1}, d_{1,-1} = 0, d_{1,0} = 0, d_{1,1} = d_{1,1}\}$$

```
>S[7];
```

$$\{d_{-1,-1} = d_{-1,-1}, d_{-1,0} = d_{-1,0}, d_{-1,1} = d_{-1,1}, d_{0,-1} = d_{0,-1}, d_{0,0} = d_{0,0}, d_{0,1} = d_{0,1}, d_{1,-1} = 0, d_{1,0} = 0, d_{1,1} = 0\}$$

```
>S[8];
```

$$\{d_{-1,-1} = d_{-1,-1}, d_{-1,0} = d_{-1,0}, d_{-1,1} = d_{-1,1}, d_{0,-1} = d_{0,-1}, d_{0,0} = d_{0,0}, d_{0,1} = 0, d_{1,-1} = d_{1,-1}, d_{1,0} = 0, d_{1,1} = 0\}$$

This yields the eight step sets listed in Lemma 2.2.

An alternate approach is to use the *PolynomialIdeals* package

```
>with(PolynomialIdeals):
```

and consider the prime decomposition of the radical of the ideal

```
>J := <(coeff(DD, t, 4), coeff(DD, t, 5), coeff(DD, t, 6), coeff(DD, t, 7),
coeff(DD, t, 8), coeff(DD, t, 9), coeff(DD,t,10),coeff(DD,t,11),coeff(DD,t,12)>:
> PrimeDecomposition(J);
```

$< d_{-1,-1}, d_{-1,0}, d_{-1,1} >$, $< d_{-1,-1}, d_{-1,0]}, d_{0,-1} >$, $< d_{-1,-1}, d_{0,-1}, d_{1,-1]} >$, $< d_{-1,0}, d_{-1,1}, d_{0,1}) >$,
$< d_{-1,1}, d_{0,1}, d_{1,1} >$, $< d_{0,-1}, d_{1,-1}, d_{1,0} >$, $< d_{0,1}, d_{1,0}, d_{1,1} >$, $< d_{1,-1}, d_{1,0}, d_{1,1} >$

The *PrimeDecomposition* command lists a set of prime ideals whose intersection is the radical of the original ideal. In particular, these ideals have the property that any zero of the original ideal is a zero of one of the listed ideals and vice versa (see [CLO97, Chapter 4, Section 6]). As seen, this yields the same result as the *solve* command.

In the above calculation we assumed that DX has a double root at $[1, b]$ where b is not zero. We now consider the case where b is zero and so DX has a double root at $[1, 0]$. We will show that this case leads to models of walks already mentioned above

```
>DDX := subs(y[1] = y, y[0] = 1, DX):
```

If $y = 0$ is a double root then the coefficient of 1 and y must be zero

```
>coeff(DDX, y, 0); coeff(DDX, y, 1)
```

$$-4\,t^2 d_{-1,1} d_{1,1} + t^2 {d_{0,1}}^2$$

$$-4\,t^2 d_{-1,0} d_{1,1} - 4\,t^2 d_{-1,1} d_{1,0} + 2\,t^2 d_{0,0} d_{0,1} - 2\,t d_{0,1}$$

Taking into account that t is transcendental and the $d_{i,j}$ are algebraic, we are led to three cases

$[d_{0,1} = 0, d_{-1,1} = 0, d_{-1,0} = 0]$
$[d_{0,1} = 0, d_{-1,1} = 0, d_{1,1} = 0]$
$[d_{0,1} = 0, d_{1,1} = 0, d_{1,0} = 0]$

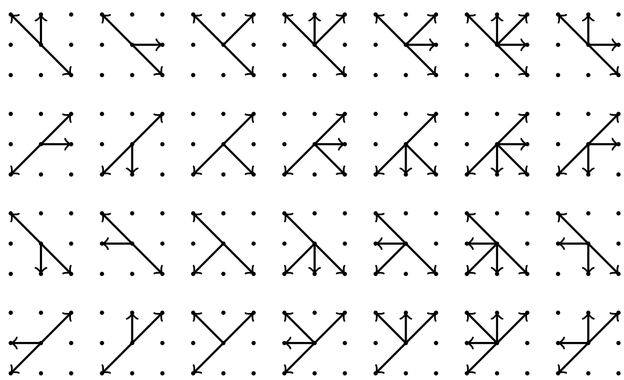
The first of these corresponds to S[4], the second to S[5], and the third to S[8]. □

Remark 2.5. The proof of Proposition 1.2 proceeds by a direct "hand calculation" while the proof of Lemma 2.2 follows from a simple MAPLE calculation. It would be interesting to have a simple MAPLE based proof of Proposition 1.2 and a hand calculation proof of Lemma 2.2.

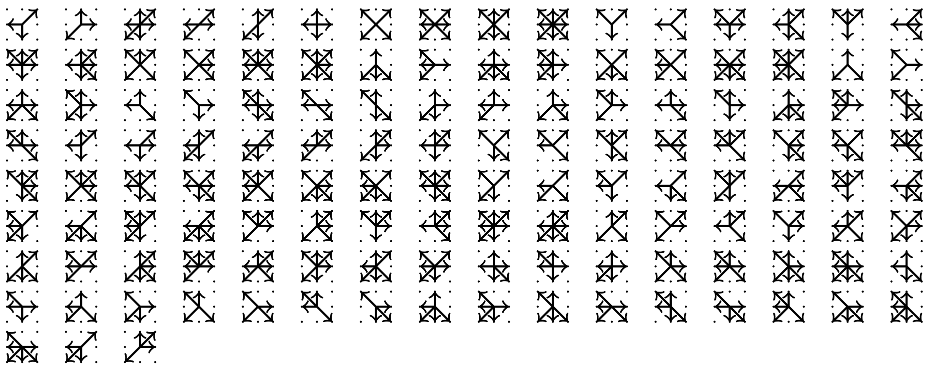
Corollary 2.6. *The following holds:*

(1) *The nondegenerate genus zero models of walks are the nondegenerate models whose step set is included in an half space. More precisely, they are the models whose step set belongs to one of the following configurations*

(G0)



(2) *The nondegenerate genus one models of walks are the models whose step set is not included in any half space. More precisely, they are the models whose the step set belongs to one of the following configurations*

(G1) 

Proof. This is a direct consequence of Proposition 1.2, Proposition 2.1, and Lemma 2.2. □

Our next aim is to give an expression for the roots of the discriminants $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ and for the singular point Ω of $\overline{E_t}$ in the genus zero case. Let us write

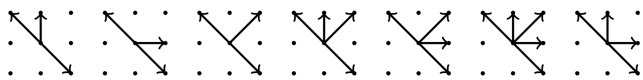
$$\Delta^x_{[x_0:x_1]} = \sum_{i=0}^{4} \alpha_i(t) x_0^i x_1^{4-i}, \quad \text{and } \Delta^y_{[y_0:y_1]} = \sum_{i=0}^{4} \beta_i(t) y_0^i y_1^{4-i},$$

where

$$\begin{array}{rcl}
\alpha_0(t) & = & d_{-1,0}^2 - 4d_{-1,1}d_{-1,-1} \\
\alpha_1(t) & = & 2d_{0,0} - 2t - 4d_{-1,1}d_{0,-1} - 4d_{0,1}d_{-1,-1} \\
\alpha_2(t) & = & d_{0,0}^2 - 2td_{0,0} + t^2 + 2d_{-1,0}d_{1,0} - 4d_{-1,1}d_{1,-1} - 4d_{0,1}d_{0,-1} - 4d_{1,1}d_{-1,-1} \\
\alpha_3(t) & = & -2td_{1,0} + 2d_{0,0}d_{1,0} - 4d_{1,1}d_{0,-1} - 4d_{0,1}d_{1,-1} \\
\alpha_4(t) & = & d_{1,0}^2 - 4d_{1,1}d_{1,-1} \\
\beta_0(t) & = & d_{0,-1}^2 - 4d_{1,-1}d_{-1,-1} \\
\beta_1(t) & = & 2d_{0,0} - 2t - 4d_{1,-1}d_{-1,0} - 4d_{1,0}d_{-1,-1} \\
\beta_2(t) & = & d_{0,0}^2 - 2td_{0,0} + t^2 + 2d_{0,-1}d_{0,1} - 4d_{1,-1}d_{-1,1} - 4d_{1,0}d_{-1,0} - 4d_{1,1}d_{-1,-1} \\
\beta_3(t) & = & -2td_{0,1} + 2d_{0,0}d_{0,1} - 4d_{1,1}d_{-1,0} - 4d_{1,0}d_{-1,1} \\
\beta_4(t) & = & d_{0,1}^2 - 4d_{1,1}d_{-1,1}.
\end{array}$$

Note that $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$) is of degree 4 and so has four roots a_1, a_2, a_3, a_4 (resp. b_1, b_2, b_3, b_4) in $\mathbb{P}^1(\mathbb{C})$ (taking into consideration multiplicities). Proposition 2.1 ensures that the discriminant $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$) has a double root; up to renumbering, we can assume that $a_1 = a_2$ (resp. $b_1 = b_2$).

Lemma 2.7. *Assume that the model of the walk has steps supported in one of the following configurations (the first line of* (G0)*)*



Then, the singular point of $\overline{E_t}$ is $\Omega = ([0:1],[0:1])$, that is, $a_1 = a_2 = [0:1]$ (resp. $b_1 = b_2 = [0:1]$) is a double root of $\Delta^x_{[x_0:x_1]}$ (resp. $\Delta^y_{[y_0:y_1]}$). The other roots are distinct and are given by

	a_3	a_4
$\alpha_4(t) \neq 0$	$\left[-\alpha_3(t) - \sqrt{\alpha_3(t)^2 - 4\alpha_2(t)\alpha_4(t)} : 2\alpha_4(t)\right]$	$\left[-\alpha_3(t) + \sqrt{\alpha_3(t)^2 - 4\alpha_2(t)\alpha_4(t)} : 2\alpha_4(t)\right]$
$\alpha_4(t) = 0$	$[1:0]$	$[-\alpha_2(t) : \alpha_3(t)]$
	b_3	b_4
$\beta_4(t) \neq 0$	$\left[-\beta_3(t) - \sqrt{\beta_3(t)^2 - 4\beta_2(t)\beta_4(t)} : 2\beta_4(t)\right]$	$\left[-\beta_3(t) + \sqrt{\beta_3(t)^2 - 4\beta_2(t)\beta_4(t)} : 2\beta_4(t)\right]$
$\beta_4(t) = 0$	$[1:0]$	$[-\beta_2(t) : \beta_3(t)]$

Remark 2.8. We can extend Lemma 2.7 to the other configurations in (G0) by using the following remarks:

(1) Replacing $([x_0:x_1],[y_0:y_1])$ by $([x_0:x_1],[y_1:y_0])$, which corresponds to the variable change $(x,y) \mapsto (x,y^{-1})$, amounts to consider a weighted model of walk with weights $d'_{i,j} := d_{i,-j}$. This can be used to extend Lemma 2.7 to the second line of (G0); for instance, the singular point of $\overline{E_t}$ is $\Omega = ([0:1],[1:0])$ in that case.
(2) Replacing $([x_0:x_1],[y_0:y_1])$ by $([x_1:x_0],[y_1:y_0])$ amounts to consider a weighted model of walk with weights $d'_{i,j} := d_{-i,-j}$. This can be used to extend Lemma 2.7 to the third line of (G0); for instance, the singular point of $\overline{E_t}$ is $\Omega = ([1:0],[1:0])$ in that case.
(3) Replacing $([x_0:x_1],[y_0:y_1])$ by $([x_1:x_0],[y_0:y_1])$ amounts to consider a weighted model of walk with weights $d'_{i,j} := d_{-i,j}$. This can be used to extend Lemma 2.7 to the fourth line of (G0); for instance, the singular point of $\overline{E_t}$ is $\Omega = ([1:0],[0:1])$ in that case.

Remark 2.9. Note that if we consider the x_3, x_4 (resp. y_3, y_4) defined in [FIM17, Chapter 6], we have the equality of sets $\{a_3, a_4\} = \{x_3, x_4\}$ and $\{b_3, b_4\} = \{y_3, y_4\}$, but do not have necessarily $a_i = x_i$, $b_j = y_j$, with $3 \leq i, j \leq 4$.

Proof of Lemma 2.7. We shall prove the lemma for $\Delta^y_{[y_0:y_1]}$, the proof for $\Delta^x_{[x_0:x_1]}$ being similar. By assumption, $d_{-1,-1} = d_{-1,0} = d_{0,-1} = 0$. Then, $\alpha_0(t) = \alpha_1(t) = 0$. Therefore, the discriminant $\Delta^y_{[y_0:y_1]}$ has a double root at $[0:1]$ and we can write

$$\Delta^y_{[y:1]} = \beta_4(t)y^4 + \beta_3(t)y^3 + \beta_2(t)y^2.$$

Since t is transcendental and the $d_{i,j}$ are in $\overline{\mathbb{Q}}$, we see that the coefficient of y^2 is nonzero. Therefore $[0:1]$ is precisely a double root of $\Delta^y_{[y_0:y_1]}$. To see that b_3 and b_4 are distinct, we calculate the discriminant of $\Delta^y_{[y:1]}/y^2$, which is almost the same as the one we considered in the proof of Lemma 2.2. This is a polynomial of degree four in t with the following coefficients:

term	coefficient
t^4	$-16(4d_{-1,1}d_{1,-1}d_{1,1} - d_{1,-1}d_{1,0}^2 - d_{0,0}^2 d_{1,1} + d_{0,0}d_{0,1}d_{1,0} - d_{0,1}^2 d_{1,-1})d_{-1,1}$
t^3	$-16(2d_{0,0}d_{1,1} - d_{0,1}d_{1,0})d_{-1,1}$
t^2	$16d_{-1,1}d_{1,1}$
t	0
1	0

If $\Delta^y_{[y_0:y_1]}$ has a double root different to $[0:1]$, all the above coefficients must be zero. From the coefficient of t^2 (recalling that $d_{-1,1}d_{1,-1} \neq 0$), we must have $d_{1,1} = 0$. From the coefficient of t^3, we have that $d_{0,1} = 0$ or $d_{1,0} = 0$. From the coefficient of t^4, we get in both cases $d_{0,1} = d_{1,0} = 0$. This implies that the model of the walk would be degenerate, a contradiction. The formulas for b_3 and b_4 follow from the quadratic formula. $\square$

3. Involutive automorphisms of the kernel curve

Following [BMM10, Section 3], [KY15, Section 3] or [FIM17], we consider the involutive rational functions[¶]

$$i_1, i_2 : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$$

given by

$$i_1(x,y) = \left(x, \frac{A_{-1}(x)}{A_1(x)y}\right) \text{ and } i_2(x,y) = \left(\frac{B_{-1}(y)}{B_1(y)x}, y\right).$$

Note that i_1, i_2 are "only" rational functions in the sense that they are *a priori* not defined when the denominators vanish. The rational functions i_1, i_2 induce involutive rational functions

$$\iota_1, \iota_2 : \overline{E_t} \dashrightarrow \overline{E_t}$$

given by

$$\begin{array}{rrcl} & \iota_1([x_0:x_1],[y_0:y_1]) & = & \left([x_0:x_1], \left[\dfrac{A_{-1}(\frac{x_0}{x_1})}{A_1(\frac{x_0}{x_1})\frac{y_0}{y_1}} : 1\right]\right), \\ \text{and} & \iota_2([x_0:x_1],[y_0:y_1]) & = & \left(\left[\dfrac{B_{-1}(\frac{y_0}{y_1})}{B_1(\frac{y_0}{y_1})\frac{x_0}{x_1}} : 1\right], [y_0:y_1]\right). \end{array}$$

Again, these functions are *a priori* not defined where the denominators vanish. However, the following result shows that, actually, this is only an "apparent problem": ι_1 and ι_2 can be extended into morphisms of $\overline{E_t}$. We recall that a rational map $f : \overline{E_t} \dashrightarrow \overline{E_t}$ is a morphism if it is regular at any $P \in \overline{E_t}$, *i.e.* if f can be represented in suitable affine charts containing P and $f(P)$ by a rational function with nonvanishing denominator at P.

Proposition 3.1. *The rational maps $\iota_1, \iota_2 : \overline{E_t} \dashrightarrow \overline{E_t}$ can be extended into involutive automorphisms of $\overline{E_t}$.*

[¶]In what follows, we use the classical dashed arrow notion to denote rational maps; *a priori*, such functions may not be defined everywhere.

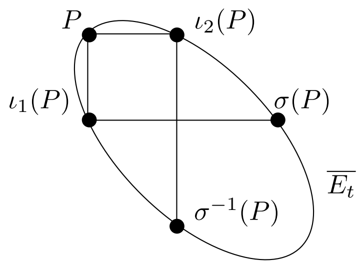


FIGURE 1. The maps ι_1, ι_2 restricted to the kernel curve $\overline{E_t}$

Proof. We have to prove that $\iota_1, \iota_2 : \overline{E_t} \dashrightarrow \overline{E_t}$ can be extended into endomorphisms of $\overline{E_t}$. According to Proposition 2.1, if the curve $\overline{E_t}$ has genus one, then it is smooth and the result follows from [Har77, Proposition 6.8, p. 43].

It remains to study the case when $\overline{E_t}$ has genus zero. In that case, Proposition 2.1 ensures that $\overline{E_t}$ has a unique singularity Ω. It follows from [Har77, Proposition 6.8, p. 43] that ι_1 and ι_2 can be uniquely extended into morphisms $\overline{E_t} \setminus \{\Omega\} \to \overline{E_t}$ still denoted by ι_1 and ι_2. It remains to study ι_1 and ι_2 at Ω. Let us first assume that the walk under consideration belongs to the first line of (G0). Lemma 2.7 ensures that $\Omega = ([0:1],[0:1])$. For $([x:1],[y:1]) \in \overline{E_t}$, the equation $K(x,y,t)=0$ ensures that

$$\frac{A_{-1}(x)}{A_1(x)y} = \frac{1}{tA_1(x)} - \frac{A_0(x)}{A_1(x)} - y = \frac{x}{t\widetilde{A}_1(x)} - \frac{\widetilde{A}_0(x)}{\widetilde{A}_1(x)} - y \tag{3.1}$$

where $\widetilde{A}_0(x) = xA_0(x) = d_{-1,0} + d_{0,0}x + d_{1,0}x^2$ and $\widetilde{A}_1(x) = xA_1(x) = d_{-1,1} + d_{0,1}x + d_{1,1}x^2$. Since $d_{-1,1} \neq 0$, $\widetilde{A}_1(x)$ does not vanish at $x=0$. So, (3.1) shows that ι_1 is regular at Ω and that $\iota_1(\Omega) = \Omega$. The argument for ι_2 is similar.

The other cases listed in (G0) can be treated similarly using Remark 2.8. □

We also consider the automorphism of $\overline{E_t}$ defined by

$$\sigma = \iota_2 \circ \iota_1.$$

It is easily seen that ι_1 and ι_2 are the vertical and horizontal switches of $\overline{E_t}$ (see Figure 1), *i.e.*, for any $P=(x,y) \in \overline{E_t}$, we have

$$\{P, \iota_1(P)\} = \overline{E_t} \cap (\{x\} \times \mathbb{P}^1(\mathbb{C})) \text{ and } \{P, \iota_2(P)\} = \overline{E_t} \cap (\mathbb{P}^1(\mathbb{C}) \times \{y\}).$$

We now give a couple of lemmas for later use.

Lemma 3.2. *A point $P = ([x_0:x_1],[y_0:y_1]) \in \overline{E_t}$ is fixed by ι_1 (resp. ι_2) if and only if $\Delta^x_{[x_0:x_1]} = 0$ (resp. $\Delta^y_{[y_0:y_1]} = 0$).*

Proof. Assume that P is fixed by ι_1. Then, the polynomial $[y_0:y_1] \mapsto \overline{K}(x_0,x_1,y_0,y_1,t)$ has a double root, meaning that the discriminant is zero. This is exactly $\Delta^x_{[x_0:x_1]} = 0$. Conversely, $\Delta^x_{[x_0:x_1]} = 0$ implies that $[y_0:y_1] \mapsto \overline{K}(x_0,x_1,y_0,y_1,t)$ has a double root and therefore P is fixed by ι_1. The proof for ι_2 is similar. □

Lemma 3.3. *Let $P \in \overline{E_t}$. The following statements are equivalent:*

1. *P is fixed by ι_1 and ι_2;*
2. *P is a singular point of $\overline{E_t}$;*
3. *P is fixed by $\sigma = \iota_2 \circ \iota_1$.*

Proof. Let $P = ([a : b], [c : d]) \in \overline{E_t}$. From Proposition 2.1, we have that P is a singular point if and only if $\Delta^x_{[x_0:x_1]}$ and $\Delta^y_{[y_0:y_1]}$ vanish at $[a : b]$ and $[c : d]$ respectively. We conclude with Lemma 3.2, that (1) is equivalent to (2).

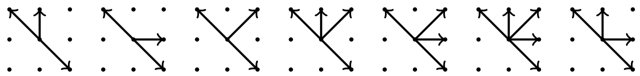
Clearly, (1) implies (3). It remains to prove that (3) implies (1). Assume that $P = (a_1, b_1)$ is fixed by σ. Since $\iota_1(P) = (a_1, b_1')$ and $\iota_2(\iota_1(P)) = (a_1', b_1')$, it is clear that $\sigma(P) = P$ implies successively $\iota_1(P) = P$ and $\iota_2(P) = P$. □

4. Parameterization of the kernel curve

We still consider a weighted model of nondegenerate walk. The aim of this section is to give an explicit parametrization of $\overline{E_t}$. Thanks to Proposition 2.1, the latter may have genus zero or one. Let us start with the genus zero case.

4.1. Genus zero case.

Let us consider a nondegenerate weighted model of walks of genus zero. Thank to Corollary 2.6 combined with Remark 2.8, it suffices to consider the situation where the model of walks arises from the following set of steps



Genus zero curves may be parameterized with maps $\phi : \mathbb{P}^1(\mathbb{C}) \to \overline{E_t}$ which are bijective outside a finite set. The aim of this subsection, achieved with Proposition 4.6, is to find such a parametrization explicitly. Although we could have just written down the formula for this parametrization and verified its properties, we have preferred to explain how the formula arises. This requires a preliminary study of σ, ι_1 and ι_2 (and, more precisely, of the automorphism of $\mathbb{P}^1(\mathbb{C})$ obtained by pulling back these maps by ϕ), which is done with a series of lemmas preceding Proposition 4.6.

According to Lemma 2.7, $\overline{E_t}$ has a unique singular point $\Omega = (a_1, b_1) = ([0 : 1], [0 : 1])$. Moreover $\Delta^x_{[x_0:x_1]}$ has degree four with a double root at $a_1 = [0 : 1]$ and the remaining two roots a_3, a_4 are distinct. We let $S_3 = (a_3, *)$ and $S_4 = (a_4, *)$ be the points of $\overline{E_t}$ with first coordinates a_3 and a_4 respectively. Similarly, $\Delta^y_{[y_0:y_1]}$ has degree four with a double root at $b_1 = [0 : 1]$ and the remaining two roots b_3, b_4 are distinct. We let $S_3' = (*, b_3)$ and $S_4' = (*, b_4)$ be the points of $\overline{E_t}$ with second coordinates b_3 and b_4 respectively.

Since $\overline{E_t}$ has genus zero, there is a rational parameterization of $\overline{E_t}$ [Ful89, Page 198, Ex.1], *i.e.* there exists a birational map

$$\phi = (x, y) : \mathbb{P}^1(\mathbb{C}) \dashrightarrow \overline{E_t}.$$

This ϕ is actually a surjective morphism of curves (as is any nonconstant rational map from $\mathbb{P}^1$ to a projective curve, see [Ful89, Corollary 1, Page 160]). More precisely, since Ω is the unique singular point of $\overline{E_t}$, ϕ induces a bijection between $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$ and $\overline{E_t} \setminus \{\Omega\}$. The maps $x, y : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ are surjective morphisms of curves as well.

We let $s_3, s_4 \in \mathbb{P}^1(\mathbb{C})$ (resp. $s_3', s_4' \in \mathbb{P}^1(\mathbb{C})$) be such that $S_3 = \phi(s_3)$ and $S_4 = \phi(s_4)$ (resp. $S_3' = \phi(s_3')$ and $S_4' = \phi(s_4')$).

We will need the cardinality of $x^{-1}(P)$ (resp. $y^{-1}(P)$) for $P \in \mathbb{P}^1(\mathbb{C})$. This quantity might depend on P but it is a general fact about morphisms of curves that the cardinality of $x^{-1}(P)$ (resp. $y^{-1}(P)$) is constant for P outside a finite subset of $\mathbb{P}^1(\mathbb{C})$. This common value is called the degree of x (resp. y).

Lemma 4.1. *The morphisms $x, y : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ have degree two.*

Proof. This is a consequence of the fact that $\overline{E_t}$ is a biquadratic curve. Indeed, let us consider $V = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1\}$. Note that the preimage by ϕ of any element of $\overline{E_t}$ of the form $(a, *)$ with $a \in V$ has one element (simply because ϕ induces a bijection between $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$ and $\overline{E_t} \setminus \{\Omega\}$). Let U be the set of $a \in \mathbb{P}^1(\mathbb{C})$ such that the intersection of $\{a\} \times \mathbb{P}^1(\mathbb{C})$ with $\overline{E_t}$ has exactly two elements. This is also the set of $a \in \mathbb{P}^1(\mathbb{C})$ such that $\Delta^x_a \neq 0$ and, hence, $U = \mathbb{P}^1(\mathbb{C}) \setminus \mathcal{S}$ for some finite set $\mathcal{S}$. Then, for any $a \in U \cap V$, $x^{-1}(a)$ has exactly two elements (indeed, we have $x^{-1}(a) = \phi^{-1}((\{a\} \times \mathbb{P}^1(\mathbb{C})) \cap \overline{E_t})$, moreover the fact that a belongs to U ensures that $(\{a\} \times \mathbb{P}^1(\mathbb{C})) \cap \overline{E_t}$ has two elements and the fact that a belongs to V ensures that $\phi^{-1}((\{a\} \times \mathbb{P}^1(\mathbb{C})) \cap \overline{E_t})$ has two elements as well). So, x has degree two. The argument for y is similar. □

We will now follow the ideas contained in [FIM17] to produce an explicit "automorphic parameterization" of $\overline{E_t}$.

The involutive automorphisms ι_1, ι_2 of $\overline{E_t}$ induce involutive automorphisms $\tilde{\iota}_1, \tilde{\iota}_2$ of $\mathbb{P}^1(\mathbb{C})$ *via* ϕ. Similarly, σ induces an automorphism $\tilde{\sigma}$ of $\mathbb{P}^1(\mathbb{C})$. In other words, we have the commutative diagrams

$$\begin{array}{ccc} \overline{E_t} & \xrightarrow{\iota_k} & \overline{E_t} \\ \phi \uparrow & & \uparrow \phi \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow[\tilde{\iota}_k]{} & \mathbb{P}^1(\mathbb{C}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \overline{E_t} & \xrightarrow{\sigma} & \overline{E_t} \\ \phi \uparrow & & \uparrow \phi \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow[\tilde{\sigma}]{} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

Note that since ϕ induces a bijection between $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$ and $\overline{E_t} \setminus \{\Omega\}$, the group generated by ι_1 and ι_2 is isomorphic to the group generated by $\tilde{\iota}_1$ and $\tilde{\iota}_2$. Thus we recover the same group as in [BMM10] for instance. We summarize some remarks in the following lemmas.

Lemma 4.2. *We have $x = x \circ \tilde{\iota}_1$ and $y = y \circ \tilde{\iota}_2$.*

Proof. We obtain $x = x \circ \tilde{\iota}_1$ by equating the first coordinates in the equality $\phi \circ \tilde{\iota}_1 = \iota_1 \circ \phi = (x, *)$ and we obtain $y = y \circ \tilde{\iota}_2$ by equating the second coordinates in the equality $\phi \circ \tilde{\iota}_2 = \iota_2 \circ \phi = (*, y)$. □

Lemma 4.3. *Let $P = \phi(s) \in \overline{E_t}$ and let $k \in \{1, 2\}$. We have:*

- *if $\tilde{\iota}_k(s) = s$, then $\iota_k(P) = P$;*
- *if $P \neq \Omega$ and $\iota_k(P) = P$, then $\tilde{\iota}_k(s) = s$.*

Proof. We have $\iota_k(P) = \iota_k(\phi(s)) = \phi(\tilde{\iota}_k(s))$. The first assertion is now clear, and the second one follows from the fact that ϕ is injective on $\overline{E_t} \setminus \phi^{-1}(\Omega)$. □

Lemma 4.4. *The premimage of Ω by ϕ has two elements: $\phi^{-1}(\Omega) = \{s_1, s_2\}$ with $s_1 \neq s_2$.*

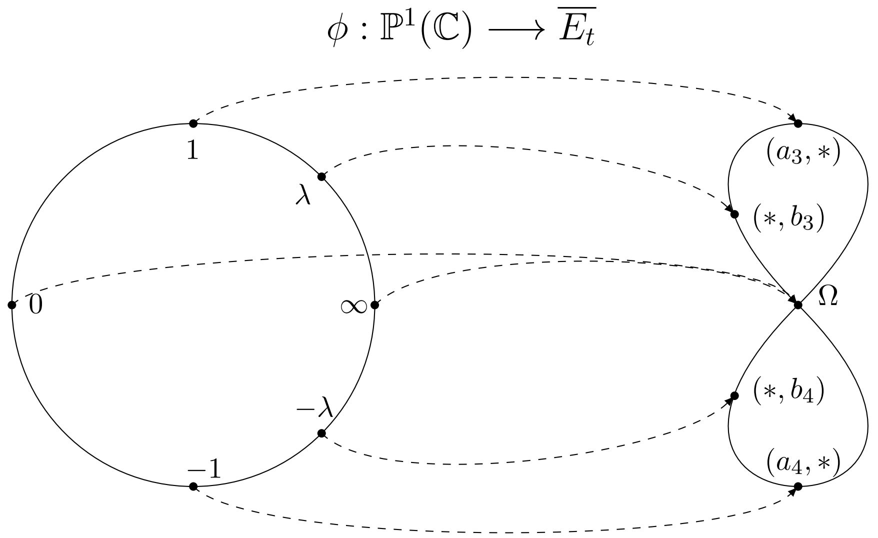


FIGURE 2. The uniformization map

Proof. We know that $x, y : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ have degree two, so $\phi^{-1}(\Omega)$ has one or two elements. Suppose to the contrary that $\phi^{-1}(\Omega) = \{s_1\}$ has 1 element. Since $\phi(\tilde{\iota}_1(s_1)) = \iota_1(\phi(s_1)) = \iota_1(\Omega) = \Omega$, we have $\tilde{\iota}_1(s_1) = s_1$. Moreover, since $S_3, S_4 \neq \Omega$ are fixed by ι_1, Lemma 4.3 ensures that s_3 and s_4 are fixed by $\tilde{\iota}_1$. Therefore, $\tilde{\iota}_1$ is an automorphism of $\mathbb{P}^1(\mathbb{C})$, *i.e.* an homography, with at least three fixed points, so $\tilde{\iota}_1$ is the identity. This is a contradiction. □

Lemma 4.5. *The map $\tilde{\iota}_1$ (resp. $\tilde{\iota}_2$) has exactly two fixed points, namely s_3 and s_4 (resp. s'_3 and s'_4), and interchanges s_1 and s_2. The map $\tilde{\sigma}$ has exactly two distinct fixed points, s_1 and s_2.*

Proof. Let $s \in \mathbb{P}^1(\mathbb{C})$ be a fixed point of $\tilde{\iota}_1$. Lemma 4.3 ensures that $\phi(s)$ is fixed by ι_1. So, $\phi(s) = \Omega$, S_3 or S_4. If $\phi(s) \neq \Omega$, then $s = s_3$ or s_4 (recall that ϕ induces a bijection between $\mathbb{P}^1(\mathbb{C}) \setminus \phi^{-1}(\Omega)$ and $\overline{E_t} \setminus \{\Omega\}$) and s_3 and s_4 are indeed fixed by $\tilde{\iota}_1$. Moreover, we have $\phi(s) = \Omega$ if and only if $s = s_1$ or s_2 and the equality $\iota_1(\phi(s)) = \phi(\tilde{\iota}_1(s))$ shows that $\tilde{\iota}_1$ induces a permutation of $\phi^{-1}(\Omega) = \{s_1, s_2\}$. If s_1 and s_2 were fixed by $\tilde{\iota}_1$, then $\tilde{\iota}_1$ would be an automorphism of $\mathbb{P}^1(\mathbb{C})$, *i.e.* an homography, with at least four fixed points (s_1, s_2, s_3, s_4) and, hence, would be the identity. This is a contradiction. So, $\tilde{\iota}_1$ interchanges s_1 and s_2.

The proof for $\tilde{\iota}_2$ is similar.

As any homography which is not the identity, $\tilde{\sigma}$ has at most two fixed points in $\mathbb{P}^1(\mathbb{C})$. It only remains to prove that s_1 and s_2 are fixed by $\tilde{\sigma}$, and this is indeed the case because $\tilde{\sigma} = \tilde{\iota}_2 \circ \tilde{\iota}_1$ and $\tilde{\iota}_1, \tilde{\iota}_2$ interchange s_1 and s_2. □

We are now ready to give an explicit expression of ϕ. Let us recall that $\alpha_2(t), \alpha_3(t), \alpha_4(t), \beta_2(t), \beta_3(t), \beta_4(t)$ are the coefficients of the discriminants given by

$$\begin{array}{lcl} \alpha_2(t) & = & 1 - 2td_{0,0} + t^2 d_{0,0}^2 - 4t^2 d_{-1,1} d_{1,-1} \\ \alpha_3(t) & = & 2t^2 d_{1,0} d_{0,0} - 2td_{1,0} - 4t^2 d_{0,1} d_{1,-1} \\ \alpha_4(t) & = & t^2(d_{1,0}^2 - 4d_{1,1} d_{1,-1}) \\ \beta_2(t) & = & 1 - 2td_{0,0} + t^2 d_{0,0}^2 - 4t^2 d_{1,-1} d_{-1,1} \\ \beta_3(t) & = & 2t^2 d_{0,1} d_{0,0} - 2td_{0,1} - 4t^2 d_{1,0} d_{-1,1} \\ \beta_4(t) & = & t^2(d_{0,1}^2 - 4d_{1,1} d_{-1,1}). \end{array}$$

Proposition 4.6. *An explicit parameterization $\phi : \mathbb{P}^1(\mathbb{C}) \to \overline{E_t}$ such that*

$$\tilde{\iota}_1(s) = \frac{1}{s},\ \tilde{\iota}_2(s) = \frac{\lambda^2}{s} = \frac{q}{s} \text{ and } \widetilde{\sigma}(s) = qs$$

for a certain $\lambda \in \mathbb{C}^*$ is given by

$$\phi(s) = \left(\frac{4\alpha_2(t)}{\sqrt{\alpha_3(t)^2 - 4\alpha_2(t)\alpha_4(t)}(s + \frac{1}{s}) - 2\alpha_3(t)}, \frac{4\beta_2(t)}{\sqrt{\beta_3(t)^2 - 4\beta_2(t)\beta_4(t)}(\frac{s}{\lambda} + \frac{\lambda}{s}) - 2\beta_3(t)} \right).$$

Moreover, we have, see Figure 2

$$\begin{array}{lll} x(0) = x(\infty) = a_1, & x(1) = a_3, & x(-1) = a_4, \\ y(0) = y(\infty) = b_1, & y(\lambda) = b_3, & y(-\lambda) = b_4. \end{array}$$

Remark 4.7. When $t = 1$, we recover the uniformization of [FIM17, Section 6.4.3].

Proof of Proposition 4.6. According to Lemma 4.5, $\tilde{\iota}_1$ is an involutive homography with fixed points s_3 and s_4, so there exists an homography h such that $h(s_3) = 1$, $h(s_4) = -1$ and $h \circ \tilde{\iota}_1 \circ h^{-1}(s) = 1/s$. Up to replacing ϕ by $\phi \circ h$, we can assume that $s_3 = 1$, $s_4 = -1$ and $\tilde{\iota}_1(s) = \frac{1}{s}$. Since $s_1 \neq s_2$, we can assume up to renumbering that $s_1 \neq \infty$. Moreover, up to replacing ϕ by $\phi \circ k$ where k is the homography given by $k(s) = \frac{s - s_1}{-s_1 s + 1}$, we can also assume that $s_1 = 0$ and $s_2 = \infty$ (note that k commutes with $\tilde{\iota}_1$, so changing ϕ by $\phi \circ k$ does not affect $\tilde{\iota}_1$). Lemma 4.1 and Lemma 4.2 ensure that the morphism $x : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ has degree two and satisfies $x(s) = x(1/s)$ for all $s \in \mathbb{P}^1(\mathbb{C})$. It follows that

$$x(s) = \frac{a(s + 1/s) + b}{c(s + 1/s) + d}$$

for some $a, b, c, d \in \mathbb{C}$. We have $x(s_1) = x(0) = a_1 = 0$, $x(s_2) = x(\infty) = a_1 = 0$, $x(s_3) = x(1) = a_3$ and $x(s_4) = x(-1) = a_4$. The equality $x(\infty) = 0$ implies $a = 0$. The equalities $x(1) = a_3$ and $x(-1) = a_4$ imply

$$x(s) = \frac{4a_3 a_4}{(a_4 - a_3)(s + \frac{1}{s}) + 2(a_3 + a_4)}.$$

The known expressions for a_3 and a_4 given in Lemma 2.7 lead to the expected expression for $x(s)$.

According to Lemma 4.5, $\tilde{\iota}_2$ is an homography interchanging 0 and ∞, so $\tilde{\iota}_2(s) = \frac{\lambda^2}{s}$ for some $\lambda \in \mathbb{C}^*$. Up to renumbering, we have $s_3' = \lambda$ and $s_4' = -\lambda$. Using the fact that the morphism $y : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ has degree two and is invariant by $\tilde{\imath}_2$, and arguing as we did above for x, we see that there exist $\alpha, \beta, \gamma, \eta \in \mathbb{C}$ such that

$$y(s) = \frac{\alpha(\frac{s}{\lambda} + \frac{\lambda}{s}) + \beta}{\gamma(\frac{s}{\lambda} + \frac{\lambda}{s}) + \eta}.$$

The equality $y(\infty) = 0$ implies $\alpha = 0$. Using the equalities $y(s'_3) = y(\lambda) = b_3$ and $y(s'_4) = y(-\lambda) = b_4$, and arguing as we did above for x, we obtain the expected expression for $y(s)$. □

Remark 4.8. (1) The uniformization is not unique. More precisely, the possible uniformizations are of the form $\phi \circ h$, where h is an homography. However, if one requires that h fixes setwise $0, \infty$ then q is uniquely defined up to its inverse.
(2) The real q or q^{-1} specializes for $t = 1$ to the real ρ^2 in [FIM17, Page 178].

The following proposition determines q up to its inverse.

Proposition 4.9 ([DHRS20], Proposition 1.7, Corollary 1.10)**.** *One of the two complex numbers q or q^{-1} is equal to*

$$\frac{-1 + d_{0,0}t - \sqrt{(1 - d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}{-1 + d_{0,0}t + \sqrt{(1 - d_{0,0}t)^2 - 4d_{1,-1}d_{-1,1}t^2}}.$$

Furthermore, $q \in \mathbb{R} \setminus \{\pm 1\}$.

Remark 4.10. This implies that σ and $\tilde{\sigma}$ have infinite order (see also [BMM10, FR11]). It follows that the group of the walk introduced in [BMM10], which is by definition the group generated by i_1 and i_2, has infinite order (because σ is induced on $\overline{E_t}$ by $i_1 \circ i_2$, so if σ has infinite order then $i_1 \circ i_2$ has infinite order as well). Note that this was proved in [BMM10] using a valuation argument.

4.2. **Genus one case.** In this section, we give an overview of [DR19]. Let us consider a nondegenerate model of walk of genus one. This corresponds to one of the configurations listed in (G1). By Proposition 2.1, $\overline{E_t}$ is a smooth curve of genus one and, hence, it is biholomorphic to $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ for some lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ of $\mathbb{C}$ *via* some $(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$-periodic holomorphic map

$$\begin{array}{rccc} \Lambda : & \mathbb{C} & \to & \overline{E}_t \\ & \omega & \mapsto & (\mathfrak{q}_1(\omega), \mathfrak{q}_2(\omega)), \end{array} \tag{4.1}$$

where $\mathfrak{q}_1, \mathfrak{q}_2$ are rational functions of $\wp$ and its derivative $d\wp/d\omega$, and $\wp$ is the Weierstrass function associated with the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$:

$$\wp(\omega) = \wp(\omega; \omega_1, \omega_2) := \frac{1}{\omega^2} + \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(\omega + \ell_1\omega_1 + \ell_2\omega_2)^2} - \frac{1}{(\ell_1\omega_1 + \ell_2\omega_2)^2} \right). \tag{4.2}$$

Then, the field of meromorphic functions on $\overline{E_t}$ is isomorphic to the field of meromorphic functions on $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$, which is itself isomorphic to the field of meromorphic functions on $\mathbb{C}$ that are (ω_1, ω_2)-periodic. The latter field is equal to $\mathbb{C}(\wp, \wp')$ (see [WW96]).

The maps ι_1, ι_2 and σ may be lifted to the ω-plane $\mathbb{C}$. We denote these lifts by $\tilde{\iota}_1$, $\tilde{\iota}_2$ and $\widetilde{\sigma}$ respectively. So we have the commutative diagrams

$$\begin{array}{ccc} \overline{E_t} & \xrightarrow{\iota_k} & \overline{E_t} \\ \Lambda \uparrow & & \uparrow \Lambda \\ \mathbb{C} & \xrightarrow[\tilde{\iota}_k]{} & \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} \overline{E_t} & \xrightarrow{\sigma} & \overline{E_t} \\ \Lambda \uparrow & & \uparrow \Lambda \\ \mathbb{C} & \xrightarrow[\widetilde{\sigma}]{} & \mathbb{C} \end{array}$$

The following result has been proved in [FIM17, Section 3.3] when $t = 1$, in [Ras12] in the unweighted case, and in [DR19, Proposition 18] in the weighted case, with general $0 < t < 1$, not necessarily transcendental. In what follows, we set $D(x) = \Delta^x_{[x:1]}$. Let us introduce $z = 2A(x)y + B(x)$, where $A(x) = t(d_{-1,1} + d_{0,1}x + d_{1,1}x^2)$, and $B(x) = t(d_{-1,0} - \frac{1}{t}x + d_{0,0}x + d_{1,0}x^2)$.

Proposition 4.11. *An explicit uniformization $\Lambda : \mathbb{C} \to \overline{E_t}$ such that*

$$\tilde{\iota}_1(\omega) = -\omega, \quad \tilde{\iota}_2(\omega) = -\omega + \omega_3 \quad \textit{and} \quad \widetilde{\sigma}(\omega) = \omega + \omega_3,$$

for a certain $\omega_3 \in \mathbb{C}^*$ is given by

$$\Lambda(\omega) = (x(\omega), y(\omega))$$

where $x(\omega)$ and $y(\omega)$ are given by

	$x(\omega)$	$z(\omega)$
$a_4 \neq [1:0]$	$\left[a_4 + \frac{D'(a_4)}{\wp(\omega) - \frac{1}{6}D''(a_4)} : 1\right]$	$\left[\frac{D'(a_4)\wp'(\omega)}{2(\wp(\omega) - \frac{1}{6}D''(a_4))^2} : 1\right]$
$a_4 = [1:0]$	$[\wp(\omega) - \alpha_2/3 : \alpha_3]$	$[-\wp'(\omega) : 2\alpha_3]$

A suitable choice for the periods (ω_1, ω_2) is given by the elliptic integrals

$$\omega_1 = \mathbf{i}\int_{a_3}^{a_4} \frac{dx}{\sqrt{|D(x)|}} \in \mathbf{i}\mathbb{R}_{>0} \quad \textit{and} \quad \omega_2 = \int_{a_4}^{a_1} \frac{dx}{\sqrt{D(x)}} \in \mathbb{R}_{>0}. \tag{4.3}$$

Note that, according to [DR19, Section 2],

$$\omega_3 = \int_{a_4}^{X_\pm(b_4)} \frac{dx}{\sqrt{D(x)}} \in]0, \omega_2[.$$

Remark 4.12. Contrary to the genus zero situation, the map σ may have finite order.

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